On the sub-Riemannian area of submanifolds

Conference on "Geometric Analysis in Control and Vision Theory"

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Outline

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1. Known area formulas
2. Hausdorff measure in sub-Riemannian manifolds
3. Homogeneous groups
4. Degree of vectors and intrinsic area
5. Intrinsic measure of submanifolds
6. Measure theoretic area formula
7. Intrinsic singular points of smooth submanifolds
8. Blow-up and sub-Riemannian area formula
9. Existence of blow-ups

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Hausdorff and spherical measure

The idea of this notion is elementary. Let $E \subset X$ be a subset of a metric space X. Then for some fixed number $c_\alpha > 0$ and $k\alpha > 0$ we define

$$\mathcal{H}^\alpha(E) = \sup_{\delta>0} \inf \left\{ \sum_{j=0}^{\infty} c_\alpha \mathrm{diam}(S_j)^k : E \subset \bigcup S_j, \ \mathrm{diam}(S_j) \le \delta \right\}$$

$$\mathcal{S}^\alpha(E) = \sup_{\delta>0} \inf \left\{ \sum_{j=0}^{\infty} c_\alpha \mathrm{diam}(S_j)^\alpha : E \subset \bigcup S_j, \ \mathrm{diam}(S_j) \le \delta \right.$$

$$\left. \text{and } S_j \text{ is a ball} \right\}$$

The *Hausdorff dimension* of $E \subset X$ is the number

$$\dim_H E = \inf \{\alpha > 0 : \ \mathcal{S}^\alpha(E) = 0\} \in [0, +\infty].$$

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Hausdorff measure on Riemannian manifolds

If M is an n-dimensional Riemannian manifold with metric g, (U, ψ) is a local chart of M and with $\psi : A \to U$ with an open set $A \subset \mathbb{R}^n$, then we have

$$\mathcal{H}^n_\rho(U) = \int_A \sqrt{\det(g_{ij})}dx,$$

where ρ is the Riemannian distance associated to g.

If $\Sigma \subset M$ is a k-dimensional submanifold of (M, g), then for a local chart $\psi : \Omega \to \Sigma$ of Σ, we have

$$\mathcal{H}^k_\rho(\Sigma \cap \psi(\Omega)) = \int_\Omega \sqrt{\det(\tilde{g}_{ij})}dx,$$

where $\tilde{g}_{ij}$ is the induced metric on Σ.

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Hausdorff measure in metric spaces

More generally, if $f : A \to X$ is a Lipschitz mapping taking values in a metric space, then

$$\mathcal{H}^k(f(A)) = \int_A Jf(x)dx$$

where $A \subset \mathbb{R}^k$ is a Lebesgue measure subset and $k \le n$. This result was proved in [B. Kirchheim, Proceeding of AMS, 1994].

Definition (Rectifiable sets)

A subset $E \subset X$ in a metric space X is called rectifiable *if there exists a Lipschitz mapping $f : A \to E$, where $A \subset \mathbb{R}^k$ such that $E = f(A)$.*

We can conclude that as soon as our subset is *rectifiable*, then we can compute its area.

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Gromov's formula for Hausdorff dimension of submanifolds

M. Gromov [1996, Carnot-Carathéodory spaces seen from within] introduced a natural formula for the Hausdorff dimension of submanifolds, that works for *generic* submanifolds of equiregular Carnot-Carathéodory spaces.

Let $\Sigma \subset \mathbb{M}$ be a smooth submanifold of an *equiregular sub-Riemannian manifold M* and fix $p \in M$. If $\mathcal{D}$ is the horizontal distribution of M, we define the following flag

$$T_p^1 M = \mathcal{D}_p, \quad T_p^2 M = \mathcal{D}_p + [\mathcal{D}, \mathcal{D}]_p, \quad \dots,$$
$$\dots \quad T_p^\iota M = \mathcal{D}_p + [\mathcal{D}, \mathcal{D}]_p + \cdots + [[\cdots[\mathcal{D}, \mathcal{D}], \mathcal{D}], \dots, \mathcal{D}]_p$$

at the point p.

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at the point p.

We consider the induced flag on $T_p\Sigma$ defining

$$T_p^j\Sigma = T_p\Sigma \cap T_p^j M$$

for each $j = 1, \ldots, \iota$.

We define we define a kind of "pointwise Hausdorff dimension" at p as

$$D'(p) = \sum_{j=0}^{\iota} j \dim \left(T_p^j\Sigma / T_p^{j-1}\Sigma \right)$$

Finally, we define the integer $D_H(\Sigma) = \max_{p\in\Sigma} D'(p)$.

Gromov 1996

Smooth submanifolds $\Sigma \subset \mathbb{M}$ have generically Hausdorff dimension $D_H(\Sigma)$.

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Absence of Lipschitz parametrizations in sub-Riemannian manifolds

Let us now pick any submanifold $\Sigma \subset M$ of topological dimension k and Hausdorff dimension N with respect to the sub-Riemannian distance, where $\mathrm{N} > k$.

Assume by contradiction that there exists a Lipschitz mapping $f : \Omega \to \Sigma$, where $\Omega \subset \mathbb{R}^k$ is an open set, that is also *surjective*. From the standard property of Lipschitz mappings, we get

$$\dim_{CC} \Sigma \leq \dim_{Eucl} \Omega = k,$$

that is a contradiction.

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Let us now pick any submanifold $\Sigma \subset M$ of topological dimension k and Hausdorff dimension N with respect to the sub-Riemannian distance, where $\mathrm{N} > k$.

Assume by contradiction that there exists a Lipschitz mapping $f : \Omega \to \Sigma$, where $\Omega \subset \mathbb{R}^k$ is an open set, that is also *surjective*. From the standard property of Lipschitz mappings, we get

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Sub-Riemannian area

Once we know that N is the Hausdorff dimension of a submanifold $\Sigma \subset \mathbb{G}$, we can immediately consider the natural notion of "surface measures" that is

$$\mathcal{H}_\rho^{\mathrm{N}} \llcorner \Sigma \quad \text{or} \quad \mathcal{S}_\rho^{\mathrm{N}} \llcorner \Sigma,$$

where ρ here is the sub-Riemannian distance.

`How can we compute these measures?`
`Are they finite and positive?`

`The exact formula for` $\mathcal{H}^{\mathrm{N}}(\Sigma)$ `is not yet unknown.`
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Homogeneous groups

- We think of a *homogeneous group*

$$\mathbb{G} = H^1 \oplus \cdots \oplus H^\iota$$

as a graded vector space equipped with both a group operation and a Lie product with the properties

$$H^{i+j} \subset [H^i, H^j] \qquad (1)$$

for every $i, j \geq 1$, where we have set $H^j = \{0\}$ whenever $j > \iota$. The algebraic grading (1) gives the nilpotence of the group and it allows us for compatible dilations.

- This grading of $\mathbb{G}$ allows us to define for each $r > 0$ the dilation

$$\delta_r : \mathbb{G} \to \mathbb{G}, \quad \delta_r x = r^j x \quad \text{for all } x \in H^j$$

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Homogenous distance and SR distance

A distance function $d : \mathbb{G} \times \mathbb{G} \to \mathbb{R}$ which satisfies

$$d(xz, xw) = d(z, w) \quad \text{and} \quad d(\delta_r z, \delta_r w) = r d(z, w)$$

for each $x, z, w \in \mathbb{G}$ and $r > 0$ is a *homogeneous distance.*
It can be proved that for every compact set $K \subset \mathbb{G}$ there exists $C > 0$ such that the powers in the following estimates are optimal

$$C^{-1}|x - y| \le d(x, y) \le C\,|x - y|^{1/\iota} \tag{2}$$

whenever $x, y \in K$.
The *sub-Riemannian metric* is also a homogeneous distance and all homogeneous distances are globally bi-Lipschitz equivalent with respect to each other

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Anisotropic geometry of homogeneous groups

The homogeneity of any homogeneous distance implies different behaviors along different directions of the group.

If $v \in H^j$ is not vanishing, then for each $t \in \mathbb{R}$ we get

$$d(0, tv) = \sqrt[j]{|t|}\, d(0, v).$$

So, the distance is Hölder continuous with respect to the Euclidean distance. As a result, if $L \subset H^j$ is a one dimensional subspace, then

$$\mathcal{H}^j_d \llcorner L = \mathcal{H}^1_E \quad \text{and} \quad \dim_H L = j,$$

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The homogeneity of any homogeneous distance implies different behaviors along different directions of the group.

If $v \in H^j$ is not vanishing, then for each $t \in \mathbb{R}$ we get

$$d(0, tv) = \sqrt[j]{|t|}\, d(0, v).$$

So, the distance is Hölder continuous with respect to the Euclidean distance. As a result, if $L \subset H^j$ is a one dimensional subspace, then

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Let us consider the Heisenberg group $\mathbb{H} = H^1 \oplus H^2$, where $\dim H^1 = 2$ *and* $\dim \mathbb{H}^2 = 1$. *It is well known in fact that*

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Local grading of homogeneous groups

What allows us to overcome the previuos problem is the idea that degrees can be summed only at **infinitesimal level.**

The idea is to look at the manifold at very large scale, to detect the precise nature of a point. We then look at vectors in the tangent space.

The grading of $\mathbb{G} = H^1 \oplus \cdots \oplus H^\iota$ automatically induces a grading at every tangent space $T_p\mathbb{G}$. We can canonically identify $T_0\mathbb{G}$ with $\mathbb{G}$, since $\mathbb{G}$ is a linear space. Then we have a grading of the Lie algebra

$$\mathrm{Lie}(\mathbb{G}) = \mathcal{V}_1 \oplus \cdots \oplus \mathcal{V}_\iota,$$

where denoting by $\exp : \mathrm{Lie}(\mathbb{G}) \to \mathbb{G}$ the exponential mapping, we get

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Definition (Degree of homogeneous vector fields)

We say that $X \in \mathrm{Lie}(\mathbb{G})$ is a (homogeneous) left invariant vector field of degree j, and set $\deg(X) = j$, if and only if X belongs to the subspace $\mathcal{V}_j \subset \mathrm{Lie}(\mathbb{G})$.

Definition (Degree of homogeneous k-vector fields)

Let us consider $Z_1, \ldots, Z_k \in \mathrm{Lie}(\mathbb{G})$, having degrees $n_1, \ldots, n_k$, respectively. Then we set

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These definitions allow us to introduce a *pointwise notion* of degree for submanifolds, using their tangent space.

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We say that $X \in \mathrm{Lie}(\mathbb{G})$ is a (homogeneous) left invariant vector field of degree j, and set $\deg(X) = j$, if and only if X belongs to the subspace $\mathcal{V}_j \subset \mathrm{Lie}(\mathbb{G})$.

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Let us consider $Z_1, \ldots, Z_k \in \mathrm{Lie}(\mathbb{G})$, having degrees $n_1, \ldots, n_k$, respectively. Then we set

$$\deg(Z_1 \wedge \cdots \wedge Z_k) = n_1 + \cdots + n_k.$$

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Graded coordinates

Once we have assign a degree, or *weight*, to any direction of any tangent space, it is convenient to introduce special *adapted coordinates*.

Definition (Graded basis in $\mathbb{G}$)

If $\mathbb{G} = H^1 \oplus \cdots \oplus H^\iota$, then we define $\mathrm{m}_j = \sum_{i=1}^{j} \dim H^i$ and $\mathrm{m}_0 = 0$. We say that a basis $(e_1, \ldots, e_\mathrm{q})$ of $\mathbb{G}$ is graded *if*

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Degree of non-homogeneous vectors and of submanifolds

As for polynomials, once we have assigned a degree to homogeneous k-vector fields, then we can assigne a degree to all of them.

Projections

If $\xi \in \Lambda_k(\mathrm{Lie}(\mathbb{G}))$ is a k-vector field with

$$\xi = \sum_{I \in \mathcal{I}_k} c_I X_I$$

then we define the N-*projection of* ξ as follows

$$\pi_{\mathrm{N}}(\xi) = \sum_{\substack{I \in \mathcal{I}_k \\ d(X_I) = \mathrm{N}}} c_I X_I \,.$$

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Definition (Degree of submanifolds)

If $\Sigma \subset \mathbb{G}$ is an n-dimensional smooth submanifold of $\mathbb{G}$ and $p \in \Sigma$, then we define a tangent n-vector of Σ at p as follows

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Intrinsic measure of submanifolds

Let us fix any auxiliary Riemannian metric $\tilde{g}$ on $\mathbb{G}$, such that τ_Σ is a *unit* tangent n-vector with respect to this metric. Let N be the degree of Σ. Then we introduce the *intrinsic measure* of Σ as follows

$$\mu_\Sigma = |\pi_{\mathrm{N}}(\tau_\Sigma)|\, vol_{\tilde{g}} \llcorner \Sigma.$$

This measure, introduced in [M. and D. Vittone, J. Reine Ang. Math., 2008], is the natural

`sub-Riemannian area of submanifolds.`

In fact, we will see that using recent differentiation theorems for measures, it is possible to show that this measure *coincides with the spherical Hausdorff measure*.

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Let us fix any auxiliary Riemannian metric $\tilde{g}$ on $\mathbb{G}$, such that τ_Σ is a *unit* tangent n-vector with respect to this metric. Let N be the degree of Σ. Then we introduce the *intrinsic measure* of Σ as follows

$$\mu_\Sigma = |\pi_{\mathrm{N}}(\tau_\Sigma)| \, vol_{\tilde{g}} \llcorner \Sigma.$$

This measure, introduced in [M. and D. Vittone, J. Reine Ang. Math., 2008], is the natural

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Measure theoretic area formula

In a recent paper [M., Proc. Royal. Soc. Ed., 2015.] a rather general area-type formula has been established in a general metric space X. If $\Sigma \subset X$ is a Borel set, μ is a Borel regular measure concentrated on Σ, that is absolutely continuous with respect to $\mathcal{S}^\alpha \llcorner \Sigma$, then

$$\mu = \theta^\alpha(\mu, \cdot)\, \mathcal{S}^\alpha \llcorner \Sigma. \tag{3}$$

Some technical assumptions on X are needed, as for instance that $\mathrm{diam}(\mathbb{B}(x,r)) = \varphi(r)$ for each $x \in X$, where φ is continuous.
The key of this formula is the explict representation of the density θ^α, namely the `Federer density`:

$$\theta^\alpha(\mu, x) = \inf_{\varepsilon>0} \sup \left\{ \frac{\mu(\mathbb{B})}{c_\alpha (2r)^\alpha} : \mathbb{B} \in \mathcal{F}_b,\ x \in \mathbb{B},\ \mathrm{diam}\mathbb{B} \le \varepsilon \right\}.$$

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Intrinsic singular points of smooth submanifolds

The previous formula is general, so in which cases we are able to compute the Federer density?

Singular points

The notion of degree we have seen could be seen somehow as a "pointwise Hausdorff dimension". According to this fact, we consider the following set of points

$$C(\Sigma) = \{p \in \Sigma : d_\Sigma(p) < \mathrm{N}\},$$

as the *singular set* of the submanifold, with respect to the distance of the group, where $\mathrm{N} = d(\Sigma)$.

Negligibility problem

Find the minimal regularity of Σ, such that

$$\mathcal{S}^{\mathrm{N}}(C(\Sigma)) = 0.$$

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$C^{1,1}$ blow-up

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$$\delta_{1/r}(p^{-1}\Sigma) \to S_p\Sigma \tag{4}$$

locally as $r \to 0^+$ with respect to the Hausdorff convergence of sets. $S_p\Sigma$ is a *subgroup of* $\mathbb{G}$.

Application to area-type formula

If the blow-up holds at some point, then we can compute the Federer density at that point. Joining this fact with the negligibility condition and taking a *suitable homogeneous distance*, we get

$$\mu_\Sigma = \mathcal{S}^N \llcorner \Sigma. \tag{5}$$

For $C^{1,1}$ smooth submanifolds in two step groups the negligibility condition holds, [M., JGA, 2010], then the formula (5).

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If the blow-up holds at some point, then we can compute the Federer density at that point. Joining this fact with the negligibility condition and taking a *suitable homogeneous distance*, we get

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For $C^{1,1}$ smooth submanifolds in two step groups the negligibility condition holds, [M., JGA, 2010], then the formula (5).

$C^{1,1}$ blow-up

In the paper [M. and D. Vittone, J. Reine Ang. Math., 2008], we have proved that at each point *of maximum degree* $p \in \Sigma$, if Σ is a $C^{1,1}$ smooth submanifold, then

$$\delta_{1/r}(p^{-1}\Sigma) \to S_p\Sigma \tag{4}$$

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Questions

For higher step groups the validity of the negligibility condition for $C^{1,1}$ submanifolds is not yet established. Do we have blow-up for C^1 smooth submanifolds?

According to the classical situation, it is natural to expect an area formula for smooth submanifolds of class C^1.
This is the case for generic submanifolds.

Transversal submanifold

For each dimension $\mathrm{n} \le \mathrm{q} = \dim \mathbb{G}$, we define

$$D(\mathrm{n}) = \max \{ d(\Sigma) : \Sigma \text{ is an } \mathrm{n}\text{-dimensional submanifold of } \mathbb{G} \}.$$

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Is it possible to define an intrinsic tangent space of a submanifold $\Sigma \subset \mathbb{G}$ before we know the existence of a blow-up?

Algebraic tangent space

Let $\Sigma \subset \mathbb{G}$ be a C^1 smooth submanifold of dimension n and let $d_\Sigma(p) = \mathrm{N}$, where $p \in \Sigma$. We first consider the unique left invariant n-vector field ξ such that $\xi(p) = \tau_\Sigma(p)$ and $\tau_\Sigma(p)$ is a tangent n-vector of Σ at p. We project ξ on the space of left invariant n-vector fields ξ with degree N, that is

$$\xi_\mathrm{N} = \pi_\mathrm{N}(\xi).$$

Then we define the n-vector $\tau_{\Sigma,\mathrm{N}}(p) = \xi_\mathrm{N}(0) \in \Lambda_\mathrm{n}(\mathbb{G})$. Thus, we define the *algebraic tangent space of* Σ *at* p as the n-dimensional subspace

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Algebraically regular points of $C^{1,1}$ submanifolds

In [M. and D. Vittone, J. Reine Ang. Math., 2008] we proved that points of maximum degree are *algebraically regular*.

Sketch of the proof

If $X_j(0), X_i(0) \in A_p\Sigma$, we can find an adapted basis $(X_1, \ldots, X_{\mathrm{q}})$ and special Lipschitz vector fields on Σ of the form

$$v_j^k = X_j + \sum_{d(X_r)\le k} \phi_r\, X_r \quad \text{and} \quad v_i^l = X_i + \sum_{d(X_s)\le l} \psi_s\, X_s.$$

where $\deg(X_j) = k$ and $\deg(X_i) = l$ and we have Lipschitz functions ϕ_r, ψ_s, which vanish at x whenever $d(r) = k$ or $d(s) = l$.

By the $C^{1,1}$ condition, the following equality holds almost everywhere.

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&= [X_j, X_i] + \sum_{d(X_r)\le k} \phi_r\,[X_r, X_i] + \sum_{d(X_s)\le l} \psi_s\,[X_j^k, X_s] \\
&+ \sum_{d(X_r)\le k, d(X_s)\le l} \phi_r\,\psi_s\,[X_r, X_s] \\
&+ \sum_{d(X_s)\le l} (X_j\psi_s)X_s - \sum_{d(X_r)\le k} (X_i\phi_r)X_r \\
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By Frobenius theorem we know that this vector is tangent to Σ a.e. Then this Lie product is a linear combination of $v_1, \ldots, v_n$, spanning the tangent space.

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The Lie product lies in $V_1 \oplus \cdots \oplus V_{k+l}$, hence the special form of these vector fields implies

$$[v_j, v_i] = \sum_{\deg(v_r) \le k+l} a_r v_r$$

almost everywhere on Σ.

Projecting both sides of the previous identity onto V_{k+l}, we get

$$[X_j, X_i] + \sum_{d(X_r)=k} \phi_r [X_r, X_i] + \sum_{d(X_s)=l} \psi_s [X_j, X_s] + \\ + \sum_{d(X_r)=k, d(X_s)=l} \phi_r \psi_s [X_r, X_s] = \sum_{\sigma(v_r)=k+l} a_r \pi_{k+l}(v_r).$$

Finally we pass to the limit by a converging sequence $a_r(p_k)$ where $p_k \to p$, where all ϕ_r, ϕ_s vanish at p for $\deg(X_r) = k$ and $\deg(X_s) = l$. We get

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Theorem (M. 2016 and previous works)

If Σ is a C^1 smooth submanifold and $p \in \Sigma$ has maximum degree, then

$$\delta_{1/r}(p^{-1}\Sigma) \to S_p\Sigma \quad as \quad r \to 0^+ \tag{8}$$

holds under one of the following conditions.

1. *$A_p\Sigma$ is a horizontal subgroup.*
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Let Σ be a C^1 smooth submanifold of homogeneous group and let $p \in \Sigma$ be a point of maximum degree. *Is p algebraically regular? In other words, is the algebraic tangent space at p a subgroup of $\mathbb{G}$?*

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If p is *algebraically regular*, then can we always prove the existence of the blow-up limit?

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Theorem (M. 2016 and previous work.)

If d is a suitable distance *and Σ is a C^1 smooth submanifold of degree* N, *then we have*

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Further comments I

We have limited out presentation to special distances to construct the spherical measure, without giving more details.

In the assumption of the previous theorem, we also have

$$\theta^{\mathrm{N}}(\mu_\Sigma, p) = \max_{d(y,0)\le 1} \mathcal{H}^{\mathrm{n}}_{|\cdot|}\big((y^{-1}A_p)\Sigma \cap \mathbb{B}(0,1)\big),$$

where the right hand side con be defined a priori as a *metric factor*, depending on the distance.
We denote this factor with respect to p as $\beta(d,p)$.

Under the previous assumptions, we get a more general formula

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It is always possible to construct a *special* homogeneous distance d on any homogeneous group $\mathbb{G}$ whose unit ball is an Euclidean ball with sufficiently small radius, see [W. Hebisch and A. Sikora, Studia Math. 1990].

From this special symmetric distance d, the metric factor

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From this special symmetric distance d, the metric factor

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