A LOW RANK PROPERTY AND NONEXISTENCE OF HIGHER DIMENSIONAL HORIZONTAL SOBOLEV SETS

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Abstract. We establish a "low rank property" for Sobolev mappings that pointwise solve a first order nonlinear system of PDEs, whose smooth solutions have the so-called "contact property". As a consequence, Sobolev mappings from an open set of the plane, taking values in the first Heisenberg group $\mathbb{H}^1$ and that have almost everywhere maximal rank must have images with positive 3-dimensional Hausdorff measure with respect to the sub-Riemannian distance of $\mathbb{H}^1$. This provides a complete solution to a question raised in a paper by Z. M. Balogh, R. Hoefer-Isenegger and J. T. Tyson. Our approach differs from the previous ones. Its technical aspect consists in performing an "exterior differentiation by blow-up", where the standard distributional exterior differentiation is not possible. This method extends to higher dimensional Sobolev mappings, taking values in higher dimensional Heisenberg groups.

Contents

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1. Introduction

It is well known that every noninvolutive tangent distribution on a manifold does not admit any integral submanifold. One of the simplest cases is given by the nonintegrable tangent distribution of the first Heisenberg group $\mathbb{H}^1$, identified by $\mathbb{R}^3$ with coordinates associated to the left invariant vector fields

$$X_1 = \partial_{x_1} - x_2\partial_{x_3} \quad \text{and} \quad X_2 = \partial_{x_2} + x_1\partial_{x_3}\,. \tag{1.1}$$

At each point of the space, these vector fields linearly span a subspace of the tangent space, hence a tangent distribution is defined, corresponding to the so-called "horizontal subbundle". Although no smooth surfaces in $\mathbb{H}^1$ can be everywhere tangent to $H\mathbb{H}^1$, one may still wonder whether there exist more general "2-dimensional sets" that can be still considered "tangent" to this distribution in a broad sense. This problem is amazingly related to the study of the Hausdorff dimension of sets with respect to the sub-Riemannian distance, in short SR-distance, that is associated to $H\mathbb{H}^1$.

In this connection, Z. M. Balogh and J. T. Tyson have constructed an interesting example of "horizontal fractal", called the *Heisenberg square* Q_H, [3]. The 2-dimensional Hausdorff measure of Q_H with respect to both the SR-distance and the Euclidean distance is finite and positive, see [3, Theorem 1.10]. As proved in [4], it is possible to find a BV function $g : (0,1)^2 \to \mathbb{R}$, whose graph G is contained in Q_H and satisfies

$$0 < \mathcal{H}^2_d(G) < +\infty. \tag{1.2}$$

The symbol $\mathcal{H}^2_d$ denotes the Hausdorff measure with respect to the SR-distance d of $\mathbb{H}^1$. Condition (1.2) never holds for graphs of smooth functions. It can be interpreted as a "metric definition" of horizontality for lower regular sets. In fact, in the general Heisenberg group $\mathbb{H}^n$, represented by $\mathbb{R}^{2n+1}$ equipped by the left invariant vector fields

$$X_i = \partial_{x_i} - x_i\partial_{x_{2n+1}}, \quad X_{n+i} = \partial_{x_{n+i}} + x_i\partial_{x_{2n+1}} \quad \text{and} \quad i = 1, \ldots, n, \tag{1.3}$$

spanning $H\mathbb{H}^n$, every C^1 smooth m-dimensional submanifold $\Sigma \subset \mathbb{H}^n$ that is everywhere tangent to $H\mathbb{H}^n$ must have the measure $\mathcal{H}^m_d \llcorner \Sigma$ locally finite. On the other hand, from Contact Topology, it is well known that the nonintegrability of $H\mathbb{H}^n$ is stronger than the noninvolutivity condition of Frobenius Theorem, since not only hypersurfaces but rather all sufficiently smooth submanifolds $\Sigma \subset \mathbb{H}^n$ of dimension m, with $n < m \le 2n$, cannot be everywhere tangent to $H\mathbb{H}^n$, in short $T\Sigma \not\subseteq H\mathbb{H}^n$, see for instance [8, Proposition 1.5.12]. Thus, when $m > n$ there must exist at least a point $x \in \Sigma$ such that $T_x\Sigma \not\subseteq H_x\mathbb{H}^n$.

This fact has an important metric implication, since the density of $\mathcal{H}^{m+1}_d \llcorner \Sigma$ with respect to the Euclidean surface measure $\mathcal{H}^m_{|\cdot|} \llcorner \Sigma$ is proportional to the length of the "vertical tangent m-vector" $\tau_{\Sigma,\mathcal{V}}$ and this vector vanishes only at those points $x \in \Sigma$, called *horizontal points*, that are characterized by the condition $T_x\Sigma \subset H_x\mathbb{H}^n$.

When Σ is C^1 smooth, the absolute continuity of $\mathcal{H}^{m+1}_d \llcorner \Sigma$ with respect to $\mathcal{H}^m_{|\cdot|}$ is mainly a consequence of a higher codimensional negligibility result, [12], joined with a blow-up at nonhorizontal points, [7, 13, 14]. The m-vector $\tau_{\Sigma,\mathcal{V}}$ is defined as the projection of the unit tangent p-vector of Σ onto the orthogonal subspace to the linear space $\Lambda_m(H\mathbb{H}^n)$ of horizontal m-vectors, see [13] for more details and related references. Such results imply that for each smooth m-dimensional submanifolds $\Sigma \subset \mathbb{H}^n$ with $m > n$, there holds

$$\mathcal{H}_d^{m+1}(\Sigma) > 0\,. \tag{1.4}$$

In the case $n = 1$ and $m = 2$, the non-horizontality condition (1.4) for nonsmooth sets has been shown in [4], where Σ is a 2-dimensional Lipschitz graph of $\mathbb{H}^1$. Here the authors raise the interesting question on the existence of horizontal sets in the sense of (1.2) having regularity between Lipschitz and BV.

A first answer to this question is given in [15], where it is shown that 2-dimensional $W^{1,1}_{\rm loc}$ Sobolev graphs Σ in $\mathbb{H}^1$ have to satisfy (1.4), with $m = 2$. This approach relies on the fact that for a smooth local parametrization $f : \Omega \to \Sigma$, where $\Omega \subset \mathbb{R}^2$, the equation

$$df^3 = f^1 df^2 - f^2 df^1 \tag{1.5}$$

only holds at those points $y \in \Omega$ such that $T_{f(y)}\Sigma \subset H\mathbb{H}^1$ and (1.5) cannot hold everywhere, since its exterior differentiation would imply that the rank of Df is everywhere less than two. To see this fact when $f \in W^{1,1}_{\rm loc}(\Omega, \mathbb{H}^1)$ and it is defined by the graph of a real-valued Sobolev function, the point is to show that the almost everywhere validity of (1.5) allows us to take its distributional exterior differential, obtaining that the rank of Df cannot be almost everywhere maximal and this conflicts with the graph structure. This is the key to establish (1.4), since the previous argument shows that (1.5) fails to hold at least on a set of positive measure and the Whitney extension theorem yields a C^1 smooth submanifold $\tilde\Sigma$ that coincides with the Sobolev graph Σ on some measurable subset $A \subset \tilde\Sigma \cap \Sigma$ of positive Euclidean surface measure, where in addition $TA \nsubseteq H\mathbb{H}^n$. As a consequence, in view of the previous comments on the density of $\mathcal{H}_d^3 \llcorner \tilde\Sigma$, we achieve

$$\mathcal{H}_d^3(\Sigma) \ge \mathcal{H}_d^3(A) > 0\,.$$

More generally, the same argument applies to all cases where we are able to show that (1.5) cannot hold almost everywhere. To show this fact in other cases of low regular sets, we need the summability of both f and Df to allow for the distributional exterior differentiation of (1.5). The distributional exterior differential of $f^1 df^2 - f^2 df^1$ is exactly twice the distributional Jacobian of the mapping (f^1, f^2), hence assuming for instance that $(f^1, f^2) \in W^{1,p}_{\rm loc}(\Omega, \mathbb{R}^2)$ with $p \ge 4/3$, we obtain that this distributional Jacobian is well defined. As a consequence, every image Σ of a mapping in $W^{1,p}_{loc}(\Omega, \mathbb{R}^3$ with $p \ge 4/3$ and whose Jacobian matrix has almost everywhere maximal rank must satisfy (1.4) with $m = 2$, [15]. The validity of this result in the case $1 \le p < 4/3$ was left open, since the distributional Jacobian cannot be defined. The following theorem answers this question.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^2$ be open, let $f \in W^{1,1}_{\rm loc}(\Omega, \mathbb{R}^3)$ be such that the Jacobian matrix Df has almost everywhere maximal rank and define $\Sigma = f(\Omega)$. It follows that $\mathcal{H}_d^3(\Sigma) > 0$.*

This completes the answer to the previously mentioned question raised in [4]. Our approach differs from the previous ones and it can be applied to every Heisenberg group $\mathbb{H}^n$, that we identify with $\mathbb{R}^{2n+1}$ as a linear space. We consider $f : \Omega \to \mathbb{R}^{2n+1}$, where Ω is an open set of $\mathbb{R}^m$. In this case, the horizontality condition for f is given by the equation

$$df^{2n+1} = \sum_{j=1}^{n} \left(f^j df^{j+n} - f^{j+n} df^j \right). \tag{1.6}$$

The previous arguments apply if we are able to show that the almost everywhere validity of (1.6) implies a *low rank property*, namely, Df must have rank less than $n+1$ almost everywhere in Ω. Clearly, we will apply such a result in the nontrivial case $n+1 \le m \le 2n$. We will assume that $f \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^{2n+1})$. Let us summarize the main idea of the proof. First, assume that $m = 2$. We perform a kind of "exterior differentiation by blow-up", rescaling f at Lebesgue points $z \in \Omega$ of both f and Df. The rescaled functions $f_{z,\rho}$, introduced in Definition 4.1, are defined on the unit ball $\mathbb{B}$ of $\mathbb{R}^2$ for all $\rho > 0$ sufficiently small and converge to the linear mapping $u : y \mapsto Df(z) \cdot y$ in $W^{1,1}(\mathbb{B})$ as $\rho \to 0_+$. The almost everywhere pointwise validity of (1.6) implies that the one-form

$$\sum_{j=1}^{n} \left(f^j_{z,\rho}\, df^{j+n}_{z,\rho} - f^{j+n}_{z,\rho}\, df^j_{z,\rho} \right) \tag{1.7}$$

is "weakly exact" in the sense that it is a.e. equal to dw_ρ for some $w_\rho \in W^{1,1}(\mathbb{B})$, see Lemma 4.1. We exploit this fact by integrating (1.7) on the Euclidean sphere $\partial B(0,r)$ for almost every $r \in (0,1)$ and pass to the limit with respect to ρ as it goes to zero by a suitable positive infinitesimal sequence (ρ_k). Since the blow-up limit has the form

$$\sum_{j=1}^{n} \left(u^j\, du^{j+n} - u^{j+n}\, du^j \right)$$

with $u(y) = Df(z)\cdot y$, we obtain that its oriented integral on almost every sphere vanishes, hence the Stokes theorem implies that

$$\sum_{j=1}^{n} df^j(z) \wedge df^{j+n}(z) = 0\,. \tag{1.8}$$

Now, if $m > 2$, we obtain (1.8) by a slicing argument, so that the whole range $m \ge 2$ is provided. We will deduce from (1.8) that the rank of $Df(z)$ is less than $n+1$, so this rank condition holds almost everywhere, eventually leading us to our Theorem 6.1. According to this theorem, Sobolev mappings that satisfy the horizontality condition (1.6) almost everywhere must satisfy a "low rank property". This fact should be seen somehow as a "differential obstruction". It is worth to compare this obstruction with the "Lipschitz obstructions" appearing in the study of Lipschitz homotopy groups of the Heinsenberg group, [5]. The main application of Theorem 6.1 is the following result.

Theorem 1.2. *Let $\Omega \subset \mathbb{R}^m$ be an open set, let $n < m \le 2n$ and let $f \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^{2n+1})$. Suppose that the Jacobian matrix Df has rank equal to m almost everywhere and set $\Sigma = f(\Omega)$. Then $\mathcal{H}^{m+1}_d(\Sigma) > 0$.*

We remark that in the case $m = 2$ and $n = 1$, this theorem exactly yields Theorem 1.1. In ending, we wish to point out a curious observation on the graph G of the BV function g mentioned above, since we can translate the metric horizontality of (1.2) into a somehow "tangential condition". In fact, as a byproduct of our techniques, one can easily observe that the approximate differential of the graph mapping $f = (x_1, x_2, g)$ must satisfy (1.5) almost everywhere, hence $\mathrm{ap}\,\nabla g = (-x_2, x_1)$ almost everywhere, see Theorem 6.2. This can be seen as a tangential condition in the sense of Geometric Measure Theory.

2. Slicing

For the reader's convenience, in this section we recall some well known facts about Sobolev sections, that will be used in the subsequent part of the paper. Let m be a positive integer and denote by $(\boldsymbol{e}_1, \dots, \boldsymbol{e}_m)$ the canonical basis of $\mathbb{R}^m$. If $\Gamma \subset \{1, \dots, m\}$ is a set of indices, then V_Γ is the linear span of $\{\boldsymbol{e}_j : j \in \Gamma\}$ and $V_\Gamma^\perp$ is the linear span of $\{\boldsymbol{e}_j : j \in \{1, \dots, m\} \setminus \Gamma\}$. We introduce the orthogonal projections

$$\pi_\Gamma(x) = \sum_{j \in \Gamma} x_j \boldsymbol{e}_j \quad \text{and} \quad \hat{\pi}_\Gamma(x) = x - \pi_\Gamma(x)$$

where $x \in \mathbb{R}^m$, $\pi_\Gamma : \mathbb{R}^m \to V_\Gamma$ and $\hat{\pi}_\Gamma : \mathbb{R}^m \to V_\Gamma^\perp$. Let Q be an open m-dimensional interval in $\mathbb{R}^m$, namely the product of m open intervals, and fix a nonempty subset $\Gamma \subsetneq \{1, \dots, m\}$. We define the projected intervals

$$Q_\Gamma = \pi_\Gamma(Q) \quad \text{and} \quad \hat{Q}_\Gamma = \hat{\pi}_\Gamma(Q).$$

If $u \colon Q \to \mathbb{R}$ is a function and $z \in \hat{Q}_\Gamma$, we define the *section* $u^z : Q_\Gamma \to \mathbb{R}$ as

$$u^z(y) = u(z + y), \qquad y \in Q_\Gamma.$$

Definition 2.1. We say that a sequence $\{u_h\}$ in a Banach space $(X, \|\cdot\|)$ converges *fast* to $u \in X$, or that it is *fast convergent*, if $\sum_{h=1}^\infty \|u_h - u\| < \infty$.

We wish to point out that the fast convergence in $W^{1,1}$ is just the joint fast convergence in L^1 of functions and their gradients. As a consequence of both Fubini's theorem and Beppo Levi's convergence theorem for series, we get the next proposition.

Proposition 2.1. *Let $\{u_h\} \subset W^{1,1}(Q)$ be a sequence which converges fast to $u \in W^{1,1}(Q)$. Then for each $k = 1, \dots, m$ and for almost every $z \in \hat{Q}_\Gamma$ we have $u^z, (\partial_{y_k} u)^z, u_h^z, (\partial_{y_k} u_h)^z \in L^1(Q_\Gamma)$, $h \in \mathbb{N}$, further, $\{u_h^z\}$ converges fast to u^z in $L^1(Q_\Gamma)$ and $(\partial_{y_k} u_h)^z$ converges fast to $(\partial_{y_k} u)^z$ in $L^1(Q_\Gamma)$.*

Each $u \in W^{1,1}(Q)$ is a limit of a fast convergent sequence of smooth functions. Applying Proposition 2.1 we obtain the following consequence.

Proposition 2.2. *Let $u \in W^{1,1}(Q)$. Then for almost every $z \in \hat{Q}_\Gamma$ we have $u^z \in W^{1,1}(Q_\Gamma)$ and*

$$\partial_{y_k} u^z = (\partial_{y_k} u)^z \quad \text{a.e. in } Q_\Gamma, \qquad k = 1, \dots, m. \tag{2.1}$$

Summarizing Propositions 2.1 and 2.2 we obtain the following.

Proposition 2.3. *Let $\{u_h\} \subset W^{1,1}(Q)$ be a sequence which converges fast to $u \in W^{1,1}(Q)$. Then for almost every $z \in \hat{Q}_\Gamma$ we have $u^z, u_h^z \in W^{1,1}(Q_\Gamma)$, $h \in \mathbb{N}$, and $\{u_h^z\}$ converges fast to u^z in $W^{1,1}(Q_\Gamma)$.*

3. Oriented integration on the circle

The idea of slicing can be also applied to behavior of Sobolev functions on a.e. sphere. However, for our purposes it is enough to perform this analysis in $\mathbb{R}^2$ only, so that we will study Sobolev spaces on circles. The open ball in $\mathbb{R}^2$ with center at x and radius r is denoted by $B(x, r)$.

Definition 3.1 (Function spaces on the circle)**.** Consider the circle $\partial B(x,r)$ and its parametrization

$$\psi(t) = (x_1 + r\cos t,\ x_2 + r\sin t), \qquad t \in \mathbb{R}. \tag{3.1}$$

We define $\psi_- = \psi\lfloor_{(-\pi,\pi)}$ and $\psi_+ = \psi\lfloor_{(0,2\pi)}$, hence (ψ_+,ψ_-) is an oriented atlas of $\partial B(x,r)$. This atlas automatically defines function spaces on $\partial B(x,r)$. Let X be a generic function space symbol which may refer e.g. to $W^{1,p}$, L^p or C. We say that $u : \partial B(x,r) \to \mathbb{R}$ belongs to $X(\partial B(x,r))$ if $u \circ \psi_-$ belongs to $X((-\pi,\pi)$ and $u \circ \psi_+$ belongs to $X(0,2\pi))$.

Definition 3.2 (Integrable forms on the circle)**.** Let us consider $u, v : \partial B(x,r) \to \mathbb{R}$. Then the *oriented integral* of the differential form $u\,dv$ is defined as follows

$$\int_{\partial B(x,r)} u\,dv = \int_{-\pi}^{\pi} (u\circ\psi)(t)\,(v\circ\psi)'(t)\,dt,$$

whenever this expression has a good sense, if e.g. $u \in L^\infty(\partial B(x,r))$, $v \in W^{1,1}(\partial B(x,r))$ and $(v\circ\psi)'$ is the distributional derivative of $v\circ\psi$.

The following lemma relates the fast convergence with the convergence of oriented integrals on spherical sections.

Lemma 3.1. *Let $u, u_h, v, v_h \in W^{1,1}(B(x,\rho))$, $h \in \mathbb{N}$, and suppose that both $u_h \to u$ and $v_h \to v$ fast in $W^{1,1}(B(x,\rho))$. Then for almost every $0 < r < \rho$ the restrictions of u, u_h, v, v_h to $\partial B(x,r)$ belong to $W^{1,1}(\partial B(x,r))$ and*

$$\int\limits_{\partial B(x,r)} u_h\,dv_h \to \int\limits_{\partial B(x,r)} u\,dv\,. \tag{3.2}$$

Proof. We use the polar coordinates given by $\Psi(r,t) = (x_1 + r\cos t,\ x_2 + r\sin t)$ and the notation $\Psi^r = \Psi(r,\cdot)$, $r \in (0,\rho)$. First, we observe that given $w \in W^{1,1}(B(x,\rho))$, then $w \circ \Psi$ belongs to $W^{1,1}((\delta,\rho)\times(-2\pi,2\pi))$ for each $\delta \in (0,\rho)$. The fast convergence of both $\{u_h\}$ and $\{v_h\}$ in $W^{1,1}(B(x,r))$ implies that $u_h\circ\Psi$ and $v_h\circ\Psi$ are fast convergent in $W^{1,1}\big((\delta,\rho)\times(-2\pi,2\pi)\big)$ with limits equal to $u\circ\Psi$ and $v\circ\Psi$, respectively. By Proposition 2.3, for a.e. $r \in (\delta,\rho)$ we have that $u_h\circ\Psi^r, v_h\circ\Psi^r, u\circ\Psi^r, v\circ\Psi^r \in W^{1,1}\big((-2\pi,2\pi)\big)$ and both $u_h\circ\Psi^r$ and $v_h\circ\Psi^r$ converge fast in $W^{1,1}\big((-2\pi,2\pi)\big)$ to $u\circ\Psi^r$ and $v\circ\Psi^r$, respectively.

Fix such a good radius r. Then $u\circ\Psi^r$, $u_h\circ\Psi^r$ are absolutely continuous up to a modification on a null set. Using the one-dimensional Sobolev embedding and passing to absolutely continuous representatives, we obtain a uniform convergence $u_h\circ\Psi^r \to u\circ\Psi^r$. Joining with the L^1-convergence $(v_h\circ\Psi^r)' \to (v\circ\Psi^r)'$ we conclude that

$$\begin{aligned}\int_{\partial B(x,r)} u_h\,dv_h &= \int_{-\pi}^{\pi} (u_h\circ\Psi^r)(t)(v_h\circ\Psi^r)'(t)\,dt \to \int_{-\pi}^{\pi} (u\circ\Psi^r)(t)(v\circ\Psi^r)'(t)\,dt \\ &= \int_{\partial B(x,r)} u\,dv\end{aligned}$$

as required. By the arbitrary choice of $\delta > 0$, we have proved that (3.2) holds for a.e. $r \in (0,\rho)$. □

Lemma 3.2. *Let $v \in W^{1,1}(B(x,\rho))$. For almost every $r \in (0,\rho)$, the oriented integral $\displaystyle\int_{\partial B(x,r)} dv$ is well defined and equal to zero.*

Proof. Again, we use the polar coordinates as in the preceding proof. By Proposition 2.2, for a.e. $r \in (0,\rho)$, the section $v \circ \Psi^r$ belongs to $W^{1,1}(-2\pi, 2\pi)$. If $\bar{v} \circ \Psi^r$ is the absolutely continuous representative of $v \circ \Psi^r$, we have

$$\int_{\partial B(x,r)} dv = \int_{-\pi}^{\pi} (v \circ \Psi^r)'(t)\, dt = \bar{v} \circ \Psi^r(\pi) - \bar{v} \circ \Psi^r(-\pi) = 0,$$

as $\bar{v} \circ \Psi^r$ is obviously 2π-periodic. □

4. An exterior differentiation by blow up

Throughout this section, we fix an open set $\Omega \subset \mathbb{R}^2$, a mapping $f \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^{2n+1})$ and a point $z \in \Omega$ that is a Lebesgue point of both f and Df. Recall that z is a Lebesgue point for a measurable function u if

$$\lim_{r \to 0_+} r^{-n} \int_{B(z,r)} |u(y) - u(z)|\, dy = 0$$

and that almost every point is a Lebesgue point of u if $u \in L^1_{loc}(\Omega)$. As already pointed out in the introduction, $\mathbb{H}^n$ is identified with $\mathbb{R}^{2n+1}$ equipped with the vector fields of (1.3). We fix $\rho > 0$ such that $\overline{B(z,\rho)} \subset \Omega$. Finally, the open unit ball in $\mathbb{R}^2$ centered at the origin will be denoted by $\mathbb{B}$.

Definition 4.1. Let $0 < r < \rho$ and define the *rescaled function* $f_{z,r} : \mathbb{B} \to \mathbb{R}^{2n+1}$ as

$$f_{z,r}(y) := \frac{f(z + ry) - f(z)}{r}\,.$$

Obviously, $f_{z,r} \in W^{1,1}(\mathbb{B}, \mathbb{R}^{2n+1})$ is well defined whenever $0 < r \le \rho$. We use the assumption that z is a Lebesgue point of both f and Df to conclude that

$$\lim_{r \to 0_+} \int_{\mathbb{B}} |f_{z,r}(y) - Df(z) \cdot y|\, dy = 0, \tag{4.1}$$

cf. e.g. [16, Theorem 3.4.2]. The next lemma provides us with important information on the rescaled function $f_{z,\rho}$.

Lemma 4.1. *If* (1.6) *holds almost everywhere, then there exists $w \in W^{1,1}(\mathbb{B})$ such that*

$$dw(y) = \sum_{j=1}^{n} f^j_{z,\rho}(y) df^{j+n}_{z,\rho}(y) - f^{j+n}_{z,\rho}(y)\, df^j_{z,\rho}(y) \quad \textit{for a.e.} \quad y \in \mathbb{B}\,.$$

Proof. In view of (1.6), it follows that

$$\nabla f^{2n+1}_{z,\rho}(y) = \nabla f^{2n+1}(z + \rho y) = \sum_{j=1}^{n} f^j(z+\rho y) \nabla f^{j+n}(z + \rho y) - f^{j+n}(z+\rho y) \nabla f^j(z + \rho y)$$

for a.e. $y \in \mathbb{B}$. We add and subtract all terms of the form $f^j(z)\nabla f^{j+n}(z+\rho y)$, getting

$$\begin{aligned}
\nabla f^{2n+1}_{z,\rho}(y) &= \sum_{j=1}^n f^j(z+\rho y)\nabla f^{j+n}(z+\rho y) - f^{j+n}(z+\rho y)\nabla f^j(z+\rho y) \\
&= \sum_{j=1}^n \left(f^j(z+\rho y) - f^j(z)\right)\nabla f^{j+n}(z+\rho y) - \left(f^{j+n}(z+\rho y) - f^{j+n}(z)\right)\nabla f^j(z+\rho y) \\
&+ \sum_{j=1}^n f^j(z)\nabla f^{j+n}(z+\rho y) - f^{j+n}(z)\nabla f^j(z+\rho y)\ .
\end{aligned}$$

Dividing by ρ, we can rewrite the previous equation as follows

$$\begin{aligned}
&\frac{1}{\rho}\left\{\nabla f^{2n+1}_{z,\rho}(y) - \sum_{j=1}^n \left(f^j(z)\nabla f^{j+n}(z+\rho y) - f^{j+n}(z)\nabla f^j(z+\rho y)\right)\right\} \\
&\qquad = \sum_{j=1}^n f^j_{z,\rho}(y)\nabla f^{j+n}(z+\rho y) - f^{j+n}_{z,\rho}(y)\nabla f^j(z+\rho y)\ .
\end{aligned}$$

Since $\nabla f(z+\rho y) = \nabla f_{z,\rho}(y)$, this immediately leads to the conclusion. □

Next, we show that, under sufficient integrability conditions, it is possible to take somehow the differential of both sides of (1.6), achieving the following theorem.

Lemma 4.2. *If* (1.6) *holds almost everywhere, then we have*

$$\sum_{j=1}^n df^j(z)\wedge df^{j+n}(z) = 0\ .$$

Proof. We choose $\rho_h \searrow 0$ such that $\rho_1 < \rho$ and set $u_h = f_{z,\rho_h}$. By Lemma 4.1, there exists $w_h \in W^{1,1}(\mathbb{B})$ such that for $\mathcal{L}^2$-almost every $y \in \mathbb{B}$ we have

$$dw_h(y) = \sum_{j=1}^n u_h^j(y)\, du_h^{j+n}(y) - u_h^{j+n}(y)\, du_h^j(y)\ .$$

Furthermore, since z is a Lebesgue point of both f and Df, it follows that

$$u_h \to u \ \text{ in } \ W^{1,1}(\mathbb{B}), \quad \text{where} \quad u(y) = \nabla f(z)\cdot y, \quad y \in \mathbb{B}. \tag{4.2}$$

We may assume that the sequence ρ_h is defined in such a way that the convergence in (4.2) is fast. Lemma 3.1 implies that for almost every $r \in (0,1)$ the integral

$$\int_{\partial B(0,r)} \Big(\sum_{j=1}^n u_h^j\, du_h^{j+n} - u_h^{j+n}\, du_h^j\Big)$$

is well defined and equal to $\int_{\partial B(0,r)} dw_h$. Thus, in view of Lemma 3.2 we have

$$\int_{\partial B(0,r)} \Big(\sum_{j=1}^n u_h^j\, du_h^{j+n} - u_h^{j+n}\, du_h^j\Big) = \int_{\partial B(0,r)} dw_h = 0$$

for all h and almost every $r \in (0,1)$. Taking into account both (4.2) and Lemma 3.1, for almost every $r \in (0,1)$ we have

$$0 = \int_{\partial B(0,r)} \Big(\sum_{j=1}^{n} u_h^j \, du_h^{j+n} - u_h^{j+n} \, du_h^j \Big) \to \int_{\partial B(0,r)} \Big(\sum_{j=1}^{n} u^j \, du^{j+n} - u^{j+n} \, du^j \Big).$$

It is enough to pick one such a radius, so that by Stokes theorem, we obtain

$$\int_{B(0,r)} \sum_{j=1}^{n} du^j \wedge du^{j+n} = 0. \tag{4.3}$$

The equation (4.3) yields

$$\mathcal{L}^2(B(0,r)) \sum_{j=1}^{n} \det\big(\nabla f^j(z), \nabla f^{j+n}(z)\big) = 0\,.$$

Thus, we have $\sum_{j=1}^{n} \det\big(\nabla f^j(z), \nabla f^{j+n}(z)\big) = 0$ which gives our claim. □

5. The m-dimensional case

In this section we treat the general case $m \geq 2$.

Theorem 5.1. *Let $\Omega \subset \mathbb{R}^m$ be open, let $m \geq 2$ and set and $f \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^{2n+1})$. If (1.6) holds almost everywhere, then almost everywhere we have*

$$\sum_{j=1}^{n} df^j \wedge df^{j+n} = 0\,. \tag{5.1}$$

Proof. It is enough to verify (5.1) on an arbitrary m-dimensional open cube $Q \subset\subset \Omega$. Fix $1 \leq k < l \leq m$. We set $\Gamma = \{k, l\}$ and use the notation of Section 2, with the exception that now we use the *subscript* z to denote the section

$$f_z(y) = f(z+y), \qquad y \in Q_\Gamma.$$

By Proposition 2.2, for a.e. $z \in \hat{Q}_\Gamma$ we have that $f_z \in W^{1,1}(Q_\Gamma)$ and

$$\frac{\partial f_z}{\partial x_k} = \Big(\frac{\partial f}{\partial x_k}\Big)_z, \qquad \frac{\partial f_z}{\partial x_l} = \Big(\frac{\partial f}{\partial x_l}\Big)_z \quad \text{a.e. in } Q_\Gamma. \tag{5.2}$$

In particular, we have

$$df_z^{2n+1} = \sum_{j=1}^{n} \big(f_z^j df_z^{j+n} - f_z^{j+n} df_z^j \big) \quad \text{a.e. in } Q_\Gamma.$$

Then use Lemma 4.2 on Q_Γ to infer that

$$\sum_{j=1}^{n} df_z^j \wedge df_z^{j+n} = 0 \quad \text{a.e. in } Q_\Gamma.$$

Using Fubini's theorem and (5.2) we obtain that

$$\sum_{j=1}^{n} \det \begin{pmatrix} \frac{\partial f^j}{\partial x_k}, & \frac{\partial f^j}{\partial x_l} \\ \frac{\partial f^{j+n}}{\partial x_k}, & \frac{\partial f^{j+n}}{\partial x_l} \end{pmatrix} = 0 \quad \text{a.e. in } Q.$$

By the arbitrary choice of k and l, the equality (5.1) holds a.e. in Q. □

6. Non-horizontality of higher dimensional Sobolev sets

In this section, the positive integers m and n will be assumed to satisfy the condition $n+1 \le m \le 2n$.

Lemma 6.1. *Let $\boldsymbol{u}_1, \dots, \boldsymbol{u}_{2n} \in \mathbb{R}^m$. Assume that*

$$\sum_{j=1}^{n} \boldsymbol{u}_j \wedge \boldsymbol{u}_{j+n} = 0.$$

Then the matrix B with rows $\boldsymbol{u}_1, \dots, \boldsymbol{u}_{2n}$ has rank at most n.

Proof. We denote the inner product in $\mathbb{R}^{2n}$ by $\langle \cdot, \cdot \rangle$. Further, $(\boldsymbol{e}_1, \dots, \boldsymbol{e}_{2n})$ is the canonical basis of $\mathbb{R}^{2n}$ and I_n is the $n \times n$ identity matrix. We consider the $2n \times 2n$ matrix

$$J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}.$$

Choose $\boldsymbol{v} = (v_1, \dots, v_m)$, $\boldsymbol{w} = (w_1, \dots, w_m) \in \mathbb{R}^m$. We have

$$B\boldsymbol{w} = \sum_{i=1}^{n} \sum_{k=1}^{m} (b_i^k w_k \boldsymbol{e}_i + b_{i+n}^k w_k \boldsymbol{e}_{i+n}) \ , \qquad JB\boldsymbol{v} = \sum_{j=1}^{n} \sum_{l=1}^{m} (b_j^l v_l \boldsymbol{e}_{j+n} - b_{j+n}^l v_l \boldsymbol{e}_j)$$

and this implies that

$$\langle B\boldsymbol{w},\ JB\boldsymbol{v} \rangle = \sum_{k,l=1}^{m} \sum_{i,j=1}^{n} \langle b_i^k w_k \boldsymbol{e}_i + b_{i+n}^k w_k \boldsymbol{e}_{i+n},\ b_j^l v_l \boldsymbol{e}_{j+n} - b_{j+n}^l v_l \boldsymbol{e}_j \rangle.$$

The summands are nonzero only for $i = j$, in which case

$$\langle b_i^k w_k \boldsymbol{e}_i + b_{i+n}^k w_k \boldsymbol{e}_{i+n},\ b_i^l v_l \boldsymbol{e}_{i+n} - b_{i+n}^l v_l \boldsymbol{e}_i \rangle = w_k v_l \det \begin{pmatrix} b_i^l, & b_i^k \\ b_{i+n}^l, & b_{i+n}^k \end{pmatrix} \ ,$$

so that

$$\langle B\boldsymbol{w},\ JB\boldsymbol{v} \rangle = \sum_{k,l=1}^{m} w_k v_l \sum_{i=1}^{n} \det \begin{pmatrix} b_i^l, & b_i^k \\ b_{i+n}^l, & b_{i+n}^k \end{pmatrix} = \sum_{k,l=1}^{m} w_k v_l \Big(\sum_{i=1}^{n} \boldsymbol{u}_i \wedge \boldsymbol{u}_{i+n} \Big)_{l,k} = 0.$$

Then the images of B and of JB are orthogonal subspaces of $\mathbb{R}^{2n}$, having the same dimension, hence the rank of B cannot be greater than n. □

Theorem 6.1. *Let $\Omega \subseteq \mathbb{R}^m$ be an open set and consider $f \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^{2n+1})$ which almost everywhere satisfies (1.6). It follows that the Jacobian matrix of f almost everywhere has rank at most n.*

Proof. This is a consequence of Theorem 5.1 and Lemma 6.1. □

By Theorem 6.1, the proof of Theorem 1.2 follows essentially the same lines of [15]. Next, for the sake of the reader, we adapted this proof to our setting.

Proof of Theorem 1.2. By Theorem 6.1, the equation (1.6) fails to hold for f on a set $E \subset \Omega$ of positive $\mathcal{L}^m$-measure. We can assume that E is bounded, made by density points, that everywhere on E the approximate differentiable of f exists and equals its distributional differential and they have everywhere rank equal to m. Up to taking a smaller piece of E, we can also assume that f is Lipschitz. Then we consider a Lipschitz extension of $f|_E$ to all of $\mathbb{R}^m$ and apply Whitney extension theorem, hence finding a subset E_0 of E with positive measure and $g \in C^1(\mathbb{R}^m, \mathbb{R}^{2n+1})$ such that $g|_{E_0} = f|_{E_0}$ and the approximate differential of f and the differential of g coincide on E_0. We choose $y_0 \in E_0$ and notice that for a fixed $r_0 > 0$ sufficiently small, we have $\mathcal{L}^m(B_{y_0,r_0} \cap E_0) > 0$ and $\Sigma_0 = g(B_{y_0,r_0})$ is an m-dimensional embedded manifold of $\mathbb{R}^{2n+1}$. By the properties of g and the classical area formula, we have

$$\Sigma_1 = f(B_{y_0,r_0} \cap E_0) = g(B_{y_0,r_0} \cap E_0) \subset \Sigma_0 \cap \Sigma \quad \text{and} \quad \mathcal{H}^m_{|\cdot|}(\Sigma_1) > 0.$$

Since (1.6) does not hold on E_0, for any $y \in B_{y_0,r_0} \cap E_0$, we have $T_{f(y)}\Sigma_0 \not\subset H_y\mathbb{H}^n$, therefore

$$\tau_{\Sigma_0,\mathcal{V}}(f(y)) \neq 0,$$

where we have used the notation $\tau_{\Sigma_0,\mathcal{V}}(x)$ with $x \in \Sigma_0$ to indicate the *vertical tangent p-vector* to Σ_0 at x, see [13, Definition 2.14]. This m-vector vanishes exactly at those points x where $T_x\Sigma_0 \subset H_x\mathbb{H}^n$, see [13, Proposition 3.1]. From both [12] and [13], the spherical Hausdorff measure $\mathcal{S}^{m+1}_d \llcorner \Sigma_0$ is equivalent, up to geometric constants, to the measure $|\tau_{\Sigma_0,\mathcal{V}}|\, \mathcal{H}^m_{|\cdot|} \llcorner \Sigma_0$, hence in particular $\mathcal{S}^{m+1}_d(\Sigma_1) > 0$, therefore

$$\mathcal{H}^{m+1}_d(\Sigma) \geq \mathcal{H}^{m+1}_d(\Sigma_1) > 0\,,$$

so the proof is complete. □

6.1. **Formal horizontality of some BV graphs.** Our previous arguments also allow us to establish a kind of "generalized horizontal tangency" of BV functions whose graph satisfies the metric constraint (1.2), as explained in the introduction. In fact, by the arguments in the proof of Theorem 1.2, it is not difficult to establish the following result.

Theorem 6.2. *Let $2 \leq \alpha < 3$ and let $g : (0,1)^2 \to \mathbb{R}$ be a BV function such that its graph*

$$G = \{(x_1, x_2, g(x)) : \ 0 < x_1, x_2 < 1\} \quad \textit{satisfies} \quad \mathcal{H}^\alpha_d(G) < +\infty\,,$$

where d is the SR-distance of $\mathbb{H}^1$, identified with $\mathbb{R}^3$ by the coordinates associated to the vector fields of (1.1). Then the approximate gradient $\operatorname{ap}\nabla g$ almost everywhere satisfies

$$\operatorname{ap}\nabla g(x) = (-x_2, x_1)\,. \tag{6.1}$$

Remark 6.2. As already mentioned in the introduction, the existence of BV functions that satisfy the assumptions of Theorem 6.2 with $\alpha = 2$ has been proved by Z. M. Balogh, R. Hoefer-Isenegger and J. T. Tyson, [4]. The existence of BV functions whose absolutely continuous part of the distributional gradient almost everywhere equals a vector field with nonvanishing curl is a special instance of a general result due to G. Alberti, [1].

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