Max Algebras

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Max-algebra has existed as a form of linear algebra for almost half a century. We have seen a massive development in this area especially in the last 15 years. This is indicated by numerous papers published in leading journals, 5 books, and a good number of conferences or special sessions. This minisymposium provides state-of-the-art research presentations by established researchers in the field.

Representation of maxitive measures

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Thu 12:15, Room Fermi

A maxitive measure is the analogue of a finitely additive measure or charge, in which the usual addition of reals is replaced by the supremum operation in a partially ordered set (poset). This notion was first introduced by Shilkret (1971), and reintroduced by many authors with different purposes such as capacity theory and large deviations, idempotent analysis and max-plus algebra, fuzzy set theory, optimisation, or fractal geometry.

A completely maxitive measure has a cardinal density, which means that there exists a map c such that the measure of any set is the supremum of c on that set. This property is related to the theory of residuation or Galois connections, or dualities.

We shall present and compare various representation results of this type. For instance, a countably maxitive measure on the topology of a separable metric space has a density (see [1] together with a max-plus linear form version shown by Kolokoltsov and Maslov (1988), see e.g. [2]). Barron, Cardaliaguet, and Jensen (2000) have shown a similar representation using essential suprema with respect to a usual positive measure, generalized in [3]. More generally, a maxitive measure can be decomposed as the supremum of a maxitive measure with density, and a residual maxitive measure that is null on compact sets [4].

[1] M. Akian. Densities of idempotent measures and large deviations. *Trans. Amer. Math. Soc.*, 351(11):4515–4543, 1999.
[2] V. Kolokoltsov and V. Maslov. *Idempotent analysis and applications*. Kluwer Acad. Publisher, 1997.
[3] P. Poncet. A note on two-valued possibility (σ-maxitive) measures and Mesiar's hypothesis. *Fuzzy Sets and Systems*, 158(16):1843–1845, 2007.
[4] P. Poncet. A decomposition theorem for maxitive measures, 2009. Accepted for publication in LAA, see also arXiv:0912.5178.

Joint work with P. Poncet

Tropical approximation of matrix eigenvalues

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Mon 15:00, Auditorium

We establish several inequalities of log-majorization type, relating the moduli of the eigenvalues of a complex matrix with certain combinatorial objects, the tropical eigenvalues, which depend only on the moduli of the entries of the matrix. In nondegenerate cases, the orders of magnitude of the different eigenvalues of the complex matrix turn out to be given by the tropical eigenvalues. We use this information to perform a preprocessing (diagonal scaling) to improve the numerical accuracy of eigenvalue computations.

Joint work with M. Akian, M. Sharify

Characterization of non-strictly-monotone interval eigenvectors in max-min algebra

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Mon 16:45, Auditorium

The interval eigenproblem in max-min algebra for non-strictly-monotone eigenvectors is studied. For given real matrices $\underline{A}, \overline{A}$ of type (n, n), the matrix interval $\mathbf{A} = [\underline{A}, \overline{A}]$ is defined as the set of all matrices A, for which the inequalities $\underline{A} \le A \le \overline{A}$ hold true. For given real vectors $\underline{x}, \overline{x}$ of type $(n, 1)$ the vector interval $\mathbf{X} = [\underline{x}, \overline{x}]$ is the set of all vectors x with $\underline{x} \le x \le \overline{x}$. The interval eigenproblem $\mathbf{A} \otimes \mathbf{X} = \mathbf{X}$ is the problem of finding a solution to the equation $A \otimes x = x$ in max-min algebra, with additional conditions that the coefficient matrix A belongs to the given interval $\mathbf{A}$, and the eigenvector x belongs to $\mathbf{X}$. Six types of solvability of the interval eigenproblem are introduced, according to various combinations of quantifiers applied to $A \in \mathbf{A}$ and $x \in \mathbf{X}$. The characterization of all six types in the form of necessary and sufficient condition is given, with restriction to non-strictly-monotone eigenvectors. The conditions can be verified in polynomial time. All true implications between the solvability types are presented, and the false implications are illustrated by counter-examples.

Joint work with Ján Plavka (Technical University in Košice) and Hana Tomášková (University of Hradec Králové)

Tropical Rank and Beyond

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Mon 15:25, Auditorium

Rank functions over various classes of semirings are intensively investigated during the last decades. Among the other rank functions the following two are very important.

Let $(S, \oplus, \otimes)$ be a semiring, Σ_k be the permutation group on $\{1, \dots, k\}$, $A_k \subset \Sigma_k$ be the subgroup of even permutations.

A matrix $A = [a_{ij}] \in M_k(S)$ is said to be *tropically singular* if there exists a subset $T \in \Sigma_k$ such that

$$\bigoplus_{\sigma \in T} a_{1\sigma(1)} \otimes \cdots \otimes a_{k\sigma(k)} = \bigoplus_{\sigma \in \Sigma_k \setminus T} a_{1\sigma(1)} \otimes \cdots \otimes a_{k\sigma(k)}.$$

Note that for tropical semirings this definition coincides with the classical one: the minimum in the permanent expression

$$\begin{aligned} \mathrm{per}(A) &:= \bigoplus_{\sigma \in S_k} a_{1\sigma(1)} \otimes \cdots \otimes a_{k\sigma(k)} \\ &= \min\{a_{1\sigma(1)} + \ldots + a_{k\sigma(k)} : \sigma \in S_k\} \end{aligned}$$

is attained at least twice.

Otherwise A is *tropically non-singular*.

Tropical rank of $M \in M_n(S)$ is the largest r such that M has a tropically non-singular $r \times r$ minor.

A matrix $A = [a_{ij}] \in M_k(R)$ is said to be *d-singular* if

$$\bigoplus_{\sigma \in A_k} a_{1\sigma(1)} \otimes \cdots \otimes a_{k\sigma(k)} = \bigoplus_{\sigma \in \Sigma_k \setminus A_k} a_{1\sigma(1)} \otimes \cdots \otimes a_{k\sigma(k)}.$$

Otherwise A is *d-non-singular*.

Determinantal rank of $M \in M_n(S)$ is the largest r such that M has a d-non-singular $r \times r$ minor.

This talk is devoted to our recent investigations of these two rank functions and their interrelations.

On the dual product and the dual residuation over idempotent semiring of intervals

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Mon 17:35, Auditorium

An idempotent semiring $\mathcal{S}$ can be endowed with a partial order relation defined as $a \preceq b \Leftrightarrow a \oplus b = b \Leftrightarrow a \wedge b = a$, in other words the sum operator $\oplus$ corresponds to the least upper bound of the set $\{a, b\}$. According to this order relation it is possible to obtain the greatest solution of equation $A \otimes X \preceq B$ where A, X and B are matrices of proper dimension and $(A \otimes X)_{ij} = \bigoplus_{k=1 \ldots n} (a_{ik} \otimes x_{kj})$. The greatest solution is obtained by considering residuation theory and is practically given by $(X)_{kj} = \bigwedge_{i=1 \ldots m} (a_{ik} \ldiv b_{ij})$, where $a_{ik} \ldiv b_{ij}$ is the greatest solution of the scalar equation $a_{ik} \otimes x_{kj} \preceq b_{ij}$. In this talk we will consider the dual matrix product $A \odot X$ defined as $(A \odot X)_{ij} = \bigwedge_{k=1 \ldots n} (a_{ik} \otimes x_{kj})$, and the dual residuation to deal with computation of the smallest solution of inequality $A \odot X \succeq B$. Due to the lack of distributivity of operator $\otimes$ over the infimum operator $\wedge$, the existence of this unique solution is not always ensured. A sufficient condition is obtained when all the elements of the semiring admit an inverse, *i.e.* $\forall a \in \mathcal{S}$, $\exists b$ such that $a \otimes b = e$ where e is the identity element of the semiring. This condition is fulfilled in (max-plus) algebra, and allows to deal with opposite semimodules in [1], but it is not the case in semirings of intervals such as introduced in [2]. Nevertheless a sufficient condition allowing to compute this smallest solution in this algebraic setting will be given.

[1] G. Cohen, S. Gaubert, and J.P. Quadrat. Duality and separation theorems in idempotent semimodules. Linear Algebra and its Applications, 379:395–422, 2004.

[2] L. Hardouin, B. Cottenceau, M. Lhommeau, and E. Le Corronc. Interval systems over idempotent semiring. Linear Algebra and its Applications, 431(5-7):855–862, August 2009.

Joint work with B. Cottenceau (University of Angers) and E. Le Corronc (University of Angers)

Nonlinear Markov games

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Thu 11:00, Room Fermi

I will discuss a new class of stochastic games that I call nonlinear Markov games, as they arise as a (competitive) controlled version of nonlinear Markov processes (an emerging field of intensive research, see e.g. [1]-[3]. This class of games can model a variety of situation for economics and epidemics, statistical physics, and pursuit - evasion processes. Further discussion of this topic will be given in author's monograph [4].

Roughly speaking, a nonlinear Markov process is defined by the property that its future behavior depends on the past not only via its present position, but also its present distribution. A nonlinear Markov semigroup can be considered as a nonlinear deterministic dynamic system, though on a weird state space of measures (notwithstanding the fact that the specific structure of generators allows for a nontrivial stochastic interpretation of the evolution, which can be thought of as solving integral equation based on a path integral). Thus, as the stochastic control theory is a natural extension of the deterministic control, we are going to further extend it by turning back to deterministic control, but of measures. In particular, as introducing stochasticity in control destroys the max-plus linearity of the Bellman operator, the introduction of a nonlinear control can restore this linearity.

[1] V. Kolokoltsov. Nonlinear Markov Semigroups and Interacting Lévy Type Processes. Journ. Stat. Physics **126:3** (2007), 585-642.

[2] T.D. Frank. Nonlinear Markov processes. Phys. Lett. A **372:25** (2008), 4553-4555.

[3] M. Zak. Quantum Evolution as a Nonlinear Markov Process. Foundations of Physics Letters **15:3** (2002), 229-243.

[4] V. N. Kolokoltsov. Nonlinear Markov processes and kinetic equations. Monograph. To appear in Cambridge University Press 2010.

Supervisory control of a class of implicit systems

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Thu 11:25, Room Fermi

We aim at addressing the positive invariance of polytopic regions for implicit systems of the form

$$F\dot{w}(t) = \sum_{k=0}^{\nu} G_k x(t - \delta_k) \ . \tag{1}$$

Such a system arises from network models, in particular for a class of timed Petri nets, called linear nets [1]. A particularly important problem in this context is that of the supervisory control . For Petri nets [2], it consists of guarantying a bound on the variable $w(t)$, specified in terms of a polytopic region $\mathcal{P}(H, h) = \{w | Hw \leq h\}$. Thus, the supervisory control problem comes down to the positive invariance of the polytope $\mathcal{P}(H, h)$.

An important feature of system (1) is that in general, it is not regular, since the matrices F, and G_k, are rectangular. We aim at addressing the positive invariance of polytopic regions for such a system, trying to generalize the known results, that concern square regular systems [3].

[1] L. Libeaut, Sur l'utilisation des dioïdes pour la commande des systèmes à événements discrets, PhD thesis, Ecole Centrale de Nantes, France, 1996.

[2] M.V. Iordache and P.J. Antsaklis, A survey on the supervision of Petri nets, DES Workshop PN 2005, Miami, FL, June 21, 2005.

[3] S. Tarbouriech and E.B. Castelan, Positively invariant sets for singular discrete-time systems, Int. J. of Systems Science, vol.24, no.9, pp.1687-1705, 1993.

Joint work with J.J. Loiseau (IRCCyN, CNRS, Nantes, France)

An Idempotent Approach to Continuous-Time Stochastic Control Using Projection Operations

William M. McEneaney, University of California San Diego, USA
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Wed 11:00, Room Fermi

It is now well-known that many classes of deterministic control problems may be solved by max-plus or min-plus (more generally, idempotent) numerical methods. It has recently been discovered that idempotent methods are applicable to stochastic control and games. The methods are related to the curse-of-dimensionality-free idempotent methods for deterministic control. The first such methods for stochastic control were developed only for discrete-time problems. The key tools enabling their development were the idempotent distributive property and the fact that certain solution forms are retained through application of the dynamic programming semigroup operator. Using this technology, the value function can be propagated backwards with a representation as a pointwise minimum of quadratic or affine forms.

Here, we will remove the severe restriction to discrete-time problems. We obtain approximate solutions to the problems through repeated application of approximate backward dynamic programming operators. A generalization of the min-plus distributive property, applicable to continuum versions will be obtained. This will allow interchange of expectations over normal random variables with infimum operators. At each time-step, the solution will be represented as an infimum over a set of quadratic forms. Backward propagation is reduced to simple standard-sense linear algebraic operations for the coefficients in the representation. The difficulty with the approach is an extreme curse-of-complexity, wherein the number of terms in the min-plus expansion grows very rapidly as one propagates. The complexity growth will be attenuated via projection onto a lower dimensional min-plus subspace at each time step. At each step, one desires to project onto the optimal subspace relative to the solution approximation.

Joint work with Hidehiro Kaise (Nagoya University, Japan)

Max-plus linear systems

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Mon 15:50, Auditorium

Stochastic max-plus linear systems are defined as iterates of a random max-plus linear map. In this talk, we will present their long term behaviour and compare it to the one of positive linear systems. The first order results (law of large numbers [1,3]) are based on subadditivity and approximation by linear systems while the second order ones (Central limit theorems, [2]) rely on the geometric properties of action of the maps on the max-plus projective space and the approximation by sum of independant real variables.

[1] T. Bousch et Jean Mairesse, Finite-range topical functions and uniformly topical functions, Dynamical Systems 21, 1 (2006), pp. 73-114.
[2] G. Merlet, A central limit theorem for stochastic recursive sequences of topical operators, Ann. Appl. Probab. 17 (2007), no. 4, 1347-1361.
[3] G. Merlet, Cycle time of stochastic max-plus linear systems, Electronic Journal of Probability 13 (2008), Paper 12, 322-340.

Convex structures and separation in max-min (fuzzy) algebra

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We present classification and separation results in max-min convexity. Separation by hyperplanes/halfspaces is a standard tool in convex geometry and its tropical (max-plus) analogue. Several separation results in max-min convex geometry are based on semispaces [1]. A counterexample to separation by hyperplanes in max-min convexity is shown in [2]. In the talk we answer the question which semispaces are hyperplanes and when it is possible to separate by hyperplanes in max-min convex geometry: a point can be separated from a convex set that does not contain it, if and only if the point belongs to the main diagonal. Further new separation results are presented, such as separation of a closed box from a max-min convex set by max-min semispaces. This can be regarded as an interval extension of the known separation results by semispaces [1]. We give a constructive proof of the separation in the case when the box satisfies a certain condition, and we show that the separation is never possible when the condition is not satisfied. These results hold in arbitrary finite dimension. We also study the separation of two max-min convex sets by a box and by a box and a semispace. These results hold only in the 2-dim case, and we provide counterexamples in the 3-dim case. The talk is based on [3] and [4].

[1] V. Nitica, The structure of max-min hyperplanes, Linear Algebra Appl., 432, pp. 402-429, 2010.
[2] V. Nitica and I. Singer, Contributions to max-min convex geometry. II. Semispaces and convex sets, Linear Algebra Appl., 428, pp. 2085–2115, 2008.
[3] V. Nitica and S. Sergeev, On hyperplanes and semispaces in max-min convex geometry, to appear in Kybernetika. arXiv:math/0910.0557
[4] V. Nitica and S. Sergeev, An interval version of separation by semispaces in max-min convexity, submitted to Linear Algebra Appl. arXiv:math/0910.0566

Joint work with S. Sergeev, University of Birmingham, UK

On the maximum cycle geometric mean

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Wed 12:15, Room Fermi

The maximum cycle geometric mean $\mu(A)$ of a $n \times n$ non-negative matrix A plays a role of the spectral radius in max algebra. We generalize the notion of the maximum circuit geometric mean to infinite non-negative matrices and provide several descriptions under certain conditions. In particular, we provide the max algebra description of $\mu(K)$, which provides connection to Bonsall's spectral radius and thus to max-eigenvalues.

If time allows, we will also consider some problems about submultiplicativity, subadditivity and the generalized spectral radius in max algebra.

Efficient algorithms for checking of the robustness and for computing the greatest eigenvector of a matrix in a fuzzy algebra

Ján Plavka, Technical University in Košice, Slovak Republic
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Mon 17:10, Auditorium

Let $(B, \leq)$ be a nonempty, bounded, linearly order set and $a \oplus b = \max(a, b)$, $a \otimes b = \min(a, b)$ for $a, b \in B$. A vector x is said to be a λ-eigenvector of a square matrix A if $A \otimes x = \lambda \otimes x$ for some $\lambda \in B$. We introduce some properties of the greatest λ-eigenvector of a given matrix A and in this context derive the $O(n^2 \log n)$ algorithm for computing the greatest λ-eigenvector [1]. A given matrix A is called (strongly) λ-robust if for every x the vector $A^k \otimes x$ is an (greatest) eigenvector of A for some natural number k. We present a characterization of λ-robust and strongly λ-robust matrices. As a consequence, an efficient algorithm for checking the λ-robustness and strong λ-robustness of a given matrix is introduced [2].

[1] M. Gavalec, J. Plavka, J. Polák: On the $O(n^2 \log n)$ algorithm for computing the greatest λ-eigenvector in fuzzy algebra (in preparation).

[2] J. Plavka: On the λ-robustness of matrices in a fuzzy algebra (submitted).

Joint work with Martin Gavalec (University of Hradec Králové, Czech Republic) and Ján Polák (Technical University in Košice, Slovak Republic)

Fundamental Traffic Diagrams : A Maxplus Point of View

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Wed 11:25, Room Fermi

Following Daganzo we discuss the variational formulation of the Lighthill-Witham-Richards equation describing the traffic on a road. First, we consider the case of a circular road with a very simple dynamics which is minplus linear. We extend it to the case of two roads with a junction with the right priority. The equation obtained is no more an Hamilton-Jacobi-Bellman equation. To study the eigenvalue problem extending the standard minplus one, we consider a space discretization of the equation. The discrete problem can be solved analytically, it gives the eigenvalue as function of the car density. The limit when the discretization step goes to zero gives a very simple formula. This eigenvalue gives a good approximation of what we call "the global fundamental traffic diagram" (the relation between the density and the average flow in the system). This global fundamental diagram must be distinguished from the standard fundamental diagram which is local, obtained empirically and, in the concave case, can be seen as an hamiltonian of a control problem.

[1] C. F. Daganzo: A variational formulation of kinematic waves: Basic theory and complex boundary conditions. Transportation Research part B, 39(2), 187-196, 2005.
[2] J. Lighthill, J. B. Whitham: On kinetic waves: II A theory of traffic Flow on long crowded roads, Proc. Royal Society A229 p. 281-345, 1955.
[3] N. Farhi: Modélisation minplus et commande du trafic de villes régulière, thesis dissertation, University Paris 1 Panthéon - Sorbonne, 2008.

Joint work with N. Farhi (INRIA-Grenoble) & M. Goursat (INRIA-Rocquencourt)

The level set method for the two-sided eigenproblem in max-plus algebra

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Wed 11:50, Room Fermi

We consider the max-plus analogue of the eigenproblem for matrix pencils, $Ax = \lambda Bx$. We show that the spectrum of (A, B) (i.e., the set of possible values of λ) is a finite union of intervals, which can be computed by a pseudo-polynomial number of calls to an oracle that computes the value of a mean payoff game. The proof relies on the introduction of a spectral function, which we interpret in terms of the least Chebyshev distance between Ax and λBx. The spectrum is obtained as the zero level set of this function.

Joint work with S. Gaubert (INRIA and École Polytechnique, France)

Optimization Problems under (max, min)-Linear Two-sided Equality Constraints

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Wed 12:40, Room Fermi

We consider the following optimization problem:

$$\text{minimize } f(x) \equiv \max_{j \in J} f_j(x_j)$$

subject to

$$\max_{j \in J}(a_{ij} \wedge x_j) = \max_{j \in J}(b_{ij} \wedge x_j) \ ,$$

$$\underline{x}_j \leq x_j \leq \overline{x}_j \ , \quad j \in J \ ,$$

where I, J are finite index sets, $f_j : R^1 \to R^1$ are continuous unimodal functions, $a_{ij}, b_{ij}, \underline{x}_j, \overline{x}_j$ are real numbers, and $\alpha \wedge \beta \equiv \min\{\alpha, \beta\}$ for any $\alpha, \beta \in R^1$.
An iteration method for solving the optimization problem is proposed. The method is based on a method for finding the maximum element of the set of feasible solutions of the given optimization problem combined with a bisection iterations. As a result an approximate solution of the given problem is obtained. Possibilities of applications of the considered class of problems are presented. The method is further used to approximate minimization of Lipschitzian objective functions under the given constraints. Extensions and generalizations of the presented results are briefly discussed.

Joint work with Martin Gavalec (University of Hradec Králové)
