Period rings

1 Witt vectors

1.1 Idea

From $\mathbb{F}_p$ one can reconstruct $\mathbb{Z}_p$. How? Consider $\mathbb{F}_p$ as $\mathbb{Z}/p\mathbb{Z}$; by *reconstruct*, we mean *define sum and product on* $\mathbb{Z}_p$*, starting from those of* $\mathbb{F}_p$.

So, start with (say) the sum on $\mathbb{Z}/p\mathbb{Z}$, and consider how to extend them to $\mathbb{Z}/p^2\mathbb{Z}$. Take two elements $a_0 + a_1 p$, $b_0 + b_1 p$ where a_i, b_i are taken in a set of representatives $T_p = \{\alpha_0, \ldots, \alpha_{p-1}\}$ which we will fix later[1]. We want to express the sum

$$(a_0 + b_0) + (a_1 + b_1)p$$

in terms of our representatives. Say that $(a_0 + b_0) \equiv (a_0 + b_0)_{\bmod p} \pmod p$, with $(a_0 + b_0)_{\bmod p} \in T_p$; then we write

$$(a_0 + b_0) + (a_1 + b_1)p = (a_0 + b_0)_{\bmod p} + (a_0 + b_0 - (a_0 + b_0)_{\bmod p}) + (a_1 + b_1)p.$$

Observation 1.1. *Observe that*

$$(a_0 + b_0) \equiv (a_0 + b_0)_{\bmod p} \pmod p$$

implies

$$(a_0 + b_0)^p \equiv (a_0 + b_0)^p_{\bmod p} \pmod{p^2}.$$

Hence, if we choose the representative T_p *to be the Teichmüller representatives (i.e. those that satisfy* $\alpha_i^p = \alpha_i$*), we obtain the following expression:*

$$(a_0 + b_0)^p \equiv (a_0 + b_0)^p_{\bmod p} = (a_0 + b_0)_{\bmod p}$$

One may use a similar argument to "construct", starting from the operations of sum and product on $\mathbb{Z}/p^{n-1}\mathbb{Z}$, the ones on $\mathbb{Z}/p^n/\mathbb{Z}$. Hence, working by induction and then taking the inverse limit $\lim \mathbb{Z}/p^n/\mathbb{Z}$, there should be an (algebraic) way to construct the operations of sum and product on $\mathbb{Z}_p$, starting from those of $\mathbb{F}_p$. This will be made more formal in the next paragraphs.

[1] In the end, our choice is going to be that of the Teichmüller representatives.

1.2 Definition

Definition 1.2. *In the context of the Witt vectors of* $\mathbb{F}_p$*, given an element* $\alpha \in \mathbb{F}_p$*, we write* $[\alpha]$ *for its Teichmüller lift in* $\mathbb{Z}_p$*.*

More generally, let R be a **perfect** ring of characteristic p.

Definition 1.3. *We define* $W(R)$*, the* ***Witt vectors of*** R*, as:*

$$W(R) := \left\{ \sum_{k\geq 0} [\alpha_k] p^k | \alpha_k \in R \right\},$$

with sum and product defined analogously to the above, namely

$$+\left(\sum_{k\geq 0} [a_k] p^k, \sum_{h\geq 0} [b_h] p^h \right) = \sum_{i\geq 0} [S_i(a_k, b_h)] p^i$$

$$\times\left(\sum_{k\geq 0} [a_k] p^k, \sum_{h\geq 0} [b_h] p^h \right) = \sum_{i\geq 0} [P_i(a_k, b_h)] p^i$$

where the $S_i, P_i \in R[a_k^{\frac{1}{p^n}}, b_h^{\frac{1}{p^n}}]_{n\in\mathbb{N}}$ *are polynomials, and the* $[a_k]$ *are formal elements of* $W(R)$*.* [2]*.*

One may construct the polynomials S_i and P_i as follows. Let $W_n : W(R) \to R$, the *Witt polynomials*, be defined as:

$$W_n\left(\sum_{k\geq 0} [\alpha_k] p^k \right) := \sum_{k\leq n} p^k \alpha_k^{p^{n-k}}.$$

Then, the polynomials S_i and P_i are the unique polynomials that satisfy the following equalities in $\mathbb{Z}[a_k^{p^{-m}}, b_h^{p^{-m}}]_{m\in\mathbb{N}}$, for each $n \in \mathbb{N}$:

$$W_n\left(\sum_{i\geq 0} [S_i(a_k, b_h)^{p^{-n+i}}] p^i \right) = W_n\left(\sum_{k\geq 0} [a_k^{p^{-n+k}}] p^k \right) + W_n\left(\sum_{k\geq 0} [b_k^{p^{-n+k}}] p^k \right),$$

and:

$$W_n\left(\sum_{i\geq 0} [P_i(a_k, b_h)^{p^{-n+i}}] p^i \right) = W_n\left(\sum_{k\geq 0} [a_k^{p^{-n+k}}] p^k \right) \cdot W_n\left(\sum_{k\geq 0} [b_k^{p^{-n+k}}] p^k \right).$$

Remark 1.4. The polynomials $S_i(a_k, b_h)$ involve fractional powers of the variables: for example,

$$S_1(a_0, b_0, a_1, b_1) = \frac{a_0 + b_0 - (a_0^{1/p} + b_0^{1/p})^p}{p} + a_1 + b_1.$$

Moreover, S_i (and P_i) depend at most on the variables to the power $1/p^i$, and the variables that appear are all the a_j and all the b_k with $j, k \leq i$.

[2] Which will turn out to be, by construction the Teichmüller representatives (see below)

1.3 Teichmüller representatives

Let A be a ring of characteristic 0, complete for the topology induced by a maximal ideal $\mathfrak{m}$, and let $R := A/\mathfrak{m}$ be of characteristic p and perfect. One defines the Teichmüller representative as follows: for $x \in R$, we set

$$[x] := \lim_{k \to \infty} \widetilde{x^{-p^k}}^{p^k} \in A,$$

where $\tilde{r}$ denotes any lift of r to A.

Remark 1.5.
- $[x]$ exists and does not depend on the choice of the lift (by a 'lifting the exponent'-kind of lemma),
- $[x] \equiv x \pmod{\mathfrak{m}}$,
- $[xy] = [x][y]$.

1.4 Properties of the Witt vectors

- $W(R)$ is complete wrt the topology generated by the ideal (p); the quotient $W(R)/pW(R)$ is isomorphic to R.
- $\alpha_k \in R \Rightarrow [\alpha_k]$ is exactly the Teichmüller representative of α_k.
- For $R = \mathbb{F}_{p^k}$, $W(R)$ is the ring of integers of K, the unique unramified extension of $\mathbb{Q}_p$ of degree k.

Lemma 1.6. *Let R be a perfect ring of characteristic p, A complete wrt an ideal $I < A$, $\mathrm{char}(A/I) = p$. Given a homomorphism $\varphi : R \to A/I$, there exists a unique homomorphism $\tilde{\varphi} : W(R) \to A$ that lifts φ.*

Corollary 1.7. *Let $p \in \mathbb{N}$ be a prime, and let A be a ring such that $R = A/(p)$, is a perfect ring of characteristic p, A is complete and Hausdorff in the (p)-topology, and p is not a zero-divisor (or, equivalently, when R is a field, p is not nilpotent). Then $A \cong W(R)$, and the isomorphism is compatible with the equality $R = A/(p)$ and isomorphism $W(R)/pW(R) \cong R$.*

2 Period rings

2.1 Idea

A way to think about the comparison theorem between De Rham and Betti cohomology is the following:

Theorem 2.1. *Let X be a smooth manifold/$\mathbb{R}$. There is a perfect pairing*

$$H_k(X,\mathbb{R}) \times H^k_{dR}(X,\mathbb{R}) \to \mathbb{R},$$

given by $(\gamma,[\omega]) \mapsto \int_\gamma \omega$. As a consequence, $H^k_{dR}(X,\mathbb{R}) \cong H^k(X,\mathbb{R})$.

We'd like something similar for p-adic varieties:

$$H^k_{dR}(X,\mathbb{Q}_p) \cong H^k_{\text{crys}}\left(\overline{X},\mathbb{Q}_p\right),$$

where $\overline{X} = X \times_{\operatorname{Spec}\mathbb{Z}_p} \operatorname{Spec}\mathbb{F}_p$.

Unfortunately, although these two are going to be isomorphic (in fact, we will prove they are vector spaces of the same dimension), the isomorphism is not going to be functorial. The comparison isomorphism will become natural only after extension to some (huge) ring.

This kind of phenomena may be seen also in the isomorphism $H^k_{dR}(X,\mathbb{Q}) \cong H^k(X,\mathbb{Q})$, which exists (by dimension counting), but in order to make the isomorphism functorial one needs to basechange to $\mathbb{R}$.

2.2 Periods for the cyclotomic character

2.2.1 First attempt: $\mathbb{C}_p$ (Sen)

Let $\mathbb{C}_p := \widehat{\overline{\mathbb{Q}}_p}$.

Theorem 2.2. *(Ax-Sen-Tate) Let $H < G_{\mathbb{Q}_p}$ be a closed subgroup. Then $\mathbb{C}_p^H = \widehat{\overline{\mathbb{Q}}_p^H}$.*

From now on, K is a finite extension of $\mathbb{Q}_p$, and F is the maximal unramified subextension of K

Theorem 2.3. *(Tate) Let $\psi : G_K \to \mathbb{Z}_p^\times$ be a character such that*

- *$\psi|_{H_K} = 1$, where $H_K = \operatorname{Gal}\left(\overline{K}/K_\infty\right)$ and $K_\infty = \bigcup_{n\geq 0} K(\mu_{p^n})$.*
- *ψ does not factor through a finite group*

Then $H^0(G_K,\mathbb{C}_p(\psi^{-1})) = \{0\}$, where

$$H^0(G_K,\mathbb{C}_p(\psi^{-1})) = \{x \in \mathbb{C}_p : gx = \psi(g)x,\ \forall g \in G_K\}$$

Definition 2.4. *The cyclotomic character $\chi : G_K \to \mathbb{Z}_p^\times$ is the only character such that $g(\zeta) = \zeta^{\chi(g)}$ for all p^n-th root of unity ζ.*

The cyclotomic character (and its powers) is an example of character that satisfies the hypothesis of Theorem 2.3.

Observe that Tate's theorem suggests that $\mathbb{C}_p$ is *not* the right period ring: the cyclotomic character arises from geometry, so we would like our period ring to contain periods for the cyclotomic character – in other words, if R is a 'good' period ring, one should have:

$$\dim H^0(G_K, R(\psi^{-1})) = 1.$$

Theorem 2.5. *(Sen)*

$$H^1(H_K, \mathrm{GL}(d, \mathbb{C}_p)) = \{0\},$$

which implies the following: if V is a G_K-representation, then $(V \otimes_{\mathbb{Q}_p} \mathbb{C}_p)^{H_K}$ is a $\widehat{K_\infty}$-vector space of dimension $\dim_K V$.

What we *want* is an analogous theorem for all representations (or at least all representations coming from geometry).

2.3 Hodge-Tate representations

Definition 2.6. *V a G_K-representation is* ***Hodge-Tate*** *if*

$$\left(V \otimes_K \bigoplus_{i=-\infty}^{\infty} \mathbb{C}_p(i) \right)^{G_K}$$

is a K-vector space of dimension $\dim_K V$. We write $(h_j)_{j=1}^{\dim_K V}$ for the Hodge-Tate weights of this representation, that is, integers such that

$$(V \otimes_K \bigoplus \mathbb{C}_p(i))^{G_K} \cong \bigoplus_{j=1}^{\dim_K V} K(-h_j)$$

2.4 Construction of $\mathbb{B}^{\mathrm{dR}}$

$$\tilde{E}^+ := \varprojlim_{x \to x^p} \mathcal{O}_{\mathbb{C}_p} = \{(x^{(0)}, x^{(1)}, x^{(2)}, \cdots) : (x^{(i+1)})^p = x^{(i)}\}$$

Fix once and for all an element $\varepsilon = (\mu_{p^n})$, where μ_{p^n} is a primitive p^n-th root of unity.

We make $\tilde{E}^+$ into a ring by setting

$$(x+y)^{(i)} = \lim_{k \to \infty} \left(x^{(i+k)} + y^{(i+k)} \right)^{p^k}$$

and

$$(xy)^{(i)} = x^{(i)} y^{(i)}.$$

Properties of $\tilde{E}^+$

- $\mathrm{char}\, \tilde{E}^+ = p$
- $\mathbb{F}_p((\varepsilon - 1)) \subset \tilde{E} := \tilde{E}^+[(\varepsilon - 1)^{-1}]$
- $\tilde{E}^+ \cong \varprojlim_{x \to x^p} \mathcal{O}_{\mathbb{C}_p}/(p)$[3]
- $\tilde{E}^+$ is perfect (since raising to the p-th power is nothing but a shift).

[3] This is a ring with the obvious componentwise operations

We will denote the fraction field of $\tilde{E}^+$ by $\tilde{E}$.

Definition 2.7. *We define a valuation on $\tilde{E}$, as $v_E(x) := v_p(x^{(0)})$, for $x \in \tilde{E}$.*

Remark 2.8. $v_p(\varepsilon - 1) = \frac{p}{p-1}$. Indeed,

$$(\varepsilon - 1)^{(0)} = \lim_{n \to \infty} (\mu_{p^n} - 1)^{p^n},$$

and $(\mu_{p^n} - 1)^{p^n}$ has valuation $p^n \frac{1}{\varphi(p^n)} = \frac{p}{p-1}$.

Definition 2.9. $\tilde{A}^+ := W(\tilde{E}^+)$ *and*

$$\tilde{B}^+ := \tilde{A}^+[1/p] = \left\{ \sum_{k \gg -\infty} p^k [x_k] : x_k \in \tilde{E}^+ \right\}$$

Remark 2.10. These objects depend only on p and not on the field K (finite extension of $\mathbb{Q}_p$).

Remark 2.11. There is a canonical embedding (in particular, this is Galois equivariant) of $\bar{\mathbb{F}}_p$ in $\tilde{E}^+$ (this is simply given by taking Teichmüller representatives). One may then lift (through Lemma 1.6) this embedding to a (Galois equivariant) embedding of $\mathcal{O}_{K^{nr}}$ in $\tilde{B}^+$. In partiucar, we have that $\mathcal{O}_F \hookrightarrow (\tilde{B}^+)^{G_K}$.

Definition 2.12.

$$\begin{array}{cccc} \overline{\theta}: & \tilde{E}^+ & \to & \mathcal{O}_{\mathbb{C}_p}/(p) \\ & (x^{(i)}) & \mapsto & \overline{x^{(0)}} \end{array}$$

By the universal property of the Witt vectors (Lemma 1.6), there are lifts $\theta : \tilde{A}^+ \to \mathbb{O}_{\mathbb{C}_p}$ and $\theta : \tilde{B}^+ \to \mathbb{C}_p$.

Remark 2.13.

- θ is surjective (obvious).
- $\theta(\sum p^k [x_k]) = \sum p^k x_k^{(0)}$. To see this, notice that:

$$\theta([x_k]) = \lim_{n \to \infty} (x_k^{(n)})^{p^n} = x_k^{(0)}.$$

- Recall $\varepsilon = (\mu_{p^n})$. Define $\varepsilon_1 = (\mu_{p^{n+1}})$, so that $\varepsilon_1^p = \varepsilon$. We then have that:

$$\theta\left(1 + [\varepsilon_1] + \ldots + [\varepsilon_1]^{p-1}\right) = 0,$$

  because $\theta([\varepsilon_1])$ is a p-th root of unity and $1 + x + \cdots + x^{p-1}$ is the p-th cyclotomic polynomial. Here $[\cdot]$ is the Teichmüller lift from $\tilde{E}^+$ to $\tilde{A}^+$.

Proposition 2.14. *Let $\omega := 1 + [\varepsilon_1] + \ldots + [\varepsilon_1]^{p-1} = \frac{[\varepsilon]-1}{[\varepsilon_1]-1}$. Then, considering θ as a map $\tilde{B}^+ \to \mathbb{C}_p$, we have*

$$\ker \theta = (\omega).$$

Proof. $\ker\bar\theta = \{x \in \tilde{E}^+ : v_E(x) \geq 1\}$. Moreover, $v(\omega) = 1$, so $\omega\tilde{A}^+ \subseteq \ker\theta$ (where now $\theta : \tilde{A}^+ \to \mathcal{O}_{\mathbb{C}_p}$), and we know $\bar\omega\tilde{E}^+ = \ker\overline{\theta}$. Hence the map

$$\omega\tilde{A}^+ \hookrightarrow \ker\theta$$

is surjective modulo p, and since everything is p-adically complete it's surjective *tout court*. □

Hance, we have a morphism

$$\theta : \tilde{B}^+ \to \mathbb{C}_p$$

with kernel generated by ω.

Definition 2.15.

$$\mathbb{B}_+^{dR} = \varprojlim_{k\to\infty} \tilde{B}^+/(\omega)^k$$

and

$$\mathbb{B}^{\mathrm{dR}} = \mathbb{B}_+^{\mathrm{dR}}[1/\omega]$$

One can show that $\mathbb{B}^{\mathrm{dR}}$ is a field.

Definition 2.16. $\mathbb{B}^{\mathrm{dR}}$ *contains the following element, which one has to think as "*$\log[\varepsilon]$*",*

$$t := -\sum_{k\geq 1} \frac{(1-[\varepsilon])^k}{k}.$$

Remark 2.17. We have a period for the cyclotomic character!

$$g(t) = g\log[\varepsilon] = \log[\varepsilon^{\chi(g)}] = \chi(g)\log[\varepsilon] = \chi(g)\cdot t$$

Moreover, $(\omega) = (t)$ in $\mathbb{B}_+^{\mathrm{dR}}$ (the ratio is invertible in the ω-adic completion)

Definition 2.18.

$$\mathrm{Fil}^i\,\mathbb{B}^{\mathrm{dR}} = t^i\mathbb{B}_+^{\mathrm{dR}}$$

We have

$$\mathrm{gr}\,\mathbb{B}^{\mathrm{dR}} = \bigoplus_{i=-\infty}^{\infty} \mathbb{C}_p(i),$$

because t generates the kernel of θ, whose quotient is $\mathbb{C}_p$, and we know the action of Galois on t.

Proposition 2.19. [4]

$$(\mathbb{B}^{\mathrm{dR}})^{G_K} = K$$

Definition 2.20. *Let $V/\mathbb{Q}_p$ be a G_K-representation. We say that V is* ***de Rham admissible*** *if*

$$\dim_K \left(V \otimes_{\mathbb{Q}_p} \mathbb{B}^{\mathrm{dR}}\right)^{G_K} = \dim_K V.$$

Remark 2.21. If V is de Rham admissible, by passing to the associated graded ring, we see that V is also Hodge-Tate.

[4] we're not sure about the embedding of K in $\mathbb{B}^{\mathrm{dR}}$

2.5 $\mathbb{B}_{\text{cris}}$

Motivation: on $\tilde{E}^+$ there is a natural Frobenius φ. This can be lifted to $\tilde{\varphi} : \tilde{A}^+ \to \tilde{A}^+$ and therefore to $\tilde{B}^+ \to \tilde{B}^+$. However,

$$\tilde{\varphi}(\omega) = \tilde{\varphi}(1 + [\varepsilon_1] + \ldots + [\varepsilon_1]^{p-1}) = 1 + [\varepsilon] + \ldots + [\varepsilon]^{p-1} \equiv p \pmod{(\omega)}$$

so $\tilde{\varphi}$ does not preserve the ω-adic topology, hence it does not extend to the completion. We can do better:

Definition 2.22.

$$\mathbb{B}^+_{\text{cris}} := \left\{ x \in \mathbb{B}^+_{\text{dR}} : \exists x_k \to 0 \textit{ in } \tilde{B}^+ \textit{ such that } x = \sum_k x_k \frac{\omega^k}{k} \right\},$$

where convergence is with respect to the valuation v_E. We further define

$$\mathbb{B}_{\text{cris}} = \mathbb{B}^+_{\text{cris}}[1/t]$$

The definition is given in such a way that $\tilde{\varphi}$ extends naturally to $\mathbb{B}^+_{\text{cris}}$.

Remark 2.23. Notice that we invert t and not ω (is it really different?). Also, $\mathbb{B}_{\text{cris}}$ is not a field: $\omega - p$ is not invertible.

Proposition 2.24.

$$(\mathbb{B}_{\text{cris}})^{G_K} = F$$

Definition 2.25. *Let V be a $\mathbb{Q}_p$-representation. We set*

$$\mathbb{D}_{\text{cris}}(V) = (V \otimes_{\mathbb{Q}_p} \mathbb{B}_{\text{cris}})^{G_K}.$$

Remark 2.26. $\mathbb{D}_{\text{cris}}(V)$ has a σ_F-semilinear automorphism; moreover, $\mathbb{D}_{\text{cris}}(V) \otimes_F K$ has a natural filtration coming from $\mathbb{B}_{\text{dR}}$.

Remark 2.27. Frobenius is **not** an automorphism of $\mathbb{B}_{\text{cris}}$.

Exercise 2.28. *Fix a sequence $\{r_n\} \subseteq \mathbb{Z}$. There exists $x_r \in \bigcap_{k \geq 0} \varphi^k \mathbb{B}_{\text{cris}}$ such that $\varphi^{-n}(x_r) \in \operatorname{Fil}^{r_n} \mathbb{B}_{\text{dR}} \setminus \operatorname{Fil}^{r_n+1} \mathbb{B}_{\text{dR}}$.*

References