Lezioni 7 & 8

Varietà pseudo-Riemanniana: (M, g) $\quad M$ varietà

g è un TENSORE METRICO, cioè $g \in \mathcal{T}^0_2(M)$ $\quad (0,2)$

cioè $\forall p \quad g(p) : T_pM \times T_pM \to \mathbb{R}$ t.c.

PRODOTTO SCALARE
$$\begin{cases} 1) \text{ è simmetrico} & g(p)(v,w) = g(p)(w,v) \\ 2) \text{ è non degenere } \forall p & \Rightarrow \text{Ha una SEGNATURA } (p,m) \text{ t.c. } p+m=n \end{cases}$$

Base ortonormale per V^n con (p,m)

$\mathcal{B} = \{v_1, \dots, v_n\}$ t.c. $\langle v_i, v_j \rangle = \pm\delta_{ij}$

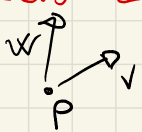
con $\langle v_i, v_i \rangle = -1$ per $1 \le i \le m$
$\qquad\qquad\qquad\;\; = +1$ " $m+1 \le i \le n$

Riemanniana: $(p,m) = (n,0)$

Lorentziana: $(p,m) = (n-1,1)$

Spazio Euclideo: $(\mathbb{R}^n, g^E)$ $\quad g(p) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$

$$(x,y) \mapsto \sum_{i=1}^n x_i y_i = {}^t x \cdot y$$



$g_{ij} = \delta_{ij} \qquad g(p)(x,y) = g_{ij} x^i y^j = {}^t x \cdot S \cdot y$

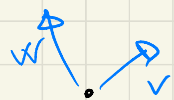
Spazio di Minkowski: $(\mathbb{R}^n, \eta)$

$g_{ij} = \eta_{ij}$

$$g = \begin{pmatrix} +1 & & \\ & \ddots & \\ & & +1 \end{pmatrix} \qquad \eta = \begin{pmatrix} -1 & & & \\ & +1 & & \\ & & \ddots & \\ & & & +1 \end{pmatrix}$$

$\eta_{00} = -1$ $0, \dots, n-1$

Ci interessa $n=4$ $\mathbb{R}^{1,3} = (\mathbb{R}^4, \eta)$ $\eta_{00} = -1$



$$\eta = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

Vettori tangenti di tipo tempo, luce e spazio:

$p \in M$ Lorentziana

T_pM $\eta(p)$ segnatura $(n-1, 1)$

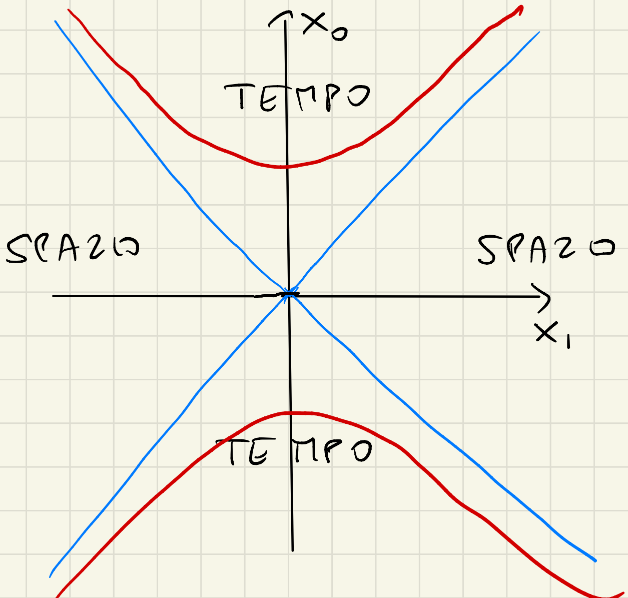
$v \in T_pM$ $v \neq 0$

$\boxed{c=1}$

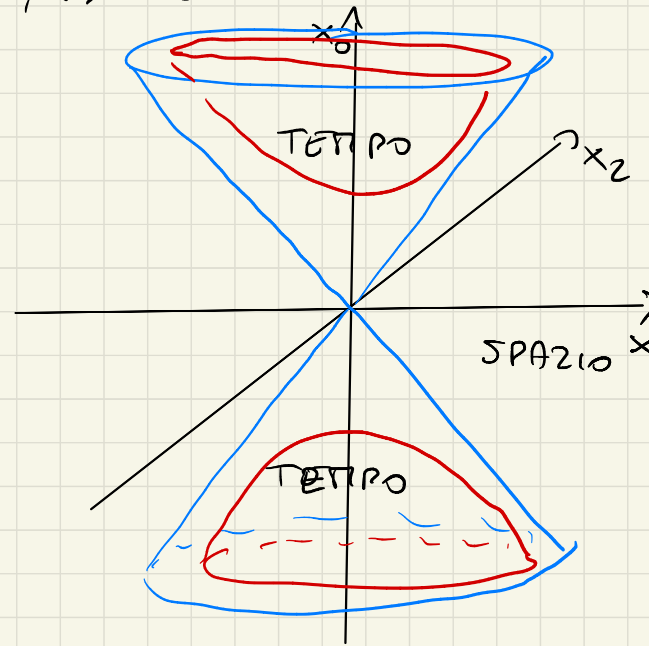
v è di tipo TEMPO se $\langle v, v \rangle < 0$

SPAZIO se $\langle v, v \rangle > 0$

LUCE se $\langle v, v \rangle = 0$



$x_0 = t, x_1, \dots, x_{n-1}$

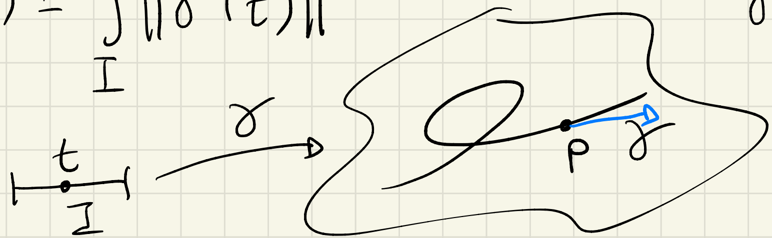


$$-x_0^2 + x_1^2 + \dots + x_{n-1}^2 = 0$$

Curve e loro lunghezza:

(M, g) pR $\qquad \gamma : I \to M$ curva $\qquad I \subseteq \mathbb{R}$

$$L(\gamma) = \int_I \|\gamma'(t)\|$$



M

$$\gamma'(t) = d\gamma_t(1) \in T_pM$$

$$d\gamma_t : T_t I \to T_p M \qquad T_tI = \mathbb{R} \qquad p = \gamma(t)$$

$$v \in T_pM \qquad \|v\| = \sqrt{|\langle v, v \rangle|}$$

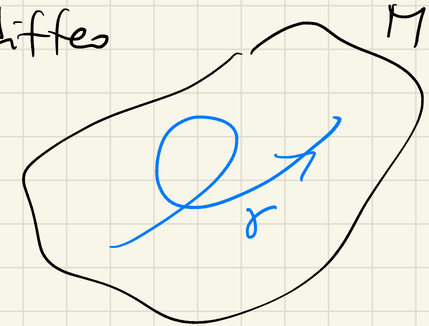
$$\langle v, w \rangle = g(p)(v, w)$$

γ è di tipo SPAZIO / LUCE / TEMPO se $\gamma'(t)$ è di tipo SPAZIO / LUCE / TEMPO

γ è di tipo luce $\iff L(\gamma) = 0$

Riparametrizzazione: $\gamma : I \to M$ $\quad \psi : J \xrightarrow{\sim} I$ diffeo

$\gamma \dashrightarrow \gamma \circ \psi : J \to M$



Supporto di γ è $\gamma(I)$

Supponiamo sempre: $\gamma'(t) \neq 0 \ \forall t$ (REGOLARE)

e luce spazio o temp

Prop: $L(\gamma)$ non dipende dalla parametrizzazione

Parametrizzazione per lunghezza d'arco (P.L.A.):

Se γ è tempo o spazio (no LUCE!) $\exists$ parametrizzazione canonica t.c. $\|\gamma'(t)\| = 1 \ \forall t \in I$

dim: $I = (a,b)$ $\quad \gamma : I \to M$ $\quad \gamma(t) \in M \quad t \in (a,b)$

$$s(t) = \int_a^t \|\gamma'(t)\| \, dt \qquad t(s) \qquad J = s(I)$$

$$\gamma : J \to M$$

Linea di universo: (M, g) Lorentziana $\left(\begin{array}{c}\text{orientata} \\ \text{time-orientata}\end{array}\right)$

$\gamma : I \to M$ tipo tempo orientata positivamente nel tempo

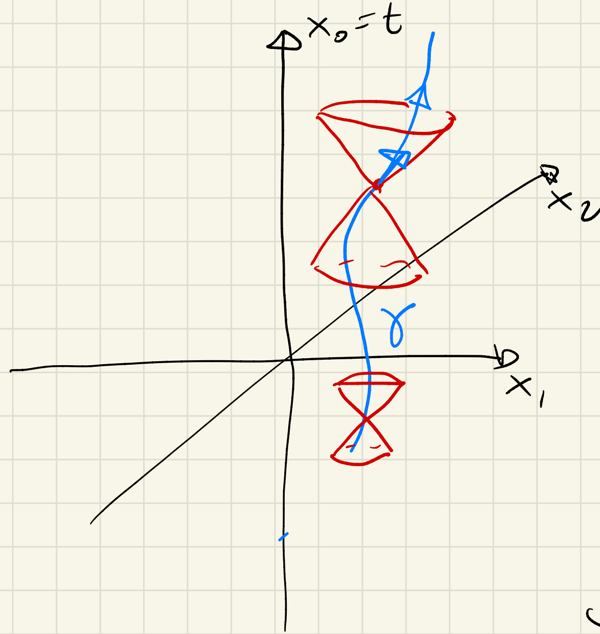
__Def__: (M, g) Lorentziana:

1) M come varietà può essere orientabile oppure no (ori su T_pM $\forall p \in M$ loc. coerente)

2) M è TIME-ORIENTED se $\forall p \in M$ è fissata una TIME ORIENTATION su T_pM loc. coerente

$T_pM = V \supseteq \{\text{tempo}\}$ ha 2 c.c. Ne scelgo una e la chiamo FUTURO

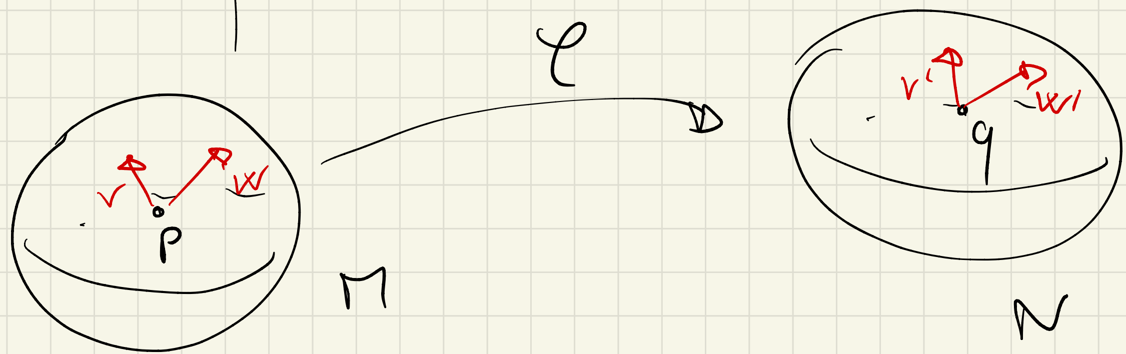
Es: $\mathbb{R}^{1,3}$ è orientato e time-orientato



EVENTI

Linea di universo

P.L.A.



Isometria: (M,g) (N,h) $\varphi: M \to N$ Diffeom è una ISOMETRIA se

$\forall p \in M$ $d\varphi_p : T_pM \xrightarrow{\sim} T_{\varphi(p)}N$ isometria cioè

$\forall v,w \in T_pM$ $\langle v,w \rangle = \langle d\varphi_p(v), d\varphi_p(w) \rangle$

Trasformazioni di Lorentz:

$(\mathbb{R}^n, g^M = \eta_{ij})$

$A : {}^tAJA = J$

$$J = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

$${}^tv J w = {}^t(Av) J (Aw) \quad \forall v,w$$

$$= {}^tv\, {}^tAJA\, w$$

Isometrie vettoriali euclidee:

$(\mathbb{R}^n, g^E = \delta_{ij})$

A matrice ortogonale: ${}^tA \cdot A = I$

$\varphi(x) = A \cdot x$ è una isometria

$$d\varphi_p : T_p\mathbb{R}^2 \to T_p\mathbb{R}^2$$

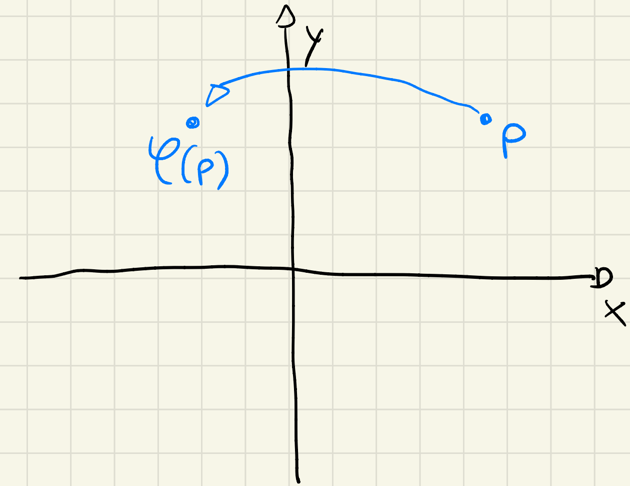
$$\mathbb{R}^2 \xrightarrow{A} \mathbb{R}^2$$

$$O(n-1,1) = \{ A : {}^tA\, J A = J \}$$

Trasformazioni di Lorentz

$$\varphi : \mathbb{R}^{1,n-1} \to \mathbb{R}^{1,n-1}$$
$$x \longmapsto A \cdot x$$

è un'isometria



$$A = \begin{pmatrix} \cos\vartheta & -\operatorname{sen}\vartheta \\ \operatorname{sen}\vartheta & \cos\vartheta \end{pmatrix} = \mathrm{Rot}_\vartheta$$

$$O(n) = \{ A \text{ ortogonali } n \times n \}$$

<u>$n=3$</u>:

$$J = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

<u>Esempi</u>:

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\vartheta & -\operatorname{sen}\vartheta & 0 \\ 0 & \operatorname{sen}\vartheta & \cos\vartheta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in O(3,1)$$

$$A = \begin{pmatrix} \cosh t & -\operatorname{senh} t & & \\ \operatorname{senh} t & \cosh t & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

Lorentz boost

$\hat{L}$ $\quad {}^tAJA = J \quad (Ex)$

Gruppo di Poincaré:

$A \in O(n-1,1)$ $\qquad b \in \mathbb{R}^n$

$\varphi(x) = A \cdot x + b$

Isometrie affini euclidee:

A ortogonale

$\varphi(x) = A \cdot x + b \qquad b \in \mathbb{R}^n$

$$\left\{ \varphi(x) = Ax + b \;\middle|\; A \in O(3,1),\ b \in \mathbb{R}^4 \right\}$$

Teo: $\qquad$ Poincaré $= \mathrm{Isom}(\mathbb{R}^{1,3})$

$\{\text{isom. affini}\} = \mathrm{Isom}(\mathbb{R}^n)$

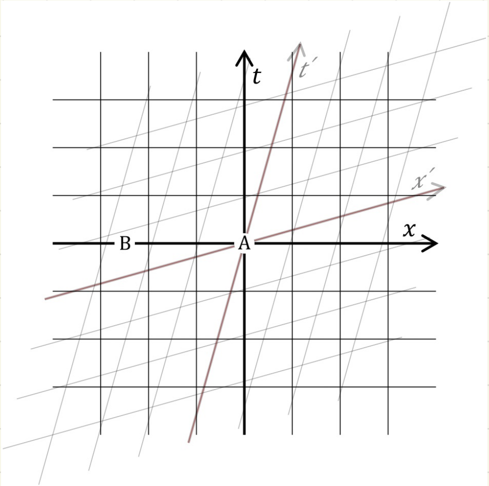
Def: (M,g) pR

$\mathrm{Isom}(M) = \{\varphi : M \to M \text{ isometrie}\}$

è un gruppo

No simultaneità

$\varphi : \mathbb{R}^{3,1} \to \mathbb{R}^{3,1}$

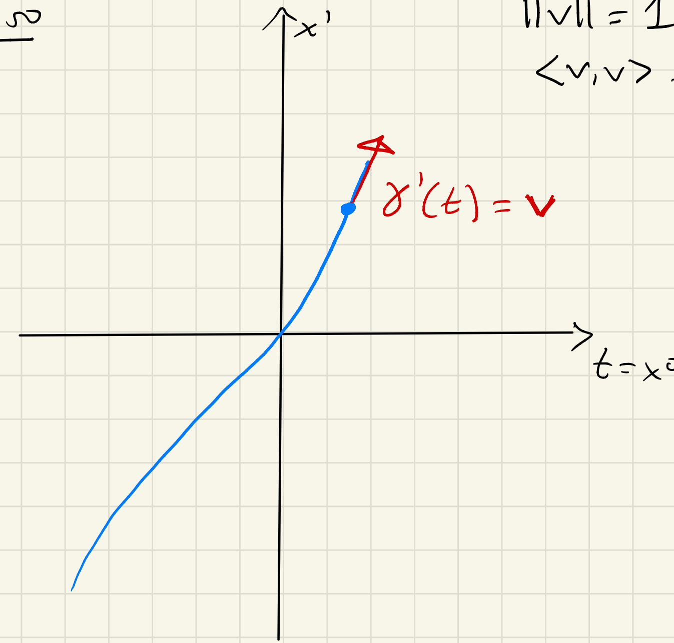


$\mathbb{R}^{1,1}$ $\mathbb{R}^{3,1}$

Quadimpulso

$\|v\| = 1$

$\langle v, v \rangle = -1$



$$\mathbf{p} = m\mathbf{v}$$

$$\mathbf{p} = (\underset{\text{ENERGIA}}{E}, \underbrace{p_x, p_y, p_z}_{\text{IMPULSO}})$$

$$\|\mathbf{p}\| = m = E^2 - p^2$$

$$p = \sqrt{p_x^2 + p_y^2 + p_z^2}$$

$$E = \sqrt{m^2 + p^2} = \sqrt{m^2 + m^2 v^2} = m\sqrt{1 + v^2} = m + \frac{1}{2} m v^2 + \dots$$

Base Lorentziana per T_pM

v_0, v_1, v_2, v_3 ortonormale con $\langle v_0, v_0 \rangle = -1$

$$\underset{i<j}{dx_i \wedge dx_j} = \begin{array}{cc} & j \\ i & \begin{pmatrix} \ddots & & 1 \\ & \ddots & \\ -1 & & \ddots \end{pmatrix} \end{array}$$

Elettromagnetismo

Tensore elettromagnetico $\mathbf{F} \in \Omega^2(\mathbb{M})$ $\qquad \mathbb{M} = (\mathbb{R}^4, \eta)$

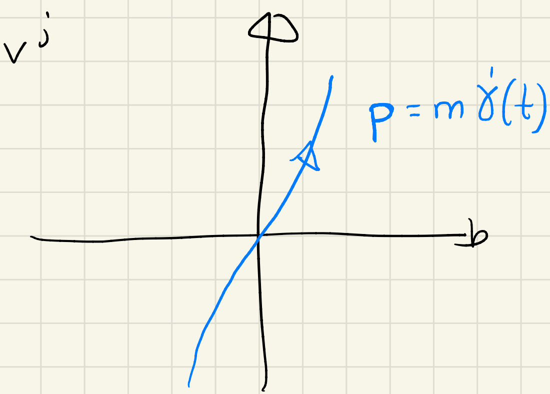
Minkowski

$$\mathbf{F} = \underset{F_{ij}}{\begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{pmatrix}}$$

$t = x_0, x_1, x_2, x_3$

$$\mathbf{F} = -E_1\, dt \wedge dx_1 - E_2\, dt \wedge dx_2 - E_3\, dt \wedge dx_3 + B_3\, dx_1 \wedge dx_2 + B_2\, dx_3 \wedge dx_1 + B_1\, dx_2 \wedge dx_3$$

Legge di Lorentz: $\dfrac{d\mathbf{P}}{dt} = q\mathbf{F}(\mathbf{v}) = q F^i{}_j v^j$



$$F^i{}_j = F_{kj}\,\eta^{ki}$$

$\mathbf{P} = m\mathbf{v} \qquad \mathbf{v} = \gamma'(t)$

$$F^i{}_j = F_{kj}\,\eta^{ki}$$

$$\eta = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

g_{ij} $\quad g^{ij}$ è l'inversa di g_{ij}

$$= F_{ij} \quad \text{se } i>0$$

$$-F_{ij} \quad \text{se } i=0$$

cioè

$$g_{ij}\, g^{jk} = \delta_i{}^k$$

$$g^{ij}\, g_{jk} = \delta^i{}_k$$

$$F^i{}_j = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{pmatrix}$$

$$\frac{dP}{dt} = q\,F^i{}_j v^j = q\,\mathbf{F}(v)$$

$$\frac{dE}{dt} = q\,F^0{}_j v^j = q\,\mathbf{E}\cdot v$$

$$m\frac{dv_1}{dt} = mq\,F^1{}_j v^j = mq\left(\frac{E}{m}\cdot E_1 + B_3 v^2 - B_2 v^3\right)$$

Ex: È equivalente a:

$$P = \left(E, \; m\mathbf{u} \right)$$
$$\mathbf{v} = \left(\frac{E}{m}, \mathbf{u} \right)$$

$$\begin{cases} \dfrac{d\mathbf{u}}{dt} = \dfrac{q}{m} \left(\frac{E}{m} \mathbf{E} + \mathbf{u} \times \mathbf{B} \right) \\ \dfrac{dE}{dt} = q \mathbf{E} \cdot \mathbf{u} \end{cases}$$

Equazioni di Maxwell

$$dF = 0$$

Ex: È equivalente a:

$$\begin{cases} \text{rot } \mathbf{E} = - \dfrac{\partial \mathbf{B}}{\partial t} \\ \text{div } \mathbf{B} = 0 \end{cases}$$

$$F = -E_1\, dt\wedge dx_1 - E_2\, dt\wedge dx_2 - E_3\, dt \wedge dx_3$$
$$+ B_3\, dx_1 \wedge dx_2 + B_2\, dx_3 \wedge dx_1 + B_1\, dx_2 \wedge dx_3$$

$$dF = -\frac{\partial E_1}{\partial x_2} dx_2 \wedge dt \wedge dx_1 - \frac{\partial E}{\partial x_3} dx_3 \wedge dt \wedge dx_1$$
$$-\frac{\partial E_2}{\partial x_1} dx_1 \wedge dt \wedge dx_2 - \frac{\partial E_2}{\partial x_3} dx_3 \wedge dt \wedge dx_2$$
$$-\frac{\partial E_3}{\partial x_1} dx_1 \wedge dt \wedge dx_3 - \frac{\partial E_3}{\partial x_2} dx_2 \wedge dt \wedge dx_3$$
$$+\frac{\partial B_3}{\partial t} dt \wedge dx_1 \wedge dx_2 + \frac{\partial B_3}{\partial x_3} dx_3 \wedge dx_1 \wedge dx_2$$
$$+\frac{\partial B_2}{\partial t} dt \wedge dx_3 \wedge dx_1 + \frac{\partial B}{\partial x_2} dx_2 \wedge dx_3 \wedge dx_1$$

$$+ \frac{\partial B_1}{\partial t} dt \wedge dx_2 \wedge dx_3 + \frac{\partial B_1}{\partial x_1} dx_1 \wedge dx_2 \wedge dx_3 = 0$$

$$\left(-\frac{\partial E_1}{\partial x_2} + \frac{\partial E_2}{\partial x_1} + \frac{\partial B_t}{\partial t} \right) \quad dt \wedge dx_1 \wedge dx_2 \quad +$$

$$- \quad - \quad - \qquad dt \wedge dx_1 \wedge dx_3 \quad +$$

$$- \quad - \quad - \qquad dt \wedge dx_2 \wedge d_3 \quad +$$

$$- \quad - \qquad dx_1 \wedge dx_2 \wedge d_3$$

Hodge $*$

V spazio vett dim n g prodotto scalare con segnatura (p,m) $\quad p+m=n$

- Induce prodotto scalare su $T^h_k(V)$

g_{ij} g .... g g^{ij}

$T : V \xrightarrow{\sim} V^*$

$v \longmapsto (w \longmapsto g(v,w))$

$v_1, \ldots, v_n$ per V

$v^1, \ldots, v^n$ per V^*

$v_{i_1} \otimes \ldots \otimes v_{i_h} \otimes v^{j_1} \otimes \ldots \otimes v^{j_k}$ per $T^h_k(V)$

$v^*, w^* \in V^*$ $\qquad g(v^*, w^*) = g(v,w)$

$v = T^{-1}(v^*)$

$w = T^{-1}(w^*)$

In generale, $\quad T^a{}_{bc} \qquad U^i{}_{jk}$

$$g(T,U) = T^a{}_{bc}\, U^i{}_{jk}\, g_{ai}\, g^{bj} g^{ck}$$

Ex: g^{ij} è inverso di g_{ij}

$$C(T \otimes U \otimes g \otimes \bar{g} \otimes \bar{g})$$

Ex: $\langle v_1 \otimes \dots \otimes v_h \otimes v^1 \otimes \dots \otimes v^k, w_1 \otimes \dots \otimes w_h \otimes w^1 \otimes \dots \otimes w^k \rangle$

$$= \prod_i \langle v_i, w_i \rangle \prod_j \langle v^j, w^j \rangle$$

Cor: $\{ v_{i_1} \otimes \dots \otimes v_{i_h} \otimes v^{j_1} \otimes \dots \otimes v^{j_k} \}$ base ortonormale se $v_1, \dots, v_n$ lo è

Cor: $\langle v^1 \wedge \dots \wedge v^k, w^1 \wedge \dots \wedge w^k \rangle = k! \det \langle v^i, w^j \rangle$

$\Lambda^k(V) \subseteq T_0^k(V)$

Def: $\langle \alpha, \beta \rangle^{new} := \dfrac{1}{k!} \langle \alpha, \beta \rangle^{old}$

Cor: $\{ v^{i_1} \wedge \dots \wedge v^{i_k} \}_{i_1 < \dots < i_k}$ ortonormale se $v_1, \dots, v_n$ lo è

Es: $\mathbb{R}^n$ Eucl. oppure $\mathbb{R}^{1,3}$ $\to$ $\{ dx_{i_1} \wedge \dots \wedge dx_{i_k} \}$ base ortonormale

$V, g,$ orientazione $\longrightarrow \omega \in \Lambda^n(V) \cong \mathbb{R}$

$\omega(v_1, \ldots, v_n) = 1$ <u>Ex</u>: È ben definita.

ortonormale
positiva

<u>dim</u>:

$\Rightarrow \omega(v_1', \ldots, v_n'$ orton. positiva$) = 1$

<u>Cor</u>: (M, g) v.R. M orientata $\Rightarrow \omega \in \Omega^n(M)$

forma volume

$\mathbb{R}^n$ euclideo: $\quad *(dx^1 \wedge \ldots \wedge dx^k) = dx^{k+1} \wedge \ldots \wedge dx^n$

$\mathbb{R}^{1,3}$ $\quad *dt = -dx^1 \wedge dx^2 \wedge dx^3$

$$*: \Lambda^k(V) \to \Lambda^{n-k}(V)$$
$$\beta \longmapsto *\beta$$

ω generatore canonico di $\Lambda^n(V)$

$$\alpha \wedge (*\beta) = \langle \alpha, \beta \rangle \omega \quad \forall \alpha \in \Lambda^k(V)$$

<u>Teo</u>: È ben definita $*\beta$

<u>Ex</u>: $v^1, \dots, v^n$ ortonormale positiva

- $*(v^1 \wedge \dots \wedge v^k) = (-1)^{m'} v^{k+1} \wedge \dots \wedge v^n$

  m': # el. negativi in $v^1, \dots, v^k$

- $**\beta = (-1)^{k(n-k)+m} \beta$

M varietà pseudo-Riemanniana:

$$*: \Omega^k(M) \to \Omega^{n-k}(M)$$

Codifferenziale:

$$\delta: \Omega^k(M) \to \Omega^{k-1}(M)$$

$$\delta = (-1)^k * d *$$

Ex: $\delta^2 = 0$

$$* d \cancel{*} \cancel{*} d *$$

$$= \cancel{*} d d \cancel{*} = 0$$

$$*F = \begin{pmatrix} 0 & B_1 & B_2 & B_3 \\ -B_1 & 0 & E_3 & -E_2 \\ -B_2 & -E_3 & 0 & E_1 \\ -B_3 & E_2 & -E_1 & 0 \end{pmatrix}$$

Hodge-* in Minkowski space:

$$\begin{aligned}
*1 &= dx \wedge dy \wedge dz \wedge dt \\
*dx &= dy \wedge dz \wedge dt \\
*dy &= dz \wedge dx \wedge dt \\
*dz &= dx \wedge dy \wedge dt \\
*dt &= dx \wedge dy \wedge dz \\
*(dx \wedge dy) &= dz \wedge dt \\
*(dz \wedge dx) &= dy \wedge dt \\
*(dy \wedge dz) &= dx \wedge dt \\
*(dx \wedge dt) &= -dy \wedge dz \\
*(dy \wedge dt) &= -dz \wedge dx \\
*(dz \wedge dt) &= -dx \wedge dy \\
*(dx \wedge dy \wedge dz) &= dt \\
*(dx \wedge dy \wedge dt) &= dz \\
*(dz \wedge dx \wedge dt) &= dy \\
*(dy \wedge dz \wedge dt) &= dx \\
*(dx \wedge dy \wedge dz \wedge dt) &= -1
\end{aligned}$$

Quadricorrente: $J = (\underbrace{\rho}_{\text{DENSITÀ DI CARICA}}, \underbrace{J_x, J_y, J_z}_{\text{DENSITÀ DI CORRENTE}})$ campo vettoriale in $\mathbb{R}^{1,3}$

$$J_i = J^j \eta_{ij} \quad \rightsquigarrow \quad J = -\rho\, dt + J_x\, dx + J_y\, dy + J_z\, dz$$

$$*J = \rho\, dx \wedge dy \wedge dz - J_x\, dt \wedge dy \wedge dz - J_y\, dt \wedge dz \wedge dx - J_z\, dt \wedge dx \wedge dy$$

Maxwell: $\boxed{\delta F = J}$ $\qquad d(*F) = *J$

$$\Downarrow$$

$$\underbrace{dd(*F)}_{=0} = d(*J)$$

Ex: È equivalente a:

$J = (\rho, j)$

$$\begin{cases} \mathrm{rot}\, \mathbf{B} = j + \dfrac{\partial \mathbf{E}}{\partial t} \\ \mathrm{div}\, \mathbf{E} = \rho \end{cases}$$

$F \in \Omega^2(M)$ $\qquad$ (M,g) Lorentziana $(3,1)$

Invarianti di F $\qquad$ Esempio: $\mathcal{C}^\infty(M)$

$$\langle F, F \rangle = \frac{1}{2} F_{ij} F_{ke} \, g^{ik} g^{je}$$

Caso Minkowski: $g^{ij} = \eta^{ij}$ $\qquad$ "$\langle F, F \rangle = \frac{1}{2} \sum_{i,j=1}^{4} F_{ij}^2$"

$$F = \begin{pmatrix} 0 & \pm E_1 & \pm E_2 & \pm E_3 \\ & 0 & \pm B_2 & \pm B_3 \\ & & 0 & \pm B_1 \\ & & & 0 \end{pmatrix}$$

$$(Ex) = B_1^2 + B_2^2 + B_3^2 - E_1^2 - E_2^2 - E_3^2$$

$$= \boxed{B^2 - E^2}$$

Ex: $\langle F, *F \rangle = 2\,\mathbf{E} \cdot \mathbf{B}$ $\qquad$ $\mathbf{E} = (E_1, E_2, E_3)$

Conseguenze:

- Eq. continuità

$$*J = \rho\, dx \wedge dy \wedge dz - J_x\, dt \wedge dy \wedge dz - J_y\, dt \wedge dz \wedge dx$$
$$- J_z\, dt \wedge dx \wedge dy$$

$$0 = d(*J) = \underbrace{\left(\frac{\partial \rho}{\partial t} + \frac{\partial J_x}{\partial x} + \frac{\partial J_y}{\partial y} + \frac{\partial J_z}{\partial z} \right)}\, dt \wedge dx \wedge dy \wedge dz$$

$$\frac{\partial \rho}{\partial t} + \operatorname{div} j = 0$$

Stokes: $D \subseteq M$ Lorentziana

dominio compatto

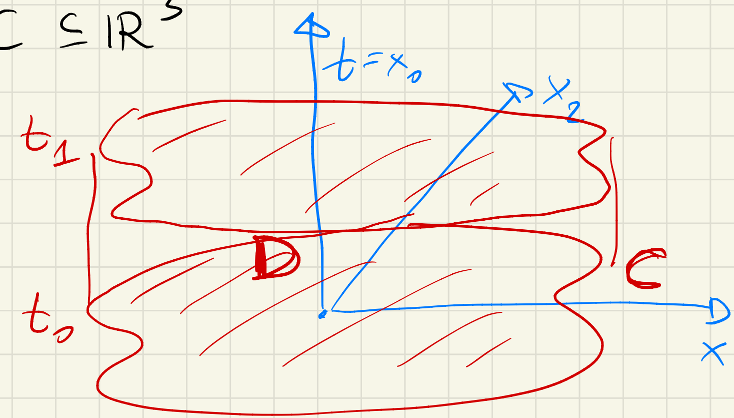
$$\int_{\partial D} *J = \int_D d(*J) = 0$$

Es:

$$D = C \times [t_0, t_1] \qquad M = \mathbb{R}^{3,1} = \mathbb{M}$$

Minkowski

$$C \subseteq \mathbb{R}^3$$



$$\int_{\partial D} *J = 0$$

$$\| $$

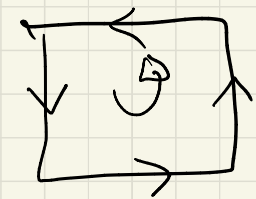
$$\int_{\partial D^{or}} *J + \int_{\partial D^{vert}} *J$$

$$\partial D^{or} = C \times \{t_0\} \sqcup C \times \{t_1\}$$

$$= \overbrace{\int_{C \times \{t_0\}} *J}^{A} + \overbrace{\int_{C \times \{t_1\}} *J}^{B} + \overbrace{\int_{\partial D^{vert}} *J}^{C}$$

$$*J = \rho \, dx_1 \wedge dx_2 \wedge d_3 - J_1 \, dt \wedge dx_2 \wedge dx_3 - J_2 \, dt \wedge dx_3 \wedge dx_1 - J_3 \, dt \wedge dx_1 \wedge dx_2$$

$$A = \int_{C \times \{t_0\}} \star J = \int_{C \times \{t_0\}}$$



Def: M^n varietà $\quad \omega \in \Omega^k(M)$

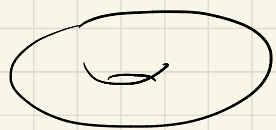
ω è **CHIUSA** se $d\omega = 0 \in \Omega^{k+1}(M)$

ω è **ESATTA** se $\exists\, \eta \in \Omega^{k-1}(M)$ t.c. $\omega = d\eta$

$d \circ d = 0 \Rightarrow$ Se ω è esatta, allora è chiusa

$$\omega = d\eta \Rightarrow d\omega = d(d\eta) = 0$$

Esempi: M^n varietà cpt senza bordo orientata



ω forma volume su M

$(\omega \in \Omega^n(M)$ t.c.

$\omega(p)(v_1, \dots, v_n) > 0$

$\forall p \in M, \ \forall v_1, \dots, v_n \in T_pM$ base $+$

Teo: Qualsiasi M ha varie metriche Riemanniane

$\Downarrow$

forma volume (se M orientata)

- Se ω forma volume:

$d\omega \in \Omega^{n+1}(M)$ $\quad n+1>n \quad \Rightarrow d\omega = 0$

$\{0\}$

$\Downarrow$

ω è chiusa

- ω non è MAI esatta:

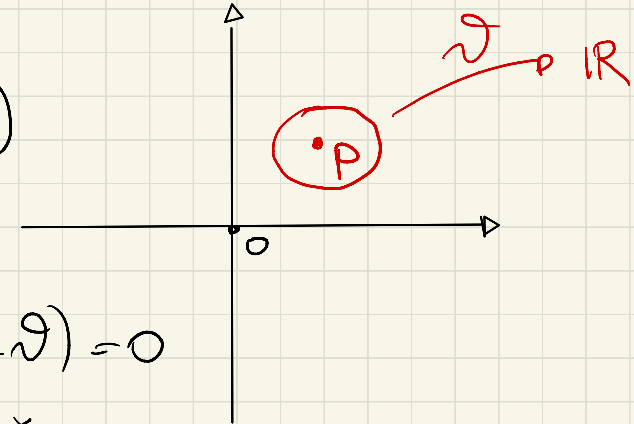
p. assurdo $\omega = d\eta \quad \Rightarrow \int_M d\eta = \int_{\partial M = \emptyset} \eta = 0$

$\Rightarrow \int_M \omega = 0$ $\quad$ però $\int_M \omega = \mathrm{Vol}(M) > 0$

In generale $\forall S \subseteq M \quad \mathrm{Vol}(S) := \int_M \omega$

Ex: Se S dominio $\mathrm{Vol}(S) > 0$

Es: $M = \mathbb{R}^2 \setminus \{0\}$



(r, ϑ) $\qquad \omega = d\vartheta \in \Omega^1(\mathbb{R}^2 \setminus \{0\})$

è loc. esatta

$\Downarrow$

chiusa $\qquad d\omega = d(d\vartheta) = 0$

Altro modo: 1) $d\vartheta = \dfrac{x\,dy - y\,dx}{x^2+y^2} = \omega$

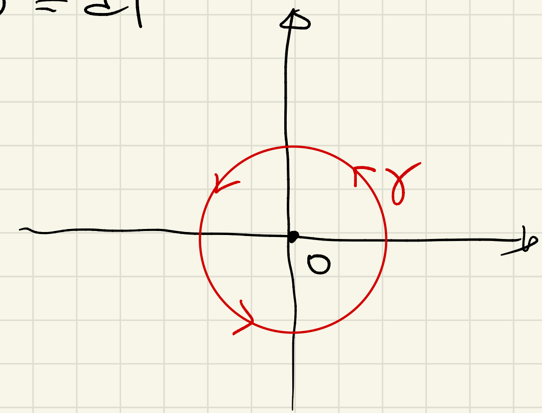
2) $\qquad$ è chiusa (verifica)

$$\omega = \frac{x}{x^2+y^2}\,dy - \frac{y}{x^2+y^2}\,dx$$

$$d\omega = \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) dx \wedge dy - \frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right) dy \wedge dx$$

(es.)

ω <u>non</u> è esatta. Cioè $\nexists\, f \in C^\infty(\mathbb{R}^2 \setminus \{0\})$ t.c. $\omega = df$



Infatti, se $\exists f$

<u>Stokes</u>: $\displaystyle\int_\gamma df = \int_{\partial\gamma = \emptyset} f = 0$

$$\int_\gamma \omega = 0$$

Assurdo: $\displaystyle\int_\gamma \omega = \int_\gamma d\vartheta \underset{(es)}{=} 2\pi$

Lemma di Poincaré: In $\mathbb{R}^n$ ogni k-forma chiusa con $k \geq 1$ è anche esatta

Vale per ogni M CONTRATTILE

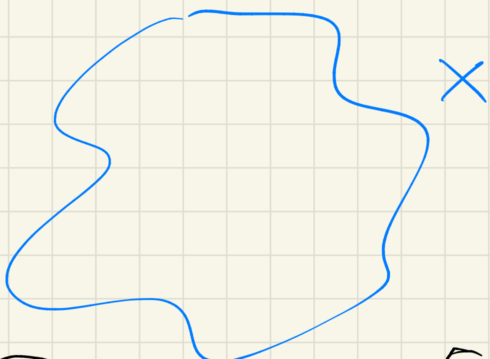
Def: X sp. top. è contrattile se

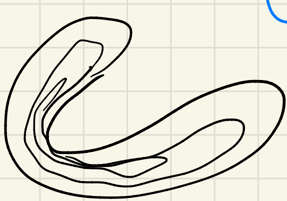
$\exists$ una omotopia fra $id : X \to X$

$f_0 = f : X \to X$ costante

$f_1 = f(x) = x_0 \in X$

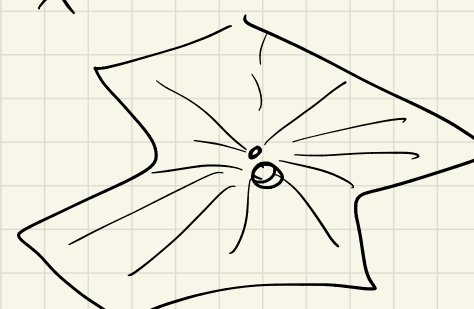
$f_t : X \to X$





Es: $X \subseteq \mathbb{R}^n$ stellato

$f_t(v) = (1-t) \cdot v$



COOMOLOGIA DI DE RHAM

M^n varietà

Def: $\Omega^k(M) \supseteq Z^k(M) = \{k\text{-forme chiuse}\}$

dim ∞ dim ∞

$ \cup\!|$

$ B^k(M) = \{k\text{-forme esatte}\}$

spesso dim ∞

Es: sono sottospazi vettoriale

$$H^k(M) = Z^k(M) \big/ B^k(M)$$

$W \subseteq V$

spazio quoz. V/W

Il k-esimo gruppo di coomologia di De Rham

$b^k(M) = \dim H^k(M)$ numeri di Betti

"numero di buchi k-dimensionali in M"

Esempi: $\odot$ $k=0$

$$H^0(M) = Z^0(M) / B^0(M) = \{0\text{-forme chiuse}\} / \{0\text{-forme esatte}\}$$

$$= \{f \in \mathcal{C}^\infty(M) : df = 0\} / \{0\} = \{f \in \mathcal{C}^\infty(M) : df = 0\}$$

$$= \{f \in \mathcal{C}^\infty(M) \text{ LOC. COSTANTI}\} = \mathbb{R}^{c.c.(M)}$$

Def: $f : M \to \mathbb{R}$ è LOC. COSTANTE se $\forall p \in M \ \exists U(p)$ t.c. $f|_{U(p)}$ è costante

Cor: $H^0(M) = \mathbb{R}^{c.c.(M)}$ $\quad b^0(M) = c.c.(M)$

- Lemma di Poincaré: $b^i(\mathbb{R}^n) = \begin{cases} 1 & \text{se } i=0 \\ 0 & \text{se } i>0 \end{cases}$

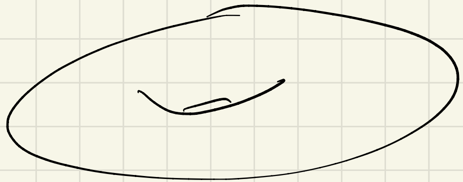
- $\underset{d\theta}{b^i(S^1)} = \begin{cases} 1 & i=0 \\ 1 & i=1 \end{cases}$

- $b^i(M^n) = 0$ se $i>n$

- M^n ori, senza ∂, cpt $\Rightarrow b^n(M) \geq 1$
  (forme volume)

- $b^i(S^n) = \begin{cases} 1 & i=0 \\ 0 & 1 \leq i \leq n-1 \\ 1 & i=n \end{cases}$

- $b^i(S^1 \times S^1) = \begin{cases} 1 & i=0 \\ 2 & i=1 \\ 1 & i=2 \end{cases}$



$d\vartheta_1 \qquad d\vartheta_2$

$d\vartheta_1 \wedge d\vartheta_2$

<u>Cor</u>: $\quad \mathbf{F} \in \Omega^2(\mathbb{R}^{3,1}) \qquad \Omega^2(D)$

$d\mathbf{F} = 0 \quad \Rightarrow \quad \mathbf{F} = d\mathbf{A}$

$\mathbf{A} \in \Omega^1(D)$

potenziale