Lezioni 7 & 8

Varietà pseudo-Riemanniana: (M, g) M varietà

g è un **TENSORE METRICO**, cioè $g \in \mathcal{T}^0_2(M)$ $(0,2)$

cioè $\forall p$ $g(p) : T_pM \times T_pM \to \mathbb{R}$ t.c.

PRODOTTO SCALARE
$$\begin{cases} 1) \text{ è simmetrico} & g(p)(v,w) = g(p)(w,v) \\ 2) \text{ è non degenere } \forall p & \Rightarrow \text{Ha una SEGNATURA } (p,m) \text{ t.c. } p+m=n \end{cases}$$

Base ortonormale per V^n con (p,m)

$\mathcal{B} = \{v_1, \dots, v_n\}$ t.c. $\langle v_i, v_j \rangle = \pm \delta_{ij}$

con $\langle v_i, v_i \rangle = -1$ per $1 \leq i \leq m$

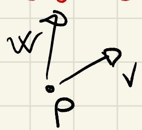
$= +1$ " $m+1 \leq i \leq n$

Riemanniana: $(p,m) = (n,0)$

Lorentziana: $(p,m) = (n-1, 1)$

Spazio Euclideo: $(\mathbb{R}^n, g^E)$ $g(p) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$

$(x, y) \mapsto \sum_{i=1}^n x_i y_i = {}^t x \cdot y$



$g_{ij} = \delta_{ij}$ $g(p)(x,y) = g_{ij} x^i y^j = {}^t x \cdot S \cdot y$

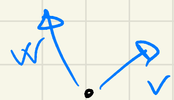
Spazio di Minkowski: $(\mathbb{R}^n, \eta)$

$$g_{ij} = \eta_{ij}$$

$$g = \begin{pmatrix} +1 & & \\ & \ddots & \\ & & +1 \end{pmatrix} \qquad \eta = \begin{pmatrix} -1 & & & \\ & +1 & & \\ & & \ddots & \\ & & & +1 \end{pmatrix}$$

$$\eta_{00} = -1 \qquad 0, \dots, n-1$$

Ci interessa $n=4$ $\quad \mathbb{R}^{1,3} = (\mathbb{R}^4, \eta) \quad \eta_{00} = -1$



$$\eta = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

Vettori tangenti di tipo tempo, luce e spazio:

$p \in M$ Lorentziana

$T_pM \qquad \eta(p)$ segnatura $(n-1, 1)$

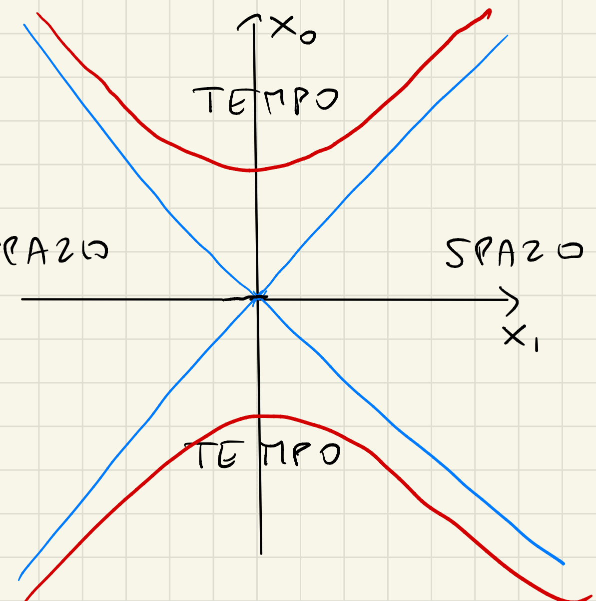
$$\boxed{c = 1}$$

$v \in T_pM \quad v \neq 0$

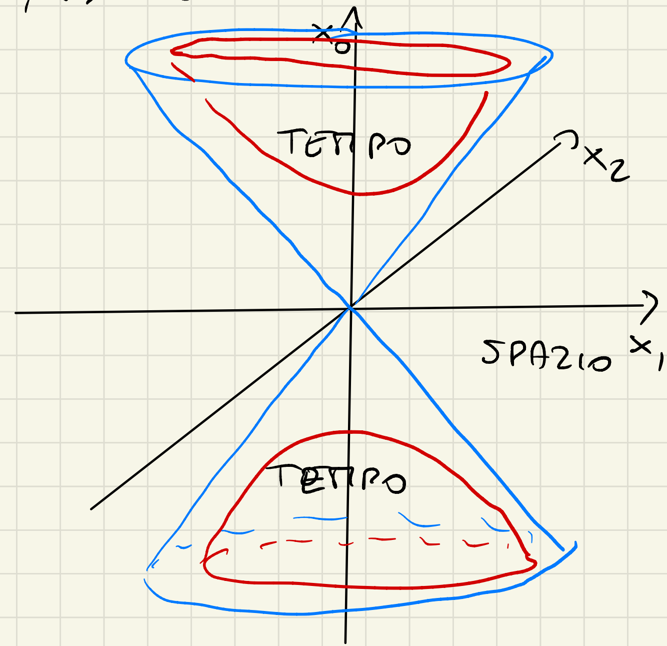
v è di tipo TEMPO se $\langle v, v \rangle < 0$

SPAZIO se $\langle v, v \rangle > 0$

LUCE se $\langle v, v \rangle = 0$



$x_0 = t, x_1, \dots, x_{n-1}$



$$-x_0^2 + x_1^2 + \dots + x_{n-1}^2 = 0$$

Curve e loro lunghezza:

(M, g) pR $\qquad \gamma : I \to M$ curva

$I \subseteq \mathbb{R}$

$$L(\gamma) = \int_I \|\gamma'(t)\|$$

$$\gamma'(t) = d\gamma_t(1) \in T_pM$$

$$d\gamma_t : T_tI \to T_pM$$

$T_tI = \mathbb{R}$ $\qquad p = \gamma(t)$

$v \in T_pM \qquad \|v\| = \sqrt{|\langle v, v\rangle|}$

$$\langle v, w\rangle = g(p)(v, w)$$

γ è di tipo SPAZIO se $\gamma'(t)$ è di tipo SPAZIO

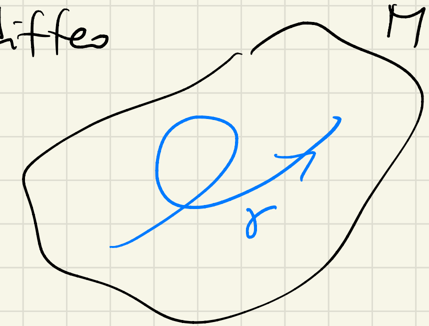
LUCE — LUCE

TEMPO — TEMPO

γ è di tipo luce $\iff L(\gamma) = 0$

Riparametrizzazione: $\gamma : I \to M$ $\quad \psi : J \xrightarrow{\sim} I$ diffeo

$\gamma \dashrightarrow \gamma \circ \psi : J \to M$



Supporto di γ è $\gamma(I)$

Supponiamo sempre: $\gamma'(t) \neq 0 \ \forall t$ (REGOLARE)
e luce spazio o tempo

Prop: $L(\gamma)$ non dipende dalla parametrizzazione

Parametrizzazione per lunghezza d'arco (P.L.A.):

Se γ è tempo o spazio (no LUCE!) $\exists$ parametrizzazione canonica t.c. $\|\gamma'(t)\| = 1 \ \forall t \in I$

dim: $I = (a, b)$ $\quad \gamma : I \to M$ $\quad \gamma(t) \in M \quad t \in (a,b)$

$$s(t) = \int_a^t \|\gamma'(t)\| \, dt \qquad t(s) \qquad J = s(I)$$

$\gamma : J \to M$

Linea di universo: (M, g) Lorentziana $\left(\begin{array}{c}\text{orientata} \\ \text{time-orientata}\end{array}\right)$

$\gamma : I \to M$ tipo tempo orientata positivamente nel tempo

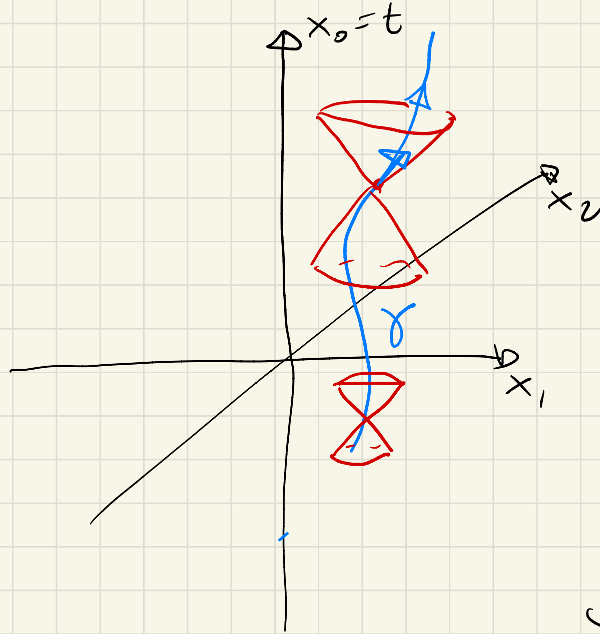
<u>Def</u>: (M, g) Lorentziana:

1) M come varietà può essere orientabile oppure no (ori su T_pM $\forall p \in M$ loc. coerente)

2) M è TIME-ORIENTED se $\forall p \in M$ è fissata una TIME ORIENTATION su T_pM loc. coerente

$T_pM = V \supseteq \{\text{tempo}\}$ ha 2 c.c. Ne scelgo una e la chiamo FUTURO

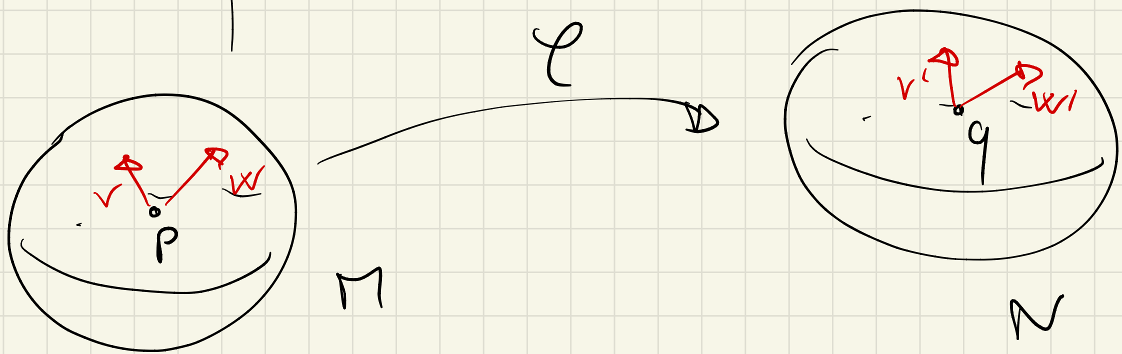
Es: $\mathbb{R}^{1,3}$ è orientato e time-orientato



EVENTI

Linea di universo

P.L.A.



Isometria: (M, g) (N, h) $\varphi : M \to N$ diffeom è una **ISOMETRIA** se

$$\forall p \in M \quad d\varphi_p : T_pM \xrightarrow{\sim} T_{\varphi(p)}N \text{ isometria cioè}$$

$$\forall v, w \in T_pM \qquad \langle v, w \rangle = \langle d\varphi_p(v), d\varphi_p(w) \rangle$$

Trasformazioni di Lorentz:

$$(\mathbb{R}^n, g^M = \eta_{ij})$$

$$A : \quad {}^tA J A = J$$

$$J = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

$${}^tv J w = {}^t(Av) J (Aw) \quad \forall v, w$$
$$= {}^tv\, {}^tA J A\, w$$

Isometrie vettoriali euclidee:

$$(\mathbb{R}^n, g^E = \delta_{ij})$$

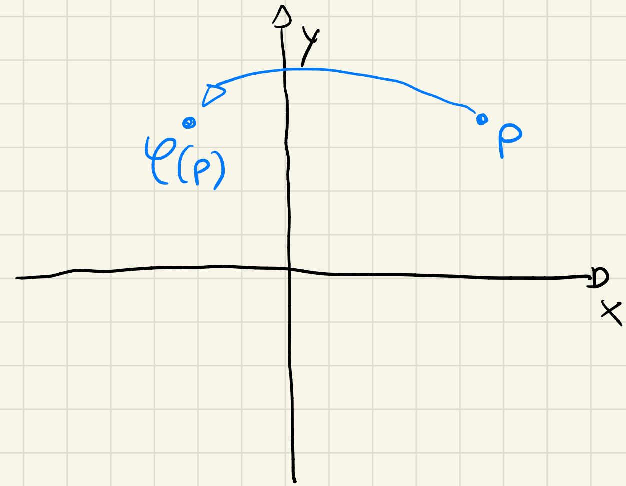
A matrice ortogonale: ${}^tA \cdot A = I$

$\varphi(x) = A \cdot x$ è una isometria

$$d\varphi_p : T_p\mathbb{R}^2 \to T_p\mathbb{R}^2$$
$$\qquad\quad \parallel \qquad\qquad \parallel$$
$$\qquad\quad \mathbb{R}^2 \xrightarrow{A} \mathbb{R}^2$$

$$O(n-1,1) = \{ A : {}^tA J A = J \}$$

Trasformazioni di Lorentz



$$\varphi : \mathbb{R}^{1,n-1} \to \mathbb{R}^{1,n-1}$$
$$x \longmapsto A \cdot x$$

è un'isometria

$\underline{n=3}$:

$$J = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

$$A = \begin{pmatrix} \cos\vartheta & -\operatorname{sen}\vartheta \\ \operatorname{sen}\vartheta & \cos\vartheta \end{pmatrix} = Rot_\vartheta$$

$$O(n) = \{ A \text{ ortogonali } n \times n \}$$

Esempi:

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\vartheta & -\operatorname{sen}\vartheta & 0 \\ 0 & \operatorname{sen}\vartheta & \cos\vartheta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in O(3,1)$$

$$A = \begin{pmatrix} \cosh t & -\operatorname{senh} t & & \\ \operatorname{senh} t & \cosh t & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

Lorentz boost

$\hookrightarrow \; {}^tAJA = J \quad (Ex)$

Gruppo di Poincaré:

$A \in O(n-1,1)$ $\qquad b \in \mathbb{R}^n$

$\varphi(x) = A \cdot x + b$

Isometrie affini euclidee:

A ortogonale

$\varphi(x) = A \cdot x + b \qquad b \in \mathbb{R}^n$

$$\left\{ \varphi(x) = Ax + b \;\middle|\; A \in O(3,1),\ b \in \mathbb{R}^4 \right\}$$

Teo: $\qquad$ Poincaré $= \mathrm{Isom}(\mathbb{R}^{1,3})$

$\{\text{isom. affini}\} = \mathrm{Isom}(\mathbb{R}^n)$

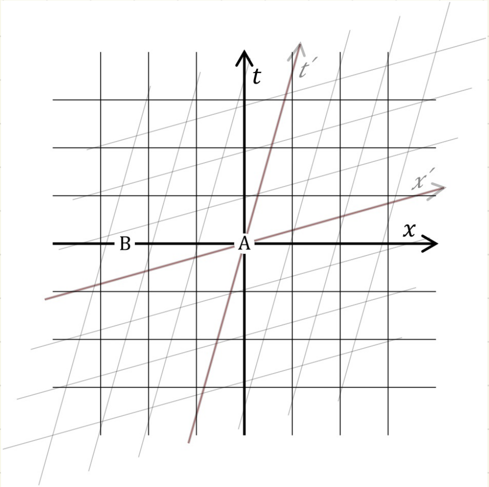
Def: (M,g) pR

$\mathrm{Isom}(M) = \{\varphi : M \to M \text{ isometria}\}$

è un gruppo

No simultaneità

$\varphi : \mathbb{R}^{3,1} \to \mathbb{R}^{3,1}$

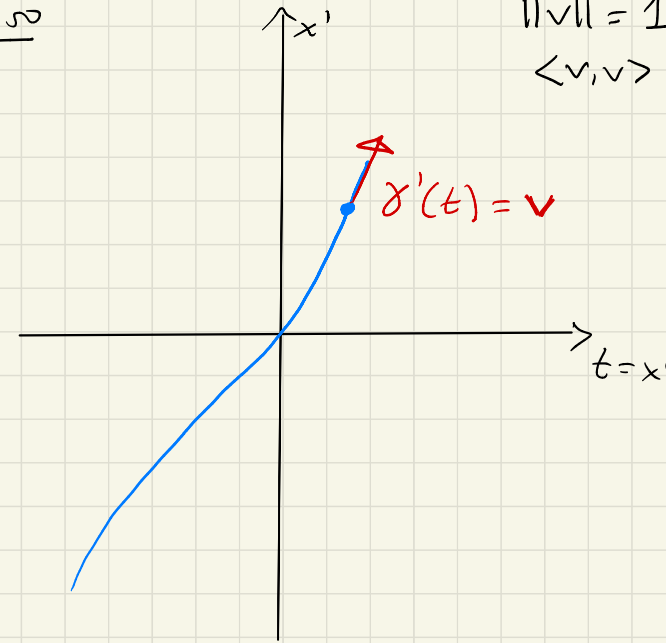


$\mathbb{R}^{1,1}$ $\mathbb{R}^{3,1}$

Quadrimpulso

$\|v\| = 1$

$\langle v, v \rangle = -1$



$$\mathbf{p} = m\mathbf{v}$$

$$\mathbf{p} = (\underbrace{E}_{\text{ENERGIA}}, \underbrace{p_x, p_y, p_z}_{\text{IMPULSO}})$$

$$\|\mathbf{p}\| = m = E^2 - p^2$$

$$p = \sqrt{p_x^2 + p_y^2 + p_z^2}$$

$$E = \sqrt{m^2 + p^2} = \sqrt{m^2 + m^2 v^2} = m\sqrt{1 + v^2} = m + \frac{1}{2} m v^2 + \dots$$

Base Lorentziana per $T_p M$

v_0, v_1, v_2, v_3 ortonormale con $\langle v_0, v_0 \rangle = -1$

$$\underset{i<j}{dx_i \wedge dx_j} = \begin{array}{cc} & j \\ i & \begin{pmatrix} \ddots & & 1 \\ & \ddots & \\ -1 & & \ddots \end{pmatrix} \end{array}$$

Elettromagnetismo

Tensore elettromagnetico $\quad \mathbf{F} \in \Omega^2(\mathbb{M}) \qquad \mathbb{M} = (\mathbb{R}^4, \eta)$

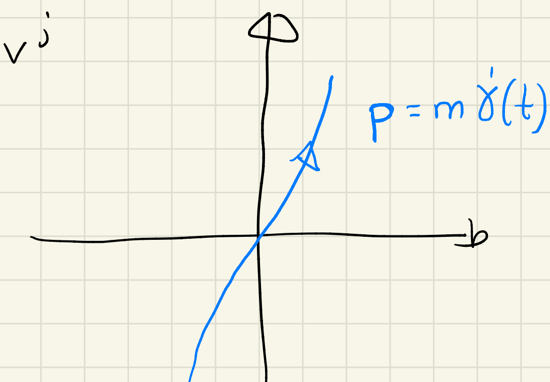
Minkowski

$$\mathbf{F} = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{pmatrix} \qquad F_{ij}$$

$$t = x_0, x_1, x_2, x_3$$

$$\mathbf{F} = -E_1\, dt \wedge dx_1 - E_2\, dt \wedge dx_2 - E_3\, dt \wedge dx_3 + B_3\, dx_1 \wedge dx_2 + B_2\, dx_3 \wedge dx_1 + B_1\, dx_2 \wedge dx_3$$

Legge di Lorentz: $\quad \dfrac{d\mathbf{p}}{dt} = q\mathbf{F}(\mathbf{v}) = q F^i{}_j v^j$



$$F^i{}_j = F_{kj}\, \eta^{ki}$$

$$\mathbf{p} = m\mathbf{v} \qquad \mathbf{v} = \gamma'(t)$$

$$F^i{}_j = F_{kj}\,\eta^{ki}$$

$$\eta = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

g_{ij} $\quad g^{ij}$ è l'inversa di g_{ij}

$$= F_{ij} \quad \text{se } i>0$$

$$-F_{ij} \quad \text{se } i=0$$

cioè

$$g_{ij}\,g^{jk} = \delta_i{}^k$$

$$g^{ij}\,g_{jk} = \delta^i{}_k$$

$$F^i{}_j = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{pmatrix}$$

$$\frac{d\mathbf{P}}{dt} = q\,F^i{}_j\,v^j = q\,\mathbf{F}(v)$$

$$\frac{dE}{dt} = q\,F^0{}_j\,v^j = q\,\mathbf{E}\cdot v$$

$$m\frac{d v_1}{dt} = mq\,F^1{}_j\,v^j = mq\left(\frac{E}{m}\cdot E_1 + B_3 v^2 - B_2 v^3\right)$$

Ex: È equivalente a:

$$p = \left(E, \ m\mathbf{u} \right)$$

$$\mathbf{v} = \left(\frac{E}{m}, \mathbf{u} \right)$$

$$\begin{cases} \dfrac{d\mathbf{u}}{dt} = \dfrac{q}{m} \left(\frac{E}{m} \mathbf{E} + \mathbf{u} \times \mathbf{B} \right) \\ \dfrac{dE}{dt} = q \mathbf{E} \cdot \mathbf{u} \end{cases}$$

Equazioni di Maxwell

$$dF = 0$$

Ex: È equivalente a:

$$\begin{cases} \operatorname{rot} \mathbf{E} = - \dfrac{\partial \mathbf{B}}{\partial t} \\ \operatorname{div} \mathbf{B} = 0 \end{cases}$$

$$F = -E_1\, dt \wedge dx_1 - E_2\, dt \wedge dx_2 - E_3\, dt \wedge dx_3$$
$$+ B_3\, dx_1 \wedge dx_2 + B_2\, dx_3 \wedge dx_1 + B_1\, dx_2 \wedge dx_3$$

$$dF = -\frac{\partial E_1}{\partial x_2} dx_2 \wedge dt \wedge dx_1 - \frac{\partial E}{\partial x_3} dx_3 \wedge dt \wedge dx_1$$
$$-\frac{\partial E_2}{\partial x_1} dx_1 \wedge dt \wedge dx_2 - \frac{\partial E_2}{\partial x_3} dx_3 \wedge dt \wedge dx_2$$
$$-\frac{\partial E_3}{\partial x_1} dx_1 \wedge dt \wedge dx_3 - \frac{\partial E_3}{\partial x_2} dx_2 \wedge dt \wedge dx_3$$
$$+\frac{\partial B_3}{\partial t} dt \wedge dx_1 \wedge dx_2 + \frac{\partial B_3}{\partial x_3} dx_3 \wedge dx_1 \wedge dx_2$$
$$+\frac{\partial B_2}{\partial t} dt \wedge dx_3 \wedge dx_1 + \frac{\partial B}{\partial x_2} dx_2 \wedge dx_3 \wedge dx_1$$

$$+ \frac{\partial B_1}{\partial t} dt \wedge dx_2 \wedge dx_3 + \frac{\partial B_1}{\partial x_1} dx_1 \wedge dx_2 \wedge dx_3 = 0$$

$$\left(- \frac{\partial E_1}{\partial x_2} + \frac{\partial E_2}{\partial x_1} + \frac{\partial B_t}{\partial t} \right) \quad dt \wedge dx_1 \wedge dx_2 \quad +$$

$$- \quad - \quad — \qquad dt \wedge dx_1 \wedge dx_3 \quad +$$

$$- \quad — \quad — \qquad dt \wedge dx_2 \wedge d_3 \quad +$$

$$- \quad —— \qquad dx_1 \wedge dx_2 \wedge d_3$$

Hodge $*$

V spazio vett dim n g prodotto scalare con segnatura (p,m) $\quad p+m=n$

- Induce prodotto scalare su $T^h_k(V)$

g_{ij} $\quad g \cdots\cdots \to g \quad g^{ij}$

$T: V \xrightarrow{\sim} V^*$

$v \longmapsto (w \longmapsto g(v,w))$

$v_1, \ldots, v_n$ base V

$v^1, \ldots, v^n$ per V^*

$v_{i_1} \otimes \cdots \otimes v_{i_h} \otimes v^{j_1} \otimes \cdots \otimes v^{j_k}$ per $T^h_k(V)$

$v^*, w^* \in V^*$ $\quad g(v^*, w^*) = g(v,w)$

$v = T^{-1}(v^*)$

$w = T^{-1}(w^*)$

In generale, $\quad T^a{}_{bc} \quad U^i{}_{jk}$

$$g(T,U) = T^a{}_{bc}\, U^i{}_{jk}\, g_{ai}\, g^{bj} g^{ck}$$

Ex: g^{ij} è inverso di g_{ij}

$$C(T \otimes U \otimes g \otimes \bar{g} \otimes \bar{g})$$

Ex: $\langle v_1 \otimes \cdots \otimes v_h \otimes v^1 \otimes \cdots \otimes v^k, w_1 \otimes \cdots \otimes w_h \otimes w^1 \otimes \cdots \otimes w^k \rangle$

$$= \prod_i \langle v_i, w_i \rangle \prod_j \langle v^j, w^j \rangle$$

Cor: $\left\{ v_{i_1} \otimes \cdots \otimes v_{i_h} \otimes v^{j_1} \otimes \cdots \otimes v^{j_k} \right\}$ base ortonormale se $v_1, \dots, v_n$ lo è

Cor: $\langle v^1 \wedge \cdots \wedge v^k, w^1 \wedge \cdots \wedge w^k \rangle = k! \det \langle v^i, w^j \rangle$

$\Lambda^k(V) \subseteq T_0^k(V)$

Def: $\langle \alpha, \beta \rangle^{new} := \frac{1}{k!} \langle \alpha, \beta \rangle^{old}$

Cor: $\left\{ v^{i_1} \wedge \cdots \wedge v^{i_k} \right\}_{i_1 < \cdots < i_k}$ ortonormale se $v_1, \dots, v_n$ lo è

Es: $\mathbb{R}^n$ Eucl. oppure $\mathbb{R}^{1,3}$ $\to$ $\left\{ dx_{i_1} \wedge \cdots \wedge dx_{i_k} \right\}$ base ortonormale

$V, g,$ orientazione $\longrightarrow \omega \in \Lambda^n(V) \cong \mathbb{R}$

$\omega(v_1, \dots, v_n) = 1$ Ex: È ben definita.

ortonormale

positiva

dim:

$\Rightarrow \omega(v_1', \dots, v_n'$ orton. positiva $) = 1$

Cor: (M, g) p.R. M orientata $\Rightarrow \omega \in \Omega^n(M)$

forma volume

$\mathbb{R}^n$ euclideo: $*(dx^1 \wedge \dots \wedge dx^k) = dx^{k+1} \wedge \dots \wedge dx^n$

$\mathbb{R}^{1,3}$ $*dt = -dx^1 \wedge dx^2 \wedge dx^3$

$$*: \Lambda^k(V) \to \Lambda^{n-k}(V)$$
$$\beta \longmapsto *\beta$$

ω generatore canonico di $\Lambda^n(V)$

$$\alpha \wedge (*\beta) = \langle \alpha, \beta \rangle \omega \quad \forall \alpha \in \Lambda^k(V)$$

Teo: È ben definita $*\beta$

Ex: $v^1, \dots, v^n$ ortonormale positiva

- $*(v^1 \wedge \dots \wedge v^k) = (-1)^{m'} v^{k+1} \wedge \dots \wedge v^n$

  m': # el. negativi in $v^1, \dots, v^k$

- $**\beta = (-1)^{k(n-k)+m} \beta$

M varietà pseudo-Riemanniana:

$$*: \Omega^k(M) \to \Omega^{n-k}(M)$$

Codifferenziale:

$$\delta: \Omega^k(M) \to \Omega^{k-1}(M)$$

$$\delta = (-1)^k * d *$$

<u>Ex</u>: $\delta^2 = 0$

$$* d \cancel{*} \cancel{*} d *$$

$$= * d d * = 0$$

$$*F = \begin{pmatrix} 0 & B_1 & B_2 & B_3 \\ -B_1 & 0 & E_3 & -E_2 \\ -B_2 & -E_3 & 0 & E_1 \\ -B_3 & E_2 & -E_1 & 0 \end{pmatrix}$$

Hodge-$*$ in Minkowski space:

$$\begin{aligned}
*1 &= dx \wedge dy \wedge dz \wedge dt \\
*dx &= dy \wedge dz \wedge dt \\
*dy &= dz \wedge dx \wedge dt \\
*dz &= dx \wedge dy \wedge dt \\
*dt &= dx \wedge dy \wedge dz \\
*(dx \wedge dy) &= dz \wedge dt \\
*(dz \wedge dx) &= dy \wedge dt \\
*(dy \wedge dz) &= dx \wedge dt \\
*(dx \wedge dt) &= -dy \wedge dz \\
*(dy \wedge dt) &= -dz \wedge dx \\
*(dz \wedge dt) &= -dx \wedge dy \\
*(dx \wedge dy \wedge dz) &= dt \\
*(dx \wedge dy \wedge dt) &= dz \\
*(dz \wedge dx \wedge dt) &= dy \\
*(dy \wedge dz \wedge dt) &= dx \\
*(dx \wedge dy \wedge dz \wedge dt) &= -1
\end{aligned}$$

Quadricorrente: $J = (\underset{\substack{\uparrow \\ \text{DENSITÀ} \\ \text{DI CARICA}}}{\rho}, \underbrace{J_x, J_y, J_z}_{\substack{\text{DENSITÀ} \\ \text{DI CORRENTE}}})$ campo vettoriale in $\mathbb{R}^{1,3}$

$$J_i = J^j \eta_{ij} \quad \rightsquigarrow \quad J = -\rho\, dt + J_x dx + J_y dy + J_z dz$$

$$*J = \rho\, dx \wedge dy \wedge dz - J_x\, dt \wedge dy \wedge dz - J_y\, dt \wedge dz \wedge dx - J_z\, dt \wedge dx \wedge dy$$

Maxwell: $\boxed{\delta \mathbf{F} = J}$ $\quad d(*F) = *J$

Ex: È equivalente a:

$J = (\rho, j)$

$$\begin{cases} \operatorname{rot} \mathbf{B} = j + \dfrac{\partial \mathbf{E}}{\partial t} \\ \operatorname{div} \mathbf{E} = \rho \end{cases}$$