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# Blow-Up Estimates at Horizontal Points and Applications

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**Abstract** Horizontal points of smooth submanifolds in stratified groups play the role of singular points with respect to the Carnot-Carathéodory distance. When we consider hypersurfaces, they coincide with the well known characteristic points. In two step groups, we obtain pointwise estimates for the Riemannian surface measure at all horizontal points of $C^{1,1}$ smooth submanifolds. As an application, we establish an integral formula to compute the spherical Hausdorff measure of any $C^{1,1}$ submanifold. Our technique also shows that $C^2$ smooth submanifolds everywhere admit an intrinsic blow-up and that the limit set is an intrinsically homogeneous algebraic variety.

**Keywords** Stratified groups · Submanifolds · Area formula · Hausdorff measure · Horizontal points

**Mathematics Subject Classification (2000)** 28A75 · 22E25

## 1 Introduction

The geometric properties of one-codimensional sets in stratified groups have been investigated under different perspectives, [3, 6, 7, 9–11, 13–16, 19, 31, 32]. This is an attractive area of research, where several important issues still deserve further investigations. Along with this research stream, some works on the geometry of higher codimensional sets also have started to appear, [4, 5, 20, 25, 27–29].

This paper can be thought of as a continuation of the project started in [27–29], related to the intrinsic measure of arbitrary submanifolds in stratified groups. Higher

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codimensional submanifolds somehow possess an elusive nature, due to the higher degree of freedom of their tangent spaces. The classical Frobenius theorem implies that a smooth hypersurface must have points that are transversal to the horizontal subbundle of the group and this implies in turn a precise Hausdorff dimension of the submanifold, along with an area type formula, [27]. However, this theorem does not suffice to tackle higher codimensional surfaces, whose Hausdorff dimension depends on an interesting interaction between their tangent bundle and the grading of the group, according to Gromov's formula in 0.6 B of [22].

Using the notion of "degree", this interaction has been made more suitable for computations in [29], where the density of their spherical Hausdorff measure has been computed under a "negligibility condition". In broad terms, the degree of a submanifold $\Sigma$ at a fixed point $x$ is a certain weight $d_\Sigma(x)$ assigned to the tangent space $T_x\Sigma$, which depends on its intersections with the flag generated by the grading; see Sect. 2 and Remark 4.2. The maximum over all pointwise degrees is the degree $d(\Sigma)$ of the submanifold. If $d$ is the degree of a submanifold, then the negligibility condition amounts to the $\mathcal{S}^d$-negligibility of points of degree lower than $d$. Here $\mathcal{S}^d$ is the spherical Hausdorff measure constructed with respect to a fixed homogeneous distance of the group, which is equivalent to and generalizes the so-called Carnot-Carathéodory distance. This negligibility condition holds for $C^1$ non-horizontal $k$-codimensional submanifolds, along with an area-type formula for their $\mathcal{S}^{Q-k}$-measure, where $Q$ is the group's Hausdorff dimension; see [27, 28]. One of the main results of this paper is the negligibility condition for all submanifolds in two step groups. This is a consequence of a more general result, namely, the following blow-up estimates.

**Theorem 1.1** *Let $\mathbb{G}$ be a two step stratified group and let $\Sigma \subset \mathbb{G}$ be a $p$-dimensional $C^{1,1}$ smooth submanifold. Then for every $x \in \Sigma$ there exist a neighborhood $U$ of $x$ and positive constants $c_1, c_2$ and $r_0$ depending on $U \cap \Sigma$ such that*

$$c_1 r^{d_\Sigma(x)} \le \tilde{\mu}_p(\Sigma \cap B_{z,r}) \le c_2 r^{d_\Sigma(x)} \tag{1}$$

*for every $z \in \Sigma \cap U$ with $d_\Sigma(z) = d_\Sigma(x)$ and every $0 < r < r_0$.*

We have denoted by $\tilde{\mu}_p$ the Riemannian surface measure induced on the $p$-dimensional submanifold $\Sigma$ by a fixed Riemannian metric $\tilde{g}$ on the group. Notice that $\tilde{g}$ here has an auxiliary role and need not be left invariant. Before presenting some applications of (1), we wish to discuss these estimates in more detail.

First of all, the degree of $x$ in Theorem 1.1 plays an important role. In fact, in the special case $x$ is non-horizontal, namely, it has degree equal to $Q-k$, where $k$ is the codimension of $\Sigma$, then from results of [28], adopting the same approach of this paper, one achieves (1) with $C^1$ smoothness of $\Sigma$. Similarly, if $x$ has maximum degree, then results of [29] apply and lead to (1). The point of Theorem 1.1 is that we do not assume any restriction on $x$. Thus, our estimates also hold at horizontal points, which play the role of characteristic points in higher codimension and actually coincide with them in codimension one, as discussed in [28].

To compare these estimates with the existing literature in codimension one, we first notice that the degree of characteristic points in two step groups is exactly $Q-2$;

see Remark 2.9. Then our blow-up estimates include (20) of [26] and extend it to arbitrary submanifolds. Other interesting estimates have been recently obtained in [9], especially for the perimeter measure, in connection with boundary regularity of domains for the Dirichlet problem in CC-spaces. Next, we discuss a series of applications of (1).

**Corollary 1.2** *Let $\Sigma$ be a $p$-dimensional $C^{1,1}$ smooth submanifold of degree $d$ in a two step stratified group. Then we have*

$$\mathcal{S}^d\left(\left\{x \in \Sigma \mid d_\Sigma(x) < d\right\}\right) = 0 \tag{2}$$

*and the following formula holds*

$$\int_\Sigma \theta(\tau_\Sigma^d(x))d\mathcal{S}^d(x) = \int_\Sigma |\tau_\Sigma^d(x)|d\tilde{\mu}_p(x). \tag{3}$$

The area-type formula (3) is an immediate consequence of (2), after the blow-up at points of maximum degree. For a proof and a discussion of this formula, we address the reader to [29], where the negligibility condition (2) was announced. Furthermore, Theorem 1.1 allows us to distinguish different types of horizontal points, to get estimates on their Hausdorff dimension depending on their degree.

**Corollary 1.3** *Let $\Sigma$ be a $C^{1,1}$ submanifold of degree $d$ in a two step stratified group and let $\delta$ be a positive integer. Then the subset $Z_\delta = \{x \in \Sigma \mid d_\Sigma(x) \leq \delta\}$ is countably $\mathcal{H}^\delta$-finite. In particular, its Hausdorff dimension is less than or equal to $\delta$.*

Remark 4.1 shows how Corollary 1.2 easily implies that the degree of a $C^{1,1}$ submanifold coincides with its Hausdorff dimension. Then Corollary 1.3 gives new information if $\delta < d(\Sigma)$. Again, taking into account Remark 2.9, this extends (26) of [26] to arbitrary submanifolds. Our observation on the Hausdorff dimension of submanifolds in two step groups fits into Gromov's formula pertaining to the Hausdorff dimension of a smooth submanifold in equiregular Carnot-Carathéodory spaces, provided that the functions $D'(x)$ and $D_H(\Sigma)$ introduced in 0.6 B of [22] coincide with the pointwise degree $d_\Sigma(x)$ and the degree $d(\Sigma)$, respectively. Although this is not difficult to check, in Remark 4.2 we show this fact.

It is then natural to search all possible Hausdorff dimensions of submanifolds whose topological dimension is fixed. This is the so-called "Gromov's dimension comparison problem", recently raised in [5], where the authors solve an interesting variant of this problem, replacing the topological dimension of a submanifold with the Euclidean Hausdorff dimension of a set. Finding the Hausdorff dimension of a submanifold can be turned to the problem of finding its degree. On the other hand, this is not necessarily a simpler problem. Since finding a submanifold of given degree corresponds to solving a system of partial differential equations, and could be tackled with Partial Differential Relations techniques, as it is mentioned in 0.5 C and 0.6 B of [22]. Some computations in this vein can be found in Sect. 4 of [29], to find submanifolds of given degree in the Engel group. The extension of (2) to higher step groups is an intriguing open question, where also regularity of the submanifold is expected to play a role. In the Engel group, using some ad hoc arguments, this negligibility condition has been recently achieved for arbitrary submanifolds, solving Gromov's dimension comparison problem in this group, [25]. It is worth stressing that also in this case $C^{1,1}$ regularity of submanifolds suffices, hence it would be interesting to understand whether this regularity suffices for all higher step groups. If we strengthen our regularity assumptions to $C^2$ smoothness, then we also get the existence of the blow-up set, which corresponds to the limit of the rescaled manifold.

**Theorem 1.4** *Let $\mathbb{G}$ be a two step stratified group and let $\Sigma \subset \mathbb{G}$ be a $C^2$ smooth submanifold. Then for every $x \in \Sigma$ the rescaled set $\delta_{1/r} l_{x^{-1}} \Sigma$ locally converges with respect to the Hausdorff distance of sets to an algebraic variety, which is the graph of a homogeneous polynomial function.*

Notice that at points of maximum degree the intrinsic blow-up set is precisely a subgroup and $C^{1,1}$ regularity suffices, [29]. On the other hand, it is easy to find simple cases where this limit set is not a linear subspace; see for instance Example 4.3. It is rather surprising that Theorem 1.4 holds only in two step groups. In fact, in the Engel group it is already possible to find a point of low degree in a 2-dimensional submanifold whose blow-up is a half plane with boundary; see Remark 4.5 of [29]. This case marks how even smooth submanifolds, when embedded in higher step groups, allow for points with more intricate intrinsic singularities.

This naturally leads us to discuss our assumptions on the step of the ambient space. In fact, the previous example is not the only case that shows the significant dichotomy between the geometry in step less than or equal to two and that in higher step. For instance, in two step groups, $C^{1,1}$ domains satisfy the Sobolev-Poincaré inequality and are furthermore NTA domains with respect to the Carnot-Carathéodory distance, but this does not hold in higher step groups; see [8, 9, 13, 21, 23, 24, 30] and the references therein. In two step stratified groups the classical De Giorgi rectifiability theorem holds, [1, 19], but the fine technology of its proof does not apply in higher step, [19]. Substantial progress has been recently obtained in [3], where the difficulty arising in higher step is also emphasized.

Finally, we point out that Theorem 1.1 precisely holds in the larger class of two step graded groups. In fact, our proofs do not use the Lie bracket generating condition of the first layer. Then one can extend our results to all nilpotent groups of step two, since they admit infinitely many gradings that make them graded groups, according to Remark 2.1.

## 2 Basic Notions

We begin this section by defining the class of groups with which we shall be dealing in this paper. These are connected, simply connected, real Lie groups of finite dimension. If one of these groups $\mathbb{G}$ has a graded Lie algebra $\mathcal{G}$, then we will say that $\mathbb{G}$ is a *graded group*. The algebra $\mathcal{G}$ is said to be graded if it can be written as direct sum of subspaces $\mathcal{G} = V_1 \oplus \cdots \oplus V_\iota$, with the property $[V_i, V_j] \subset V_{i+j}$, for any $i, j \geq 1$ and $V_i = \{0\}$ iff $i > \iota$.

*Remark 2.1* Let $\mathfrak{g}$ be a two step nilpotent algebra and let $\mathfrak{g}^2 = [\mathfrak{g}, \mathfrak{g}]$, which is a non-null subspace. Let $\mathfrak{h}$ be a subspace of $\mathfrak{g}$ such that $\mathfrak{h} \oplus \mathfrak{g}^2 = \mathfrak{g}$. Then setting $V_1 = \mathfrak{h}$ and $V_2 = \mathfrak{g}^2$ makes $\mathfrak{g}$ a graded group. There are infinitely many $\mathfrak{h}$'s that are complementary to $\mathfrak{g}^2$.

The stronger assumption on the subspaces $[V_1, V_j] = V_{j+1}$ implies that the group $\mathbb{G}$ is *stratified*, see [18] for more information. The subspace $V_1$ of $\mathcal{G}$ defines at any point $x \in \mathbb{G}$ the horizontal subspace

$$H_x\mathbb{G} = \{X(x) \mid X \in V_1\}.$$

Left translations of the group are denoted by $l_x : \mathbb{G} \to \mathbb{G}$, $l_x(y) = xy$. The graded structure defines a one-parameter group of dilations $\delta_r : \mathcal{G} \to \mathcal{G}$, where $r > 0$. Precisely, we have

$$\delta_r\left(\sum_{j=1}^{\iota} v_j\right) = \sum_{j=1}^{\iota} r^j v_j,$$

where $\sum_{j=1}^{\iota} v_j = v$ and $v_j \in V_j$ for each $j = 1, \dots, \iota$. To any element of $V_j$ we associate the integer $j$, which is called the *degree* of the vector. Under our assumptions we have that $\exp : \mathcal{G} \to \mathbb{G}$ is an analytic diffeomorphism, hence there is a canonical way to transpose dilations from $\mathcal{G}$ to $\mathbb{G}$. We will use the same symbol to denote dilations of the group. We have the standard properties

(1) $\delta_r(x \cdot y) = \delta_r x \cdot \delta_r y$ for any $x, y \in \mathbb{G}$ and $r > 0$,
(2) $\delta_r(\delta_s x) = \delta_{rs} x$ for any $r, s > 0$ and $x \in \mathbb{G}$.

To provide a metric structure on the group, we will fix a *graded metric* on $\mathbb{G}$, namely a left invariant metric such that all the subspaces $V_j$ of the Lie algebra $\mathcal{G}$ are orthogonal to each other. The *sub-Riemannian structure* of graded groups is given by any *homogeneous distance*, which is a continuous distance $\rho : \mathbb{G} \times \mathbb{G} \to \mathbb{R}$ that satisfies the properties

(1) $\rho(x, y) = \rho(ux, uy)$ for every $u, x, y \in \mathbb{G}$,
(2) $\rho(\delta_r x, \delta_r y) = r\rho(x, y)$ for every $r > 0$.

In fact, if the group is stratified, then the well known Carnot-Carathéodory distance provides the foremost example of homogeneous distance. When the group is graded it is still possible to introduce a homogeneous distance; see the appendix of [19]. However, the group equipped with this distance might not be connected by rectifiable curves.

*Example 2.2* We consider the Heisenberg group $\mathbb{H}^1$ expressed in coordinates $(x, y, t)$, satisfying the group law $(x, y, t)(x', y', t') = (x + x', y + y', t + t' + xy' - yx')$ and having dilations acting as $\delta_r(x, y, t) = (rx, ry, r^2 t)$. Then we define the vertical subgroup $\Pi$ of $\mathbb{H}^1$ corresponding to the subspace $\{(0, y, t) \mid y, t \in \mathbb{R}\}$. It is immediately realized that $d(y, t) = |y| + |t|^{1/2}$ is a homogeneous distance on the graded group $\Pi \subset \mathbb{H}^1$. In fact, the group law restricted to $\Pi$ is the commutative sum of vectors.

On the other hand, all homogeneous distances are bi-Lipschitz equivalent and induce the topology of $\mathbb{G}$. This is an easy consequence of properties (1) and (2), following the classical argument for norms of finite dimensional vector spaces.

**Definition 2.3** We define the subsets $B_{x,r}$ and $D_{x,r}$ of $\mathbb{G}$ as the open and closed ball, respectively, of center $x$ and radius $r > 0$ with respect to a homogeneous distance. We will omit the center of the ball if it coincides with the unit element of the group.

**Definition 2.4** (Adapted bases, graded bases and graded coordinates) Let us set $m_j = \dim V_j$ for any $j = 1, \dots, \iota$, $n_0 = 0$ and $n_i = \sum_{j=1}^{i} m_j$ for any $i = 1, \dots, \iota$. We say that $(W_1, \dots, W_q)$ is an *adapted basis* of $\mathcal{G}$ if

$$(W_{n_{j-1}+1}, W_{n_{j-1}+2}, \dots, W_{n_j})$$

is a basis of $V_j$ for any $j = 1, \dots, \iota$. Let us consider the mapping $F : \mathbb{R}^q \to \mathbb{G}$ defined by

$$F(y) = \exp\left(\sum_{i=1}^{q} y_i W_i\right).$$

If $(W_1, \dots, W_q)$ is also orthonormal, then we say that it is a *graded basis* and that $F$ is a system of *graded coordinates*. The *degree* of $y_i$ is set as $d_i = j$ if $W_i \in V_j$.

**Definition 2.5** (Degree of $p$-vectors) Let $(W_1, W_2, \dots, W_q)$ be an adapted basis of $\mathcal{G}$. The *degree* of the simple $p$-vector

$$W_J := W_{j_1} \wedge \cdots \wedge W_{j_p}$$

in $\Lambda_p \mathcal{G}$, with $J = (j_1, j_2, \dots, j_p)$ and $1 \le j_1 < j_2 < \cdots < j_p \le q$ is the sum $d_{j_1} + \cdots + d_{j_p}$. We denote this integer by $d_J$. Now, let $\tau \in \Lambda_p(\mathcal{G})$ be a simple $p$-vector and let $1 \le r \le Q$ be an integer. Let $\tau = \sum_J \tau_J W_J$, $\tau_J \in \mathbb{R}$, be represented with respect to the fixed adapted basis. The projection of $\tau$ with degree $r$ is defined as $(\tau)_r = \sum_{d(J)=r} \tau_J W_J$ and the *degree* of $\tau$ is defined as the integer

$$d(\tau) = \max\{d_J \mid \text{such that } \tau_J \neq 0\}.$$

**Definition 2.6** (Degree of manifolds) Let $\Sigma$ be a $C^1$ smooth $p$-dimensional submanifold and let $\tau_\Sigma(x)$ be a tangent $p$-vector of $\Sigma$ at $x \in \Sigma$. All of them are proportional. Then the *degree* of $\Sigma$ at $x$ is the positive integer

$$d_\Sigma(x) = d\big(\tau_\Sigma(x)\big).$$

The *degree* of $\Sigma$ is the number $d(\Sigma) = \max_{x \in \Sigma} d_\Sigma(x)$.

*Example 2.7* Consider the two step stratified group $\mathbb{H}^1 \times \mathbb{H}^1$, which we represent by graded coordinates $(x, y, t) = (x_1, y_1, x_2, y_2, t_1, t_2) \in \mathbb{R}^6$, satisfying the group law

$$(x, y, t)(x', y', t') = (x + x', y + y', t + t') + (0, 0, x_1 y_1' - y_1 x_1', x_2 y_2' - y_2 x_2').$$

Let $\Sigma$ be a 3-dimensional submanifold of $\mathbb{H}^1 \times \mathbb{H}^1$ passing through the origin, which is given by the zero level set of $f : \mathbb{R}^6 \to \mathbb{R}^3$,

$$f(x, y, t) = \big(t_1 + t_1^3 - x_1^2 - y_1^2 - x_2^3, x_1^2 + x_2^4 - t_2, x_2 - x_1^3\big).$$

It is easy to see that $\Sigma$ can be globally parameterized by

$$\Phi(x_1, y_1, y_2) = \big(x_1, y_1, x_1^3, y_2, \varphi(x_1^2 + y_1^2 + x_1^9), x_1^2 + x_1^{12}\big),$$

where $\varphi : \mathbb{R} \to \mathbb{R}$ is the inverse function of $t \to t + t^3$. Let

$$X_1 = \partial_{x_1} - y_1\partial_{t_1}, \qquad Y_1 = \partial_{y_1} + x_1\partial_{t_1}, \qquad X_2 = \partial_{x_2} - y_2\partial_{t_2}, \qquad Y_2 = \partial_{y_2} + x_2\partial_{t_2}$$

be the left invariant vector fields spanning the first layer $V_1$ of $\mathbb{H}^1 \times \mathbb{H}^1$. Then they all have degree equal to one and the tangent space of $\Sigma$ at the origin $T_0\Sigma = \{(x_1, y_1, 0, y_2, 0, 0) \mid x_1, y_1, y_2 \in \mathbb{R}\}$ is spanned by $X_1(0)$, $Y_1(0)$ and $Y_2(0)$. Thus, by the definition of degree, we have $d_\Sigma(0) = 3$. Fix $p = (1, 0, 1, 0, 1, 2) \in \Sigma$ and observe that $T_p\Sigma = \operatorname{span}\{\Phi_{x_1}(q), \Phi_{y_1}(q), \Phi_{y_2}(q)\}$, where $q = (1, 0, 0)$ and $\Phi(q) = p$. We define $T_1 = \partial_{t_1}$ and $T_2 = \partial_{t_2}$, which are constant left invariant vector fields spanning the second layer of $\mathbb{H}^1 \times \mathbb{H}^1$. Then one can check that

$$\begin{aligned}
&\Phi_{x_1}(q) = X_1(p) + 3X_2(p) + 11\varphi'(2)T_1(p) + 14T_2(p),\\
&\Phi_{y_1}(q) = Y_1(p) - T_1(p) \quad \text{and} \quad \Phi_{y_2}(q) = Y_2(p) - T_2(p).
\end{aligned}$$

Taking into account that $\varphi'(2) = 1/4$ and expanding the wedge product

$$\left(X_1(p) + 3X_2(p) + \frac{11}{4}T_1(p) + 14T_2(p)\right) \wedge \big(Y_1(p) - T_1(p)\big) \wedge \big(Y_2(p) - T_2(p)\big),$$

we get that all addends with maximum degree are the following:

$$\begin{aligned}
&X_1(p) \wedge T_1(p) \wedge T_2(p), \qquad 3X_2(p) \wedge T_1(p) \wedge T_2(p),\\
&-\frac{11}{4}T_1(p) \wedge Y_1(p) \wedge T_2(p), \qquad -14T_2(p) \wedge T_1(p) \wedge Y_2(p).
\end{aligned}$$

Taking into account that $d(T_1) = d(T_2) = 2$, along with Definition 2.5 and Definition 2.6, one easily realizes that $d_\Sigma(p) = d(\Sigma) = 5$.

*Remark 2.8* Let us consider the class of smooth horizontal submanifolds; see 0.7.C and 3.1 of [22]. Then it is not difficult to check that in a stratified group the pointwise degree of these submanifolds equals their topological dimension.

*Remark 2.9* Let $\Sigma$ be a $p$-dimensional $C^1$ smooth submanifold and let $h = \dim(T_x\Sigma \cap H_x\mathbb{G})$. If $\mathbb{G}$ is of step two, then the pointwise degree of $\Sigma$ at $x$ is given by

$$d_\Sigma(x) = 2p - h.$$

In fact, one can take a horizontal basis $X_1, \dots, X_h$ of $T_x\Sigma \cap H_x\mathbb{G}$ and linearly independent vectors $T_1, \dots, T_{p-h}$ such that $(X_1, \dots, X_h, T_1, \dots, T_{p-h})$ is a basis of $T_x\Sigma$. Then it is easy to observe that $d_\Sigma(x) = h + 2(p-h)$. In particular, if $\Sigma$ has codimension one, then

$$d_\Sigma(x) = \begin{cases} Q-1 & \text{if } x \text{is not characteristic,} \\ Q-2 & \text{otherwise,} \end{cases} \tag{4}$$

where $Q = m_1 + 2m_2$ is the Hausdorff dimension of $\mathbb{G}$, also called the homogeneous dimension.

**Definition 2.10** (Polynomial mappings) Let $G$ be a connected and simply connected nilpotent Lie group. A mapping of vector spaces is polynomial if it has polynomial components when it is expressed with respect to some bases both in the domain and in the range. A mapping $P : G \to \mathbb{R}^k$ is said to be *polynomial* if so is the composition $P \circ \exp : \mathcal{G} \to \mathbb{R}^k$.

*Remark 2.11* The previous definitions make sense, since the exponential mapping $\exp : \mathcal{G} \to G$ is an analytic diffeomorphism when $G$ is connected, simply connected and nilpotent. The notion of polynomial mapping of vector spaces does not depend on the fixed bases to represent the mapping; see [12] for more details.

**Definition 2.12** (Homogeneous algebraic varieties) Let $\mathbb{G}$ be a graded group. Then the set of zeros of a polynomial mapping $P : G \to \mathbb{R}^k$ for some $k \geq 1$ defines an *algebraic variety* in $G$. We say that an algebraic variety $\mathcal{A}$ is *homogeneous* if $\delta_r\mathcal{A} \subset \mathcal{A}$ for every $r > 0$.

Notice that an algebraic variety defined by a homogeneous polynomial mapping $P : \mathbb{G} \to \mathbb{R}$ is clearly a homogeneous algebraic variety.

*Example 2.13* The set $\Sigma = \{(x, y, xy) \mid x, y \in \mathbb{R}\}$ is a homogeneous algebraic variety of the Heisenberg group $\mathbb{H}^1$ equipped with the standard coordinates $(x, y, t)$ of Example 2.2. In these coordinates, the mapping $P : \mathbb{H}^1 \to \mathbb{R}$ is given by $(x, y, t) \to t - xy$.

Of course homogeneous algebraic varieties need not be regular. It suffices to consider the zero level set of $P(x, y, t) = x^2 - y^2$ in the Heisenberg group with the coordinates defined in Example 2.2.

## 3 Blow-Up Estimates and Blow-Ups

This section is devoted to the proof of all our main results.

*Proof of Theorem 1.1* We first choose a neighborhood $V$ of $x$ and a function $f : V \to \mathbb{R}^k$ such that $\Sigma \cap V = f^{-1}(0)$, the differential $df(s)$ is surjective whenever $s \in f^{-1}(0)$ and the kernel of $df(x)_{|H_s\mathbb{G}} : H_x\mathbb{G} \to \mathbb{R}^k$ has dimension $h =$

$\dim(T_x\Sigma \cap H_x\mathbb{G})$. Let $\kappa = m - h$ and notice that $x$ is non-horizontal if and only if $\kappa = k$, namely $df(x)_{|H_s\mathbb{G}}$ is surjective. In this case, the following proof holds and becomes even simpler. Then it suffices to consider the interesting case $\kappa < k$, namely, the case when $x$ is horizontal. Let $(v_{\kappa+1}, \ldots, v_m)$ be an orthonormal basis of $T_x\Sigma \cap H_x\mathbb{G}$. We choose an orthonormal basis $(v_1, \ldots, v_\kappa)$ of $(T_x\Sigma \cap H_x\mathbb{G})^\perp \cap H_x\mathbb{G}$ and define the unique left invariant orthonormal vector fields $(Y_1, \ldots, Y_m)$ of $V_1$ such that $Y_j(x) = v_j$ for every $j = 1, \ldots, m$. As a consequence, we get

$$Y_j f^i(x) = 0 \quad \text{whenever } j = \kappa + 1, \ldots, m \text{ and } i = 1, \ldots, k. \tag{5}$$

Our hypothesis on the step of $\mathbb{G}$ implies that $T_x\mathbb{G} = H_x\mathbb{G} \oplus H_x^2\mathbb{G}$, where

$$H_x^2\mathbb{G} = \{U(x) \mid U \in V_2\}.$$

In view of the surjectivity of $df(p)$, there exist orthonormal vectors $v_{m+1}, \ldots, v_{m+l} \in H_x^2\mathbb{G}$, with $l = k - \kappa$, such that

$$\dim(\operatorname{span}\{df(x)(v_1), \ldots, df(x)(v_\kappa), df(x)(v_{m+1}), \ldots, df(x)(v_{m+l})\}) = k. \tag{6}$$

As a consequence, we choose a graded basis $(Y_1, \ldots, Y_m, Y_{m+1}, \ldots, Y_q)$ of $\mathcal{G}$ such that

$$Y_j(x) = v_j \qquad \text{for every } j \in \{1, \ldots, \kappa\} \cup \{m+1, \ldots, m+l\}.$$

We fix a system of graded coordinates $F : \mathbb{R}^q \to \mathbb{G}$ defined by $F(y) = \exp(\sum_{j=1}^q y^j Y_j)$ and set $F_x(y) = l_x F(y)$. We introduce the function $\tilde{f}(y) = f(F_x(y))$ and notice that

$$\partial_{y_j}\tilde{f}(0) = Y_j f(p).$$

Then in a neighborhood $V' \subset V$ of $x$ we get

$$\partial_{y_1}\tilde{f} \wedge \partial_{y_2}\tilde{f} \wedge \cdots \wedge \partial_{y_\kappa}\tilde{f} \wedge \partial_{y_{m+1}}\tilde{f} \wedge \cdots \wedge \partial_{y_{m+l}}\tilde{f} \neq 0$$

on $F_x^{-1}(V') \subset \mathbb{R}^q$. From the implicit function theorem there exists

$$\psi(\xi) = (\varphi^1(\xi), \ldots, \varphi^\kappa(\xi), \xi_{\kappa+1}, \ldots, \xi_m, \varphi^{m+1}(\xi), \ldots, \varphi^{m+l}(\xi), \xi_{m+l+1}, \ldots, \xi_q) \tag{7}$$

such that

$$\tilde{f}(\varphi^1(\xi), \ldots, \varphi^\kappa(\xi), \xi_{\kappa+1}, \ldots, \xi_m, \varphi^{m+1}(\xi), \ldots, \varphi^{m+l}(\xi), \xi_{m+l+1}, \ldots, \xi_q) = 0.$$

Setting $\Phi(\xi) = F_x(\psi(\xi))$ with $\xi = (\xi_{\kappa+1}, \ldots, \xi_m, \xi_{m+l+1}, \ldots, \xi_q)$, we clearly have $f \circ \Phi = 0$. We wish to compute the limit of $r^{-[h+2(p-h)]}\mathrm{vol}_{\tilde{g}}(\Sigma \cap B_{x,r})$ as $r \to 0^+$. This quotient equals

$$r^{-[h+2(p-h)]} \int_{\Phi^{-1}(B_{x,r})} \left| \frac{\partial\Phi}{\partial\xi_{\kappa+1}}(\xi) \wedge \cdots \wedge \frac{\partial\Phi}{\partial\xi_m}(\xi) \wedge \frac{\partial\Phi}{\partial\xi_{m+l+1}}(\xi) \wedge \cdots \wedge \frac{\partial\Phi}{\partial\xi_q}(\xi) \right|_{\tilde{g}} d\xi. \tag{8}$$

We restrict dilations to the subspace

$$\Pi = \{\xi \in \mathbb{R}^q \mid \xi_1 = \xi_2 = \cdots = \xi_\kappa = \xi_{m+1} = \cdots = \xi_{m+l} = 0\}$$

and perform the change of variable

$$\xi = \sum_{j=\kappa+1}^{m} r\eta_j e_j + \sum_{j=m+l+1}^{q} r^2 \eta_j e_j = \tilde{\delta}_r \eta. \tag{9}$$

Observing that the Jacobian of $\tilde{\delta}_r$ restricted to $\Pi$ is

$$m - \kappa + 2(q - m - l) = h + 2(p - h),$$

the quotient (8) becomes

$$\int_{\tilde{\delta}_{1/r}\Phi^{-1}(B_{x,r})} \left| \frac{\partial \Phi}{\partial \xi_{\kappa+1}}(\tilde{\delta}_r \eta) \wedge \cdots \wedge \frac{\partial \Phi}{\partial \xi_m}(\tilde{\delta}_r \eta) \wedge \frac{\partial \Phi}{\partial \xi_{m+l+1}}(\tilde{\delta}_r \eta) \wedge \cdots \wedge \frac{\partial \Phi}{\partial \xi_q}(\tilde{\delta}_r \eta) \right|_{\tilde{g}} d\eta,$$

where $\tilde{\delta}_{1/r}\Phi^{-1}(B_{x,r})$ equals the set of points $\xi \in \mathbb{R}^p$ such that

$$\left( \frac{\varphi^1(\tilde{\delta}_r \xi)}{r}, \ldots, \frac{\varphi^\kappa(\tilde{\delta}_r \xi)}{r}, \xi_{\kappa+1}, \ldots, \xi_m, \frac{\varphi^{m+1}(\tilde{\delta}_r \xi)}{r^2}, \ldots, \frac{\varphi^{m+l}(\tilde{\delta}_r \xi)}{r^2}, \xi_{m+l+1}, \ldots, \xi_q \right) \tag{10}$$

belongs to $F^{-1}(B_1)$. Differentiating the equality $f(\Phi(\xi)) = \tilde{f}(\psi(\xi)) = 0$, we get

$$\frac{\partial \tilde{f}}{\partial \xi_i}(\psi) + \sum_{j=1}^{\kappa} \frac{\partial \tilde{f}}{\partial \xi^j}(\psi) \frac{\partial \varphi^j}{\partial \xi^i} + \sum_{j=m+1}^{m+l} \frac{\partial \tilde{f}}{\partial \xi^j}(\psi) \frac{\partial \varphi^j}{\partial \xi^i} = 0,$$

whenever $i = \kappa + 1, \ldots, m$. As a consequence of the previous equality, from (5) we get

$$\left[ \frac{\partial \tilde{f}}{\partial \xi_1} \cdots \frac{\partial \tilde{f}}{\partial \xi_\kappa} \frac{\partial \tilde{f}}{\partial \xi_{m+1}} \cdots \frac{\partial \tilde{f}}{\partial \xi_{m+l}} \right]_{|\xi=0} \left[ \frac{\partial \varphi}{\partial \xi_i} \right]_{|\xi'=0} = 0,$$

hence (6) yields

$$\frac{\partial \varphi^j}{\partial \xi^i}(0) = 0 \quad \text{whenever} \begin{cases} j \in \{1, \ldots, \kappa\} \cup \{m+1, \ldots, m+l\}, \\ i \in \{\kappa+1, \ldots, m\}. \end{cases} \tag{11}$$

As a consequence, Taylor expansion yields

$$\varphi^j(\tilde{\delta}_r \xi) = r^2 \sum_{m+l<i\leq q} \frac{\partial \varphi^j}{\partial \xi_i}(0)\xi_i + O^j(|\delta_r \xi|^2), \tag{12}$$

where $|O^j(|y|)| \leq L|y|^2$ and $L$ is the Lipschitz constant of $\nabla \varphi^j$. Thus, there exists $M > 0$, depending on $L$, such that

$$\left| r^{-2} \varphi^j(\tilde{\delta}_r \xi) \right| \leq M|\xi|$$

for $r > 0$ sufficiently small. Let us choose a product of open intervals

$$J = I_1 \times I_2 \times \cdots \times I_q \subset F^{-1}(B_1)$$

such that $0 \in J$. Now we define the open set $A_L$ formed by those points $\xi \in \mathbb{R}^p$ such that

$$(M|\xi|, \ldots, M|\xi|, \xi_{\kappa+1}, \ldots, \xi_m, M|\xi|, \ldots, M|\xi|, \xi_{m+l+1}, \ldots, \xi_q) \in J \quad \text{and}$$
$$(-M|\xi|, \ldots, -M|\xi|, \xi_{\kappa+1}, \ldots, \xi_m, -M|\xi|, \ldots, -M|\xi|, \xi_{m+l+1}, \ldots, \xi_q) \in J.$$

Clearly, the size of $A_L$ depends on $L$ and we have

$$A_L \subset \tilde{\delta}_{1/r}\Phi^{-1}(B_{x,r})$$

for $r$ sufficiently small. It follows that

$$\frac{\mathrm{vol}_{\tilde{g}}(\Sigma \cap B_{x,r})}{r^{h+2(p-h)}} \geq \mathcal{H}^d_{|\cdot|}\left(A_L \cap F^{-1}(B_1)\right)\gamma(d\Phi), \tag{13}$$

where

$$\gamma(d\Phi) = \inf_{y \in U}\left|\frac{\partial\Phi}{\partial\xi_{\kappa+1}}(y) \wedge \cdots \wedge \frac{\partial\Phi}{\partial\xi_m}(y) \wedge \frac{\partial\Phi}{\partial\xi_{m+l+1}}(y) \wedge \cdots \wedge \frac{\partial\Phi}{\partial\xi_q}(y)\right|_{\tilde{g}}$$

and $U$ is a suitable neighborhood of the origin. From Hadamard's inequality we get

$$\left|\frac{\partial\Phi}{\partial\xi_{\kappa+1}}(y) \wedge \cdots \wedge \frac{\partial\Phi}{\partial\xi_m}(y) \wedge \frac{\partial\Phi}{\partial\xi_{m+l+1}}(y) \wedge \cdots \wedge \frac{\partial\Phi}{\partial\xi_q}(y)\right|_{\tilde{g}} \tag{14}$$

$$\leq \prod_{i \in K}\sqrt{\sum_{j \in K}\left\langle\frac{\partial\Phi}{\partial\xi_i}(y), \frac{\partial\Phi}{\partial\xi_j}(y)\right\rangle^2_{\tilde{g}}} \leq \left(\sum_{j \in K}\left|\frac{\partial\Phi}{\partial\xi_j}(y)\right|^2_{\tilde{g}}\right)^p \leq p^p\|d\Phi(y)\|^{2p}, \tag{15}$$

where $K = \{\kappa+1, \ldots, m\} \cup \{m+l+1, \ldots, q\}$. As a consequence, observing that we can find a bounded set $S \subset \mathbb{R}^p$ containing $\tilde{\delta}_{1/r}\Phi^{-1}(B_{x,r})$ for every $r > 0$, we obtain

$$\frac{\mathrm{vol}_{\tilde{g}}(\Sigma \cap B_{x,r})}{r^{h+2(p-h)}} \leq \mathcal{H}^p_{|\cdot|}\left(S \cap F^{-1}(B_1)\right)p^p L_0^{2p}, \tag{16}$$

where $L_0$ is the Lipschitz constant of the local parameterization $\Phi$ of the submanifold, the estimate (1) follows, taking into account Remark 2.9. Now, we have to study the dependence of our constants in (13) and (16) on the point $x$ when it varies in the subset of points having the same degree. Let $U$ be a neighborhood of $x$ such that (6) holds, replacing $x$ with any point of $U$. If $U \cap \{z \in \Sigma \mid d_\Sigma(z) = d_\Sigma(x)\}$ coincides with $x$ there is nothing to prove, then assume that

$$U \cap \{z \in \Sigma \mid d_\Sigma(z) = d_\Sigma(x)\} \setminus \{x\} \neq \emptyset.$$

Since the degree $\Sigma \ni z \to d_\Sigma(z)$ is lower semicontinuous, then the set

$$A_x = \{z \in \Sigma \mid d_\Sigma(z) > d_\Sigma(x) - 1\}$$

is an open neighborhood of $S_x = \{z \in \Sigma \mid d_\Sigma(z) = d_\Sigma(x)\}$. Furthermore, $S_x$ is closed in $A_x$, since

$$S_x = A_x \cap \{z \in \Sigma \mid d_\Sigma(z) \le d_\Sigma(x)\}.$$

Since the degree is constant on $S'_x = S_x \cap U$, then we can find locally Lipschitz continuous vector fields $T_1, \dots, T_h$ on $S'_x$ such that

$$\operatorname{span}\{T_1(z), \dots, T_h(z)\} = H_z\mathbb{G} \cap T_z\Sigma$$

for every $z \in S'_x$; see Remark 3.1. Then one can repeat the proof of the estimates (13) and (16), replacing $x$ with $z \in S'_x$ and the basis $(v_{\kappa+1}, \dots, v_m)$ of $H_x\mathbb{G} \cap T_x\Sigma$ with $(T_1(z), \dots, T_h(z))$. It is not restrictive assuming that $T_j(x) = Y_j(x)$. The key point is that (5) is replaced by

$$T_j f^i(z) = 0 \quad \text{whenever } j = 1, \dots, h, i = 1, \dots, k \quad \text{and} \quad z \in S'_x. \tag{17}$$

Furthermore, the vectors

$$\big(Y_1(z), \dots, Y_\kappa(z), T_1(z), \dots, T_h(z), Y_{m+1}(z), \dots, Y_q(z)\big)$$

are linearly independent and locally Lipschitz continuous on $S'_x$. Thus, the corresponding local graph centered at $z$

$$\psi_z(\xi) = (\varphi_z^1(\xi), \dots, \varphi_z^\kappa(\xi), \xi_{\kappa+1}, \dots, \xi_m, \varphi_z^{m+1}(\xi), \dots, \varphi_z^{m+l}(\xi), \xi_{m+l+1}, \dots, \xi_q) \tag{18}$$

is a Lipschitz deformation of (7) and it coincides with it for $z = x$. Then the constants appearing in (13) and (16) can be taken to be independent of $z$ as it varies in a compact neighborhood $S''_x \subset S'_x$ of $x$ in the relatively open set $A_x$. This concludes the proof. □

*Remark 3.1* Let $\Sigma$ be a $C^1$ manifold with a countable basis for its topology and consider a closed subset $F$ of $\Sigma$. Let $F \ni x \to C(x) \in M_n(\mathbb{R})$ be a Lipschitz continuous matrix having constant rank equal to $s < n$ on all points of $F \subset \mathbb{R}^q$. Then one can find locally Lipschitz continuous vector fields $T_1, \dots, T_h$ on $F$, with $h = n - s$, such that

$$\operatorname{span}\{T_1(x), \dots, T_h(x)\} = \ker C(x) \quad \text{for every } x \in F.$$

Since $\Sigma$ admits a $C^1$ partition of unity that can be eventually restricted to $F$, it suffices to prove that $T_j$ can be found in any neighborhood of a point of $F$. Then it is not restrictive assuming that for instance the first $s$ columns $C_1, \dots, C_s$ of $C$ are linearly independent on a relatively open subset $A$ of $F$. Then there exists a projection $\pi : \mathbb{R}^q \to \mathbb{R}^s$, such that $\tilde{C} = [\pi C_1 \cdots \pi C_s]$ is an invertible matrix of $M_s(\mathbb{R})$ on a possibly smaller open subset $A'$ and for every $x \in A'$ we have that

$$\eta_\xi(x) = \tilde{C}(x)^{-1}\big[\pi C_{s+1}(x) \cdots \pi C_n(x)\big]\xi$$

is locally Lipschitz continuous for every $\xi \in \mathbb{R}^h$. Taking $\xi = e_j$ for every $j = 1, \dots, h$, where $(e_j)$ is the canonical basis of $\mathbb{R}^h$ and identifying this basis with a basis of $\{0\} \times \mathbb{R}^h \subset \mathbb{R}^n$, we get

$$T_j(x) = \big(\eta^1_{e_j}(x), \dots, \eta^s_{e_j}(x), 0, \dots, 0\big) + e_j$$

that are clearly linearly independent and locally Lipschitz continuous on $A'$.

*Proof of Theorem 1.4* We will use the notation of the proof of Theorem 1.1. If $x$ is non-horizontal, namely, $\kappa = k$, then the following proof holds and becomes simpler and in this special case, our claim is already explicitly shown in [28]. Then we consider the case $\kappa < k$, namely, $x$ is horizontal, and we argue exactly as in the proof of Theorem 1.1, until we have obtained the local parameterization $\Phi = l_x \circ F \circ \psi$ of $\Sigma$ around $x \in \Sigma$, where $\psi$ is defined in (7).

As a consequence of $C^2$ regularity, for every $j \in \{1, \dots, \kappa\} \cup \{m+1, \dots, m+l\}$, the expansion (12) can be written more precisely as follows

$$\begin{aligned}\varphi^j(\delta_r \xi) &= \sum_{m+l<i\le q} \varphi^j_{\xi_i}(0) r^2 \xi_i + \frac{1}{2} \sum_{\kappa<i,j\le m} \varphi^j_{\xi_i\xi_j}(0) r^2 \xi_i \xi_j + o(|\delta_r \xi|^2) \\ &= r^2 Q_j(\xi_{m+l+1}, \dots, \xi_q) + o(|\delta_r \xi|^2),\end{aligned} \tag{19}$$

where dilations $\tilde{\delta}_r$ are defined in (9) and $Q_j$ are homogeneous polynomials. Now we consider the variables

$$\xi = \sum_{j=\kappa+1}^{m} \xi_j e_j + \sum_{j=m+l+1}^{q} \xi_j e_j \quad \text{and} \quad \xi' = \sum_{j=m+l+1}^{q} \xi_j e_j$$

that vary in a $p$-dimensional subspace of $\mathbb{R}^p$ and in a $(q-m-l)$-dimensional subspace of $\mathbb{R}^q$, respectively. Then the polynomial mapping

$$T(\xi) = (0, \dots, 0, \xi_{\kappa+1}, \dots, \xi_m, Q_{m+1}(\xi'), \dots, Q_{m+l}(\xi'), \xi_{m+l+1}, \dots, \xi_q) \tag{20}$$

defines the $d$-dimensional homogeneous algebraic variety

$$\mathcal{A} = (F \circ T)(\mathbb{R}^p) \subset \mathbb{G}$$

that has no singular points, since it is the graph of a polynomial mapping. We recall that $F : \mathbb{R}^q \to \mathbb{G}$ is the system of graded coordinates defined in the proof of Theorem 1.1.

The variety $\mathcal{A}$ is the zero set of the homogeneous polynomial mapping $P : \mathbb{G} \to \mathbb{R}^k$, where

$$\begin{aligned}P \circ F(y) = \big(y_1, \dots, y_\kappa, y_{m+1} - Q_{m+1}(y_{m+l+1}, \dots, y_q), \dots, \\ y_{m+l} - Q_{m+l}(y_{m+l+1}, \dots, y_q)\big).\end{aligned}$$

Notice that $P \circ F$ has exactly $k$ components, since $\kappa + l = k$, by the definition of $l$, according to the notation in the proof of Theorem 1.1.

To show that $\delta_{1/r} l_{x^{-1}} \Sigma \cap D_R$ converges to $\mathcal{A} \cap D_R$ in the Kuratowski sense, as $r \to 0^+$, we will use Proposition 4.5.5 of [2]. Then we fix an infinitesimal sequence $r_n > 0$ and show that

(i) if $z = \lim_{n\to\infty} z_n$ for some sequence $\{z_n\}$ such that $z_n \in (\delta_{1/r_n} l_{x^{-1}} \Sigma) \cap D_R$ and $r_n \to 0$, then $z \in \mathcal{A} \cap D_R$;

(ii) if $z \in \mathcal{A} \cap D_R$, then there exist $z_n \in (\delta_{1/r_n} l_{x^{-1}} \Sigma) \cap D_R$ such that $z_n \to z$.

Consider the sequence

$$z_n \in \left(\delta_{1/r_n} l_{x^{-1}} \Sigma\right) \cap D_R,$$

converging to $z$, we have that $z_n = (F \circ \delta_{1/r_n} \circ \psi \circ \tilde{\delta}_{r_n})(\xi_n)$, for some $\xi_n$. Due to formula (10), which constitutes the explicit formula of $(\delta_{1/r_n} \circ \psi \circ \tilde{\delta}_{r_n})(\xi_n)$, the boundedness of $z_n$ implies the boundedness of $\xi_n$. Then we get a subsequence $\xi_\nu$ of $\xi_n$ converging to $\overline{\xi} \in D_R$. As a consequence of (19) and (10), it follows that $z_\nu \to F \circ T(\overline{\xi}) = z \in \mathcal{A} \cap D_R$. This shows the validity of (i).

Now, we set $U = F \circ T$ and fix $z \in \mathcal{A} \cap D_R$, hence we have $\xi$ such that $U(\xi) = z$. Let us define $\Psi_n = F \circ \delta_{1/r_n} \circ \psi \circ \delta_{r_n}$ and set $\Sigma_n = \delta_{1/r_n} l_{x^{-1}} \Sigma$. If we show that

$$\operatorname{dist}\left(\xi, \Psi_n^{-1}(D_R)\right) \to 0, \tag{21}$$

with respect to the Euclidean norm $|\cdot|$ on $\mathbb{R}^q$, then we get a sequence $\xi_n$ that is bounded, due to (10), and such that

$$D_R \ni \Psi_n(\xi_n) \to U(\xi) = z,$$

since $\Psi_n$ uniformly converges to $U$ on compact sets. Taking into account that for $n$ large

$$\Sigma_n \cap D_R = \operatorname{Im}(\Psi_n) \cap D_R,$$

it follows that $\Psi_n(\xi_n) \in \Sigma_n \cap D_R$ and (ii) follows. We are left with showing (21). By contradiction, if this limit were not true, then possibly taking a subsequence, we would get

$$\operatorname{dist}\left(\xi, \Psi_\nu^{-1}(D_R)\right) \geq \varepsilon_0 > 0.$$

We fix an arbitrary $0 < h < 1$. Since $\Psi_n$ uniformly convergences to $U$ on compact sets, then

$$U^{-1}(D_{hR}) \subset \Psi_n^{-1}(D_R)$$

whenever $n \geq n_0$, for some $n_0$ depending on $h$. In fact, $U^{-1}(D_{hR})$ is a compact set, due to formula (20). Observing that $\delta_h(U(\xi)) \in D_{hR}$ and taking into account the homogeneity $U \circ \delta_h = \delta_h \circ U$, for $\nu \geq n_0$, we achieve

$$\varepsilon_0 \leq \operatorname{dist}\left(\xi, \Psi_\nu^{-1}(D_R)\right) \leq \operatorname{dist}\left(\xi, U^{-1}(D_{hR})\right) \leq |\xi - \tilde{\delta}_h \xi|.$$

The arbitrary choice of $h$ yields a contradiction and concludes the proof of (21). $\square$

*Remark 3.2* According to the previous proof, formula (20) yields the limit set $\mathcal{A}$ as the image of $T$ in some system of graded coordinates. Since $T$ is a homogeneous polynomial mapping, then $\mathcal{A}$ is clearly a homogeneous algebraic variety. Furthermore, this variety is everywhere analytic, since it is the graph of a polynomial mapping with respect to some system of graded coordinates. In particular, $\mathcal{A}$ is an analytic manifold without boundary.

*Proof of Corollary 1.3* For each $j = 1, \dots, \delta$, we define the subsets

$$\Sigma_j = \{x \in \Sigma \mid \text{estimates (1) hold with} d_\Sigma(x) = j\}.$$

Then $Z_\delta = \bigcup_{1 \le j \le \delta}^m \Sigma_j$. Note that some $\Sigma_j$ might be empty. This is always true for instance if $\delta > Q - k$, where $k$ is the codimension of $\Sigma$ and $Q$ is the Hausdorff dimension of the group. From estimates (1) and standard differentiability theorems for measures, see for instance 2.10.17(2) and 2.10.18(1) of [17], taking into account that the Riemannian surface measure $\tilde{\mu}_p$ is countably finite, then $\Sigma_j$ is countably $\mathcal{H}^j$-finite. In particular, $Z_\delta$ is $\mathcal{H}^\delta$-countably finite. □

*Proof of Corollary 1.2* Let $d = d(\Sigma)$ be the degree of $\Sigma$. In view of Corollary 1.3 the subset $Z_{d-1}$ of points in $\Sigma$ with degree less than $d$ is countably $\mathcal{H}^{d-1}$-finite. Then in particular, (2) holds. At points of degree $d$ the blow-up limit exists. Precisely, Theorem 1.2 of [29] holds, along with the negligibility condition (2), hence formula (1.4) of [29] also holds. This formula coincides with (3). □

## 4 Some Remarks

In this section we add remarks complementary to our main results.

*Remark 4.1* The Hausdorff dimension of a $C^{1,1}$ submanifold $\Sigma$ in a two step stratified group coincides with its degree $d = d(\Sigma)$. In fact, from the lower semicontinuity of $x \to d_\Sigma(x)$ on $\Sigma$, it follows that the subset $\Sigma_d$ of degree $d$ points is an open subset of $\Sigma$. Thus, formula (3) yields

$$\int_{\Sigma_d} \theta(\tau_\Sigma^d(x)) d\mathcal{S}^d(x) = \int_{\Sigma_d} |\tau_\Sigma^d(x)| d\tilde{\mu}_p(x) > 0.$$

From the definition of $\tau_\Sigma^d(x)$ given in (2.17) of [29], using the simple argument of Remark 2.18 in [27], one easily finds two positive constants $c_1$ and $c_2$, such that

$$c_1 \le \theta(\tau) \le c_2 \quad \text{for every } \tau \in \Lambda_p(\mathcal{G}).$$

This implies that $\mathcal{S}^d(\Sigma) > 0$, taking into account that $\mathcal{S}^d \llcorner \Sigma$ is finite on compact sets. In particular, the Hausdorff dimension of $\Sigma$ is $d$.

*Remark 4.2* Let $\Sigma$ be a $p$-dimensional submanifold of a graded group. We consider the flag

$$H'_j(x) = \left(H_x^1 \mathbb{G} + \cdots + H_x^j \mathbb{G}\right) \cap T_x \Sigma,$$

on $T_x \Sigma$, where $H_x^j \mathbb{G} = \{X(x) \mid X \in V_j\}$. Let $(t_1^1, \dots, t_{m'_1(x)}^1)$ be a basis of $H'_j(x)$ and complete this basis with $(t_1^2(x), \dots, t_{m'_2(x)}^2)$ to a basis $(t_1^1, \dots, t_{m'_1(x)}^1, t_1^2, \dots, t_{m'_2(x)}^2)$

of $H'_2(x)$. We iterate this procedure to construct bases

$$(t^1_1,\dots,t^1_{m'_1(x)},\dots,t^j_1,\dots,t^j_{m'_j(x)}) \quad \text{of subspaces } H'_j(x).$$

One immediately observes that the dimensions $\dim(H'_j(x)/H'_{j-1}(x))$ coincide with $m'_j(x)$. In 0.6 B of [22], Gromov defines the number

$$D'(x)=\sum_{j=1}^{\iota} j m'_j(x).$$

Let us define the canonical projection $\pi_j : T_x\mathbb{G}\to H^j_x\mathbb{G}$ and observe that

$$\big(\pi_j(t^j_1),\dots,\pi_j(t^j_{m'_j(x)})\big)$$

must be linearly independent. If this were not the case, then we could find a nontrivial linear combination of $t^j_i$ belonging to $H'_{j-1}(x)$, but that it cannot be a linear combination of elements of this space and this is a contradiction. Setting $\tau^j_i=\pi_j(t^j_i)$ and taking into account that $\tau_j\in H^j_x\mathbb{G}$, it follows that

$$(\tau^1_1,\dots,\tau^1_{m'_1(x)},\dots,\tau^\iota_1,\dots,\tau^\iota_{m'_\iota(x)}) \quad \text{are linearly independent.} \tag{22}$$

Since $(t^i_j)$ is a basis of $T_x\Sigma$, we have

$$d_\Sigma(x)=d\big(t^1_1\wedge\cdots\wedge t^1_{m'_1(x)}\wedge\cdots\wedge t^\iota_1\cdots\wedge t^\iota_{m'_\iota(x)}\big). \tag{23}$$

It is understood that some $m_{j_0}$ may vanish for some $j_0$; this would mean that there are no vectors $t^{j_0}_i$ both in (22) and (23). We have

$$t^1_1\wedge\cdots\wedge t^1_{m'_1(x)}\wedge\cdots\wedge t^\iota_1\wedge\cdots\wedge t^\iota_{m'_\iota(x)}=\tau^1_1\wedge\cdots\wedge\tau^1_{m'_1(x)}\wedge\cdots\wedge\tau^\iota_1\cdots\wedge\tau^\iota_{m'_\iota(x)}+r,$$

where the first addend is nonvanishing due to (22) and $d(r)<d(\tau)$, where we have set

$$\tau=\tau^1_1\wedge\cdots\wedge\tau^1_{m'_1(x)}\wedge\cdots\wedge\tau^\iota_1\cdots\wedge\tau^\iota_{m'_\iota(x)}.$$

An immediate verification shows that $d(\tau)=\sum_{j=1}^{\iota} j m'_j(x)$. This shows that

$$d_\Sigma(x)=D'(x).$$

Notice that we have computed the degree $d_\Sigma(x)$ by choosing an arbitrary adapted basis. Then the previous formula also shows that the degree does not depend on the choice of the adapted basis. Since $D_H(\Sigma)=\max_{x\in\Sigma} D'(x)$, according to the definition given in 0.6 B of [22], then we obviously have $D_H(\Sigma)=d(\Sigma)$.

*Example 4.3* Let $\Sigma$ and $\varphi$ be as in Example 2.7. To determine the blow-up set of $\Sigma$ at the origin, we consider a converging sequence $(x^n, y^n, t^n) \to (x, y, t)$ and an infinitesimal sequence $r_n \to 0$. Thus, taking into account that $\varphi'(0) = 1$, the blow-up sequence

$$\begin{aligned}&\delta_{1/r_n}\big(\Phi(r_n x_1^n, r_n y_1^n, r_n y_2^n)\big)\\ &\quad = \big(x_1^n, y_1^n, r_n^2 (x_1^n)^3, y_2^n, r_n^{-2}\varphi\big(r_n^2\big((x_1^n)^2 + (y_1^n)^2 + r_n (x_2^n)^3\big)\big), r_n (x_1^n)^2\big),\end{aligned}$$

converges to $(x_1, y_1, 0, y_2, x_1^2 + y_1^2, 0)$, which defines the blow-up set

$$\big\{(x_1, y_1, 0, y_2, x_1^2 + y_1^2, 0) : x_1, y_1, y_2 \in \mathbb{R}\big\}. \tag{24}$$

This set is a homogeneous algebraic variety, according to Theorem 1.4. Recall from Example 2.7 that the degree of $\Sigma$ at zero is equal to three. Since the blow-up set at maximum degree points is a subgroup and in particular a subspace, [29], it follows that $d(\Sigma) > 3$. This agrees with the computations of Example 2.7, where it is shown that $d(\Sigma) = 5$. Another reason for which the origin of $\Sigma$ cannot be a point of maximum degree is that there are no smooth 3-dimensional horizontal submanifolds in $\mathbb{H}^1 \times \mathbb{H}^1$, namely, submanifolds of degree equal to three.

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