A property of Rio's uniform dependence coefficients

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Let $(\Omega, \mathcal{F}, P)$ be a probability space, $\mathcal{A}$ a sub-σ-algebra of $\mathcal{F}$ and X a random variable defined on $(\Omega, \mathcal{F}, P)$ and with values in a metric space $(\mathcal{G}, \delta)$. In [1] the following measure of dependence between X and $\mathcal{A}$ is introduced:

Definition. Put

$$\varphi(\mathcal{A}, X) = \sup_{f \in \mathcal{L}_1(\mathcal{G},\delta)} \big|\big|\mathbf{E}[f(X)|\mathcal{A}] - \mathbf{E}[f(X)]\big|\big|_\infty,$$

where $\mathcal{L}_1(\mathcal{G}, \delta)$ is the set of 1-lipshitzian functions defined on $(\mathcal{G}, \delta)$ and taking values in $[0, 1]$.
We call $\varphi(\mathcal{A}, X)$ the *uniform Rio mixing coefficient* between X and $\mathcal{A}$.

In the same paper [1] the *uniform dependence coefficients* of a sequence $(X_i)_{i\in\mathbb{Z}}$ of real-valued random variables defined on $(\Omega, \mathcal{F}, P)$ are defined as follows:

Definition. Let $\mathcal{F}_k$ be the σ-algebra generated by $(X_i)_{i\le k}$. Put $\varphi_0 = 1$ and, for every integer $r \ge 1$,

$$\varphi_r = \sup_{\substack{k\in\mathbb{Z} \\ r\le r_1<r_2<r_3}} \varphi\big(\mathcal{F}_k, (X_{k+r_1}, X_{k+r_2}, X_{k+r_3})\big).$$

Then $(\varphi_r)_{r\ge 0}$ is the sequence of the *uniform dependence coefficients* of $(X_i)_{i\in\mathbb{Z}}$.

Remark. The random vector $(X_{k+r_1}, X_{k+r_2}, X_{k+r_3})$ takes values in the metric space $(\mathbb{R}^3, d)$, where d is the euclidean distance. In what follows we shall write $\mathcal{L}_1(\mathbb{R}^3)$ in place of $\mathcal{L}_1(\mathbb{R}^3, d)$.

Let $(X_i)_{i\ge 1})$ be a random sequence, weakly dependent in the sense of Rio. Let $(\varphi_r)_{r\ge 0}$ be the sequence of the uniform dependence coefficient for $(X_i)_{i\ge 1})$. Let $(Y_i)_{i\ge 1}$ be an independent copy of $(X_i)_{i\ge 1}$. For a fixed integer p, define the sequence $(Z_i^{(p)})_{i\in\mathbb{Z}}$ as follows

$$Z_i^{(p)} = \begin{cases} Y_i & \text{for } i \le p \\ \\ X_i & \text{for } i \ge p+1. \end{cases}$$

Denote by $(\tilde{\varphi}_r^{(p)})_{r\ge 0}$ the uniform dependence coefficients of the sequence $(Z_i^{(p)})$. We shall prove the following (rather natural) result

(1) Proposition. *For every integer $r \ge 0$ we have*

$$\sup_{p\in\mathbb{N}} \tilde{\varphi}_r^{(p)} \le \varphi_r.$$

We need some preliminary lemmas.

(2) LEMMA. *Let U, V, W be three random vectors, such that V is independent on (U,W). Then*

$$\mathbf{E}[U|(V,W)] = \mathbf{E}[U|W]$$

PROOF. We assume for definiteness that V (resp. W) takes its values in the measurable space $(E,\mathcal{E})$ (resp. $(G,\mathcal{G})$).
Put $Z = \mathbf{E}[U|W]$. Z is measurable with respect to W, hence of the form $Z = \psi(W)$ for some measurable function ψ; since W and V are independent, also $Z = \psi(W)$ and V are independent. Moreover Z, being measurable with respect to W, is measurable with respect to (V,W) since $\sigma(W) \subseteq \sigma(V,W)$.
Let now $A \in \mathcal{E}$ and $B \in \mathcal{G}$. The statement follows from the equalities

$$\int_{\{V\in A, W\in B\}} U\,dP = \int 1_A(V)1_B(W)U\,dP = P(V\in A)\int 1_B(W)U\,dP$$
$$= P(V\in A)\int_{\{W\in B\}} U\,dP = P(V\in A)\int_{\{W\in B\}} Z\,dP = P(V\in A)\int 1_B(W)Z\,dP$$
$$= \int 1_A(V)1_B(W)Z\,dP = \int_{\{V\in A, W\in B\}} Z\,dP.$$

In the above relations, the second equality follows from the independence of V and (U,W), the fourth one from the fact that $Z = \mathbf{E}[U|W]$ and the sixth one from the independence of V and (Z,W). ■

(3) LEMMA. *Let U, V and W be three random vectors, taking values respectively in the measurable spaces $(E,\mathcal{E})$, $(F,\mathcal{F})$, $(G,\mathcal{G})$. Assume that U is independent on (V,W), and let $f : E\times G, \mathcal{E}\otimes\mathcal{G} \to (\mathbb{R},\mathcal{B}(\mathbb{R}))$ be a measurable function such that $\mathbf{E}[|f(U,W)|]$ is finite. For $y \in G$, put $g(y) = \mathbf{E}[f(U,y)]$. Then*

$$\mathbf{E}[f(U,W)|V] = \mathbf{E}[g(W)|V].$$

PROOF. It will be enough to prove that, for every $A \in \mathcal{F}$, we have

$$\int_{\{V\in A\}} f(U,W)d\,P = \int_{\{V\in A\}} g(W)d\,P.$$

Now, since U is independent on (V,W), the joint law of (U,V,W) is equal to $\mu_U \otimes \mu_{(V,W)}$ (where μ_U is the law of U and $\mu_{(V,W)}$ the law of (V,W)). Hence, by Fubini's Theorem,

$$\int_{\{V\in A\}} f(U,W)d\,P = \int f(u,w)1_A(v)\mu_U(du)\otimes\mu_{(V,W)}(dv,dw)$$
$$= \int 1_A(v)\mu_{(V,W)}(dv,dw)\int f(u,w)\mu_U(du)$$
$$= \int 1_A(v)\mu_{(V,W)}(dv,dw)\mathbf{E}[f(U,w)]$$
$$= \int 1_A(v)\mu_{(V,W)}(dv,dw)g(w) = \int 1_A(V)g(W)d\,P = \int_{\{V\in A\}} g(W)d\,P.$$

■

We pass to the proof of Proposition (1). Put $\mathcal{G}_k = \sigma(Z_i,\, i\le k)$. We must evaluate

$$\tilde\varphi_r = \sup_{\substack{k\in\mathbb{Z}\\ r\le r_1<r_2\le r_3}} \varphi(\mathcal{G}_k,(Z_{k+r_1},Z_{k+r_2},Z_{k+r_3})),$$

and we shall prove that

$$\varphi(\mathcal{G}_k,(Z_{k+r_1},Z_{k+r_2},Z_{k+r_3})) \le \varphi(\mathcal{F}_k,(X_{k+r_1},X_{k+r_2},X_{k+r_3}))$$

for every k (see relations (4), (8), (10) and (15)); we distinguish three cases

(i) $k=p$. For every function $f\in\mathcal{L}_1(\mathbb{R}^3)$ we have

$$\mathbf{E}[f(Z_{p+r_1},Z_{p+r_2},Z_{p+r_3})|\mathcal{G}_p] = \mathbf{E}[f(Z_{p+r_1},Z_{p+r_2},Z_{p+r_3})]$$

since $(Z_{p+r_1},Z_{p+r_2},Z_{p+r_3})$ is independent on $\mathcal{G}_p=\sigma(Z_i,\, i\le p)$.
It follows that

$$\varphi(\mathcal{G}_p,(Z_{p+r_1},Z_{p+r_2},Z_{p+r_3})) = 0 \le \varphi(\mathcal{F}_p,(X_{p+r_1},X_{p+r_2},X_{p+r_3})). \tag{4}$$

(ii) $k>p$. Put

$$U = (X_{k+r_1},X_{k+r_2},X_{k+r_2}) = (Z_{k+r_1},Z_{k+r_2},Z_{k+r_2}); \tag{5}$$

$$W=(X_{p+1},\dots,X_k),\qquad V=(\dots,,Y_{-1},Y_0,Y_1,\dots,Y_p).$$

Then (U,W) is independent on V, and $\sigma(V,W)=\mathcal{G}_k$. Applying Lemma (2) we find, for every $f\in\mathcal{L}_1(\mathbb{R}^3)$,

$$\begin{aligned}
&\Big|\Big|\mathbf{E}[f(U)|\mathcal{G}_k]-\mathbf{E}[f(U)]\Big|\Big|_\infty = \Big|\Big|\mathbf{E}[f(U)|(V,W)]-\mathbf{E}[f(U)]\Big|\Big|_\infty\\
&= \Big|\Big|\mathbf{E}[f(U)|W]-\mathbf{E}[f(U)]\Big|\Big|_\infty = \Big|\Big|\mathbf{E}[f(U)|X_{p+1},\dots,X_k]-\mathbf{E}[f(U)]\Big|\Big|_\infty\\
&= \Big|\Big|\mathbf{E}\big[\mathbf{E}[f(U)|\mathcal{F}_k]\big|X_{p+1},\dots,X_k\big]-\mathbf{E}[f(U)]\Big|\Big|_\infty\\
&= \Big|\Big|\mathbf{E}\big[\big\{\mathbf{E}[f(U)|\mathcal{F}_k]-\mathbf{E}[f(U)]\big\}\big|X_{p+1},\dots,X_k\big]\Big|\Big|_\infty,
\end{aligned} \tag{6}$$

since $\sigma\big(X_{p+1},\dots,X_k\big)\subseteq\mathcal{F}_k$. The last member above is less or equal to

$$\Big|\Big|\mathbf{E}[f(U)|\mathcal{F}_k]-\mathbf{E}[f(U)]\Big|\Big|_\infty \le \sup_{f\in\mathcal{L}_1(\mathbb{R}^3)}\Big|\Big|\mathbf{E}[f(U)|\mathcal{F}_k]-\mathbf{E}[f(U)]\Big|\Big|_\infty = \varphi\big(\mathcal{F}_k,(X_{k+r_1},X_{k+r_2},X_{k+r_3})\big). \tag{7}$$

From (6) and (7) we conclude that, for every $f\in\mathcal{L}_1(\mathbb{R}^3)$,

$$\Big|\Big|\mathbf{E}[f(U)|\mathcal{G}_k]-\mathbf{E}[f(U)]\Big|\Big|_\infty \le \varphi\big(\mathcal{F}_k,(X_{k+r_1},X_{k+r_2},X_{k+r_3})\big),$$

and finally, by taking the supremum with respect to $f\in\mathcal{L}_1(\mathbb{R}^3)$ in the first member (recall relation (5)),

$$\begin{aligned}\varphi\big(\mathcal{G}_k,(Z_{k+r_1},Z_{k+r_2},Z_{k+r_3})\big) &= \varphi\big(\mathcal{G}_k,(X_{k+r_1},X_{k+r_2},X_{k+r_3})\\ &\le \varphi\big(\mathcal{F}_k,(X_{k+r_1},X_{k+r_2},X_{k+r_3})\big).\end{aligned} \tag{8}$$

(iii) $k<p$. We have

$$\begin{aligned}\tilde{\varphi}_r^{(p)} &= \sup_{\substack{k\in\mathbb{Z}\\ r\le r_1<r_2\le r_3}}\varphi(\mathcal{G}_k,(Z_{k+r_1},Z_{k+r_2},Z_{k+r_3}))\\ &= \sup_{r\le r_1<r_2\le r_3}\ \sup_{k\in\mathbb{Z}}\varphi(\mathcal{G}_k,(Z_{k+r_1},Z_{k+r_2},Z_{k+r_3})).\end{aligned}$$

The inner supremum can be written as

$$\Big(\sup_{\substack{k\in\mathbb{Z}\\ k+r_1\ge p+1}}\varphi(\mathcal{G}_k,(Z_{k+r_1},Z_{k+r_2},Z_{k+r_3}))\Big)\vee\Big(\sup_{\substack{k\in\mathbb{Z}\\ k+r_1\le p}}\varphi(\mathcal{G}_k,(Z_{k+r_1},Z_{k+r_2},Z_{k+r_3}))\Big); \tag{9}$$

now, if $k+r_1\ge p+1$, we have

$$\varphi(\mathcal{G}_k,(Z_{k+r_1},Z_{k+r_2},Z_{k+r_3}))=0\le\varphi(\mathcal{F}_k,(X_{k+r_1},X_{k+r_2},X_{k+r_3})); \tag{10}$$

in fact $k+r_1\ge p+1>k+1$ and $\mathcal{G}_k=\sigma(Z_i,i\le k)=\sigma(Y_i,i\le k)$ (recall that $k<p$), so that $(Z_{k+r_1},Z_{k+r_2},Z_{k+r_3})=(X_{k+r_1},X_{k+r_2},X_{k+r_3})$ and $\mathcal{G}_k$ are independent and relation (10) follows by the same argument used in point (i).

Now we evaluate $\varphi(\mathcal{G}_k,(Z_{k+r_1},Z_{k+r_2},Z_{k+r_3}))$ for $k+r_1\le p$.
In order to fix ideas, assume for the moment that $k+r_1\le p<p+1\le k+r_2<k+r_3$, and put

$$U=(X_{k+r_2},X_{k+r_3})=(Z_{k+r_2},Z_{k+r_3}),\qquad W=Y_{k+r_1}=Z_{k+r_1}$$

$$V=(\dots,Y_{-1},Y_0,\dots,Y_k)=(\dots,Z_{-1},Z_0,\dots,Z_k).$$

Then U is independent on (V,W), $\mathcal{G}_k=\sigma(V)$, and we can apply Lemma (3), so obtaining, for every $f\in\mathcal{L}_1(\mathbb{R}^3)$,

$$\begin{aligned}\mathbf{E}[f(Z_{k+r_1},Z_{k+r_2},Z_{k+r_3})|\mathcal{G}_k]=\mathbf{E}[f(W,U)|V]=\mathbf{E}[g(W)|V]=\\ =\mathbf{E}[g(Y_{k+r_1})|\mathcal{G}_k],\end{aligned}\tag{11}$$

where $g(y)=\mathbf{E}[f(y,X_{k+r_2},X_{k+r_3})]$. By taking expectation of both members of (11), we get also

$$\mathbf{E}[f(Z_{k+r_1},Z_{k+r_2},Z_{k+r_3})]=\mathbf{E}[g(Y_{k+r_1})].\tag{12}$$

From (11) and (12) we deduce

$$\begin{aligned}&\Big|\Big|\mathbf{E}[f(Z_{k+r_1},Z_{k+r_2},Z_{k+r_3})|\mathcal{G}_k]-\mathbf{E}[f(Z_{k+r_1},Z_{k+r_2},Z_{k+r_3})]\Big|\Big|_\infty\\ &=\Big|\Big|\mathbf{E}[g(Y_{k+r_1})|\mathcal{G}_k]-\mathbf{E}[g(Y_{k+r_1})]\Big|\Big|_\infty\\ &=\Big|\Big|\mathbf{E}[g(X_{k+r_1})|\mathcal{F}_k]-\mathbf{E}[g(X_{k+r_1})]\Big|\Big|_\infty,\end{aligned}\tag{13}$$

since $\mathcal{G}_k=\sigma(Y_i,i\le k)$ and $(X_i)_{i\in\mathbb{Z}}$ and $(Y_i)_{i\in\mathbb{Z}}$ have the same law.
Since the function $\tilde{g}:(s,t,v)\mapsto g(s)$ belongs to $\mathcal{L}_1(\mathbb{R}^3)$, the last member in relation (13) is less or equal to

$$\begin{aligned}&\sup_{f\in\mathcal{L}_1(\mathbb{R}^3)}\Big|\Big|\mathbf{E}[f(X_{k+r_1},X_{k+r_2},X_{k+r_3})|\mathcal{F}_k]-\mathbf{E}[f(X_{k+r_1},X_{k+r_2},X_{k+r_3})]\Big|\Big|_\infty\\ &=\varphi\big(\mathcal{F}_k,(X_{k+r_1},X_{k+r_2},X_{k+r_3})\big).\end{aligned}\tag{14}$$

From (13) and (14) we conclude that

$$\begin{aligned}&\Big|\Big|\mathbf{E}[f(Z_{k+r_1},Z_{k+r_2},Z_{k+r_3})|\mathcal{G}_k]-\mathbf{E}[f(Z_{k+r_1},Z_{k+r_2},Z_{k+r_3})]\Big|\Big|_\infty,\\ &\le\varphi\big(\mathcal{F}_k,(X_{k+r_1},X_{k+r_2},X_{k+r_3})\big)\end{aligned}$$

and taking the supremum with respect to $f\in\mathcal{L}_1(\mathbb{R}^3)$ in the first member of the above relation, we get

$$\varphi\big(\mathcal{G}_k,(Z_{k+r_1},Z_{k+r_2},Z_{k+r_3})\big)\le\varphi\big(\mathcal{F}_k,(X_{k+r_1},X_{k+r_2},X_{k+r_3})\big).\tag{15}$$

The above argument can be repeated word by word if $k+r_1<k+r_2\le p<p+1\le k+r_3$; it becomes even easier if $k+r_1<k+r_2<k+r_3\le p$.

The proof of Proposition (1) is concluded. ■

References

[1] **Rio E.**, (1996) *Sur le théorème de Berry-Esseen pour les suites faiblement dépendantes*, Probab. Theory Relat. Fields, **104**, 255-282. MR1373378 (96k:60061)