Matrix Means

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The theory of matrix and operator means is currently an active area of research. Investigations include the theoretical study of such means, various axiomatic and variational descriptions and characterizations, computational algorithms for their approximation, geometric interpretations and connections, and applications in a variety of settings. Recent advances include various approaches to define, study, and compute a variety of multivariable means. Applications include derivations of matrix and operator inequalities, finding closed formulas and approximating algorithms for the solution of symmetric and other matrix equations. Another active direction of research is the employing of means for the purpose of averaging, with applications including the averaging of data given in matrix form.

Higher order geometric mean equations based on monotone and jointly homogeneous maps

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Fri 12:15, Auditorium

We consider the nonlinear equations based on monotone and jointly homogeneous maps on the convex cone of positive definite matrices. We'll derive the uniqueness and existence of positive definite solution by using Thompson's part metric and that the corresponding solution map is again monotone and jointly homogeneous. Let $\Omega = \Omega(k)$ be the convex cone of $k \times k$ positive definite matrices. We first show that for monotone and jointly homogeneous mappings $g : \Omega^n \to \Omega$ and $h_i : \Omega^2 \to \Omega$, the equation

$$x = g(h_1(a_1, x), h_2(a_2, x), \ldots, h_n(a_n, x))$$

has a unique solution in Ω if $\sum_{i=1}^n w_i \alpha_i \in [0, 1)$. Here, a map $g : \Omega^n \to \Omega$ is $\mathbf{w} = (w_1, w_2, \ldots, w_n)$-jointly homogeneous if $g(t_1a_1, t_2a_2, \ldots, t_na_n) = t_1^{w_1} t_2^{w_2} \cdots t_n^{w_n} g(a_1, a_2, \ldots, a_n)$ for all $t_i > 0$ and $a_i \in \Omega$. Also $h : \Omega^2 \to \Omega$ is α-homogeneous if it is $(1-\alpha, \alpha)$-jointly homogeneous. We further show that if $\sum_{i=1}^n w_i = 1$ and $\alpha_i = \alpha$ for all i, then the solution map varying over $(a_1, a_2, \ldots, a_n) \in \Omega^n$ is again order preserving and $\mathbf{w}$-jointly homogeneous. We apply our results to high order geometric mean equations of positive definite matrices.

Matrix Means in a Euclidean setting

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Wed 12:15, Auditorium

Matrix means are defined for positive semidefinite matrices, and as such are usually studied from the viewpoint of Riemannian geometry, with the set of positive definite matrices being a differentiable Riemannian manifold. In this paper, a completely different approach is taken, inspired by certain practical problems in quantum state reconstruction. To wit, we regard the set of positive definite matrices as a subset of the set of Hermitian matrices, equipped with the Hilbert-Schmidt (HS) inner product, i.e. as a real Euclidean space.

We investigate which matrix norms obey the requirement that 'their value should lie inbetween the values of their arguments'. To make sense of the term 'inbetween', we consider a) the HS distance, and b) the angle between matrices $\cos\theta(A, B) = Tr(A^*B)/\sqrt{Tr(A^*A)Tr(B^*B)}$. We define a matrix mean $C = \mu(A, B)$ to lie within A and B w.r.t. HS distance if and only if neither the distance between A and C, nor the distance between B and C exceed the distance between A and B. Similarly, we define a matrix mean to lie within A and B w.r.t. angles if and only if neither the angle between A and C, nor the angle between B and C exceed the angle between A and B.

It turns out that many matrix means do not satisfy 'inbetweenness' in neither sense. Here we show that the inbetweenness condition is satisfied by the power means and the Heinz means, for distances as well as for angles.

Interpolation, geometric mean and matrix Chebyshev inequalities

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Wed 11:25, Auditorium

The geometric mean of positive definite matrices may be defined via complex interpolation. This approach leads to to simple proofs of Ando-Hiai and Furuta inequalities. We then show how these inequalities are used to obtain new inequalities for positive linear maps, regarded as asymmetric versions of Kadison and Choi inequalities. This talk is based on a joint paper with Éric Ricard.

Operator inequalities related to weighted geometric means

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Tue 16:45, Auditorium

The geometric mean $A \sharp B$ for positive operators A and B is given by the unique positive solution of the operator equation $XA^{-1}X = B$. That is,

$$A \sharp B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\frac{1}{2}}A^{\frac{1}{2}}.$$

By virtue of the Kubo-Ando theory, it is generalized to weighted geometric means as follows: For $\alpha \in [0, 1]$

$$A \sharp_\alpha B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\alpha}A^{\frac{1}{2}}.$$

It corresponds to the Löwner-Heinz inequality:

$$A \geq B \geq 0 \implies A^\alpha \geq B^\alpha.$$

There are many useful operator inequalities related to this. A typical example is the Ando-Hiai inequality (AH):

$$A \sharp_\alpha B \leq 1 \implies A^r \sharp_\alpha B^r \leq 1 \quad \text{for} \quad r \geq 1.$$

In this talk, we discuss generalizations of (AH) and relations among obtained inequalities. Our basic inequality is as follows:

If $\log A \geq \log B$ for $A, B > 0$, then

$$A^{-r} \sharp_{\frac{r}{p+r}} B^p \leq 1$$

holds for $p, r \geq 0$.

Operator equations via an order preserving operator inequality

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Tue 15:50, Auditorium

A capital letter means a bounded linear operator on a Hilbert space. We obtained the following order preserving operator inequality closely associated with matrix means:

Theorem A. If $A \geq B \geq 0$, then the following (i) and (ii) hold for $p \geq 1$ and $r \geq 0$;

(i) $(B^{\frac{r}{2}} A^p B^{\frac{r}{2}})^{\frac{1+r}{p+r}} \geq B^{1+r}$ and (ii) $A^{1+r} \geq (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{1+r}{p+r}}$.

Let A be a positive definite operator and B be a self-adjoint operator. We discuss the existence of positive semidefinite solutions of the Lyapunov type operator equation

$$\sum_{j=1}^{n} A^{n-j} X A^{j-1} = B$$

via Theorem A and by using the solutions we give concrete and recordable examples of positive semidefinite matrices as positive semidefinite solutions of some matrix equations.

The tracial geometric mean in several variables and related trace inequalities

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Tue 15:00, Auditorium

We introduce the tracial geometric mean of several operator variables as a generalization of the geometric mean for tuples of positive numbers. It possesses a number of attractive properties, including monotonicity and concavity in the operator variables. The non-commutative Hardy inequality is used to obtain a generalization of Carleman's inequality. Other related trace inequalities are given.

Operator log-convex functions and operator means

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Tue 15:25, Auditorium

We were motivated by the question to determine $\alpha \in \mathbb{R}$ for which the functional $\log \omega(A^\alpha)$ is convex in positive operators A for any positive linear functional ω. In the course of settling the question, we arrived at the idea to characterize continuous nonnegative functions f on $(0, \infty)$ for which the operator inequality $f\left(\frac{A+B}{2}\right) \leq f(A) \# f(B)$ holds for positive operators A and B, where $A \# B$ is the geometric mean. This inequality was formerly considered by Aujla, Rawla and Vasudeva as a matrix/operator version of log-convex functions. In fact, it is natural to say that a function f satisfying the above inequality is operator log-convex, since the numerical inequality $f\left(\frac{a+b}{2}\right) \leq \sqrt{f(a)f(b)}$ for $a, b > 0$ means the convexity of $\log f$ and the geometric mean $\#$ is the most standard operator version of geometric mean. We show that a continuous nonnegative function f on $(0, \infty)$ is operator log-convex if and only if it is operator monotone decreasing, and furthermore present several equivalent conditions related to operator means for the operator log-convexity. The operator log-concavity counterpart is also considered.

Joint work with T. Ando (Hokkaido University)

Recent researches on generalized Furuta-type operator functions

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Wed 11:50, Auditorium

In what follows, A and B are positive (semidefinite) operators on a Hilbert space, and $A \geq 0$ (resp. $A > 0$) denotes that A is a positive (resp. strictly positive) operator.

Furuta inequality *"$A \geq B \geq 0$ ensures $A^{1+r} \geq (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{1+r}{p+r}}$ for $p \geq 1$ and $r \geq 0$"* is established in 1987, and also Furuta showed its generalization (called grand Furuta inequality) in 1995 as follows: *If $A \geq B \geq 0$ with $A > 0$, then for each $t \in [0, 1]$ and $p \geq 1$,*

$$F(r,s) = A^{\frac{-r}{2}} \{A^{\frac{r}{2}} (A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}})^s A^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}} A^{\frac{-r}{2}} \tag{1}$$

is decreasing for $r \geq t$ and $s \geq 1$, and also for each $t \in [0, 1]$ and $p \geq 1$,

$$A^{1-t+r} \geq \{A^{\frac{r}{2}} (A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}})^s A^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}}$$

holds for $r \geq t$ and $s \geq 1$. We remark that grand Furuta inequality is interpolating Furuta inequality and Ando-Hiai inequality which is equivalent to the main result of log majorization. Very recently, Furuta obtained a further extension of grand Furuta inequality (we call this FGF inequality here).

α-Power mean $\sharp_\alpha$ for $\alpha \in [0, 1]$ is defined by $A \sharp_\alpha B = A^{\frac{1}{2}} (A^{\frac{-1}{2}} B A^{\frac{-1}{2}})^\alpha A^{\frac{1}{2}}$ for $A > 0$ and $B \geq 0$. It is known that α-power mean is very usful for investigating Furuta inequality and its generalizations. We can express (1) by (1') with α-power mean as follows:

$$\begin{aligned} F(r,s) &= A^{-r} \sharp_{\frac{1-t+r}{(p-t)s+r}} (A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}})^s \\ &= A^{\frac{-t}{2}} \{A^{-\gamma} \sharp_{\frac{1+\gamma}{\beta+\gamma}} (A^t \natural_{\frac{\beta-t}{p-t}} B^p)\} A^{\frac{-t}{2}} \\ &= A^{\frac{-t}{2}} \hat{F}(\beta, \gamma) A^{\frac{-t}{2}}, \end{aligned} \tag{1'}$$

where $\beta = (p-t)s + t$, $\gamma = r - t$ and $A \natural_s B = A^{\frac{1}{2}} (A^{\frac{-1}{2}} B A^{\frac{-1}{2}})^s A^{\frac{1}{2}}$ for a real number s. (If $s \in [0, 1]$, then $\natural_s = \sharp_s$.)

In this talk, firstly we shall discuss complementary inequalities and related results to generalized Ando-Hiai inequality and a generalized Furuta-type operator function. Secondly we shall obtain a more precise and clear expression of FGF inequality by considering a mean theoretic proof of grand Furuta inequality.

Joint work with E. Kamei (Maebashi Institute of Technology)

The Weighted Multivariable AGH-Mean

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In this presentation we consider a weighted mean arising as the geometric mean of the weighted arithmetic and harmonic n-means of positive definite matrices, what we call the AGH-mean. This mean is readily computable and exhibits a variety of other desirable properties, which we describe. We show that it also has nice variational characterizations. We also show that it generalizes in a straightforward fashion to a one-parameter family of weighted means and that many of its properties carry over to this generalization.

Joint work with J. Lawson (Louisiana State U.), Y. Lim (Kyungpook National U.)

Weighted Ando-Li-Mathias Geometric Means

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Fri 11:50, Auditorium

In [1], Ando-Li-Mathias proposed a successful definition for geometric means of several positive definite matrices. We propose a higher order weighted geometric mean based on the Ando-Li-Mathias symmetrization procedure.

For positive real numbers s and t, $G(s,t;A,B)$ is defined by $G(s,t;A,B) = A\#_{\frac{t}{s+t}}B$. A weighted geometric mean $G(t_1,t_2,\ldots,t_n;A_1,A_2,\ldots,A_n)$ of positive definite matrices $A_1, A_2, \ldots, A_n$ and positive real numbers $t_1,t_2,\ldots,t_n$ is defined by induction as follows: Assume that the weighted geometric mean of any $(n-1)-$ tuple of matrices is defined. Let

$$G((t_j)_{j\neq i};(A_j)_{j\neq i}) = G(t_1,\ldots,t_{i-1},t_{i+1},\ldots,t_n; A_1,\ldots,A_{i-1},A_{i+1},\ldots,A_n)$$

and let $A_i^{(1)} = A_i$ and $A_i^{(r+1)} = G((t_j)_{j\neq i};(A_j^{(r)})_{j\neq i})$. Then the sequences $A_i^{(r)}$ converge to a common limit, denoted by $G(t_1,\ldots,t_n;A_1,\ldots,A_n) = \lim_{r\to\infty} A_i^{(r)}$.

We show that the weighted mean satisfies the properties given by Ando-Li-Mathias in a weighted version: consistency with scalars, joint homogeneity, permutation invariance, monotonicity, continuity, invariance under the congruence and inversion, joint concavity, self-duality, determinant identity and arithmetic-geometric-harmonic means inequality.

[1] T. Ando, C.K. Li and R. Mathias, Geometric means, Linear Algebra Appl., 385 (2004), 305-334.

Joint work with Yongdo Lim (Kyungpook National University) and T. Yamazaki (Kanagawa University)

Weighted Bini-Meini-Poloni Geometric Means

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Taking a weighted version of Bini-Meini-Poloni symmetrization procedure for a multivariable geometric mean [1], we propose a definition for a weighted geometric mean of n positive definite matrices, where the weights vary over all n-dimensional positive probability vectors. We show that the weighted mean satisfies multidimensional versions of all properties that one would expect for a two-variable weighted geometric mean;

(P1) $\mathfrak{B}_n(\omega;A_1,\ldots,A_n) = A_1^{w_1}\cdots A_n^{w_n}$ for commuting A_i's;

(P2) (Joint homogeneity);

$$\mathfrak{B}_n(\omega;a_1A_1,\ldots,a_nA_n) = a_1^{w_1}\cdots a_n^{w_n}\mathfrak{B}_n(\omega;A_1,\ldots,A_n);$$

(P3) (Permutation invariance)

$$\mathfrak{B}_n(\omega_\sigma;A_{\sigma(1)},\ldots,A_{\sigma(n)}) = \mathfrak{B}_n(\omega;A_1,\ldots,A_n)$$

for any permutation σ, where $\omega_\sigma = (w_{\sigma(1)},\ldots,w_{\sigma(n)})$;

(P4) (Monotonicity) If $B_i \leq A_i$ for all $1\leq i\leq n$, then $\mathfrak{B}_n(\omega;B_1,\ldots,B_n) \leq \mathfrak{B}_n(\omega;A_1,\ldots,A_n)$;

(P5) (Continuity) The map $\mathfrak{B}_n(\omega;\cdot)$ is continuous;

(P6) (Congruence invariance)

$$\mathfrak{B}_n(\omega;M^*A_1M,\ldots,M^*A_nM) = M^*\mathfrak{B}_n(\omega;A_1,\ldots,A_n)M;$$

(P7) (Joint concavity) For $0\leq t\leq 1$,

$$\mathfrak{B}_n(\omega;A_1+(1-t)B_1,\ldots,A_n+(1-t)Bn) \geq t\mathfrak{B}_n(\omega;A_1,\ldots,A_n)+(1-t)\mathfrak{B}_n(\omega;B_1,\ldots,B_n)$$

(P8) (Self-duality); $\mathfrak{B}_n(\omega;A_1^{-1},\ldots,A_n^{-1})^{-1} = \mathfrak{B}_n(\omega;A_1,\ldots,A_n)$;

(P9) (Determinantal identity)

$$\mathrm{Det}\mathfrak{B}_n(\omega;A_1,\ldots,A_n) = \prod_{i=1}^n(\mathrm{Det}A_i)^{w_i};$$

and

(P10) (AGH mean inequalities)

$$(\sum_{i=1}^n w_iA_i^{-1})^{-1} \leq \mathfrak{B}_n(\omega;A_1,\ldots,A_n) \leq \sum_{i=1}^n w_iA_i.$$

[1] D. Bini, B. Meini and F. Poloni, An effective matrix geometric mean satisfying the Ando-Li-Mathias properties, Math. Comp. **79** (2010), 437-452.

Joint work with Jimmie Lawson (Louisiana State University) and Hosoo Lee (Kyungpook National University)

Hermitian metrics and matrix means

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Recently there has been great interest in extending matrix means to several variables. Many authors considered more or less similar iterative methods to construct a multi-variable form for matrix means as the limit point of these iterative procedures. One of the most widely studied matrix mean is the geometric mean

$$G(A,B) = A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}.$$

This mean has several special properties. One of them is that the geometric mean is the midpoint map of the manifold of positive definite matrices endowed with the metric induced by the inner product

$$\langle U,V\rangle_p = Tr\{p^{-1}Up^{-1}V\}$$

defined for the tangent space at p. This space is also a Hermitian symmetric space.

Here we show that every matrix mean is the midpoint map of a hermitian symmetric space defined over the space of positive definite matrices [3]. In particularly we show that this is a special case of a more general phenomenon. Given a holomorphic function that has a unique fixed point and fullfills some other properties, automatically induces a hermitian metric on the space of positive definite matrices. These manifolds also turn out to be Riemannian symmetric spaces so therefore also Lie Groups.

We will show that the geometric mean, the harmonic mean and the arithmetic mean obey this construction, so we get the correct corresponding metrics.

After this we consider an iterative multi-variable extension method for means given as midpoint maps in $k-$convex metric spaces [2]. We use $k-$convexity to show that the procedure converges and we also give bounds on the rate of convergence. Later we consider the center of mass on these spaces and we give upper bounds on the distance of the center of mass of the starting points and the limit point of the iterative procedure. We will also give sufficient conditions for the two points to be identical.

Considering once again the $k-$convexity condition we leave this general setting and move back to the case of hermitian metrics on the space of positive definite matrices. As a conclusion we use this machinery given for $k-$convex metric spaces on these symmetric spaces to extend two-variable matrix means to several variables similarly as in [1].

[1] M. Pálfia, Iterative multi-variable extensions to the two-variable mean of positive-definite matrices, SIAM J. Matrix Anal. Appl., to appear.
[2] M. Pálfia, Midpoint maps in metric spaces and the center of mass, preprint.
[3] M. Pálfia, Hermitian symmetric spaces and means of positive definite matrices, in preparation.

Pólya-Szegö inequality for the chaotically geometric mean

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Wed 11:00, Auditorium

Greub-Rheinboldt showed the generalized Pólya-Szegö inequality, which is equivalent to the Kantorovich inequality: Let A and B be commuting positive operators on a Hilbert space H such that $mI \leq A, B \leq MI$ for some scalars $0 < m < M$. Then $\sqrt{(Ax,x)(Bx,x)} \leq \frac{M+m}{2\sqrt{Mm}}(A^{\frac{1}{2}}B^{\frac{1}{2}}x,x)$ for every unit vector $x \in H$. Fujii, Izumino, Nakamoto and Seo showed the non-commutative version: Let A and B be positive operators on H such that $mI \leq A, B \leq MI$ for some scalars $0 < m < M$. Then $\sqrt{(Ax,x)(Bx,x)} \leq \frac{M+m}{2\sqrt{Mm}}(A\sharp Bx,x)$ for every unit vector $x \in H$, where the geometric mean $A\sharp B$ of A and B in the sense of Kubo-Ando is defined by $A\sharp B = A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^{1/2}A^{1/2}$. Ando-Li-Mathias defined the geometric mean of n-operators and by using it Yamazaki showed an n-variable version of Pólya-Szegö inequality. Moreover, Lawson-Lim defined the weighted geometric mean of n-operators, which extends to the Ando-Li-Mathias geometric mean.

Let $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$ be a weight vector if $\sum_{i=1}^n \alpha_i = 1$ and $\alpha_i \geq 0$ for all $i = 1, \cdots, n$. For positive invertible operators $A_1, A_2, \cdots A_n$ on a Hilbert space H, the chaotically geometric mean of $A_1, A_2, \cdots A_n$ for a weight vector α is defined by $\diamondsuit_\alpha(A_1, \cdots, A_n) = \exp\left(\sum_{i=1}^n \alpha_i \log A_i\right)$. If $A_1, A_2, \cdots A_n$ mutually commute, then $\diamondsuit_\alpha(A_1, \cdots, A_n) = A_1^{\alpha_1}\cdots A_n^{\alpha_n}$. The geometric mean $\sharp$ have a monotone property and the chaotically geometric mean does not have a monotone property.

In this talk, we show the chaotically geometric mean version of Pólya-Szegö inequality: Let $A_1, A_2, \cdots A_n$ be positive invertible operators on a Hilbert space H such that $mI \leq A_i \leq MI$ for some scalars $0 < m < M$ and $i = 1, \cdots, n$. Put $h = \frac{M}{m}$. Then for each weight vector α

$$\begin{aligned}\frac{1}{S(h)}(\diamondsuit_\alpha(A_1, \cdots, A_n)x, x) &\leq (A_1x,x)^{\alpha_1}\cdots(A_nx,x)^{\alpha_n} \\ &\leq S(h)(\diamondsuit_\alpha(A_1, \cdots, A_n)x, x)\end{aligned}$$

for every unit vector $x \in H$, where the Specht ratio $S(h)$ is defined by $S(h) = \frac{(h-1)h^{\frac{1}{h-1}}}{e \log h}$ $(h \neq 1, h > 0)$ and $S(1) = 1$.

Operator Monotone Functions, Positive Definite Kernels and Majorization

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Let $f(t)$ be a real continuous function on an interval, and consider the operator function $f(X)$ defined for Hermitian operators X. We will show that if $f(X)$ is increasing w.r.t. the operator order, then for $F(t) = \int f(t)dt$ the operator function $F(X)$ is convex. Let $h(t)$ and $g(t)$ be C^1 functions defined on an interval I. Suppose $h(t)$ is non-decreasing and $g(t)$ is increasing. Then we will define the continuous kernel function $K_{h,\,g}$ by $K_{h,\,g}(t,s) = (h(t)-h(s))/(g(t)-g(s))$, which is a generalization of the Löwner kernel function. We will see that it is positive definite if and only if $h(A) \leq h(B)$ whenever $g(A) \leq g(B)$ for Hermitian operators A, B, and give a method to construct a lot of infinitely divisible kernel functions.

[1] M. Uchiyama, Operator Monotone Functions, Positive Definite Kernels and Majorization, to appear PAMS
[2] M. Uchiyama, A new majorization between functions, polynomials, and operator inequalities II, J. Math. Soc. Japan 60(2008) no. 1, 291–310

On properties of geometric mean of n-operators via Riemannian metric

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Tue 17:10, Auditorium

For positive matrices $A_1, \cdots, A_n$, arithmetic mean $\mathfrak{A}(A_1, \cdots, A_n) = \frac{A_1+\cdots+A_n}{n}$ of $A_1, \cdots, A_n$ can be defined by

$$\mathfrak{A}(A_1, \cdots, A_n) = \text{arcmin} \sum_{i=1}^n \|A_i - X\|^2,$$

where arcmin$f(X)$ means the point X_0 at which the function $f(X)$ attains its minimum value and $\|\cdot\|$ means operator norm. If we use Riemannian metric in the above definition instead of operator norm, geometric mean $\mathfrak{G}_\delta(A_1, \cdots, A_n)$ can be considered as

$$\mathfrak{G}_\delta(A_1, \cdots, A_n) = \text{arcmin} \sum_{i=1}^n \delta_2^2(A_i, X).$$

In this talk, we shall introduce properties of geometric mean of n-operators from the view point of operator inequality.
