Outline of previous research and research statement

Davide Lombardo

1 Outline of previous research

Let K be a number field and A be a K-abelian variety. For each prime ℓ there is a natural representation

$$\rho_{\ell^\infty} : \operatorname{Gal}\left(\overline{K}/K\right) \to \operatorname{Aut} T_\ell(A)$$

whose image G_{ℓ^∞} one would like to describe, along with the image G_∞ of the adelic representation

$$\rho_\infty : \operatorname{Gal}\left(\overline{K}/K\right) \to \prod_\ell \operatorname{Aut} T_\ell(A) \cong \operatorname{GL}_{2g}(\hat{\mathbb{Z}}).$$

In my past work I have dealt with the problem of giving effective estimates on the size of the groups G_{ℓ^∞} and G_∞, and with various connected questions.

1.1 Effective results in the vertical setting

In [15] I've considered the adelic representation attached to an elliptic curve without CM:

Theorem 1 *([15]) Let E/K be an elliptic curve without CM. We have*

$$\left[\operatorname{GL}_2(\widehat{\mathbb{Z}}) : G_\infty\right] < C_1 \cdot [K : \mathbb{Q}]^{C_2} \cdot \max\left\{1, h(E), \log[K : \mathbb{Q}]\right\}^{2C_2},$$

where $C_1 = \exp(6 \cdot 10^{29527})$, $C_2 = 4.9 \cdot 10^{10}$, and $h(E)$ is the stable Faltings height of E.

This is the first result of its kind which is both *adelic* and *explicit*, after the results for large ℓ proven by Masser and Wüstholz in [22] and the semi-effective adelic approach of Zywina [35]. I've then extended this theorem ([13]) to give an explicit lower bound on the size of G_∞ for products of elliptic curves without CM. The proof of these results led me to formulate extensions of Pink's results from [24] to more general situations (specifically, to subgroups of $\operatorname{SL}_2(\mathbb{Z}_\ell)^n$ that are not necessarily pro-ℓ), see [21].

The case of CM abelian varieties is relatively well understood, but obtaining effective estimates still requires some effort. In [17] I have shown:

Theorem 2 *Let K be a number field and A/K be an absolutely simple abelian variety of dimension g admitting complex multiplication over K by an order in the CM field E. Let $\operatorname{MT}(A)$ be the Mumford-Tate group of A. There exists an (explicit) constant $M = M(g, K)$ such that for all primes ℓ the following holds:*

$$[\operatorname{MT}(A)(\mathbb{Z}/\ell^n\mathbb{Z}) : \operatorname{Gal}\left(K(A[\ell^n])/K\right)] \leq M.$$

1.2 Effective results for large ℓ

For various classes of higher-dimensional abelian varieties I have obtained descriptions of the groups G_{ℓ^∞} for all primes ℓ exceeding certain explicit bounds. In particular, in [14] I've completely solved this problem for geometrically simple abelian surfaces A/K and for GL_2-varieties. For example one has:

Theorem 3 *There exists an explicit polynomial $p(d,h)$ with the following property. Let K be a number field and A/K be an abelian surface such that $\operatorname{End}_{\overline{K}}(A)=\mathbb{Z}$. Let ℓ be a prime that is divisible by at least one place of K at which A has semistable reduction. If ℓ is unramified in K and larger than $p([K:\mathbb{Q}],h(A))$, then the equality $G_{\ell^\infty}=\mathrm{GSp}_4(\mathbb{Z}_\ell)$ holds.*

In [16] I have then considered the corresponding problem for $g=\dim A\geq 3$. I have shown an effective result analogous to the previous one for the case $g=3$, and also a semi-effective result in all dimensions which can be roughly stated as follows. Suppose that $\operatorname{End}_{\overline{K}}(A)=\mathbb{Z}$ and that $G_{\ell^\infty}=\mathrm{GSp}_{2g}(\mathbb{Z}_\ell)$ for some (unspecified) prime ℓ. Then we have $G_{\ell^\infty}=\mathrm{GSp}_{2g}(\mathbb{Z}_\ell)$ for all primes ℓ larger than a bound $\ell_0(A/K)$ which can be computed explicitly in terms of K, g, $h(A)$ (the stable Faltings height of A), and the order of the residue field of a place v of K such that the Frobenius at v acts on A in a "maximally generic" way.

1.3 Product varieties

Let $\mathcal{G}_\ell$ be the $\mathbb{Q}_\ell$-Zariski closure of G_{ℓ^∞} in $\mathrm{GL}_{T_\ell(A)\otimes\mathbb{Q}_\ell}$. In [18] I've developed techniques to study the algebraic groups $\mathcal{G}_\ell(A\times B)$ attached to a product $A\times B$ of two abelian varieties (defined over an arbitrary finitely generated field). This has allowed me to prove an ℓ-adic analogue of a Hodge-theoretic result of Ichikawa [10], and to provide an analogue for Tate classes of the computation – due to Moonen and Zarhin [23] – of the space of Hodge classes on nonsimple complex abelian varieties of small dimension.

1.4 Other contributions

In [19] I have given a proof of the uniform Rasmussen-Tamagawa conjecture (see [27]) for CM abelian varieties of arbitrary dimension, extending a recent result of Bourdon [5] concerning CM elliptic curves.

I have also shown [20] that a certain weak version of the property called μ in the series of papers [7] [8] [9] is true for all abelian varieties, while a slightly stronger statement is false in general, even for abelian varieties that satisfy $\operatorname{End}_{\overline{K}}(A)=\mathbb{Z}$.

2 Research statement

I shall describe below some problems which seem to me to be both interesting and within reach given the present state of our knowledge, and which I feel would be suitable and realistic objectives for a research programme spanning a few years.

2.1 Endomorphism algebras of abelian surfaces

Let A/K be an abelian surface whose endomorphism ring is not known (this is often the case in practice, for example if A is given as the Jacobian of a genus 2 curve). An interesting (and in general hard) problem is that of determining $\operatorname{End}_K(A)$, or even just $\operatorname{End}_{\overline{K}}(A)\otimes\mathbb{Q}$.

Computing characteristic polynomials of Frobenius one can easily give "upper bounds" on these rings, but proving that an abelian variety has nontrivial endomorphisms is a challenging problem.

In [14] I have obtained effective results concerning the Galois representations attached to abelian surfaces which can be useful in this context, because they give *a priori* bounds on the number of characteristic polynomials of Frobenius that it is necessary to compute in order to establish the existence of a nontrivial endomorphism. This leads to an effective algorithm to compute the structure of $\operatorname{End}_{\overline{K}}(A) \otimes \mathbb{Q}$ when A is an abelian surface (with some restrictions in the case of quaternionic multiplication). At present, however, such an algorithm is not efficient enough to be used in practice, but it seems possible to improve it by complementing it with a study of the subvarieties of the moduli space of genus-2 curves that parametrize curves whose Jacobians have additional endomorphisms. This approach might lead to a practical algorithm for computing $\operatorname{End}_{\overline{K}}(A) \otimes \mathbb{Q}$, and in any case it involves some interesting theoretical questions regarding the moduli space of genus 2 curves whose solution could provide further insight in its structure.

This project would answer a question of Poonen [26] and provide a theoretical and systematic verification of the (conjectural) results obtained in [4] via numerical methods.

2.2 Modulo-p reduction of points of infinite order on algebraic groups

In [11], Jones and Rouse have introduced a notion of "arboreal Galois representation" associated with the dynamical system given by the orbit of a point of a commutative algebraic group $\mathcal{G}$ under the action of an endomorphism of $\mathcal{G}$, and they have studied this notion in some special cases. It would be interesting to understand to which extent the hypotheses in their results are optimal, and to extend such results to more general situations (in particular going beyond the case $\dim \mathcal{G} = 1$). This will certainly involve doing Kummer theory over $\mathcal{G}$, a task which is already nontrivial (and interesting) when $\mathcal{G}$ is an abelian variety.

The notion of arboreal Galois representations has many immediate arithmetical applications: it provides for example a framework in which to understand some classical results of Hasse ("Let n be an integer not of the form $\pm m^2, \pm 2m^2$. Then the set of primes p such that the order of n modulo p is odd admits a natural density, which is equal to 1/3"), and can also be used to prove some new, interesting ones, concerning for example the modulo-p reduction of points of infinite order on elliptic curves.

This project is a recently-started collaboration with Antonella Perucca (Universität Regensburg).

2.3 Uniformity problems for the representations attached to rational elliptic curves

Let $E/\mathbb{Q}$ be an elliptic curve. The group $\operatorname{Aut} T_\ell(E)$ is isomorphic to $\mathrm{GL}_2(\mathbb{Z}_\ell)$, and a short argument (see for example [35]) shows that, for every prime ℓ, the index $[\mathrm{GL}_2(\mathbb{Z}_\ell) : G_{\ell^\infty}]$ is bounded by a constant c_ℓ independent of E. This constant has been determined for $\ell = 2$ ($c_2 = 96$, see [29]), but it would be interesting to compute c_ℓ for other primes ℓ (this will require computing the rational points on a large number of modular curve, a problem which is also very interesting in itself). There are at least two reasons why this question is worth considering:

- on one hand, knowing the distribution of these rational points for different primes could allow us to understand how the indices $[\mathrm{GL}_2(\mathbb{Z}_\ell) : G_{\ell^\infty}]$ for different primes ℓ interact: is it possible to have very small 2-adic and 3-adic representations at the same time? Or is there a "repulsion phenomenon", which forces G_{3^∞} to be very large whenever G_{2^∞} is very small?

- adelic uniformity: Serre's uniformity question is whether the equality $G_{\ell^\infty} = \mathrm{GL}_2(\mathbb{Z}_\ell)$ holds for every elliptic curve $E/\mathbb{Q}$ and every $\ell > 37$. If the answer to this question is positive, the arguments of [35] show that the adelic index $[\mathrm{GL}_2(\hat{\mathbb{Z}}) : G_\infty]$ is bounded by a constant independent of $E/\mathbb{Q}$. The value of this constant is obviously closely linked to the constants c_ℓ.

2.4 Abelian surfaces and semi-stability

In [14] I've considered the ℓ-adic representations attached to abelian surfaces A/K with $\mathrm{End}_{\overline{K}}(A) = \mathbb{Z}$ under the assumption that A/K is semistable at least at one place v of K of residual characteristic ℓ. This hypothesis is needed to apply a theorem of Raynaud [28] which describes the action of the tame inertia at v on the group of ℓ-torsion points of A. As it is well-known, A acquires semistable reduction over an extension K' of K of controlled degree, but the prime ℓ becomes ramified in K', which complicates the argument. When the reduction of A at v is potentially purely toric, the deep results of [6] give fine information on the structure of $A[\ell]$ as a Galois module, and in this case one can probably suppress the semistability assumption. At the other end of the spectrum, when the reduction at v is bad but potentially good, the situation seems more complicated.

A better understanding of these issues would be the first step towards a generalisation in dimension 2 of Serre's very precise results [30, §1.11 and §1.12] concerning the action of inertia on the Galois representations arising from elliptic curves. These questions are especially interesting because answering them would give results that are *uniform* in the surface A.

2.5 Local-global principle for isogenies

In [33] and [1] the authors have considered whether a certain local-global principle holds for isogenies between elliptic curves over number fields. The analogous question for abelian surfaces, or even higher-dimensional abelian varieties, is certainly an interesting one, and it seems likely that the group-theoretic arguments of the aforementioned papers can be generalized to some extent. As in the case of elliptic curves, one expects the existence of counter-examples to the local-global principle, at least over number fields. Unlike the case of elliptic curves, however, it is not easy to actually get a hold on these examples, mainly due to our poor understanding of the moduli spaces of higher dimensional abelian varieties. Producing interesting examples that violate the local-global principle, or – ideally – classifying them completely, would greatly improve our understanding of the arithmetic of abelian varieties of dimension at least 2.

One could also hope that such an investigation might be the first step in a much more ambitious projects, namely the systematic study of rational points on the moduli spaces of abelian varieties whose associated Galois representations have interesting properties – starting with, but not limited to, the moduli spaces of abelian surfaces with a torsion point of order N, in connection with the well-known uniform boundedness of torsion conjecture.

2.6 Adelic representations in dimension at least 2. Integral Lie algebras.

A key ingredient for the arguments of [15] is a (nontrivial) modification of Pink's theory of the integral Lie algebras attached to the pro-p subgroups of $\mathrm{SL}_2(\mathbb{Z}_p)$ [24]. As Pink himself points out in his paper, it would be extremely useful to have a similar theory for the subgroups of $\mathrm{SL}_n(\mathbb{Z}_p)$, or even more generally for the subgroups of $G(\mathbb{Z}_p)$, where G is a reductive group over $\mathbb{Z}_p$. If such a theory were available, possibly obtained by refining the methods developed by Lazard in [12], it might yield explicit results for the adelic representations attached to some higher-dimensional varieties. One would then start by considering the case of abelian surfaces, where one may possibly exploit the relatively simple structure of the lattice of Lie subalgebras of $\mathfrak{sp}_{4,\mathbb{Q}_\ell}$.

2.7 New cases of the Mumford-Tate conjecture

It seems likely that the methods of Pink's paper [25] can be used to show new cases of the Mumford-Tate conjecture for abelian varieties. In fact, the only abelian varieties A/K considered in [25] are those that satisfy $\mathrm{End}_{\overline{K}}(A) = \mathbb{Z}$; for them, Pink shows:

Theorem 4 *Let K be a number field and A/K be an abelian variety of dimension g. Assume $\mathrm{End}_{\overline{K}}(A) = \mathbb{Z}$ and*

$$g \notin \mathcal{E} := \left\{ \frac{1}{2}\binom{4n+2}{2n+1} \mid n \in \mathbb{N}^+ \right\} \cup \left\{ \frac{1}{2}(2n)^{2k+1} \mid n, k \in \mathbb{N}^+ \right\}.$$

The Mumford-Tate conjectures holds for A, and we have $\mathrm{MT}(A) = \mathrm{GSp}_{2g,\mathbb{Q}}$.

Using part of Pink's techniques, the authors of [2] and [3] have shown the Mumford-Tate conjecture and computed the Mumford-Tate group for abelian varieties whose *relative dimension* is odd. Now Pink's theorem generalises a previous theorem of Serre [31][32], which dealt with abelian varieties of *odd* dimension, and shows that the set of "exceptional" dimensions one needs to exclude can be shrunk from the set of all even numbers to the much thinner set $\mathcal{E}$.

It seems possible that (at least in some cases) the results of [2] and [3] can similarly be generalized by replacing "odd relative dimension" with "relative dimension outside an exceptional set $\mathcal{E}'$", for some $\mathcal{E}'$ of density zero.

Finally, these generalisations (perhaps combined with Vasiu's paper [34]) could also lead to an extension of the results of [18] to a much wider class of abelian varieties.

References

[1] S. Anni. A local-global principle for isogenies of prime degree over number fields. *J. Lond. Math. Soc. (2)*, 89(3):745–761, 2014.

[2] G. Banaszak, W. Gajda, and P. Krasoń. On the image of ℓ-adic Galois representations for abelian varieties of type I and II. *Doc. Math., Extra Volume: John H. Coates Sixtieth Birthday*, pages 35–75.

[3] G. Banaszak, W. Gajda, and P. Krasoń. On the image of Galois ℓ-adic representations for abelian varieties of type III. *Tohoku Mathematical Journal*, 62:163–189, 2010.

[4] A. R. Booker, J. Sijsling, A. V. Sutherland, J. Voight, and D. Yasaki. A database of genus 2 curves over the rational numbers. *ArXiv e-prints*, February 2016.

[5] A. Bourdon. A uniform version of a finiteness conjecture for CM elliptic curves. *Math. Res. Lett.*, 22(2):403–416, 2015.

[6] A. Grothendieck, M. Raynaud, and D. S. Rim. *Groupes de monodromie en géométrie algébrique. I.* Lecture Notes in Mathematics, Vol. 288. Springer-Verlag, 1972. Séminaire de Géométrie Algébrique du Bois-Marie 1967–1969 (SGA 7 I).

[7] M. Hindry and N. Ratazzi. Torsion dans un produit de courbes elliptiques. *J. Ramanujan Math. Soc.*, 25(1):81–111, 2010.

[8] M. Hindry and N. Ratazzi. Points de torsion sur les variétés abéliennes de type GSp. *J. Inst. Math. Jussieu*, 11(1):27–65, 2012.

[9] M. Hindry and N. Ratazzi. Torsion pour les variétés abéliennes de type I et II. *ArXiv e-prints*, May 2015.

[10] T. Ichikawa. Algebraic groups associated with abelian varieties. *Mathematische Annalen*, 289(1):133–142, 1991.

[11] R. Jones and J. Rouse. Galois theory of iterated endomorphisms. *Proc. Lond. Math. Soc. (3)*, 100(3):763–794, 2010. Appendix A by Jeffrey D. Achter.

[12] M. Lazard. Groupes analytiques p-adiques. *Inst. Hautes Études Sci. Publ. Math.*, (26):389–603, 1965.

[13] D. Lombardo. An explicit open image theorem for products of elliptic curves. *Accepted for publication in the Journal of Number Theory subject to minor modifications.*

[14] D. Lombardo. Explicit surjectivity for Galois representations attached to abelian surfaces and GL_2-varieties. *Accepted for publication in the Journal of Algebra.*

[15] D. Lombardo. Bounds for Serre's open image theorem for elliptic curves over number fields. *Algebra & Number Theory*, 9-10:2347–2395, 2015.

[16] D. Lombardo. Explicit open image theorems for abelian varieties with trivial endomorphism ring. *ArXiv e-prints (submitted)*, August 2015.

[17] D. Lombardo. Galois representations attached to abelian varieties of CM type. *ArXiv e-prints (submitted)*, June 2015.

[18] D. Lombardo. On the ℓ-adic Galois representations attached to nonsimple abelian varieties. *Annales de l'Institut Fourier (to appear)*, 2015.

[19] D. Lombardo. On the uniform Rasmussen-Tamagawa conjecture in the CM case. *ArXiv e-prints (submitted)*, November 2015.

[20] D. Lombardo. Roots of unity and torsion points of abelian varieties. *ArXiv e-prints (submitted)*, July 2015.

[21] D. Lombardo. Pink-type results for general subgroup of $\mathrm{SL}_2(\mathbb{Z}_\ell)^n$. *Accepted for publication in the Journal de Théorie des Nombres de Bordeaux subject to minor modifications*, 2016.

[22] D. W. Masser and G. Wüstholz. Galois properties of division fields of elliptic curves. *Bull. London Math. Soc.*, 25(3):247–254, 1993.

[23] B. J. J. Moonen and Yu. G. Zarhin. Hodge classes on abelian varieties of low dimension. *Math. Ann.*, 315(4):711–733, 1999.

[24] R. Pink. Classification of pro-p subgroups of SL_2 over a p-adic ring, where p is an odd prime. *Compositio Math.*, 88(3):251–264, 1993.

[25] R. Pink. ℓ-adic algebraic monodromy groups, cocharacters, and the Mumford-Tate conjecture. *J. reine angew. Math*, (495), 1998.

[26] B. Poonen. Computational aspects of curves of genus at least 2. In *Algorithmic number theory (Talence, 1996)*, volume 1122 of *Lecture Notes in Comput. Sci.*, pages 283–306. Springer, Berlin, 1996.

[27] C. Rasmussen and A. Tamagawa. Arithmetic of abelian varieties with constrained torsion. *Transactions of the AMS (to appear)*, 2015.

[28] M. Raynaud. Schémas en groupes de type $(p, \ldots, p)$. *Bull. Soc. Math. France*, 102:241–280, 1974.

[29] J. Rouse and D. Zureick-Brown. Elliptic curves over $\mathbb{Q}$ and 2-adic images of Galois. *Research in Number Theory (accepted)*, 2015.

[30] J.-P. Serre. Propriétés galoisiennes des points d'ordre fini des courbes elliptiques. *Invent. Math.*, 15(4):259–331, 1972.

[31] J.-P. Serre. Résumé des cours de 1985-1986, Annuaire du Collġe de France, 1986.

[32] J.-P. Serre. Lettre à M-F. Vigneras du 10/2/1986. In *Œuvres. Collected papers. IV.* Springer-Verlag, Berlin, 2000.

[33] Andrew V. Sutherland. A local-global principle for rational isogenies of prime degree. *J. Théor. Nombres Bordeaux*, 24(2):475–485, 2012.

[34] A. Vasiu. Some cases of the Mumford-Tate conjecture and Shimura varieties. *Indiana Univ. Math. J.*, 57(1):1–75, 2008.

[35] D. Zywina. Bounds for Serre's open image theorem. *ArXiv e-prints*, 1102.4656, February 2011.