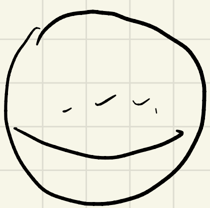
Teo (CLASSIFICAZIONE DELLE SUPERFICI)

Ogni superficie Σ connessa cpt orientabile senza ∂

$\|$ 2-varietà

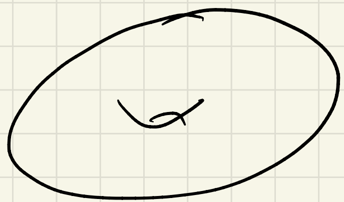
è diffeomorfa a una di queste:



S^2

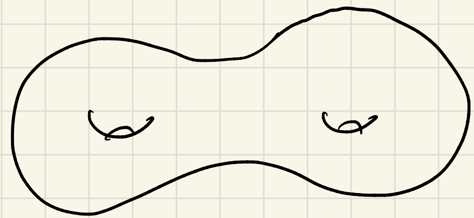
$g = 0$

GENERE

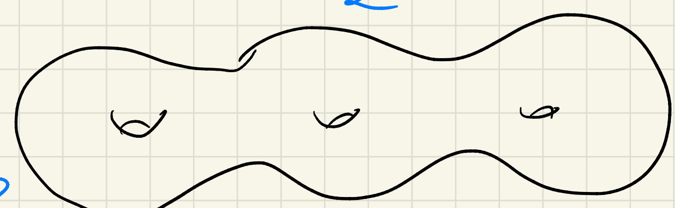


$S^1 \times S^1$

1



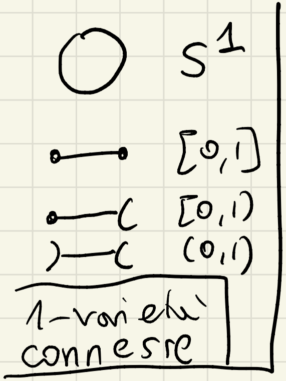
2

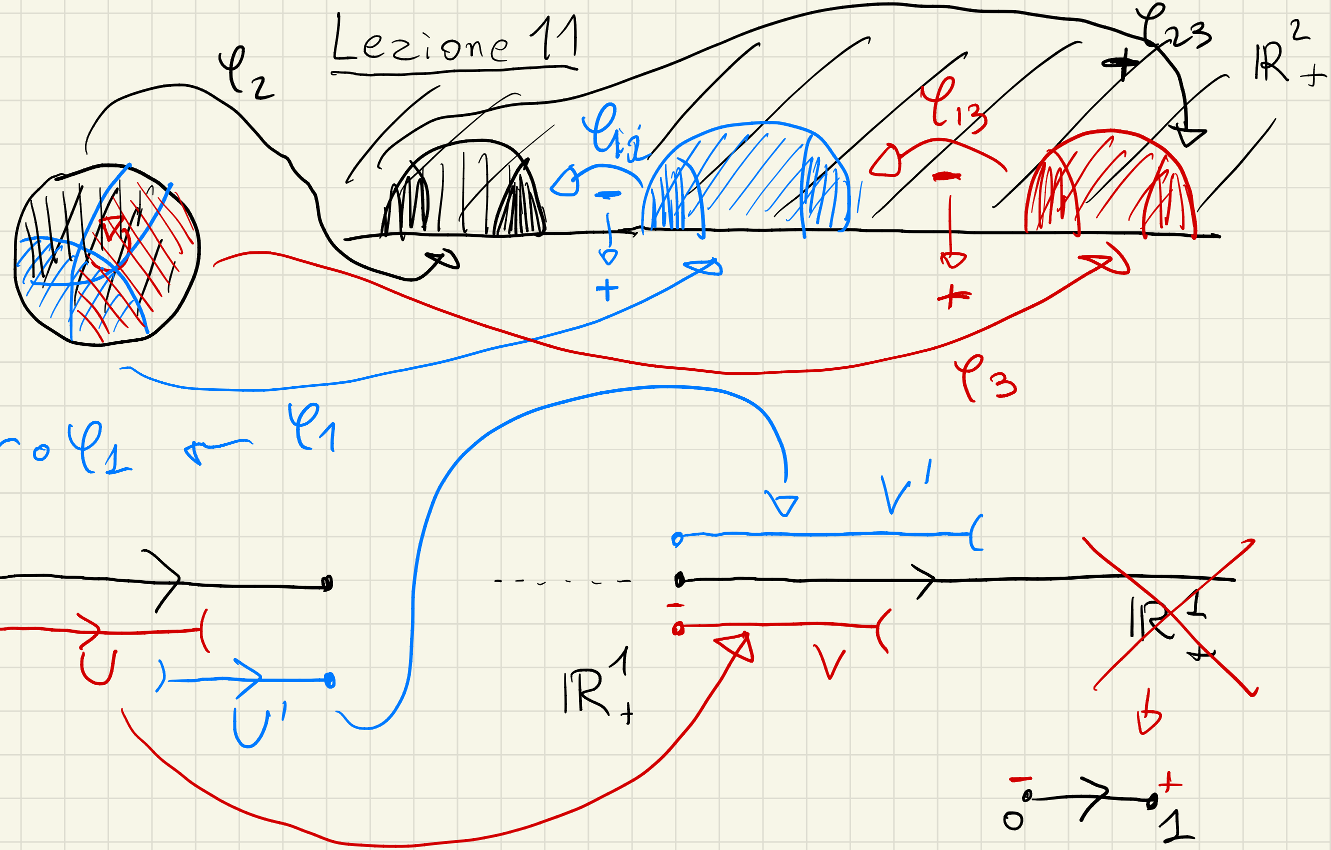


3

. . . .

Lezione 11

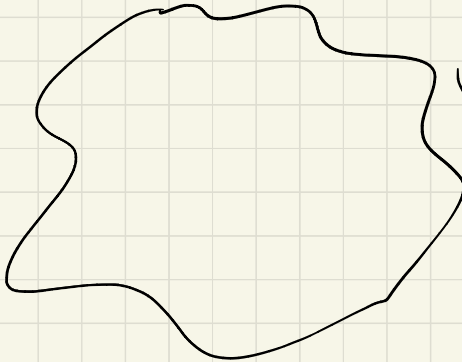




$r(x,y) = (-x,y)$ $\qquad$ $r : \mathbb{R}^2_+ \xrightarrow{\sim} \mathbb{R}^2_+$ inverte orientazione

$\mathbb{R}^n_+$ è SPECCHIABILE

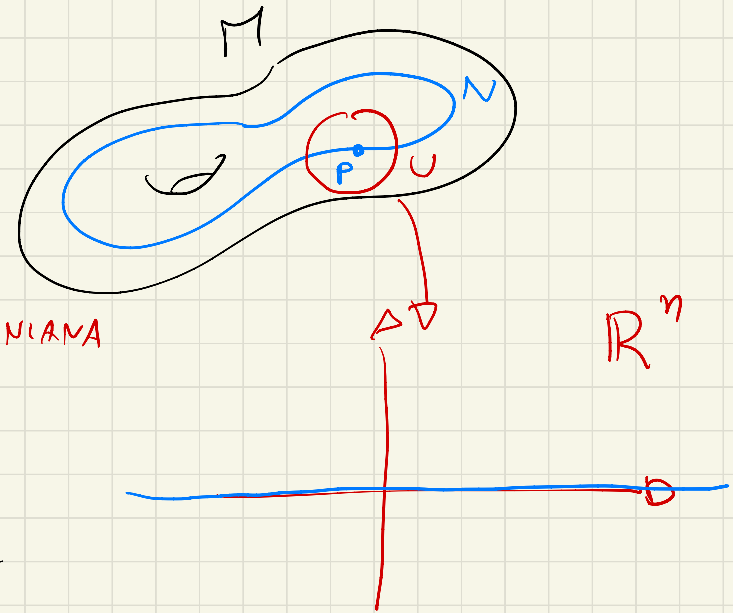
cioè $\exists\ \varphi : \mathbb{R}^n_+ \xrightarrow{\sim} \mathbb{R}$ diffeo che inverte ori



$U \subseteq \mathbb{R}^n$

$U = B^n$

$$g^H = \left(\frac{2}{1-\|x\|^2}\right)^2 g^E$$



Def: (M^n, g) varietà pR

Sia $N \subseteq M$ sottovarietà liscia connessa

Lei è SOTTOVARIETÀ PSEUDO-RIEMANNIANA

se $\forall p \in N$

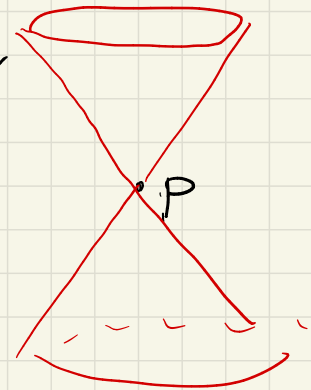
$T_pN \subseteq T_pM$ sottospazio

$g|_{T_pN}$ è NON DEGENERE

<u>Oss</u>: Se g è def+ questo è verificato sempre

<u>Es</u>: $\mathbb{R}^{2,1} \supseteq \gamma$

è sottovarietà pR
se è tipo tempo
o spazio

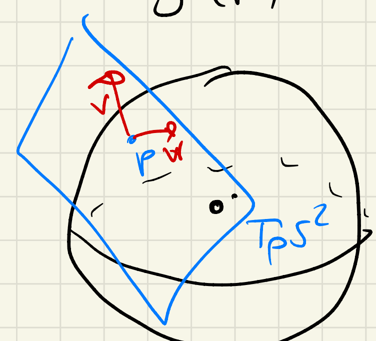


<u>Oss</u>: N è varietà pR $\quad (N, g')$

$$\forall p \in N \quad g'(p) := g|_{T_pN}$$

<u>Es</u>: $S \subseteq \mathbb{R}^n$ sottovarietà

<u>Es</u>: $S^{n-1} \subseteq \mathbb{R}^n$
$(\mathbb{R}^n, g^E)$

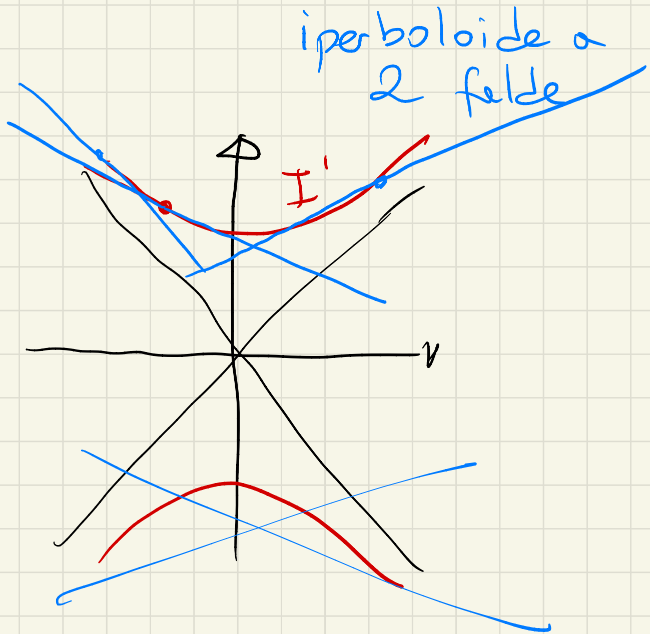


$S^2 \subseteq \mathbb{R}^3$

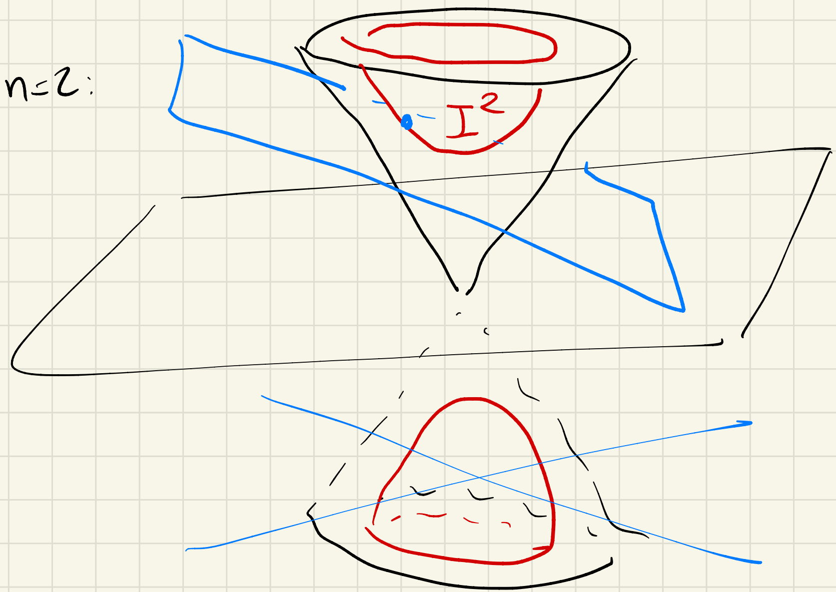
Es: $I^n \subseteq \mathbb{R}^{n,1} = (\mathbb{R}^{n+1}, \eta) \quad \eta = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$

$$I^n = \{ x \in \mathbb{R}^{n,1} : \langle x, x \rangle = -1 \} \cap \{ x_1 > 0 \}$$

$$\langle x, x \rangle = -x_1^2 + x_2^2 + \dots + x_{n+1}^2 = -1$$



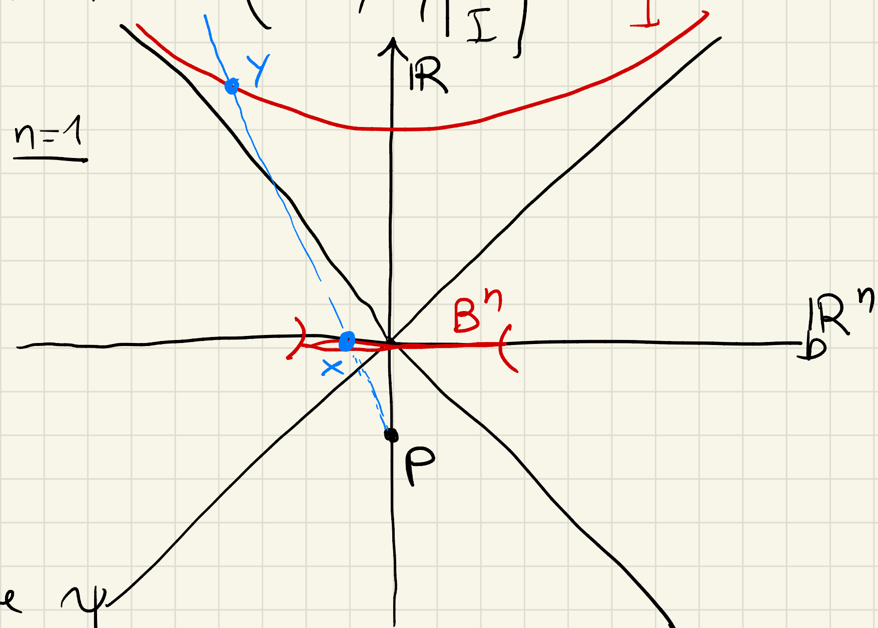
$n=2$:



Ex: $I^n \subseteq \mathbb{R}^{n,1}$ è sottovarietà pR Riemanniana

I^n è il MODELLO DELL'IPERBOLOIDE dello SPAZIO IPERBOLICO

Ex: (B^n, g^H) è isometrica a $(I^n, \eta|_I)$



$P = (-1, 0, \dots, 0)$

$\psi : B^n \xrightarrow{\sim} I^n$ diffeo

$\psi(x) = y$

Facendo i conti si vede che ψ è isometria cioè $\forall x \in B^n, \quad \forall v, w \in T_x B^n$

$$\underset{g^H}{\langle v, w \rangle} = \underset{\eta|_I}{\langle v', w' \rangle} \qquad \begin{aligned} v' &= d\psi_x(v) \\ w' &= d\psi_x(w) \end{aligned}$$

<u>Prop</u>: Ogni M ha struttura Riemanniana.

<u>dim</u>

$\mathcal{A} = \{ \varphi_i : U_i \to \mathbb{R}^n \}$ atlante $\to$ $\{U_i\}$ ric. aperto

$\to \exists \ \rho_i : M \to [0, \infty)$ partiz. unità

t.c. $\operatorname{supp} \rho_i \subseteq U_i \qquad \sum_i \rho_i(p) = 1$

Su U_i prendo $g_i := \varphi_i^*(g^E) \qquad g_i(v, w) := g^E(v', w')$

$v' = (d\varphi_i^*)_x(v) \quad \ldots$

Definisco $\forall p \in M$ $\quad g(p) := \sum_i \rho_i g_i(p)$

Si ottiene g campo tensoriale di tipo $(0,2)$ su M

simmetrico

det+

Combinaz. lineare con coeff. positivi di prod. scalari positivi

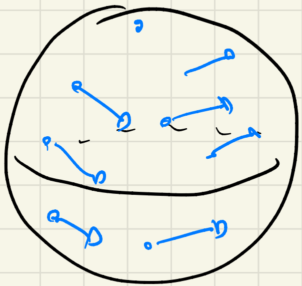
è definito positivo

infatti $\quad \underset{v \neq 0}{g(p)(v,v)} = \sum_i \underbrace{\rho_i}_{>0} \underbrace{g_i(p)(v,v)}_{>0} > 0$

$\square$

Oss: Non tutte le M hanno struttura Lorentziana

Teo: M ha struttura Lorentziana $\Longleftrightarrow$ ha un campo vettoriale mai nullo



S^2

idem per S^4

$\nexists$ campo mai nullo su S^2

$\boxed{\Leftarrow}$ (M, g) g Riemanniana X campo vettoriale mai nullo

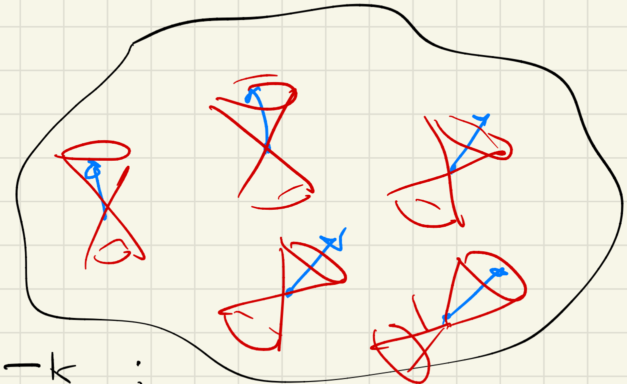
normalizzo X: $\bar{X}(p) = \dfrac{X(p)}{\|X(p)\|} = \dfrac{X(p)}{\sqrt{g(p)(X(p), X(p))}}$

$\|\bar{X}(p)\| = 1 \quad \forall p$

$$g'(p)(v,w) := g(p)(v,w) - 2\, g(p)(v, \bar{X}(p)) \cdot g(p)(\bar{X}(p), w)$$

$\forall p \in M, \ \forall v, w \in T_pM$

g' è nuovo tensore simm. $(0,2)$



In coordinate:

$$g'_{ij} v^i w^j = g_{ij} v^i w^j - 2\, g_{i\ell} v^i \bar{X}^\ell g_{kj} \bar{X}^k w^j$$

cioè

$$g'_{ij} = g_{ij} - 2\, g_{i\ell} \bar{X}^\ell g_{kj} \bar{X}^k$$

Calcolo la segnatura di g'_{ij}:

$\mathcal{B} = \{ \bar{X}(p), v_2, \dots, v_n \}$ base ortonormale di $(\mathbb{R}^n, g_{ij})$

Riscrivo g_{ij} e g'_{ij} rispetto a $\mathcal{B}$

Ottengo $g_{ij} = \delta_{ij}$ $\quad g'_{ij} = \delta_{ij} - 2\delta_{i\ell} \bar{X}^\ell \delta_{kj} \bar{X}^k$

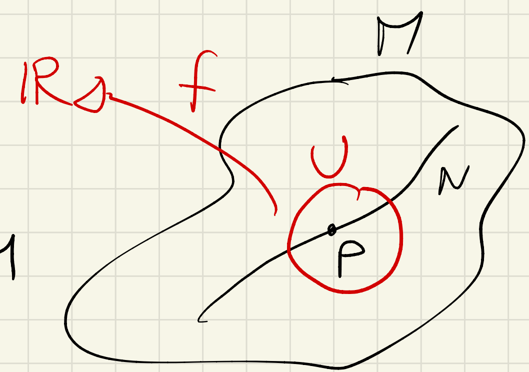
$$g'_{ij} = \delta_{ij} - 2\,\delta_{i1}\,\delta_{1j} = \eta_{ij}$$

$$= I - \begin{pmatrix} 2 & 0 & \cdots \\ 0 & 0 & \\ \vdots & & \ddots \end{pmatrix} = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix} - \begin{pmatrix} 2 & & 0 \\ & 0 & \\ 0 & & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \qquad \square$$

$N \subseteq M$ sottovarietà

$\forall p \in N \qquad T_pN \subseteq T_pM$



$$v \in T_pN \dashrightarrow ? \quad v \in T_pM$$

$$v(f) := v\left(f\big|_{N \cap U}\right) \qquad f : U(p) \to \mathbb{R}$$

ESERCIZI

Es 1 : A, B matrici quadrate $\qquad X_A(x) = Ax$ in $\mathbb{R}^n$

$$[X_A, X_B] = X_{BA-AB}$$

$$X_A(x) = Ax \qquad X_A^i = (Ax)^i = A^i_j x^j$$

$$[X,Y]^i = X^j \frac{\partial Y^i}{\partial x^j} - Y^j \frac{\partial X^i}{\partial x^j}$$

$$[X_A, X_B]^i = X_A^j \frac{\partial X_B^i}{\partial x^j} - X_B^j \frac{\partial X_A^i}{\partial x^j}$$

$$= A^j_\ell x^\ell \frac{\partial}{\partial x^j}\left(B^i_k x^k\right) - B^j_\ell x^\ell \frac{\partial}{\partial x^j}\left(A^i_k x^k\right)$$

$$= A^j_\ell x^\ell B^i_k \frac{\partial x^k}{\partial x^j} - \cdots\cdots$$

$$= A^j_\ell x^\ell B^i_K \delta_{jK} - \dots$$

$$= A^j_\ell x^\ell B^i_j - \dots$$

$$= B^i_j A^j_e x^e - \dots$$

$$= B \cdot A \cdot x - A \cdot B \cdot x = (BA - AB)x$$

<u>Ex 2</u>: $\underbrace{[[X,Y],Z](f)}_{=} + [[Y,Z],X](f) + [[Z,X],Y](f) = 0$

$\forall f \in \mathcal{C}^\infty(U) \quad \forall U \subseteq M$

1) Lavoro in coordinate

2) Scrivere f $\quad [XY - YX, Z]$

$\quad\quad\quad\quad\quad\quad\quad =$

3) Lavoro con operatori

$[XY, Z] - [YX, Z]$

$\quad = \quad\quad\quad =$

altra strada scrivendo sempre f

$XYZ - ZXY - (YXZ - ZYX)$

$XY = X^i \frac{\partial Y}{\partial x^i}$

$[[X,Y],Z](f) =$

$= [X,Y](Zf) - Z([X,Y](f))$

$[A,B](f) = ABf - BAf$ OK

Ex 3: $[fX, gY] = fg[X,Y] + f(Xg)Y - g(Yf)X$

In coordinate

$\| $

$(fX)(gY) - (gY)(fX)$

$= f(Xg)Y + fgXY - g(Yf)X - gfYX$

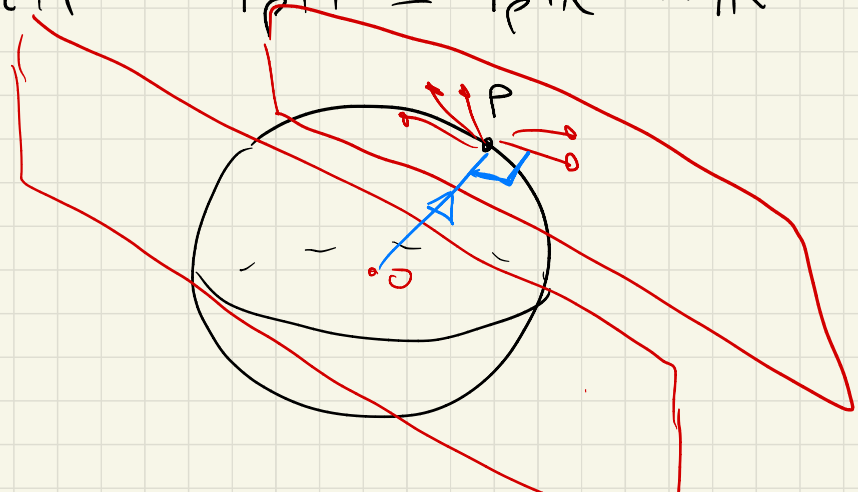
Ex 4: $S^{2n-1} \subseteq \mathbb{R}^{2n}$

$$X(x_1, \dots, x_{2n}) = (x_2, -x_1, \dots, x_{2n}, -x_{2n-1})$$

Fatti generali:

$M \subseteq \mathbb{R}^N$ sottovarietà

$\forall p \in M \qquad T_pM \subseteq T_p\mathbb{R}^N = \mathbb{R}^N$



$X \in \mathfrak{X}(M)$

$f \in C^\infty(M)$

$fX \in \mathfrak{X}(M)$

$(fX)(p) = f(p) \cdot X(p)$

$Xf \in C^\infty(M)$

$(Xf)(p) = X(p)(f)$

$T_pS^{N-1} = p^\perp$

$$(x_2, -x_1, \dots, x_{2n}, -x_{2n-1}) \in T_x S^{2n-1} \quad ?$$

$$T_x S^{2n-1} = x^\perp$$

$$\iff \langle \;''\;, x \rangle = 0$$

$$\Updownarrow$$

$$x_1 x_2 - x_2 x_1 + \dots + x_{2n-1}x_{2n} - x_{2n}x_{2n-1} = 0$$

$$\|x\| = 1 \Rightarrow \sum x_i^2 = 1 \Rightarrow \|X(x)\| = 1 \quad \forall x \in S^{2n-1}$$

$$x(t) = \gamma(t)$$

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1 \\ \vdots \\ \dot{x}_{2n-1} = x_{2n} \\ x_{2n} = -x_{2n-1} \end{cases} \qquad \left.\begin{matrix} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1 \end{matrix}\right\} \Rightarrow \ddot{x}_1 = -x_1$$

$$x_1 = A \cos t + B \operatorname{sen} t$$

$$x_2 = \dot{x}_1 = B \cos t - A \operatorname{sen} t$$

$$x_1(0) = x_1^0 \qquad x_2(0) = x_2^0$$

$$A = x_1^0 \qquad B = x_2^0$$

$$x(t) = \left(x_1^0 \cos t + x_2^0 \operatorname{sen} t,\ x_2^0 \cos t - x_1^0 \operatorname{sen} t, \dots \right)$$

NOTA: $\exists\ \forall t \in (-\infty, \infty)$

$$x(t+2\pi) = x(t)$$

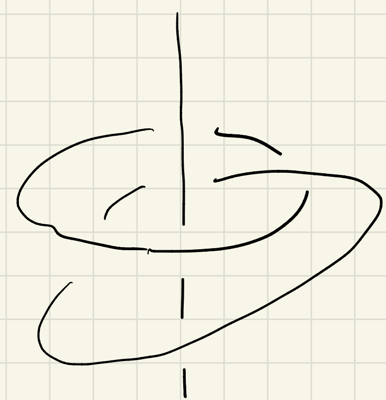
$$F_t(x) = F(x,t) = x(t)$$

S^3 è unione di circonferenze disgiunte! (FIBRAZIONE DI HOPF)

$$S^n = \mathbb{R}^n \cup \{\infty\}$$

proiez. stereografica

$$S^3 = \mathbb{R}^3 \cup \{\infty\}$$



$$S^3 \subseteq \mathbb{C}^2 \setminus \{0\} \xrightarrow{\pi} \mathbb{P}^1\mathbb{C}$$

$$(z,w) \longmapsto [z,w]$$

$$S^3 \subseteq \mathbb{R}^4 \setminus \{0\} = \mathbb{C}^2$$

$$\pi : S^3 \to \mathbb{P}^1\mathbb{C} = S^2$$
$$(z,w) \mapsto [z,w] \mapsto \frac{z}{w}$$

FIBRAZIONE DI HOPF

$$S^3 = \left\{ (z,w) : \|z\|^2 + \|w\|^2 = 1 \right\}$$

$$\mathbb{P}^1\mathbb{C} = \mathbb{C} \cup \{\infty\} = \mathbb{R}^2 \cup \{\infty\} = S^2$$

$$\mathbb{P}^1\mathbb{C} \to \mathbb{C} \cup \{\infty\}$$
$$[z,w] \mapsto \frac{z}{w}$$

$$\pi^{-1}([z,w]) = \left\{ (e^{i\vartheta} z, e^{i\vartheta} w) \mid \vartheta \in [0,2\pi) \right\}$$

$$\|z\|^2 + \|w\|^2 = 1$$

$$[z,w] = [\lambda z, \lambda w] \quad \forall \lambda \neq 0 \quad \lambda \in \mathbb{C}$$

Ex 1.5 : $M^n \longrightarrow (TM)^{2n}$ ORIENTABILE

$$\mathcal{A} = \{ \varphi_i : U_i \to V_i \subseteq \mathbb{R}^n \}$$

$\pi : TM \to M$

$$\mathcal{A}' = \{ \psi_i : \pi^{-1}(U_i) \to V_i \times \mathbb{R}^n \}$$

$$\psi_i(p, v) = (\varphi_i(p), (d\varphi_i)_p(v))$$ è orientato!

$v \in T_pM$

$$\psi_{ij} = \psi_i \circ \psi_j^{-1} \qquad (d\psi_{ij})_x = \left(\begin{array}{c|c} & \\ \hline & \end{array}\right)$$

$$\psi_{ij}(x, w) = \left(\varphi_i \circ \varphi_j^{-1}(x),\ d(\varphi_i) \circ d(\varphi_j^{-1})_x(w) \right)$$

$$= \left(\varphi_i \circ \varphi_j^{-1}(x) \,,\; d\left(\varphi_i \circ \varphi_j^{-1}\right)_x (w) \right)$$

$$\left(d\,\psi_{ij}\right)_{(x,w)} : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$$

Jacobian

$$\det \left(\begin{array}{c|c} d\left(\varphi_i \circ \varphi_j^{-1}\right)_x & 0 \\ \hline ? & d\left(\varphi_i \circ \varphi_j^{-1}\right)_x \end{array} \right)$$

$$= \left(\det d\left(\varphi_i \circ \varphi_j^{-1}\right)_x \right)^2 > 0$$

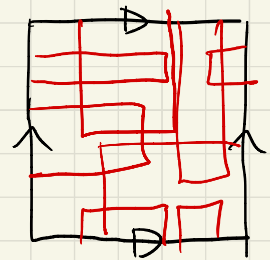
Ex 1.6

$M^m \times N^n$ (non ori) non è ori

$\mathcal{A}$ $\quad \mathcal{A}'$ $\qquad \mathcal{A} \times \mathcal{A}'$

non è orientato

$T = S^1 \times S^1$



Supponiamo p.a. che $M \times N$ orientabile

Fissiamo $p \in M$ $\qquad \underset{\text{non ori}}{\{p\} \times N} \subseteq \underset{\text{ori}}{M \times N}$

$$\{p\} \times N \cong N$$

Fisso $v_1, \dots, v_m \in T_pM$

Uso l'orientazione di $M \times N$ per definirne una su N (assurdo)

<u>Def</u>: $\forall q \in N$ $\quad w_1, \dots, w_n$ base di T_qN

è positiva se

$$\underbrace{v_1, \dots, v_m}, \underbrace{w_1, \dots, w_n} \text{ è base positiva}$$

di $T_pM \times T_qN = T_{(p,q)}M \times N$

<u>Ex</u>: $U \times V = W$ Date due orientazioni su U e W

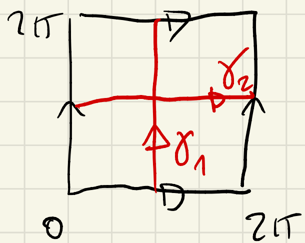
$\exists$ modo canonico di definirne una su V

1.7:

$$T = S^1 \times S^1 \qquad (\vartheta^1, \vartheta^2) \qquad \omega = d\vartheta^1$$

$$\gamma_1 = \{-1\} \times S^1 \qquad \gamma_2 = S^1 \times \{-1\}$$

$$\int_{\gamma_1} \omega = 0 \qquad \int_{\gamma_2} \omega = 2\pi$$



$$\omega = dx$$

1.8:

$$f : M \to N \qquad \omega \in \Omega^k(N),\ \eta \in \Omega^h(N)$$

$$f^*(\omega \wedge \eta) = (f^*\omega) \wedge (f^*\eta)$$

$$f^*\left(\alpha\, dx^{i_1} \wedge \dots \wedge dx^{i_k}\right)$$

$$= \alpha \circ f \, df^{i_1} \wedge \dots \wedge df^{i_k}$$

$$V \xrightarrow{f} W \qquad f^* : \mathcal{T}_0^k(W) \to \mathcal{T}_0^k(V)$$

$$\begin{array}{ccc} \mathcal{T}_0^k(W) & \to & \mathcal{T}_0^k(V) \\ \cup & & \cup \\ \Lambda^k(W) & \to & \Lambda^k(V) \\ \Psi & & \\ \alpha & & \end{array}$$

$$f^*(\alpha)(v_1, \dots, v_k) =$$
$$\alpha\left(f(v_1), \dots, f(v_k)\right)$$

$\omega \in \Lambda^k(W) \qquad \eta \in \Lambda^h(W)$

TESI

$$\boxed{f^*(\omega \wedge \eta) = f^*(\omega) \wedge f^*(\eta)}$$

$n = k + h$

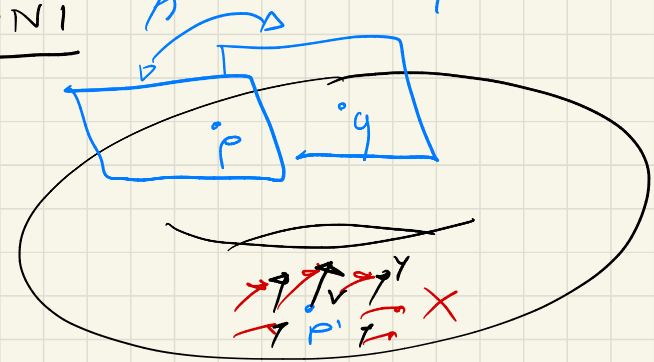
$$(\omega \wedge \eta)(v_1, \ldots, v_n) =$$

$$= \frac{1}{k!\,h!} \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)\, \omega(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) \cdot \eta(v_{\sigma(k+1)}, \ldots, v_{\sigma(n)})$$

CONNESSIONI

CONNESSIONE = DERIVATA COVARIANTE

$\nexists \; T_pM \xrightarrow{\sim} T_qM$



$\mathcal{L}_Y X = [Y, X]$

$\mathcal{L}_Y S$

Def: M varietà.

Una DERIVATA COVARIANTE

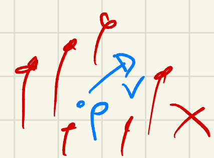
è un oggetto ∇ che

$\forall p \in M, \; \forall X \in \mathfrak{X}(U(p))$

$\forall v \in T_pM$

definisce $\nabla_v X \in T_pM$

t.c.:



LOCALE

1) Se X e Y coincidono su $U(p)$ allora $\nabla_v X = \nabla_v Y$

LINEARE

2) Lineare in v e in X:

$$\nabla_v (\lambda X + \mu Y) = \lambda \nabla_v X + \mu \nabla_v Y$$

$\lambda, \mu \in \mathbb{R}$

$X, Y \in \mathfrak{X}(U(p))$

$$\nabla_{\lambda v + \mu w} X = \lambda \nabla_v X + \mu \nabla_w X$$

LEIBNITZ

3) $$\nabla_v (f X) = \boxed{\nabla_v (f)} \cdot X(p) + f(p) \cdot \nabla_v (X)$$

$$\nabla_v (f) = v(f)$$

$f \in \mathcal{C}^\infty(U(p))$

$X \in \mathfrak{X}(U(p))$

LISCIA

4) $\nabla_v X$ "dipende in modo liscio da v e da X"

formalmente: dati $X, Y \in \mathfrak{X}(U)$

$$\nabla_Y X \in \mathfrak{X}(U)$$

$$(\nabla_Y X)(p) = \nabla_{Y(p)} X$$

In carte: $\mathbf{X} = X^i e_i = X^i \frac{\partial}{\partial x^i} = X^i \partial_i$

$\mathbf{v} \in \mathbb{R}^n$ $\qquad \mathbf{v} = v^i e_i$

$\nabla_{\mathbf{v}} \mathbf{X} \in \mathbb{R}^n$

$$= \nabla_{v^i e_i}(X^j e_j) \underset{\text{LIN}}{=} v^i \nabla_{e_i}(X^j e_j) \underset{\text{LEIB}}{=}$$

$$= v^i \left(\left(\nabla_{e_i} X^j \right) \mathbf{e}_j + X^j \nabla_{e_i} \mathbf{e}_j \right)$$

$$= v^i \left(\frac{\partial X^j}{\partial x^i} \mathbf{e}_j + X^j \Gamma_{ij}^k \mathbf{e}_K \right)$$

simbolo di Christoffel

∇ è determinato da Γ_{ij}^k

$\Gamma_{ij}^k(p)$ n^3 funzioni definite solo in carte

NON SONO TENSORI

$$\nabla_v X = \underbrace{v^i \frac{\partial X^j}{\partial x^i}}\mathbf{e}_j + v^i X^j \Gamma_{ij}^k \mathbf{e}_K \qquad *$$

$$v^i \frac{\partial X}{\partial x^i} = \frac{\partial X}{\partial v}$$

D'altra parte, se $U \subseteq \mathbb{R}^n$ e $\Gamma_{ij}^k \in \mathcal{C}^\infty(U)$

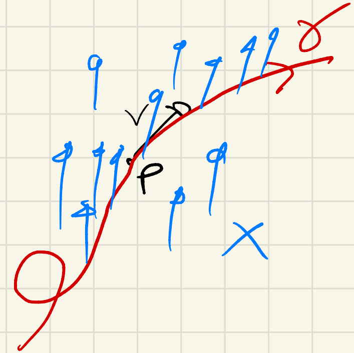
definiscono una connessione ∇ con $\circledast$

<u>Ex</u>: Effettivamente ∇ soddisfa (1) (2) (3) (4)

$(\mathbb{R}^n, \nabla^E)$ $\quad$ ∇^E connessione euclidea

$$\Gamma_{ij}^k \equiv 0$$

Prop: (M, ∇)



Sia $\gamma : I \to M$ curva

t.c. $\gamma'(t_0) = v$

$\gamma(t_0) = p$

$\forall X \in \mathfrak{X}(U(p))$, $\nabla_v X$ dipende solo dai valori di X su γ

dim:

$$\nabla_v X = \underbrace{v^i \frac{\partial X^j}{\partial x^i}}\, e_j + v^i X^j \Gamma_{ij}^k e_k$$

$$v^i \frac{\partial X}{\partial x^i} = \frac{\partial X}{\partial v}$$

dip. da $X|_\gamma$

dip. solo da $X(p)$

CAMPI SU CURVE

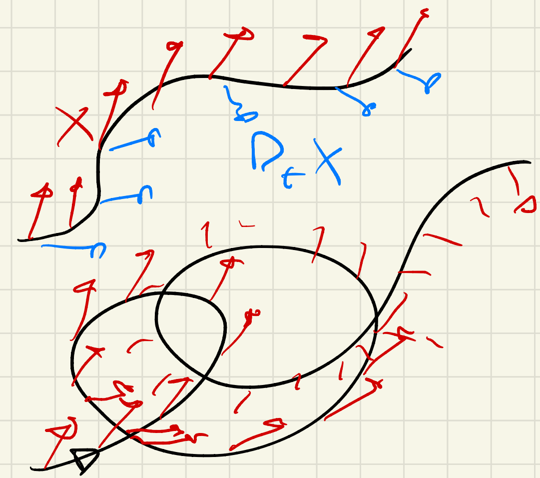
(M, ∇)

$\gamma : I \to M$ REGOLARE, cioè $\dot{\gamma}(t) \neq 0$ $\forall t$

Def: Un CAMPO SU γ è

$X : I \to TM$ t.c.

$X(t) \in T_{\gamma(t)} M$

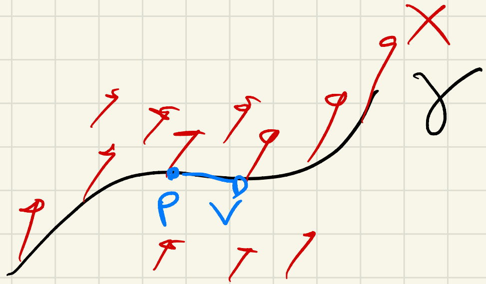


Def: Se X è un campo su γ regolare

$D_t X$ CAMPO DERIVATO LUNGO γ

è la "derivata di X lungo γ"

Ogni γ reg. è loc. iniettiva e il suo supporto è sottovarietà dim 1



$v = \gamma'(t_0)$

$p = \gamma(t_0)$

$$D_t X(t_0) = \nabla_v X$$

si estende X su $U(p)$

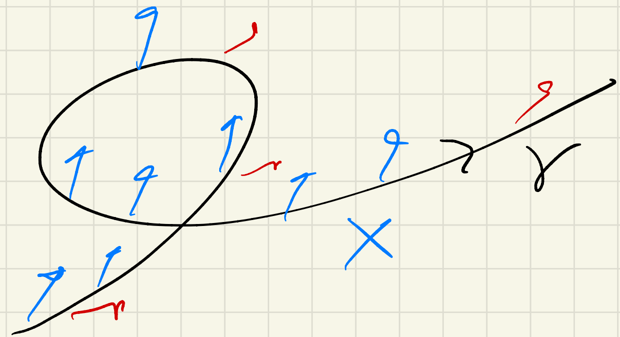
In carte: $\gamma(t) \in \mathbb{R}^n \qquad X = X^i e_i$

$$\nabla_v X = \underbrace{v^i \frac{\partial X^j}{\partial x^i}}_{} e_j + v^i X^j \Gamma_{ij}^k e_k$$

$$v^i \frac{\partial X}{\partial x^i} = \frac{\partial X}{\partial v}$$

$$v = \gamma'(t_0)$$

$$D_t X(t_0) = \frac{dX}{dt}(t_0) + (\gamma'(t_0))^i X^j_{(t_0)} \Gamma^k_{ij}(\gamma(t_0))\, e_k$$



$X(t)$ $\qquad$ $\gamma(t)$

$x(t)$

$$D_t X = \frac{dX}{dt} + \dot{x}^i X^j \Gamma^k_{ij} e_k$$

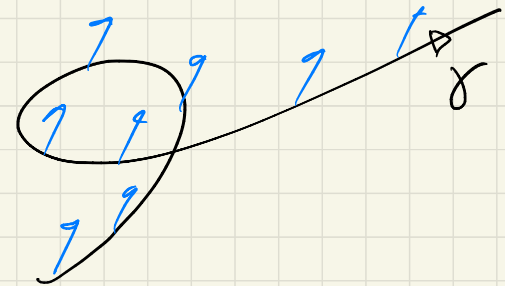
TRASPORTO PALLELO

Def: $\gamma : I \to M$ $\qquad (M, \nabla)$

$X : I \to TM$ campo su γ

X è PARALLELO se $D_t X = 0$

∇^E $\quad \Gamma_{ij}^k = 0$



$X(t) \in \mathbb{R}^n$

$\gamma(t) \in \mathbb{R}^n$

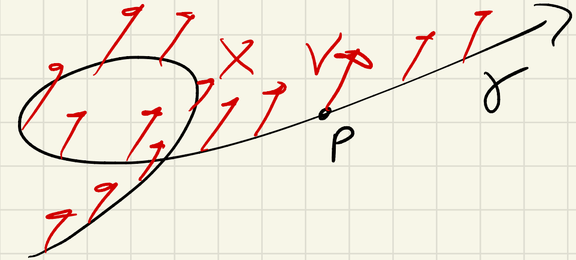
X parallelo $\iff D_t X \equiv 0 \iff X$ cost

$$D_t X = \frac{dX}{dt} + \overbrace{\underset{\overset{\shortparallel}{0}}{\Gamma_{ij}^k}}^{0}$$

<u>Prop:</u> Data (M, ∇) $\quad \gamma$ reg. $\quad p = \gamma(t_0)$

$\exists!$ X su γ parall. t.c. $X(t_0) = v$ $\quad v \in T_pM$

dim:



Basta dimostrarlo localmente

In carte: X è parallelo $\Leftrightarrow$

$$\frac{dX^k}{dt}(t) + (\gamma'(t))^i X^j_{(t)} \Gamma^k_{ij}(\gamma(t)) \equiv 0$$

sistema di eq. diff. lineari $k=1,\dots,n$

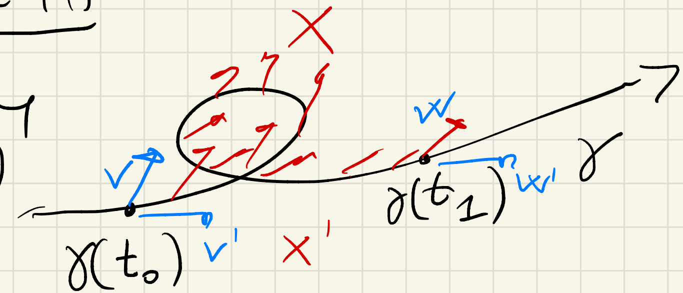
a coeff. variabili

$$X^k : I \to \mathbb{R}$$

$\exists!$ Probl. Cauchy $\Rightarrow$ $\exists!$ soluz. ~~loc.~~ su tutto I t.c. $X^k(t_0) = v$

PROPRIETÀ

$$\Gamma(\gamma)_{t_0}^{t_1} : T_{\gamma(t_0)}M \to T_{\gamma(t_1)}M$$



$$\begin{tikzcd} v \arrow[r, mapsto] \arrow[d, dashed] & w \\ X \arrow[r, dashed] & X(t_1) \arrow[u, phantom, "="] \end{tikzcd}$$

$$\lambda v + \lambda' v' \to \lambda w + \lambda' w'$$

$$\lambda X + \lambda' X'$$

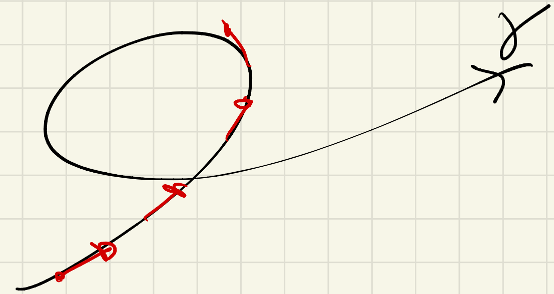
Prop: $\Gamma(\gamma)_{t_0}^{t_1}$ è un isomorfismo.

dim:

- è lineare
- $\Gamma(\gamma)_{t_0}^{t_1} \circ \Gamma(\gamma)_{t_1}^{t_0} = \Gamma(\gamma)_{t_0}^{t_0}$

GEODETICHE

$\gamma: I \to M \qquad (M, D)$



Esempio di campo lungo γ

$X(t) = \gamma'(t)$

<u>Def</u>: γ è GEODETICA se $\gamma'(t)$ è parallelo.