Nonabelian Chabauty study group

1 De Rham Unipotent Fundamental Group

Definition 1.1. *Let $Un^{\nabla}(X)$ be the category of unipotent vector bundles with connection over X.*

$\pi_{1,DR}(X;b) = \mathrm{Iso}^{\otimes}(e_b)$, *where $e_b : Un^{\nabla}(X) \to \mathrm{Vec}_{\mathbb{Q}_p}$ is the evaluation functor at b.*

Remark 1.2. Let $V = \mathbb{Q}_p << A, B >> \otimes \mathcal{O}_X^{\dagger}$ be the universal pro-bundle with connection on X. There is a natural structure of Hopf algebra on $\mathbb{Q}_p << A, B >>$ defined by $\Delta(A) = 1 \otimes A + A \otimes 1$, $\Delta(B) = 1 \otimes B + B \otimes 1$ such that $\pi_{1,DR}(X)(\mathbb{Q}_p)$ corresponds to the set of group-like elements of $\mathbb{Q}_p << A, B >>$. Recall that a group-like element is an x such that $\Delta(x) = x \otimes x$.

Moreover, $\pi_{1,DR} \cong \mathrm{Spec}(\mathbb{Q}_p << A, B >>^{\vee})$.

Definition 1.3. *The **unipotent Albanese map** of level n is*

$$UAlb_n = \pi_n \circ UAlb,$$

where $\pi_n : \pi_{1,Dr} \to [\pi_{1,DR}]_n$ is the natural projection and

$$\begin{array}{rccc} UAlb: & X(\mathbb{Q}_p) & \to & \pi_{1,DR} \\ & z & \mapsto & (\mathrm{Li}^w(z))_{w \text{ word of length } \leq n} \end{array}$$

Here

$$\mathrm{Li}^w(z) = \int \cdots \int^z w,$$

where the iterated integral over a word w is obtained by replacing $A \to \frac{dz}{z}$, $B \to \frac{dz}{1-z}$, and then integrating in order, so that for example

$$\int^z AB = \int_b^z \left(\int_b^{t_2} \frac{dt_1}{t_1} \right) \frac{dt_2}{1 - t_2}.$$

Theorem 1.4 (Kim 2005)**.** *The functions $\mathrm{Li}^w(z)$ are $\mathbb{Q}_p$-linearly independent.*

2 The fundamental diagram

Definition 2.1. *Let S be a finite set of primes, $p \notin S$, $T = S \cup \{p\}$, $\overline{\mathbb{Q}}^T$ be the maximal extension of $\mathbb{Q}$ unramified outside T, $G_p = \mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ and $\Gamma_T = \mathrm{Gal}(\overline{\mathbb{Q}}^T/\mathbb{Q})$.*

Kim's fundamental diagram is as follows:

$$\begin{tikzcd}
X(\mathbb{Z}[1/S]) \arrow[r] \arrow[d] & X(\mathbb{Q}_p) \arrow[r, "UAlb_n"] \arrow[d] & {[\pi_{1,DR}]_n(\mathbb{Q}_p)} \\
H^1_f(\Gamma_T, [\pi_{1,\acute{e}t}]_n) \arrow[r, "\mathrm{loc}_p"'] & H^1_f(G_p, [\pi_{1,\acute{e}t}]_n) \arrow[ur, "D"'] &
\end{tikzcd}$$

Unravelling the definitions one sees that this diagram is commutative.

3 The Unipotent Étale Fundamental group

Definition 3.1. *We let $Un(\overline{X})$ be the category of unipotent lisse*[1] *$\mathbb{Q}_p$-sheaves. The* ***unipotent étale fundamental group*** *$\pi_{1,\acute{e}t}(\overline{X}, b) = \mathrm{Iso}^{\otimes}(e_b)$, where e_b is again the fiber functor at b, $e_b : Un(\overline{X}) \to \mathrm{Vec}_{\mathbb{Q}_p}$. We also define the* ***fundamental groupoid****, or path torsor, as*

$$\pi_{1,\acute{e}t}(\overline{X}; b, p) = \mathrm{Iso}^{\otimes}(e_b, e_p)$$

Remark 3.2. If one ignores the Galois action, there is an isomorphism

$$\pi_{1,\acute{e}t} \cong \mathrm{Spec}\left(\mathbb{Q}_p << A, B >>^{\vee}\right),$$

where the dual is in the sense of topological vector spaces. However, $\pi_{1,\acute{e}t}$ has a natural Galois action.

More precisely, our $\pi_{1,\acute{e}t}$ is the $\mathbb{Q}_p$-unipotent completion of $\tilde{\pi}_{1,\acute{e}t}$, the usual étale fundamental group, which certainly comes equipped with a Galois action. One has for example $[\pi_{1,\acute{e}t}]_1 \cong \mathbb{Q}_p(1)A \oplus \mathbb{Q}_p(1)B$, where $\mathbb{Q}_p(1)$ is the 1-dimensional $\mathbb{Q}_p$ vector space on which Galois acts via the cyclotomic character.

We also define

$$1 \to [\pi_{1,DR}]^n \to [\pi_{1,DR}] \to [\pi_{1,DR}]_n \to 1,$$

where $[\pi_{1,DR}]^n = [[\pi_{1,DR}]^{n-1}, \pi_{1,DR}]$. The following sequence is exact by definition:

$$1 \to [\pi_{1,\acute{e}t}]^n/[\pi_{1,\acute{e}t}]^{n+1} \to [\pi_{1,\acute{e}t}]_{n+1} \to [\pi_{1,\acute{e}t}]_n \to 1,$$

and the Galois action on the first term can be computed since there is a natural surjection

$$(\mathbb{Q}_p(1)^{r_n})^{\otimes n} \cong [\pi_{1,\acute{e}t}]_1^{\otimes n} \twoheadrightarrow [\pi_{1,\acute{e}t}]^n/[\pi_{1,\acute{e}t}]^{n+1}.$$

[1] i.e. locally constant in the étale topology

This implies

$$[\pi_{1,ét}]^n/[\pi_{1,ét}]^{n+1} \cong \mathbb{Q}_p(n)^{r_n}$$

for some integer r_n.

With this technology at hand, we may define the **Kummer map**

$$\begin{array}{rcl} X(\mathbb{Z}[1/S]) & \to & H^1(\Gamma_T, [\pi_{1,ét}]) \\ P & \mapsto & [\pi_{1,ét}(\overline{X}; b, P)] \end{array}$$

3.1 Example

Let's compute $H^1(\Gamma_T, \mathbb{Q}_p(1))$ for $T = \{\ell\} \cup \{p\}$, p odd. There is a natural exact sequence

$$1 \to \mu_{p^k} \to \overline{\mathbb{Z}[1/T]}^\times \xrightarrow{p^k} \overline{\mathbb{Z}[1/T]}^\times \to 1$$

which induces

$$1 \to \mathbb{Z}[1/T]^\times/\mathbb{Z}[1/T]^{\times p^k} \xrightarrow{\sim} H^1(\Gamma_T, \mu_{p^k}) \to \mathrm{Cl}(\mathbb{Z}[1/T]) \to 1$$

Since $\mathbb{Z}[1/T]^\times/\mathbb{Z}[1/T]^{\times p^k} \cong (\mathbb{Z}/p^k\mathbb{Z})^{\oplus 2}$, by passing to the limit we find $H^1(\Gamma_T, \mathbb{Z}_p(1)) \cong \mathbb{Z}_p^2$ and $H^1(\Gamma_T, \mathbb{Q}_p(1)) \cong \mathbb{Q}_p^2$. From this, one sees that the map

$$X(\mathbb{Z}[1/S]) \to H^1(\Gamma_T, [\pi_{1,ét}]_n) \cong \mathbb{Q}_p^4$$

sends $(t, 1-t)$ to $(\log(t)/\log(\ell), \log(t)/\log(p), \log(1-t)/\log(\ell), \log(1-t)/\log(p))$.

The full diagram is as follows:

$$\begin{tikzcd} (t,1-t) \arrow[r] \arrow[d] & (t,1-t) \arrow[r, "UAlb_n"] \arrow[d] & {[\pi_{1,DR}]_n(\mathbb{Q}_p)} \\ (\log(t)/\log\ell, \log(1-t)/\log\ell) \arrow[r, "\mathrm{loc}_p"'] & (\log(t)/\log\ell, \log(1-t)/\log\ell) \arrow[ur, "D=id"'] & \end{tikzcd}$$

where I'm writing the cohomology as $\mathbb{Q}_p^2$ because of the subscript $_f$, which is there to – roughly – make sure that out of the T-units we only keep the S-units[2].

4 The morphism D (which we *do not* understand)

The group $H^1_f(G_p, [\pi_{1,ét}]_n)$ parametrizes (certain) torsors under $[\pi_{1,ét}]_n$ over a point. Let $\mathfrak{P}$ be a $\mathbb{Q}_p$-algebra whose spectrum is $\mathcal{P}$, a torsor whose class $P = [\mathcal{P}]$ lies in $H^1(G_p, [\pi_{1,ét}]_n)$. The map D is defined as follows. First, to $[\mathcal{P}]$ we associate

$$\overline{D}([\mathcal{P}]) = \mathrm{Spec}\left(\mathfrak{P} \otimes_{\mathbb{Q}_p} B^{DR}\right)^{G_p},$$

where B^{DR} is the ring of De Rham periods. This is a torsor over $[\pi_{1,DR}]_n$; the "f" condition ensures that:

[2]more on this in future talks at the study group! Maybe...

- $\overline{D}(P)$ is endowed with a canonical Frobenius action, and $D(P)^{\phi=1}$ is a single point;
- $\overline{D}(P)$ also has a Hodge filtration, and $F^0\overline{D}(P)$ also consists of a single point.

Thus it makes sense to define

$$D(\mathcal{P}) = \overline{D}(P)^{\phi=1}/F_0\overline{D}(P)$$

as the trasponder[3] from $F_0\overline{D}(P)$ to $\overline{D}(P)^{\phi=1}$.

5 Proof of Siegel's theorem by Kim's method

Claim. There exists $n \gg 0$ such that

$$\dim_{\mathbb{Q}_p} H^1_f(\Gamma_T, [\pi_{1,\acute{e}t}]_n) < \dim_{\mathbb{Q}_p}[\pi_{1,DR}]_n.$$

It is not hard to see that the $H^1(...)$ are algebraic varieties; Kim claims that the same is true for $H^1_f(...)$, but Julian and I can only prove that they are *constructible.* Luckily, this is not a big deal for the proof.

Proof. We compute the two sides separately. $[\pi_{1,DR}]_n$ is a pro-unipotent group; we work with the algebra

$$\mathbb{Q}_p << A, B >>= \mathrm{Env}(\mathrm{Lie}([\pi_{1,DR}]))^{\sim 4}$$

Standard general theory shows that

$$\bigotimes_{n=0}^{\infty} \mathrm{End}\left([\pi_{1,DR}]^n/[\pi_{1,DR}]^{n+1}\right),$$

so one gets the generating function for the dimensions $r_n = \dim[\pi_{1,DR}]^n/[\pi_{1,DR}]^{n+1}$:

$$\prod_{d\geq 1} \frac{1}{(1-z^d)^{r_d}} = \frac{1}{1-2z}.$$

From here, we get $\sum_{k|n} kr_k = 2^n$, which by Möbius inversion leads to $r_n \sim \frac{2^n}{n}$. Finally,

$$\dim_{\mathbb{Q}_p}[\pi_{1,DR}]_n = \sum_{k\leq n} r_k.$$

On the other hand, we can also estimate the dimension of $H^1_f(\Gamma_T, [\pi_{1,\acute{e}t}]_n)$ by looking at the long exact sequence in cohomology

$$\begin{aligned} H^0(...) = 0 \to\ & H^1(\Gamma_T, [\pi_{1,\acute{e}t}]^n/[\pi_{1,\acute{e}t}]^{n+1}) \to\ H^1(\Gamma_T, [\pi_{1,\acute{e}t}]_n) \to\ H^1(\Gamma_T, [\pi_{1,\acute{e}t}]_{n-1}) \\ \to\ & H^2(\Gamma_T, [\pi_{1,\acute{e}t}]^n/[\pi_{1,\acute{e}t}]^{n+1}) =\ H^2(\Gamma_T, \mathbb{Q}_p(n)^{r_n}) = 0, \end{aligned}$$

[3]the unique element in the group acting on the torsor that acts on one element and brings it to the other

[4]where $\sim$ denotes the completion wrt the augmentation ideal, and Env is the universal enveloping algebra

where the last equality follows from a theorem of Soulé (which is not necessary here, but will be necessary in a moment). One has a formula for the Euler-Poincaré characteristic which goes as follows:

$$h^0(\Gamma_T, \mathbb{Q}_p(n)) - h^1(\Gamma_T, \mathbb{Q}_p(n)) + h^2(\Gamma_T, \mathbb{Q}_p(n)) = -\dim \mathbb{Q}_p(n)^-,$$

where the $-$ superscript denotes the -1-eigenspace for complex conjugation. Now h^0 vanishes, and so does h^2 by Soulé. The error term $-\dim \mathbb{Q}_p(n)^-$ is 0 for n even and -1 for n odd, so

$$h^1(\Gamma_T, \mathbb{Q}_p(n)) = \begin{cases} 0, n \text{ even} \\ 1, n \text{ odd}, n \geq 3 \end{cases}$$

Finally, $h^1(\Gamma_T, \mathbb{Q}_p(1)) = 2 \operatorname{rank} \mathbb{Z}[1/T]^\times = 2\#T =: R$. The inequality we need to prove is

$$\dim_{\mathbb{Q}_p} H^1(\Gamma_T, [\pi_{1,\acute{e}t}]_n) < \dim_{\mathbb{Q}_p} [\pi_{1,DR}]_n,$$

that is,

$$R + r_3 + r_5 + \cdots + r_{2\lceil n/2 \rceil - 1} < r_1 + \cdots + r_n,$$

which is true for large n because the r_i go to infinity, and there are more terms on the right than on the left.

$\square$

The claim implies Siegel's theorem, because we find a nonvanishing analytic function which is zero on $X(\mathbb{Z}[1/S])$.

6 Example

In the case $S = \{\ell\}$, $T = \{\ell, p\}$, $n = 2$ the fundamental diagram

$$\begin{tikzcd}
X(\mathbb{Z}[1/S]) \arrow[r] \arrow[d] & X(\mathbb{Q}_p) \arrow[r, "UAlb_n"] \arrow[d] & {[\pi_{1,DR}]_n(\mathbb{Q}_p)} \\
H^1_f(\Gamma_T, [\pi_{1,\acute{e}t}]_n) \arrow[r, "\mathrm{loc}_p"'] & H^1_f(G_p, [\pi_{1,\acute{e}t}]_n) \arrow[ur, "D"'] &
\end{tikzcd}$$

looks like

$$\begin{tikzcd}
t \arrow[r] \arrow[d] & t \arrow[r, "UAlb_n"] \arrow[d] & \mathbb{H} \\
\left(\frac{\log t}{\log \ell}, \frac{\log(1-t)}{\log \ell}\right) \arrow[r, "\mathrm{loc}_p"'] & H^1_f(G_p, [\pi_{1,\acute{e}t}]_n) \arrow[ur, "D"'] &
\end{tikzcd}$$

where $\mathbb{H}$ is the Heisenberg group[5] and the composite map from $H^1_f(\Gamma_T, [\pi_{1,\acute{e}t}]_2) = H^1_f(\Gamma_T, [\pi_{1,\acute{e}t}]_1)$[6] to $\mathbb{H}$ sends (x, y) to $(x \log \ell, y \log \ell, \frac{1}{2}(\log \ell)^2 xy)$. It follows that the function $\mathrm{Li}_2(t) - 2(\log t)(\log(1-t))$ vanishes on $X(\mathbb{Z}[1/S])$.

[5]group of upper-triangular 3×3 matrices with 1's on the diagonal

[6]this equality follows from Soulé's theorem