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Spherical Hausdorff measure of submanifolds in Heisenberg groups

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ABSTRACT. – We present the explicit formula relating spherical Hausdorff measure and Riemannian surface measure of submanifolds in the Heiseneberg group and we discuss some consequences. Complete proofs and additional results are given in [15].

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We start recalling the structure of the Heisenberg group. This is a simply connected Lie group, whose Lie algebra $\mathfrak{h}^{2n+1}$ is endowed with a basis $(X_1, \dots, X_{2n}, Z)$ satisfying the only nontrivial bracket relations $[X_{2k-1}, X_{2k}] = 2\,Z$ for every $k = 1, \dots, n$. We will identify the Lie algebra $\mathfrak{h}^{2n+1}$ with the isomorphic Lie algebra of left invariant vector fields on $\mathbb{H}^{2n+1}$. The subset of left invariant vector fields $(X_1, \dots, X_{2n})$ spans a smooth distribution of $2n$-dimensional hyperplanes, which form the left invariant *horizontal subbundle*, denoted by $H\mathbb{H}^{2n+1}$. In other words, at each point of $\mathbb{H}^{2n+1}$ we have a given subspace of directions, defined by

$$H_x\mathbb{H}^{2n+1} = \Big\{ v \in T_x\mathbb{H}^{2n+1} \mid v = \sum_{j=1}^{2n} a_j X_j(x), \;\; a_j \in \mathbb{R} \Big\}$$

and named *horizontal subspace*. Absolutely continuous curves a.e. tangent to horizontal directions are called *horizontal curves*. The Lie algebra gener-

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ated by left invariant horizontal vector fields X_j coincides with $\mathfrak{h}^{2n+1}$, therefore the well kwnown Chow theorem implies that every couple of points is joined by at least one horizontal curve, [4]. This allows us to define a distance associated to the horizontal subbundle as follows. The *Carnot-Carathéodory distance* between two points is the infimum over lengths of horizontal curves joining these points, where the length is computed with respect to a fixed left invariant Riemannian metric. In contrast with Analysis in Euclidean spaces, where the Euclidean distance is the most natural choice, in the Heisenberg group several distances have been introduced for different purposes. However, all of these distances are *homogeneous*, namely, they are continuous, left invariant and scale well with intrinsic dilations as follows $\rho(\delta_r y, \delta_r z) = r\,\rho(y,z)$.

In order to introduce properly the family of dilations we consider global coordinates associated to $(X_1, \dots, X_{2n}, Z)$. Through the exponential map $\exp : \mathfrak{h}^{2n+1} \longrightarrow \mathbb{H}^{2n+1}$, which is a diffeomorphism, we define the mapping $F : \mathbb{R}^{2n+1} \longrightarrow \mathbb{H}^{2n+1}$ given by

$$F(y) = \exp\left(y_{2n+1} Z + \sum_{j=1}^{2n} y_j X_j \right) . \tag{1}$$

Then a natural family of dilations can be defined as follows

$$\delta_r(y) = (r y_1, r x_2, \dots, r y_{2n}, r^2 y_{2n+1}) \tag{2}$$

for every $r > 0$ and every $y \in \mathbb{R}^{2n+1}$. One can check that the Carnot-Carathéodory distance is an example of homogeneous distance. All homogeneous distances are equivalent each other, hence the Hausdorff dimension of $\mathbb{H}^{2n+1}$ with respect to an arbitrary homogeneous distance is equal to $2n+2$. In the sequel, Hasudorff dimension of sets and spherical Hausdorff measure will be understood with respect to a fixed homogeneous distance.

In the last years, several works have investigated the notion of perimeter measure in Heisenberg groups, Carnot groups, Carnot-Carathéodory spaces and in a more general class of metric spaces satisfying some abstract properties, [1], [2], [5], [8], [9], [10], and [12]. In particular, results of [9] imply that the perimeter measure of a finite perimeter set in the Heisenberg group $\mathbb{H}^{2n+1}$ can be represented by the $(2n+1)$-dimensional spherical Hausdorff measure restricted to the reduced boundary. By negligibility of characteristic points, [3], [14], one can also prove that the Hausdorff dimension with respect to the Carnot-Carathéodory distance of a C^1 submanifold of $\mathbb{H}^{2n+1}$ is $2n+1$, whereas its topological dimension is $2n$. Here a natural question comes up: is this difference preserved even for submanifolds of codimension higher than one?

One can easily find examples of submanifolds where topological and Hausdorff dimensions coincide. It suffices to consider horizontal curves and more general Legendrian submanifolds, namely, those submanifolds whose tangent space at every point is contained in a horizontal subspace. Our first aim is to detect when this equality breaks, as it happens in codimension one. We show that this occurs when we have a sufficiently "large" set of points where the tangent space of the submanifold is not contained in the corresponding horizontal subspace. We call these points *transverse points*; points which are not transverse are called *characteristic points*. A key fact, is that these points can be characterized introducing certain p-vectors associated to the tangent space of the submanifold at some point. Recall that a *tangent p-vector* to a submanifold Σ at x is defined by the wedge product $t_1 \wedge t_2 \wedge \cdots \wedge t_p$, where $(t_1, \ldots, t_p)$ is an orthonormal basis of $T_x\Sigma$. We denote this p-vector by $\tau_\Sigma(x)$. Note that a tangent p-vector is uniquely defined up to its sign. In fact, one could more precisely consider the equivalence class $\{\tau_\Sigma(x), -\tau_\Sigma(x)\}$. Clearly, in the case of oriented submanifolds the sign of the tangent p-vector is uniquely defined by the orientation. In order to determine the vertical projection of $\tau_\Sigma(x)$ we start from the following orthogonal decomposition

$$T_x\mathbb{H}^{2n+1} = H_x\mathbb{H}^{2n+1} \oplus \langle Z(x)\rangle$$

for every $x \in \mathbb{H}^{2n+1}$. A linear combination of wedge products

$$X_{j_1}(x) \wedge X_{j_2}(x) \wedge \cdots X_{j_p}(x) \ ,$$

where $1 \le j_s \le 2n$ and $j = 1, \ldots, 2n$, is a *horizontal p-vector*. We denote by $\Lambda_p(H_x\mathbb{H}^{2n+1})$ the space of these vectors. Any linear combination of wedge products

$$X_{j_1}(x) \wedge X_{j_2}(x) \wedge \cdots X_{j_{p-1}}(x) \wedge Z(x)$$

where $1 \le j_s \le 2n$ and $j = 1, \ldots, 2n$, is called *vertical p-vector*. The space of vertical p-vectors is denoted by $\mathcal{V}_p(H_x\mathbb{H}^{2n+1})$. It is a basic fact that the scalar product on $T_x\mathbb{H}^{2n+1}$ induces a scalar product on $\Lambda_p(T_x\mathbb{H}^{2n+1})$, see for instance 1.7.5 of [7]. This allows us to regard the space of vertical p-vectors $\mathcal{V}_p(T_x\mathbb{H}^{2n+1})$ as the orthogonal complement of the horizontal subspace $\Lambda_p(H_x\mathbb{H}^{2n+1})$. We have the orthogonal decomposition

$$\Lambda_p(T_x\mathbb{H}^{2n+1}) = \Lambda_p(H_x\mathbb{H}^{2n+1}) \oplus \mathcal{V}_p(T_x\mathbb{H}^{2n+1}) \ . \tag{3}$$

This formula immediately gives us the existence of a vertical projection

$$p_{\mathcal{V}} : \Lambda_p(T_x\mathbb{H}^{2n+1}) \longrightarrow \mathcal{V}_p(T_x\mathbb{H}^{2n+1})$$

defined by $p_{\mathcal{V}}(\xi) = \xi_{\mathcal{V}}$, where $\xi = \xi_H + \xi_{\mathcal{V}}$ is the unique decomposition of $\xi \in \Lambda_p(T_x\mathbb{H}^{2n+1})$ with $\xi_H \in \Lambda_p(H_x\mathbb{H}^{2n+1})$ and $\xi_{\mathcal{V}} \in \mathcal{V}_p(T_x\mathbb{H}^{2n+1})$. This vertical projection permits us to define the *vertical tangent p-vector* to a submanifold Σ at x as

$$\tau_{\Sigma,\mathcal{V}}(x) = p_{\mathcal{V}}(\tau_\Sigma(x))$$

where $\tau_\Sigma(x)$ is a tangent p-vector. For simplicity of notation, we have omitted x in the definition of the vertical projection $p_{\mathcal{V}}$. Indeed, we could think of $p_{\mathcal{V}}$ as a mapping between bundles $p_{\mathcal{V}} : \Lambda_p(\mathbb{H}^{2n+1}) \longrightarrow \mathcal{V}_p(\mathbb{H}^{2n+1})$ defined at each fiber. Here clearly $\mathcal{V}_p(\mathbb{H}^{2n+1})$ denotes the vector bundle whose fibers are $\mathcal{V}_p(T_x\mathbb{H}^{2n+1})$.

Remark 1. – Broadly speaking, the length $\tau_{\Sigma,\mathcal{V}}(x)$ measures how the submanifold is far from being Legendrian near x.

In particular, the following simple characterization holds.

Proposition 1. – *A point x of Σ is transverse if and only if and only if $\tau_{\Sigma,\mathcal{V}}(x) \neq 0$.*

We will be interested in the study of non-Legendrian submanifolds.

Remark 2. – Note that any C^1 smooth submanifold of dimension greater than n is never Legendrian. More precisely, the closed subset of characteristic points cannot be open, otherwise we would have the integrability of a horizontal distribution of dimension greater than n. In fact, horizontal distributions of dimension higher than n never satisfy the Frobenius condition for integrability.

It is possible to know more on the subset of characteristic points. In codimension one, negligibility of characteristic points in Heisenberg groups and CarnotCarathéodory manifolds has been investigated for instance in [3], [6] for different purposes. In codimension higher than one, it has been proved in [14] that the set of characteristic points of a k-codimensional submanifold in Carnot group of Hausdorff dimension Q is $\mathcal{S}^{Q-k}$-negligible. Here $\mathcal{S}^\alpha$ denotes the α-dimensional spherical Hausdorff measure with respect to a homogeneous distance. In the case of p-dimensional submanifolds in $\mathbb{H}^{2n+1}$ this result shows that the set of characteristic points is $\mathcal{S}^{p+1}$-negligible. This fact allows us to restrict our attention to transverse points.

Now, we are in the position to state the main result of [15].

THEOREM 1 (Blow-up at transverse points). – *Let Σ be a p-dimensional C^1 submanifold of Ω and let $x \in \Sigma$ be a transverse point. Then the following limit holds*

$$\lim_{r\to 0^+} \frac{\mathrm{vol}_p(\Sigma \cap B_{x,r})}{r^{p+1}} = \frac{\theta_p^\rho(\tau_{\Sigma,\mathcal{V}}(x))}{|\tau_{\Sigma,\mathcal{V}}(x)|} \ . \tag{4}$$

The open ball of center x and radius $r > 0$ with respect to ρ is denoted by $B_{x,r}$. The symbol vol_p denotes the Riemannian p-dimensional measure, that can be also written as Hausdorff measure $\mathcal{H}^p_{d_g}$, where d_g is the Riemannian distance associated to the fixed left invariant metric. The *metric factor* $\theta(\tau)$ of ρ with respect to the vertical simple p-vector τ is the number

$$\theta_p^\rho(\tau) = \mathcal{H}^p\left(F^{-1}(L \cap B_1)\right) \ ,$$

where $F : \mathbb{R}^{2n+1} \longrightarrow \mathbb{H}^{2n+1}$ defines our global chart, $\mathcal{H}^p$ denotes the p-dimensional Hausdorff measure with respect to the Euclidean distance of $\mathbb{R}^{2n+1}$, B_1 is the unit ball of $\mathbb{H}^{2n+1}$ with respect to the homogeneous distance ρ, $\mathcal{L}(\tau)$ is the unique subspace associated τ and $L = \exp \mathcal{L}(\tau)$. The choice of another system of coordinates $\tilde{F} : \mathbb{R}^{2n+1} \longrightarrow \mathbb{H}^{2n+1}$ defined by another orthonormal system of vectors would have lead us to the same number $\theta_p^\rho(\tau)$. Several known homogeneous distances in the Heisenberg group have constant metric factor, as for instance the Korányi distance and the maximum distance, see [15] for more details. In this case the metric factor becomes a dimensional constant $\theta_p^q(\cdot) \equiv \alpha$ and an intrinsic spherical Hausdorff measure $\mathcal{S}^{p+1}_{\mathbb{H}^{2n+1}} = \alpha\, \mathcal{S}^{p+1}$ can be defined. Then the previous theorem yields the following formula.

THEOREM 2. – *Let $\Sigma \subset \mathbb{H}^{2n+1}$ be a p-dimensional submanifold of class C^1. Then we have*

$$\mathcal{S}^{p+1}_{\mathbb{H}^{2n+1}}(\Sigma) = \int_\Sigma |\tau_{\Sigma,\mathcal{V}}(x)| \, d\,\mathrm{vol}_p(x) \ . \tag{5}$$

In the case Σ is a Legendrian submanifold this formula becomes the trivial identity $0 = 0$. A similar formula has been recently proved by Franchi, Serapioni and Serra Cassano, [11], in the case $n < p < 2n+1$, but considering the so-called "low codimensional $\mathbb{H}$-regular surfaces", that in general are much less regular than C^1 submanifolds. As a consequence of (5), one can establish the following formula

$$\mathcal{S}^{p+1}_{\mathbb{H}^{2n+1}}(\Sigma) = \sup_{\omega \in \mathcal{F}^p_c(\Omega)} \int_\Sigma \langle \tau_{\Sigma,\mathcal{V}}, \omega \rangle \, d\,\mathrm{vol}_p \ , \tag{6}$$

where $\mathcal{F}_c^p(\Omega)$ is the space of smooth p-forms with compact support in Ω with $|\omega| \leq 1$. Here the norm of ω is defined making the standard frame of 1-forms $(dx_1, dx_2, \dots, dx_{2n}, \tilde{\theta})$ orthonormal and extending this scalar product to p-forms. Note that θ is the contact form of $\mathbb{H}^{2n+1}$. We can introduce vertical and horizontal p-forms similarly as for p-vectors. The vertical projection of a p-form reads as

$$p_{\mathcal{V}}\Big(\sum_{1 \leq i_1 < \cdots < i_p \leq 2n+1} a_{i_1 \dots i_p}\, \alpha_{i_1} \wedge \cdots \wedge \alpha_{i_p} \Big)$$
$$= \sum_{1 \leq i_1 < \cdots < i_{p-1} \leq 2n} a_{i_1 \dots i_{p-1}(2n+1)}\, \alpha_{i_1} \wedge \cdots \wedge \alpha_{i_{p-1}} \wedge \theta \ ,$$

where we have set $\alpha_j = dx_j$ for each $j = 1, \dots, 2n$ and $\alpha_{2n+1} = \theta$. Then the identity

$$\langle p_{\mathcal{V}}(\tau_\Sigma), \omega \rangle = \langle \tau_\Sigma, p_{\mathcal{V}}(\omega) \rangle$$

allows us to achieve the following representation of the spherical Hausdorff measure

$$\mathcal{S}^{p+1}_{\mathbb{H}^{2n+1}}(\Sigma) = \sup_{\gamma \in \mathcal{V}_c^p(\Omega)} \int_\Sigma \langle \tau_\Sigma, \gamma \rangle \, d\,\mathrm{vol}_p \ , \tag{7}$$

where $\mathcal{V}_c^p(\Omega)$ denotes the space of smooth vertical p-forms γ compactly supported in Ω with $|\gamma| \leq 1$. It is interesting to observe how (7) could naturally lead us to a possible notion of mass for p-dimensional currents of the Heisenberg group. It suffices to define

$$M_{\mathcal{V}}(T) = \sup_{\gamma \in \mathcal{V}_c^p(\Omega)} T(\omega) \ ,$$

where T is a current of $\Omega \subset \mathbb{H}^{2n+1}$. In fact, if T_Σ is a current induced by a p-dimensional C^1 submanifold Σ of Ω, $T_\Sigma(\omega) = \int_\Sigma \langle \tau_\Sigma, \omega \rangle \, d\mathrm{vol}_p$, then we have

$$M_{\mathcal{V}}(T_\Sigma) = \mathcal{S}^{p+1}_{\mathbb{H}^{2n+1}}(\Sigma) \ .$$

Using the classical notion of boundary it is also possible to check that

$$M_{\mathcal{V}}(\partial T_E) = P_H(E) \ ,$$

where T_E is the current associated to a set E with H-finite perimeter and $P_H(E)$ is the H-finite perimeter of E. Note that here we have modified the classical definition of mass, but keeping the classical notion of current and of boundary operator. Recently, Franchi, Serapioni and Serra Cassano, [11], have proposed and studied an intrinsic notion of current and of boundary operator in the Heisenberg group starting from the Rumin complex, [19].

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