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HYPERBOLIC–PARABOLIC SINGULAR PERTURBATION FOR MILDLY DEGENERATE KIRCHHOFF EQUATIONS: GLOBAL-IN-TIME ERROR ESTIMATES

Marina Ghisi

Dipartimento di Matematica, University of Pisa, Italy

Massimo Gobbino

Dipartimento di Matematica Applicata, University of Pisa, Italy

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Abstract. We consider the second order Cauchy problem

$$\varepsilon u_\varepsilon'' + u_\varepsilon' + m(|A^{1/2}u_\varepsilon|^2)Au_\varepsilon = 0, \qquad u_\varepsilon(0) = u_0, \quad u_\varepsilon'(0) = u_1,$$

and the first order limit problem

$$u' + m(|A^{1/2}u|^2)Au = 0, \qquad u(0) = u_0,$$

where $\varepsilon > 0$, H is a Hilbert space, A is a self-adjoint nonnegative operator on H with dense domain $D(A)$, $(u_0, u_1) \in D(A^{3/2}) \times D(A^{1/2})$, and $m : [0, +\infty) \to [0, +\infty)$ is a function of class C^1.

We prove global-in-time estimates for the difference $u_\varepsilon(t) - u(t)$ provided that u_0 satisfies the nondegeneracy condition $m(|A^{1/2}u_0|^2) > 0$, and the function $\sigma m(\sigma^2)$ is nondecreasing in a right neighborhood of its zeroes.

The abstract results apply to parabolic and hyperbolic partial differential equations with non-local nonlinearities of Kirchhoff type.

1. **Introduction.** Let H be a real Hilbert space. For every x and y in H, $|x|$ denotes the norm of x, and $\langle x, y\rangle$ denotes the scalar product of x and y. Let A be a self-adjoint linear operator on H with dense domain $D(A)$. We always assume that A is nonnegative, namely $\langle Au, u\rangle \geq 0$ for every $u \in D(A)$. For any such operator the power A^α is defined for every $\alpha \geq 0$ in a suitable domain $D(A^\alpha)$. Let $m : [0, +\infty) \to [0, +\infty)$ be a function of class C^1.

For every $\varepsilon > 0$ we consider the second order Cauchy problem

$$\varepsilon u_\varepsilon''(t) + u_\varepsilon'(t) + m(|A^{1/2}u_\varepsilon(t)|^2)Au_\varepsilon(t) = 0, \qquad \forall t \geq 0, \tag{1.1}$$

$$u_\varepsilon(0) = u_0, \qquad u_\varepsilon'(0) = u_1. \tag{1.2}$$

This problem is just an abstract setting of the initial boundary value problem for the hyperbolic partial differential equation (PDE)

$$\varepsilon u^\varepsilon_{tt}(t,x) + u^\varepsilon_t(t,x) - m\left(\int_\Omega |\nabla u^\varepsilon(t,x)|^2 \, dx\right)\Delta u^\varepsilon(t,x) = 0 \tag{1.3}$$

in an open set $\Omega \subseteq \mathbb{R}^n$. This equation is a model for the damped small transversal vibrations of an elastic string ($n = 1$) or membrane ($n = 2$) with uniform density ε.

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We also consider the first order Cauchy problem

$$u'(t)+m(|A^{1/2}u(t)|^2)Au(t)=0, \qquad \forall t\geq 0, \tag{1.4}$$

$$u(0)=u_0, \tag{1.5}$$

obtained setting formally $\varepsilon=0$ in (1.1), and omitting the second initial condition in (1.2). In the concrete setting of (1.3) the limit problem involves a PDE of parabolic type.

These problems are called *non-degenerate* if there exists a constant $\mu>0$ such that $m(\sigma)\geq\mu$ for every $\sigma\geq 0$. They are called *mildly degenerate* if the initial condition u_0 belongs to $D(A^{1/2})$ and satisfies the non-degeneracy condition

$$m(|A^{1/2}u_0|^2)>0. \tag{1.6}$$

Existence of a global solution for the first order problem (1.4), (1.5) can be established under very general assumptions on m, A, u_0. In particular one can prove the following result (see [8] and the references quoted therein).

Theorem A. *Let A be a nonnegative operator, and let $m:[0,+\infty)\to[0,+\infty)$ be a locally Lipschitz continuous function. Let us assume that $u_0\in D(A)$ satisfies the non-degeneracy condition (1.6). Then problem (1.4), (1.5) has a unique solution*

$$u\in C^1([0,+\infty);H)\cap C^0([0,+\infty);D(A)).$$

Moreover $u\in C^1((0,+\infty);D(A^\alpha))$ for every $\alpha\geq 0$.

The standard result concerning the second order problem (1.1), (1.2) is the existence of a unique global solution provided that $(u_0,u_1)\in D(A)\times D(A^{1/2})$ satisfy (1.6) and ε is small enough. This was proved by E. De Brito [3], Y. Yamada [17], and K. Nishihara [14] in the non-degenerate case, then by K. Nishihara and Y. Yamada [15] in the mildly degenerate case with $m(\sigma)=\sigma^\gamma$ ($\gamma\geq 1$), and finally by the authors [5] with a general locally Lipschitz continuous nonlinearity $m(\sigma)\geq 0$.

The following theorem is a straightforward consequence of Theorem 2.2 of [5].

Theorem B. *Let A be a nonnegative operator, and let $m:[0,+\infty)\to[0,+\infty)$ be a locally Lipschitz continuous function. Let us assume that $(u_0,u_1)\in D(A)\times D(A^{1/2})$ satisfy the non-degeneracy condition (1.6). Then there exists $\varepsilon_0>0$ such that for every $\varepsilon\in(0,\varepsilon_0)$ problem (1.1), (1.2) has a unique global solution*

$$u_\varepsilon\in C^2([0,+\infty);H)\cap C^1([0,+\infty);D(A^{1/2}))\cap C^0([0,+\infty);D(A)).$$

The singular perturbation problem is concerned with the convergence of u_ε to u as $\varepsilon\to 0^+$. Following the approach introduced by J. L. Lions [13] in the linear case, one defines the corrector $\theta_\varepsilon(t)$ as the solution of the linear second order problem

$$\varepsilon\theta_\varepsilon''(t)+\theta_\varepsilon'(t)=0, \qquad \forall t\geq 0, \tag{1.7}$$

$$\theta_\varepsilon(0)=0, \qquad \theta_\varepsilon'(0)=u_1+m(|A^{1/2}u_0|^2)Au_0=:w_0. \tag{1.8}$$

Since $\theta_\varepsilon'(0)=u_\varepsilon'(0)-u'(0)$, this corrector keeps into account the boundary layer due to the loss of one initial condition. Then one defines the remainder $r_\varepsilon(t)$ in such a way that

$$u_\varepsilon(t)=u(t)+\theta_\varepsilon(t)+r_\varepsilon(t) \qquad \forall t\geq 0. \tag{1.9}$$

The singular perturbation problem consists in proving that $r_\varepsilon\to 0$ in some sense as $\varepsilon\to 0^+$.

As far as we know, the first result in this direction for Kirchhoff equations was obtained by B. F. Esham and R. J. Weinacht [4]. Restated in the abstract setting, they considered the *non-degenerate* case with initial data $(u_0, u_1) \in D(A^{3/2}) \times D(A)$, and they proved a *local-in-time* error estimate, namely

$$|A^{1/2} r_\varepsilon(t)| \leq C_T \varepsilon, \qquad \forall t \in [0, T], \tag{1.10}$$

where C_T is a constant which depends on T, but not on ε.

Later on this problem was considered by the second author [9] in the *mildly degenerate* case, proving the following two results.

- *Global-in-time uniform convergence*. For initial conditions $(u_0, u_1) \in D(A) \times D(A^{1/2})$ (i.e., the same space involved in the existence theorem) we have that

$$r_\varepsilon \to 0 \quad \text{uniformly in } C^0([0, +\infty); D(A)), \tag{1.11}$$

$$r'_\varepsilon \to 0 \quad \text{in } L^2([0, +\infty); D(A^{1/2})). \tag{1.12}$$

- *Local-in-time error estimates*. For more regular initial data we have that

$$|r_\varepsilon(t)| \leq C_T \varepsilon \qquad \forall t \in [0, T], \tag{1.13}$$

$$|A^{1/2} r_\varepsilon(t)| \leq C_T \varepsilon \qquad \forall t \in [0, T], \tag{1.14}$$

$$\int_0^T |r'_\varepsilon(t)|^2 \, dt \leq C_T \varepsilon^2, \tag{1.15}$$

  where C_T is a constant which depends on T, but not on ε.

In this paper we need the local-in-time error estimates (1.13), (1.14), and (1.15) for initial data in $D(A^{3/2}) \times D(A^{1/2})$. For the convenience of the reader we provide in Appendix A a self contained proof of these estimates (the proof given in [9] assumes the coerciveness of the operator A and more regularity of the initial data).

Concerning the regularity of initial data, we point out that in [6] the authors proved that $D(A^{3/2}) \times D(A^{1/2})$ is the largest space where local-in-time estimates of order ε such as (1.13) and (1.14) can be proved, even for linear equations (the case where m is constant).

In a recent paper H. Hashimoto and T. Yamazaki [10] considered once again the *non-degenerate* case providing for the first time *global-in-time* error estimates. Indeed for initial data $(u_0, u_1) \in D(A^{3/2}) \times D(A)$ they proved that

$$|r_\varepsilon(t)| \leq C\varepsilon \qquad \forall t \geq 0, \tag{1.16}$$

$$|A^{1/2} r_\varepsilon(t)| \leq \frac{C}{\sqrt{1+t}} \varepsilon \qquad \forall t \geq 0, \tag{1.17}$$

where now C doesn't depend on t (and of course on ε). They also obtained estimates for $|r'_\varepsilon(t)|$ and $|Ar_\varepsilon(t)|$, but for more regular initial data (at least $(u_0, u_1) \in D(A^2) \times D(A)$).

In this paper we prove *global-in-time* error estimates for the *mildly degenerate* case, under the additional assumption that $\sigma m(\sigma^2)$ is nondecreasing. In Lemma 3.2 we show that this assumption is equivalent to the monotonicity of the operator $u \to m(|A^{1/2}u|^2)Au$ in the sense of [2], which in turn is equivalent to say that problem (1.4), (1.5) generates a contraction semigroup.

To be more precise, we need the monotonicity of $\sigma m(\sigma^2)$ only for σ in a suitable interval (see Theorem 2.1 for the details). This weaker assumption turns out to be satisfied in most cases, for instance whenever the problem is nondegenerate, or when m is monotone in a right neighborhood of its zeroes (see Remark 2.3).

Under this monotonicity assumption we prove that

$$|r_\varepsilon(t)| \leq C\varepsilon, \qquad \forall t \geq 0, \tag{1.18}$$

$$|A^{1/2} r_\varepsilon(t)| \leq C\sqrt{\varepsilon}, \qquad \forall t \geq 0, \tag{1.19}$$

$$\int_0^{+\infty} |r'_\varepsilon(t)|^2\,dt \leq C\varepsilon, \tag{1.20}$$

where C doesn't depend on t and ε. If we assume that $(u_0, u_1) \in D(A^{3/2}) \times D(A)$ we have also better convergence rates (see the second part of Theorem 2.1).

Apart from the monotonicity assumption (which we suspect to be a necessary condition) estimates (1.18), (1.19), and (1.20) are the global-in-time extensions of (1.13), (1.14) and (1.15), but with lower convergence rates.

Comparing with [10] we have weaker assumptions on the nonlinearity (we recall once again that our monotonicity assumption is automatically satisfied in the nondegenerate case), and weaker assumptions on the initial data ($D(A^{3/2}) \times D(A^{1/2})$ instead of $D(A^{3/2}) \times D(A)$). Nevertheless we obtain the same estimate on $|r_\varepsilon|$, and we get an integral estimate on $|r'_\varepsilon|$ which in [10] requires initial data in $D(A^2)\times D(A)$. On the contrary our estimate (1.19) is weaker than (1.17), both because of the convergence rate, and because the latter contains also a time-decay estimate.

Let us conclude with a few words about the technique. The main idea in previous papers is considering (1.1) and (1.4) as linear equations with time-dependent coefficients $c_\varepsilon(t) = m(|A^{1/2}u_\varepsilon(t)|^2)$ and $c(t) = m(|A^{1/2}u(t)|^2)$. In this framework the convergence estimates follow from general results on linear equations with some a priori bounds on $c_\varepsilon(t) - c(t)$, which are the single point where the nonlinearity plays a role. The proof of Proposition A.1 is an example of these techniques.

On the contrary, in the proof of our main result we don't pursue this "linear path", but we introduce a "nonlinear approach" in order to take advantage of the monotonicity assumption. The main point is considering quantities such as

$$\langle u_\varepsilon - u, c_\varepsilon A u_\varepsilon - cAu \rangle$$

which are not quadratic forms in $u_\varepsilon - u$, but nevertheless are nonnegative due to the assumed monotonicity.

This paper is organized as follows. In section 2 we state our results, which we prove in section 3. Section 4 contains some open problems. In Appendix A we give a proof of the local-in-time error estimates.

2. Statements.

In this paper we assume for simplicity that $m : [0, +\infty) \to [0, +\infty)$ is a function of class C^1. Probably all the theory can be generalized to functions m which are locally Lipschitz continuous. We set $\sigma_0 := |A^{1/2}u_0|^2$, and $\mu_0 := m(\sigma_0)$. Since we consider mildly degenerate equations we always have that $\mu_0 \neq 0$. Let

$$\sigma_1 := \sup\{\sigma \in [0, \sigma_0] : \sigma \cdot m(\sigma) = 0\}. \tag{2.1}$$

In a few words, σ_1 is either 0 or the largest $\sigma < \sigma_0$ such that $m(\sigma) = 0$. Let us choose $\sigma_2 > \sigma_0$ in such a way that $m(\sigma) > 0$ for every $\sigma \in (\sigma_1, \sigma_2]$. We set

$$\mu_1 := \min_{\sigma\in[\sigma_1,\sigma_2]} m(\sigma), \qquad \mu_2 := \max_{\sigma\in[\sigma_1,\sigma_2]} m(\sigma),$$

and we denote by L the Lipschitz constant of m in $[\sigma_1, \sigma_2]$. We finally set

$$c(t) := m(|A^{1/2}u(t)|^2), \qquad c_\varepsilon(t) := m(|A^{1/2}u_\varepsilon(t)|^2).$$

We recall from previous literature that for every small enough ε we have that $\sigma_1 \leq |A^{1/2}u_\varepsilon(t)|^2 \leq \sigma_2$ for every $t \geq 0$ (and analogous estimates for the first order problem). This means that the behavior of $m(\sigma)$ is relevant only for $\sigma \in [\sigma_1, \sigma_2]$: in particular equation (1.1) is non-degenerate if and only if $\mu_1 > 0$, which in turn is true if and only if $\sigma_1 = 0$ and $m(0) > 0$.

The following is the main result of this paper.

Theorem 2.1. *Let A be a nonnegative operator, let $m : [0, +\infty) \to [0, +\infty)$ be a function of class C^1, and let σ_1 be defined as in (2.1). Let us assume that $(u_0, u_1) \in D(A^{3/2}) \times D(A^{1/2})$ satisfy the non-degeneracy condition (1.6).*

Let us assume that there exists $\delta > 0$ such that $\sigma m(\sigma^2)$ is nondecreasing in the interval $[\sigma_1, \sigma_1 + \delta]$. Let r_ε be defined as in (1.9). Then there exist $\varepsilon_1 > 0$ and a constant C_1 such that for every $\varepsilon \in (0, \varepsilon_1)$ we have that

$$|r_\varepsilon(t)| \leq C_1\varepsilon \qquad \forall t \geq 0, \tag{2.2}$$

$$|A^{1/2}r_\varepsilon(t)| \leq C_1\sqrt{\varepsilon} \qquad \forall t \geq 0, \tag{2.3}$$

$$\int_0^{+\infty} |r'_\varepsilon(t)|^2\, dt \leq C_1\varepsilon. \tag{2.4}$$

If in addition we assume that $u_1 \in D(A)$, then there exists a constant C_2 such that for every $\varepsilon \in (0, \varepsilon_1)$ we have that

$$|A^{1/2}r_\varepsilon(t)| \leq C_2\varepsilon^{2/3} \qquad \forall t \geq 0, \tag{2.5}$$

$$|A(u_\varepsilon(t) - u(t))| \leq C_2\varepsilon^{1/3} \qquad \forall t \geq 0, \tag{2.6}$$

$$|r'_\varepsilon(t)| \leq C_2\varepsilon^{1/3} \qquad \forall t \geq 0. \tag{2.7}$$

Estimate (2.6) is stated in terms of $u_\varepsilon - u$ because $A\theta_\varepsilon$, hence also Ar_ε, cannot be considered unless $u_0 \in D(A^2)$.

Remark 2.2. Estimates (2.2), (2.3), (2.4) in Theorem 2.1 hold true more generally if we replace the initial conditions (1.2) with $u_\varepsilon(0) = u_{0\varepsilon}$, $u'_\varepsilon(0) = u_{1\varepsilon}$, where $\{(u_{0\varepsilon}, u_{1\varepsilon})\} \subseteq D(A) \times D(A^{1/2})$ is any family such that

$$|u_{0\varepsilon} - u_0| \leq C\varepsilon, \tag{2.8}$$

$$|A^{1/2}u_{0\varepsilon}| + |Au_{0\varepsilon}| + |u_{1\varepsilon}| + \sqrt{\varepsilon}|A^{1/2}u_{1\varepsilon}| \leq C, \tag{2.9}$$

where $u_0 \in D(A^{3/2})$ is the initial condition in (1.5), and C is a suitable constant independent on ε. If in addition we have that $(u_{0\varepsilon}, u_{1\varepsilon}) \in D(A^{3/2}) \times D(A)$, and

$$|A^{3/2}u_{0\varepsilon}| + \sqrt{\varepsilon}|Au_{1\varepsilon}| \leq C, \tag{2.10}$$

then the corresponding solutions satisfy also (2.5), (2.6), and (2.7).

Remark 2.3. The monotonicity assumption on $\sigma m(\sigma^2)$ is automatically satisfied both when m is monotone in a right neighborhood of its zeroes, and in the nondegenerate case. In this case we have indeed that $\sigma_1 = 0$ and $m(0) > 0$, hence

$$\frac{\mathrm{d}}{\mathrm{d}\sigma}\left[\sigma m(\sigma^2)\right] = m(\sigma^2) + 2\sigma^2 m'(\sigma^2)$$

is positive in a neighborhood of $\sigma_1 = 0$.

3. **Proofs.**

3.1. **Technical preliminaries.**

The first result we need is the following comparison principle for ordinary differential equations (ODEs). A similar result for autonomous equations has been widely used in [5, 7, 9].

Lemma 3.1. *Let $f : [0, +\infty) \to [0, +\infty)$ be a function of class C^1, and let $g : [0, +\infty) \to [0, +\infty)$ be a continuous function. Let us assume that there exist two constants $c_1 > 0$ and $c_2 > 0$ such that*

$$f'(t) \leq -c_1 f(t) + c_2 \sqrt{f(t)} + g(t) \qquad \forall t \geq 0. \tag{3.1}$$

Then we have that

$$f(t) \leq \max\left\{f(0), (c_2/c_1)^2\right\} + \int_0^t g(s)\, ds \qquad \forall t \geq 0. \tag{3.2}$$

Proof. Let us consider the ordinary differential equation

$$u' = -c_1 u + c_2 \sqrt{u} + g(t). \tag{3.3}$$

Assumption (3.1) is equivalent to say that $f(t)$ is a subsolution of this ODE. Let $y(t)$ denote the right hand side of (3.2). Then it is clear that $y(0) \geq f(0)$. Since $y(t) \geq (c_2/c_1)^2$ it is easy to see that

$$y'(t) = g(t) \geq g(t) - c_1 y(t) + c_2 \sqrt{y(t)} \qquad \forall t \geq 0,$$

which proves that $y(t)$ is a supersolution of (3.3). Therefore (3.2) follows from the standard comparison principle between subsolutions and supersolutions. □

Now we characterize all functions m for which the operator $u \to m(|A^{1/2}u|^2)Au$ is monotone in the sense of [2], namely the following inequality

$$\langle m(|A^{1/2}u|^2)Au - m(|A^{1/2}v|^2)Av, u - v\rangle \geq 0 \tag{3.4}$$

is satisfied for every u and v in $D(A)$.

Lemma 3.2. *Let A be a nontrivial (not identically zero) nonnegative operator, and let $m : [0, +\infty) \to [0, +\infty)$ be any function. Then the operator $u \to m(|A^{1/2}u|^2)Au$ is monotone if and only if the function $\sigma \to \sigma m(\sigma^2)$ is nondecreasing.*

Proof. Let $x \in D(A)$ with $|A^{1/2}x| = 1$. Writing inequality (3.4) with $u = ax$, $v = bx$, we obtain that

$$\left[am(a^2) - bm(b^2)\right](a - b) \geq 0 \qquad \forall a \in \mathbb{R}, \quad \forall b \in \mathbb{R}, \tag{3.5}$$

which implies that $\sigma \to \sigma m(\sigma^2)$ is nondecreasing.

Conversely, let us assume that $\sigma \to \sigma m(\sigma^2)$ is nondecreasing. This means that (3.5) holds true, and this is equivalent to

$$m(a^2)a^2 + m(b^2)b^2 \geq ab(m(a^2) + m(b^2)) \qquad \forall a \in \mathbb{R}, \quad \forall b \in \mathbb{R}. \tag{3.6}$$

Now let u and v be in $D(A)$. Applying (3.6) with $a = |A^{1/2}u|$, $b = |A^{1/2}v|$, and then Cauchy-Schwarz inequality, we obtain that

$$\begin{aligned} m(|A^{1/2}u|^2)|A^{1/2}u|^2 &+ m(|A^{1/2}v|^2)|A^{1/2}v|^2 \\ &\geq |A^{1/2}u||A^{1/2}v| \left(m(|A^{1/2}u|^2) + m(|A^{1/2}v|^2)\right) \\ &\geq \langle A^{1/2}u, A^{1/2}v\rangle \left(m(|A^{1/2}u|^2) + m(|A^{1/2}v|^2)\right), \end{aligned}$$

which is equivalent to

$$\langle m(|A^{1/2}u|^2)A^{1/2}u - m(|A^{1/2}v|^2)A^{1/2}v, A^{1/2}u - A^{1/2}v\rangle \geq 0,$$

which in turn is equivalent to (3.4). □

3.2. **Estimates for the first order problem.** The following lemma collects the estimates on solutions of (1.4), (1.5) which are needed in the proof of our main result.

Lemma 3.3. *Let A be a nonnegative operator, and let $m \in C^1([0,+\infty);[0,+\infty))$. Let us assume that $u_0 \in D(A)$ satisfies the nondegeneracy condition (1.6).*

Then the solution $u(t)$ of (1.4) and (1.5) satisfies the following estimates.

(1) *For $k = 0, 1, 2$ we have that*

$$|A^{k/2}u(t)| \leq |A^{k/2}u_0| \qquad \forall t \geq 0; \tag{3.7}$$

$$\int_0^{+\infty} c(s)|A^{(k+1)/2}u(s)|^2\,ds \leq \frac{1}{2}|A^{k/2}u_0|^2. \tag{3.8}$$

Moreover

$$\int_0^{+\infty} |c'(s)|\,ds < +\infty. \tag{3.9}$$

(2) *If in addition $u_0 \in D(A^{3/2})$, then (3.7) and (3.8) hold true also with $k = 3$, and there exists a constant p_0 such that*

$$|A^{3/2}u(t)| \leq p_0|A^{1/2}u(t)| \qquad \forall t \geq 0. \tag{3.10}$$

Moreover

$$\int_0^{+\infty} |u''(s)|^2\,ds < +\infty, \tag{3.11}$$

$$\int_0^{+\infty} |u''(s)|\,ds < +\infty. \tag{3.12}$$

Proof. For every $k \geq 0$ we have that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\left|A^{k/2}u(t)\right|^2\right) = \langle A^{k/2}u(t), A^{k/2}u'(t)\rangle = -c(t)\left|A^{(k+1)/2}u(t)\right|^2 \qquad \forall t > 0, \tag{3.13}$$

from which (3.7) and (3.8) follow by integration whenever $u_0 \in D(A^{k/2})$.

Since we have that

$$|c'(t)| = |m'(|A^{1/2}u(t)|^2)\cdot 2\langle Au(t), u'(t)\rangle| \leq 2L|u'(t)|\cdot|Au(t)| = 2Lc(t)|Au(t)|^2,$$

estimate (3.9) follows from (3.8) applied with $k = 1$.

Let us consider now $u_0 \in D(A^{3/2})$, and let us prove (3.10). If $A^{1/2}u_0 = 0$ thesis is trivial (for instance with $p_0 = 1$). So we can assume that $A^{1/2}u_0 \neq 0$, hence $A^{1/2}u(t) \neq 0$ for every $t \geq 0$.

By Cauchy-Schwarz inequality we have that

$$|Au|^4 = \left(\langle A^{3/2}u, A^{1/2}u\rangle\right)^2 \leq |A^{3/2}u|^2|A^{1/2}u|^2. \tag{3.14}$$

Rewriting (3.14) with $A^{1/2}u$ instead of u we have that

$$|A^{3/2}u|^4 \leq |A^2u|^2|Au|^2. \tag{3.15}$$

Multiplying (3.14) and (3.15) we obtain that

$$|Au|^2|A^{3/2}u|^2 \leq |A^2u|^2|A^{1/2}u|^2,$$

hence

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{|A^{3/2}u|^2}{|A^{1/2}u|^2}\right] = -2\frac{c(t)}{|A^{1/2}u|^4}\left(|A^2u|^2|A^{1/2}u|^2 - |Au|^2|A^{3/2}u|^2\right) \leq 0,$$

and therefore

$$\frac{|A^{3/2}u(t)|^2}{|A^{1/2}u(t)|^2} \leq \frac{|A^{3/2}u_0|^2}{|A^{1/2}u_0|^2} =: p_0^2 \qquad \forall t \geq 0,$$

which implies (3.10).

Now let us consider u''. A simple computation shows that

$$u''(t) = -c'(t)Au - c(t)Au' = 2m'(|A^{1/2}u|^2)\cdot c(t)|Au|^2Au + c^2(t)A^2u. \tag{3.16}$$

From the boundedness of m, m', and from (3.7) applied with $k = 2$ we have that

$$\begin{aligned} |u''(t)|^2 &\leq 8|m'(|A^{1/2}u|^2)|^2\cdot c(t)|Au(t)|^4\cdot c(t)|Au(t)|^2 + 2c^3(t)\cdot c(t)|A^2u(t)|^2 \\ &\leq 8L^2\mu_2|Au_0|^4\cdot c(t)|Au(t)|^2 + 2\mu_2^3\cdot c(t)|A^2u(t)|^2. \end{aligned}$$

Applying (3.8) with $k = 1$ and $k = 3$ we obtain (3.11).

Now we consider the integral of $|u''|$. From (3.16) we deduce that

$$|u''(t)| \leq 2L|Au_0|\cdot c(t)|Au(t)|^2 + c^2(t)|A^2u(t)|.$$

The integral in $[0,+\infty)$ of the first summand in the right hand side is finite because of (3.8) applied with $k = 1$. So we have to prove that also the integral in $[0,+\infty)$ of the second summand is finite. To this end we consider separately the degenerate and the nondegenerate case.

Nondegenerate case. Since $c(t)$ is bounded, it is enough to prove that

$$|A^2u(t)| \leq \frac{\gamma_1}{\sqrt{t}+t^2} \qquad \forall t > 0 \tag{3.17}$$

for a suitable constant γ_1. In turn (3.17) holds true if we show that there exist γ_2 and γ_3 such that

$$t|A^2u(t)|^2 \leq \gamma_2 \qquad \forall t \geq 0, \tag{3.18}$$

$$t^4|A^2u(t)|^2 \leq \gamma_3 \qquad \forall t \geq 0. \tag{3.19}$$

Let us prove these inequalities. Since

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[t|A^2u|^2\right] = |A^2u|^2 - 2tc(t)|A^{5/2}u|^2 \leq |A^2u|^2,$$

integrating in $[0,t]$ and using (3.8) with $k = 3$ we have that

$$t|A^2u(t)|^2 \leq \int_0^t |A^2u(s)|^2\,ds \leq \frac{1}{\mu_1}\int_0^t c(s)|A^2u(s)|^2\,ds \leq \frac{1}{2\mu_1}|A^{3/2}u_0|^2,$$

which implies (3.18). Inequality (3.19) follows from the case $k = 4$ of the more general inequality

$$t^k|A^{k/2}u(t)|^2 + 2\int_0^t s^k c(s)|A^{(k+1)/2}u(s)|^2\,ds \leq \frac{k!}{(2\mu_1)^k}|u_0|^2 \qquad \forall t \geq 0, \tag{3.20}$$

which holds true for every $k \in \mathbb{N}$. This inequality can easily be proved by induction. The case $k = 0$ indeed follows by integrating (3.13) with $k = 0$. Let us assume now that (3.20) holds true for some k. Since

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[t^{k+1}|A^{(k+1)/2}u|^2\right] = (k+1)t^k|A^{(k+1)/2}u|^2 - 2c(t)t^{k+1}|A^{(k+2)/2}u|^2,$$

integrating in $[0,t]$ we obtain that

$$\begin{aligned} t^{k+1}|A^{(k+1)/2}u(t)|^2 + 2\int_0^t s^{k+1}c(s)|A^{(k+2)/2}u(s)|^2\,ds \\ &\leq (k+1)\int_0^t s^k|A^{(k+1)/2}u(s)|^2\,ds \\ &\leq \frac{k+1}{2\mu_1}\cdot 2\int_0^t s^k c(s)|A^{(k+1)/2}u(s)|^2\,ds \\ &\leq \frac{(k+1)!}{(2\mu_1)^{k+1}}|u_0|^2, \end{aligned}$$

which completes the induction.

Degenerate case. Since $x \leq x^2+1$ for every $x \in \mathbb{R}$, we have that

$$c^2(t)|A^2u(t)| \leq c^2(t)|A^2u(t)|^2 + c^2(t) \leq \mu_2 c(t)|A^2u(t)|^2 + c^2(t).$$

Therefore it is enough to prove that the integral in $[0,+\infty)$ of the two summands in the right hand side is finite. For the first one this is true by (3.8) applied with $k=3$.

In order to estimate the integral of $c^2(t)$ we use that $m(\sigma_1)=0$ in the degenerate case, hence

$$c(t) = m(|A^{1/2}u|^2) = m(|A^{1/2}u|^2) - m(\sigma_1) \leq L(|A^{1/2}u|^2-\sigma_1) \leq L|A^{1/2}u|^2, \quad (3.21)$$

and therefore by (3.8) with $k=0$ we have that

$$\int_0^{+\infty} c^2(s)\,ds \leq \int_0^{+\infty} c(s)\cdot L|A^{1/2}u(s)|^2\,ds \leq \frac{L}{2}|u_0|^2.$$

This completes the proof of (3.12) in the degenerate case. □

3.3. **Estimates for the second order problem.** The estimates on (1.1), (1.2) follow from the monotonicity or boundedness properties of the following energies

$$D_{\varepsilon,k} := \frac{|A^{k/2}u_\varepsilon|^2}{2} + \varepsilon\langle A^{k/2}u_\varepsilon, A^{k/2}u'_\varepsilon\rangle, \quad (3.22)$$

$$E_{\varepsilon,k} := \varepsilon\frac{|A^{k/2}u'_\varepsilon|^2}{c_\varepsilon} + |A^{(k+1)/2}u_\varepsilon|^2, \quad (3.23)$$

$$G_\varepsilon := \frac{|u'_\varepsilon|^2}{c_\varepsilon^2}. \quad (3.24)$$

The following results were proved in [7] (see statement (1) of Proposition 3.10 and statement (1) of Theorem 3.6 of [7]).

Lemma 3.4. *Let A be a nonnegative operator, and let $m \in C^1([0,+\infty);[0,+\infty))$. Let us assume that $(u_0,u_1) \in D(A)\times D(A^{1/2})$ satisfy the nondegeneracy condition (1.6). Let ε_0 be as in Theorem B.*

Then there exists $\varepsilon_1 \in (0,\varepsilon_0)$ such that for every $\varepsilon \in (0,\varepsilon_1)$ the solution $u_\varepsilon(t)$ of (1.1), (1.2) satisfies the following estimates.

(1) *There exists a constant $\delta > 0$ such that*

$$c_\varepsilon(t) \geq m(|A^{1/2}u_0|^2)\cdot e^{-\delta t} \qquad \forall t \geq 0. \quad (3.25)$$

Moreover

$$1 + \varepsilon\frac{c'_\varepsilon(t)}{c_\varepsilon(t)} \geq 0 \qquad \forall t \geq 0. \quad (3.26)$$

(2) *The energies defined by (3.22), (3.23), (3.24) satisfy the following estimates (for $k = 0, 1$):*

$$\frac{|A^{k/2}u_\varepsilon(t)|^2}{4}+\int_0^t c_\varepsilon(s)|A^{(k+1)/2}u_\varepsilon(s)|^2\,ds \leq D_{\varepsilon,k}(0)+2\varepsilon\mu_2 E_{\varepsilon,k}(0) \quad \forall t\geq 0, \tag{3.27}$$

$$E_{\varepsilon,k}(t)+\int_0^t \frac{|A^{k/2}u'_\varepsilon(s)|^2}{c_\varepsilon(s)}\,ds \leq E_{\varepsilon,k}(0) \qquad \forall t\geq 0, \tag{3.28}$$

$$G_\varepsilon(t) \leq \max\{G_\varepsilon(0), 4E_{\varepsilon,1}(0)\} \qquad \forall t\geq 0. \tag{3.29}$$

If in addition $(u_0, u_1) \in D(A^{3/2}) \times D(A)$ then (3.27) and (3.28) hold true also with $k = 2$.

(3) *Let $\psi : [\sigma_1, \sigma_2] \to [0, +\infty)$ be any strictly increasing function of class C^1 such that $\psi(\sigma) \leq \sigma m(\sigma^2)$ for every $\sigma \in [\sigma_1, \sigma_2]$.*

Then $\sigma_1 \leq E_{\varepsilon,0}(t) \leq \sigma_2$ for every $t \geq 0$, and there exists a constant C, independent on ε, such that

$$t\psi\left(E_{\varepsilon,0}(t)\right) \leq C \qquad \forall t\geq 0. \tag{3.30}$$

As a consequence of Lemma 3.4 we have the following estimates, which we need in the proof of the main result.

Lemma 3.5. *Let A be a nonnegative operator, and let $m \in C^1([0, +\infty); [0, +\infty))$. Let us assume that $(u_0, u_1) \in D(A) \times D(A^{1/2})$ satisfy the nondegeneracy condition (1.6). Let ε_1 be as in Lemma 3.4.*

Then we have the following estimates.

(1) Uniform bounds. *There exist constants $h_0, \ldots, h_6$ such that for every $\varepsilon \in (0, \varepsilon_1)$ we have that*

$$\begin{aligned} |A^{k/2}u_\varepsilon(t)| &\leq h_k \qquad \forall t\geq 0, \quad \forall k\in\{0,1,2\}, &(3.31)\\ |u'_\varepsilon(t)| &\leq h_4 \qquad \forall t\geq 0, &(3.32)\\ \sqrt{\varepsilon}|A^{1/2}u'_\varepsilon(t)| &\leq h_5 \qquad \forall t\geq 0, &(3.33)\\ \int_0^{+\infty}|c'_\varepsilon(s)|\,ds &\leq h_6. &(3.34)\end{aligned}$$

If in addition $(u_0, u_1) \in D(A^{3/2}) \times D(A)$ then (3.31) holds true also with $k = 3$.

(2) Uniform decay. *There exists a function $\gamma : [0, +\infty) \to [\sigma_1, \sigma_2]$ such that $\gamma(t) \to \sigma_1$ as $t \to +\infty$, and for every $\varepsilon \in (0, \varepsilon_1)$ we have that*

$$\sigma_1 \leq |A^{1/2}u_\varepsilon(t)|^2 \leq \gamma(t) \qquad \forall t\geq 0. \tag{3.35}$$

Proof. Applying (3.27) with $k = 0$ we have that

$$|u_\varepsilon(t)|^2 \leq 4D_{\varepsilon,0}(0)+8\varepsilon\mu_2 E_{\varepsilon,0}(0) \leq 2|u_0|^2+4\varepsilon_1|u_0|\cdot|u_1|+8\varepsilon_1\mu_2 E_{\varepsilon_1,0}(0) =: h_0^2.$$

By (3.28) applied with $k = 0, 1$ we have that

$$|A^{(k+1)/2}u_\varepsilon(t)|^2 \leq E_{\varepsilon,k}(t) \leq E_{\varepsilon,k}(0) \leq E_{\varepsilon_1,k}(0) =: h_{k+1}^2.$$

If in addition $(u_0, u_1) \in D(A^{3/2}) \times D(A)$, then we have (3.28) with $k = 2$, from which we easily deduce inequality (3.31) with $k = 3$. This completes the proof of (3.31).

From (3.29) we have that

$$|u'_\varepsilon(t)|^2 = c_\varepsilon^2(t)G_\varepsilon(t) \leq \mu_2^2\cdot\max\{G_{\varepsilon_1}(0), 4E_{\varepsilon_1,1}(0)\} =: h_4^2,$$

which proves (3.32).

From (3.28) with $k = 1$ we have that

$$\varepsilon|A^{1/2}u_\varepsilon'(t)|^2 \le c_\varepsilon^2(t)E_{\varepsilon,1}(t) \le \mu_2^2 E_{\varepsilon,1}(0) \le \mu_2^2 E_{\varepsilon_1,1}(0) =: h_5^2,$$

which proves (3.33).

Now we have that

$$\begin{aligned} |c_\varepsilon'(t)| &= \left| m'(|A^{1/2}u_\varepsilon(t)|^2) \cdot 2\langle Au_\varepsilon(t), u_\varepsilon'(t)\rangle \right| \le 2L|u_\varepsilon'(t)| \cdot |Au_\varepsilon(t)| \\ &\le L\left(\frac{|u_\varepsilon'(t)|^2}{c_\varepsilon(t)} + c_\varepsilon(t)|Au_\varepsilon(t)|^2\right). \end{aligned} \tag{3.36}$$

The integral in $[0, +\infty)$ of the first summand in (3.36) is finite because of (3.28) applied with $k = 0$. The integral in $[0, +\infty)$ of the second summand in (3.36) is finite due to (3.27) applied with $k = 1$. This proves (3.34).

From statement (3) of Lemma 3.4 we have that

$$\psi(|A^{1/2}u_\varepsilon(t)|^2) \le \psi(E_{\varepsilon,0}(t)) \le \min\left\{\psi(\sigma_2), Ct^{-1}\right\}$$

for every $t > 0$. Applying ψ^{-1} to both sides we have that

$$|A^{1/2}u_\varepsilon(t)|^2 \le \psi^{-1}\left(\min\left\{\psi(\sigma_2), Ct^{-1}\right\}\right) =: \gamma(t),$$

which proves (3.35). Since $\psi(\sigma) = 0$ if and only if $\sigma = \sigma_1$, it is easy to see that $\gamma(t) \to \sigma_1$ as $t \to +\infty$. □

Remark 3.6. All the conclusions of Lemma 3.4 and Lemma 3.5 are true also if we replace the fixed initial condition (1.2) with a family of initial conditions as in Remark 2.2. The reason is that all the constants appearing in those lemmata (including ε_0 and ε_1) depend in a continuous way on the norms $|u_0|$, $|A^{1/2}u_0|$, $|Au_0|$, $|u_1|$, $\sqrt{\varepsilon}|A^{1/2}u_1|$ (and also on $|A^{3/2}u_0|$ and $\sqrt{\varepsilon}|Au_1|$ when needed).

3.4. **Proof of Theorem 2.1.** The proof is divided into four parts. In the first part we prove (2.2), (2.3), (2.4) under the additional assumption that $\sigma m(\sigma^2)$ is nondecreasing in the whole interval $[\sigma_1, \sigma_2]$. In the second part we extend the estimates of the first part to families of initial data $(u_{0\varepsilon}, u_{1\varepsilon})$ as in Remark 2.2. In the third part we prove (2.2), (2.3), (2.4) under the original assumption that $\sigma m(\sigma^2)$ is nondecreasing in $[\sigma_1, \sigma_1 + \delta]$ for a given $\delta > 0$. Finally, in the fourth part we prove (2.5), (2.6), (2.7).

In this proof we always consider $\varepsilon \in (0, \varepsilon_1)$, where ε_1 is given by Lemma 3.4. We also consider r_ε defined by (1.9), and $\rho_\varepsilon(t) := u_\varepsilon(t) - u(t) = r_\varepsilon(t) + \theta_\varepsilon(t)$. Simple calculations show that r_ε is the solution of the Cauchy problem

$$\begin{aligned} &\varepsilon r_\varepsilon''(t) + r_\varepsilon'(t) + c_\varepsilon(t)A\rho_\varepsilon(t) = (c(t) - c_\varepsilon(t))Au(t) - \varepsilon u''(t) \\ &r_\varepsilon(0) = 0, \qquad r_\varepsilon'(0) = 0, \end{aligned}$$

while ρ_ε is the solution of the Cauchy problem

$$\begin{aligned} &\varepsilon\rho_\varepsilon''(t) + \rho_\varepsilon'(t) + c_\varepsilon(t)Au_\varepsilon(t) - c(t)Au(t) = -\varepsilon u''(t), \\ &\rho_\varepsilon(0) = 0, \qquad \rho_\varepsilon'(0) = w_0, \end{aligned} \tag{3.37}$$

where w_0 is defined in (1.8). Moreover we have that

$$\theta_\varepsilon(t) = \varepsilon w_0(1 - e^{-t/\varepsilon}). \tag{3.38}$$

Throughout the proof we introduce constants $\gamma_0, \ldots, \gamma_{10}$, all independent on ε.

Let us begin with some basic estimates. Applying (3.7) and (3.31) with $k = 0$ and $k = 2$ we have that

$$|\rho_\varepsilon(t)| \le |u_\varepsilon(t)| + |u(t)| \le \gamma_0 \qquad \forall t \ge 0, \tag{3.39}$$

$$|A\rho_\varepsilon(t)| \le |Au_\varepsilon(t)| + |Au(t)| \le \gamma_1 \qquad \forall t \ge 0. \tag{3.40}$$

Thanks to (3.40), every estimate on ρ_ε yields a corresponding estimate on $A^{1/2}\rho_\varepsilon$, as follows:

$$|A^{1/2}\rho_\varepsilon(t)|^2 = \langle A\rho_\varepsilon(t), \rho_\varepsilon(t)\rangle \le |A\rho_\varepsilon(t)| \cdot |\rho_\varepsilon(t)| \le \gamma_1 |\rho_\varepsilon(t)|. \tag{3.41}$$

3.4.1. *Proof with global monotonicity.* In this part of the proof we assume that $\sigma m(\sigma^2)$ is nondecreasing in $[\sigma_1, \sigma_2]$. Due to the explicit expression (3.38) for $\theta_\varepsilon(t)$, it is equivalent to prove estimates (2.2), (2.3), (2.4) for r_ε or for ρ_ε. So we work with ρ_ε, and we exploit a technique introduced in [6]: we prove (bootstrap argument) that any estimate on ρ_ε leads to a (possibly better) estimate for ρ_ε and then, applying inductively the bootstrap argument, we start from (3.39) and we end up with (2.2).
Bootstrap argument. We show that there exist nonnegative constants M_1, M_2, M_3, M_4 with the following property: if for some $K \ge 0$ and $\alpha \in [0, 1]$ we have that

$$|\rho_\varepsilon(t)| \le K\varepsilon^\alpha \qquad \forall t \ge 0, \tag{3.42}$$

then we have also the following two estimates:

$$\int_0^{+\infty} |\rho_\varepsilon'(s)|^2\, ds \le M_1\varepsilon + M_2 K\varepsilon^\alpha, \tag{3.43}$$

$$|\rho_\varepsilon(t)| \le (M_3 + M_4 K)^{1/2}\, \varepsilon^{(\alpha+1)/2} \qquad \forall t \ge 0. \tag{3.44}$$

Let us consider indeed

$$\mathcal{E}_\varepsilon := \varepsilon|\rho_\varepsilon'|^2 + 2\langle \rho_\varepsilon, c_\varepsilon A u_\varepsilon - cAu\rangle.$$

The second summand in the definition of $\mathcal{E}_\varepsilon$ is nonnegative because of the monotonicity assumption (3.4). Moreover from (3.37) it follows that

$$\mathcal{E}_\varepsilon' = -2|\rho_\varepsilon'|^2 - 2\varepsilon\langle \rho_\varepsilon', u''\rangle + 2\langle \rho_\varepsilon, c_\varepsilon' A u_\varepsilon - c' Au\rangle + 2\langle \rho_\varepsilon, c_\varepsilon A u_\varepsilon' - cAu'\rangle. \tag{3.45}$$

Let us estimate the terms in the right hand side of (3.45). First of all we have that

$$-2\varepsilon\langle \rho_\varepsilon', u''\rangle \le 2|\rho_\varepsilon'| \cdot \varepsilon|u''| \le |\rho_\varepsilon'|^2 + \varepsilon^2|u''|^2.$$

Applying (3.7) and (3.31) with $k = 2$, and assumption (3.42), we have that

$$2\langle \rho_\varepsilon, c_\varepsilon' A u_\varepsilon - c' Au\rangle \le 2|\rho_\varepsilon| \left(|c_\varepsilon'||Au_\varepsilon| + |c'||Au|\right) \le 2K\varepsilon^\alpha \left(h_2|c_\varepsilon'| + |Au_0| \cdot |c'|\right).$$

In order to estimate the last summand in (3.45) we write it in the form

$$2\langle \rho_\varepsilon, c_\varepsilon A u_\varepsilon' - cAu'\rangle = 2c_\varepsilon\langle \rho_\varepsilon, Au_\varepsilon' - Au'\rangle + 2(c_\varepsilon - c)\langle \rho_\varepsilon, Au'\rangle. \tag{3.46}$$

The first term in (3.46) can be rewritten as

$$2c_\varepsilon\langle \rho_\varepsilon, Au_\varepsilon' - Au'\rangle = 2c_\varepsilon\langle A^{1/2}\rho_\varepsilon, A^{1/2}\rho_\varepsilon'\rangle = \left(c_\varepsilon|A^{1/2}\rho_\varepsilon|^2\right)' - c_\varepsilon'|A^{1/2}\rho_\varepsilon|^2,$$

hence by (3.41) and (3.42) we have that

$$2c_\varepsilon\langle \rho_\varepsilon, Au_\varepsilon' - Au'\rangle \le \left(c_\varepsilon|A^{1/2}\rho_\varepsilon|^2\right)' + \gamma_1 K\varepsilon^\alpha|c_\varepsilon'|.$$

Let us estimate the second term in (3.46). First of all we have that

$$\begin{aligned} |c_\varepsilon(t) - c(t)| &= \left| m(|A^{1/2}u_\varepsilon|^2) - m(|A^{1/2}u|^2) \right| \\ &\leq L \left| |A^{1/2}u_\varepsilon|^2 - |A^{1/2}u|^2 \right| \\ &= L \left| \langle A^{1/2}(u_\varepsilon - u), A^{1/2}(u_\varepsilon + u) \rangle \right| \\ &\leq L|A^{1/2}\rho_\varepsilon| \left(|A^{1/2}u_\varepsilon| + |A^{1/2}u| \right). \end{aligned} \tag{3.47}$$

Moreover from (3.10) we have that

$$\left| \langle \rho_\varepsilon, Au' \rangle \right| = \left| \langle A^{1/2}\rho_\varepsilon, A^{1/2}u' \rangle \right| \leq |A^{1/2}\rho_\varepsilon| \cdot c(t)|A^{3/2}u| \leq p_0 c(t)|A^{1/2}\rho_\varepsilon| \cdot |A^{1/2}u|,$$

hence by (3.41) and (3.42) we obtain that

$$\begin{aligned} \left| 2(c_\varepsilon - c)\langle \rho_\varepsilon, Au' \rangle \right| &\leq 2Lp_0|A^{1/2}\rho_\varepsilon|^2 c(t)|A^{1/2}u| \left(|A^{1/2}u_\varepsilon| + |A^{1/2}u| \right) \\ &\leq 2L\gamma_1 p_0 \cdot K\varepsilon^\alpha \cdot c(t)|A^{1/2}u| \left(|A^{1/2}u_\varepsilon| + |A^{1/2}u| \right). \end{aligned}$$

Replacing all these estimates in (3.45) we have that

$$\mathcal{E}_\varepsilon' \leq -|\rho_\varepsilon'|^2 + \varepsilon^2|u''|^2 + \left(c_\varepsilon|A^{1/2}\rho_\varepsilon|^2 \right)' + K\varepsilon^\alpha g_\varepsilon, \tag{3.48}$$

where

$$g_\varepsilon(t) \leq \gamma_2|c_\varepsilon'(t)| + \gamma_3|c'(t)| + \gamma_4 c(t)|A^{1/2}u(t)| \left(|A^{1/2}u_\varepsilon(t)| + |A^{1/2}u(t)| \right).$$

Integrating in $[0, t]$ we obtain that

$$\mathcal{E}_\varepsilon(t) + \int_0^t |\rho_\varepsilon'(s)|^2\,ds \leq \mathcal{E}_\varepsilon(0) + \varepsilon^2 \int_0^t |u''(s)|^2\,ds + c_\varepsilon(t)|A^{1/2}\rho_\varepsilon(t)|^2 + K\varepsilon^\alpha \int_0^t g_\varepsilon(s)\,ds.$$

Using (3.41) and (3.42) once again we easily deduce that

$$\int_0^{+\infty} |\rho_\varepsilon'(s)|^2\,ds \leq \varepsilon \left(|w_0|^2 + \varepsilon_1 \int_0^{+\infty} |u''(s)|^2\,ds \right) + K\varepsilon^\alpha \left(\mu_2\gamma_1 + \int_0^{+\infty} g_\varepsilon(s)\,ds \right).$$

The coefficient of ε is finite because of (3.11). In order to prove (3.43) it is therefore enough to prove that the integral of $g_\varepsilon(t)$ in $[0, +\infty)$ is finite and bounded independently on ε.

By (3.9) and (3.34) this is true for the terms involving $|c_\varepsilon'(t)|$ and $|c'(t)|$. The integral of $c(t)|A^{1/2}u(t)|^2$ in $[0, +\infty)$ is finite because of (3.8) applied with $k = 0$. Finally, by inequality (3.6) applied with $a = |A^{1/2}u_\varepsilon|$ and $b = |A^{1/2}u|$ we have that

$$c(t)|A^{1/2}u(t)| \cdot |A^{1/2}u_\varepsilon(t)| \leq c(t)|A^{1/2}u(t)|^2 + c_\varepsilon(t)|A^{1/2}u_\varepsilon(t)|^2.$$

The integral in $[0, +\infty)$ of the right hand side is finite and independent on ε because of (3.8) and (3.27) applied with $k = 0$. Therefore the same is true for the left hand side. This completes the proof of (3.43).

In order to prove (3.44) we consider

$$\mathcal{D}_\varepsilon := \frac{1}{2}|\rho_\varepsilon|^2 + \varepsilon\langle \rho_\varepsilon, \rho_\varepsilon' \rangle. \tag{3.49}$$

A simple computation proves that

$$\mathcal{D}_\varepsilon' = \langle \rho_\varepsilon, \rho_\varepsilon' \rangle + \varepsilon|\rho_\varepsilon'|^2 + \langle \rho_\varepsilon, \varepsilon\rho_\varepsilon'' \rangle = \varepsilon|\rho_\varepsilon'|^2 - \langle \rho_\varepsilon, c_\varepsilon Au_\varepsilon - cAu \rangle - \varepsilon\langle \rho_\varepsilon, u'' \rangle. \tag{3.50}$$

The second term in the right hand side is less or equal than zero because of the monotonicity assumption (3.4). Therefore by (3.42) we have that

$$\mathcal{D}_\varepsilon' \leq \varepsilon|\rho_\varepsilon'|^2 + \varepsilon|\rho_\varepsilon| \cdot |u''| \leq \varepsilon|\rho_\varepsilon'|^2 + K\varepsilon^{\alpha+1}|u''|. \tag{3.51}$$

Integrating in $[0,t]$, and using (3.43) and (3.12), we obtain that

$$\mathcal{D}_\varepsilon(t) \leq \varepsilon \int_0^t |\rho_\varepsilon'(s)|^2\,ds + K\varepsilon^{\alpha+1}\int_0^t |u''(s)|\,ds \leq M_1\varepsilon^2 + M_2 K\varepsilon^{\alpha+1} + \gamma_5 K\varepsilon^{\alpha+1}$$

for a suitable constant γ_5. Therefore

$$\begin{aligned} \frac{1}{2}|\rho_\varepsilon(t)|^2 &\leq M_1\varepsilon^2 + (M_2+\gamma_5)K\varepsilon^{\alpha+1} - \varepsilon\langle\rho_\varepsilon(t), \rho_\varepsilon'(t)\rangle \\ &\leq M_1\varepsilon^2 + (M_2+\gamma_5)K\varepsilon^{\alpha+1} + \frac{1}{4}|\rho_\varepsilon(t)|^2 + \varepsilon^2|\rho_\varepsilon'(t)|^2, \end{aligned}$$

hence

$$|\rho_\varepsilon(t)|^2 \leq 4\varepsilon^2(M_1 + |\rho_\varepsilon'(t)|^2) + 4(M_2+\gamma_5)K\varepsilon^{\alpha+1}. \tag{3.52}$$

It remains to estimate $\rho_\varepsilon'(t)$. This can be easily done using (3.32) and (3.7) with $k = 2$: we obtain that

$$|\rho_\varepsilon'(t)| \leq |u_\varepsilon'(t)| + |u'(t)| = |u_\varepsilon'(t)| + c(t)|Au(t)| \leq h_4 + \mu_2|Au_0| =: \gamma_6. \tag{3.53}$$

Coming back to (3.52) we have that

$$|\rho_\varepsilon(t)|^2 \leq 4\varepsilon^{\alpha+1}\left\{(M_1+\gamma_6)\varepsilon_1^{1-\alpha} + (M_2+\gamma_5)K\right\} =: \varepsilon^{\alpha+1}(M_3 + M_4 K),$$

which proves (3.44).

Iteration argument. Let us consider the sequences α_n and k_n recursively defined by

$$\alpha_0 = 0, \qquad \alpha_{n+1} = (\alpha_n + 1)/2,$$
$$k_0 = \gamma_0, \qquad k_{n+1} = \sqrt{M_3 + M_4 k_n}.$$

Then for every $n \in \mathbb{N}$ we have that

$$|\rho_\varepsilon(t)| \leq k_n\varepsilon^{\alpha_n} \qquad \forall t \geq 0. \tag{3.54}$$

Indeed for $n = 0$ this estimate is exactly (3.39), and then (3.54) follows by induction due to the bootstrap argument (note that $\alpha_n < 1$ for every $n \in \mathbb{N}$).

As $n \to +\infty$ we have that $\alpha_n \to 1$ and $k_n \to k_\infty$, where k_∞ is the unique real number such that $k_\infty = (M_3 + M_4 k_\infty)^{1/2}$.

Passing to the limit in (3.54) we finally obtain that

$$|\rho_\varepsilon(t)| \leq k_\infty\varepsilon \qquad \forall t \geq 0, \tag{3.55}$$

which implies (2.2). At this point (2.3) immediately follows from (3.55) and (3.41). Finally, if we apply the bootstrap argument starting from (3.55), we obtain (3.43) with $\alpha = 1$ and $K = k_\infty$, which proves (2.4).

3.4.2. *Proof with a family of initial data.* We prove that the conclusion of the first part of the proof holds true also if we replace the fixed initial data (u_0, u_1) for the second order problem with a family of initial data $(u_{0\varepsilon}, u_{1\varepsilon})$ satisfying (2.8) and (2.9).

The initial data for $\rho_\varepsilon(t)$ are now

$$\rho_\varepsilon(0) = u_{0\varepsilon} - u_0, \qquad \rho_\varepsilon'(0) = u_{1\varepsilon} + m(|A^{1/2}u_0|^2)Au_0 =: w_{0\varepsilon}.$$

Due to (2.8), (2.9), and Remark 3.6, all the estimates based on Lemma 3.5 remain true. So the proof of the bootstrap argument doesn't change up to the integration of (3.48). Now in the computation of $\mathcal{E}_\varepsilon(0)$ we have to keep into account that

$\rho_\varepsilon(0) \neq 0$. However assumption (2.8) implies that $\mathcal{E}_\varepsilon(0)$ is of order ε, exactly as in the case of fixed initial data.

Finally, when integrating (3.51) we cannot ignore $\mathcal{D}_\varepsilon(0)$. However assumption (2.8) implies that $\mathcal{D}_\varepsilon(0)$ is of order ε^2, and therefore nothing changes in the bootstrap argument apart from the values of the constants. The iteration argument is exactly the same.

3.4.3. *Proof with local monotonicity.* Let us come back to the initial assumption that $\sigma m(\sigma^2)$ is nondecreasing in $[\sigma_1, \sigma_1 + \delta]$ for some $\delta > 0$.

Let us consider the decay estimate (3.35), and let $T_0 > 0$ be such that $\gamma(t) \leq \sigma_1 + \delta$ for every $t \geq T_0$. This means that for every $\varepsilon \in (0, \varepsilon_1)$ we have that

$$\sigma_1 \leq |A^{1/2}u_\varepsilon(t)|^2 \leq \sigma_1 + \delta \qquad \forall t \geq T_0. \tag{3.56}$$

In the fixed interval $[0, T_0]$ we can apply the local-in-time error estimates (1.13), (1.14), and (1.15) (see Appendix A). It follows that

$$|r_\varepsilon(t)| \leq \gamma_7\varepsilon \qquad \forall t \in [0, T_0], \tag{3.57}$$

$$|A^{1/2}r_\varepsilon(t)| \leq \gamma_7\varepsilon \qquad \forall t \in [0, T_0], \tag{3.58}$$

$$\int_0^{T_0} |r'_\varepsilon(s)|^2\, ds \leq \gamma_7\varepsilon^2, \tag{3.59}$$

where the constant γ_7 depends on T_0 (hence on δ) but is independent on ε.

Now we need similar inequalities for $t \geq T_0$. To this end we consider u_ε and u as solutions of a new singular perturbation problem with "initial" data $u_{0\varepsilon} = u_\varepsilon(T_0)$, $u_{1\varepsilon} = u'_\varepsilon(T_0)$, and of course $u_0 = u(T_0)$ for the first order problem.

By (3.57) applied with $t = T_0$ we have that the family $u_{0\varepsilon}$ satisfies (2.8). Moreover from (3.31), (3.32), and (3.33) we deduce that the family $(u_{0\varepsilon}, u_{1\varepsilon})$ satisfies (2.9). Finally by (3.56) the solutions lie in the monotonicity region of the operator for every $t \geq T_0$.

Therefore from the first and second part of the proof we have that

$$|r_\varepsilon(t)| \leq \gamma_8\varepsilon \qquad \forall t \geq T_0, \tag{3.60}$$

$$|A^{1/2}r_\varepsilon(t)| \leq \gamma_8\sqrt{\varepsilon} \qquad \forall t \geq T_0, \tag{3.61}$$

$$\int_{T_0}^{+\infty} |r'_\varepsilon(s)|^2\, ds \leq \gamma_8\varepsilon. \tag{3.62}$$

Estimates (3.57) through (3.62) are enough to prove (2.2), (2.3), and (2.4).

3.4.4. *Proof of estimates (2.5), (2.6), and (2.7).* If $(u_0, u_1) \in D(A^{3/2}) \times D(A)$ we can apply (3.7) and (3.31) with $k = 3$, and deduce that

$$|A^{3/2}\rho_\varepsilon(t)| \leq |A^{3/2}u_\varepsilon(t)| + |A^{3/2}u(t)| \leq \gamma_9 \qquad \forall t \geq 0. \tag{3.63}$$

At this point (2.5) and (2.6) follow from (2.2) and (3.63) by interpolation (once again it is equivalent to prove (2.5) for r_ε of for ρ_ε).

In order to estimate r'_ε, let us consider the function $\mathcal{G}_\varepsilon := |r'_\varepsilon|^2$. Then we have that

$$\begin{aligned}
\mathcal{G}'_\varepsilon &= -\frac{2}{\varepsilon}|r'_\varepsilon|^2 + \frac{2}{\varepsilon}\langle r'_\varepsilon, (c - c_\varepsilon)Au - \varepsilon u'' - c_\varepsilon A\rho_\varepsilon\rangle \\
&\leq -\frac{2}{\varepsilon}|r'_\varepsilon|^2 + \frac{2}{\varepsilon}|r'_\varepsilon| \cdot \big(|c - c_\varepsilon||Au| + c_\varepsilon|A\rho_\varepsilon|\big) + \frac{2}{\varepsilon}\left(\frac{|r'_\varepsilon|^2}{2} + \frac{\varepsilon^2|u''|^2}{2}\right) \\
&\leq -\frac{1}{\varepsilon}\mathcal{G}_\varepsilon + \frac{2}{\varepsilon}\sqrt{\mathcal{G}_\varepsilon}\,\big(|c - c_\varepsilon||Au| + c_\varepsilon|A\rho_\varepsilon|\big) + \varepsilon|u''|^2.
\end{aligned}$$

By (3.47) we have that

$$|c-c_\varepsilon||Au|+c_\varepsilon|A\rho_\varepsilon| \le L|A^{1/2}\rho_\varepsilon|\left(|A^{1/2}u_\varepsilon|+|A^{1/2}u|\right)|Au|+c_\varepsilon|A\rho_\varepsilon|.$$

Applying (3.7) with $k=1,2$, (3.31) with $k=1$, and estimates (2.5) and (2.6), we obtain that

$$|c-c_\varepsilon||Au|+c_\varepsilon|A\rho_\varepsilon| \le \gamma_{10}\varepsilon^{1/3},$$

hence

$$\mathcal{G}_\varepsilon' \le -\frac{1}{\varepsilon}\mathcal{G}_\varepsilon+\frac{2}{\varepsilon}\gamma_{10}\varepsilon^{1/3}\sqrt{\mathcal{G}_\varepsilon}+\varepsilon|u''|^2.$$

Since $\mathcal{G}_\varepsilon(0)=0$ (and here it is essential that we considered r_ε instead of ρ_ε), from Lemma 3.1 applied with $g(t)=\varepsilon|u''(t)|^2$ we deduce that

$$|r_\varepsilon'(t)|^2=\mathcal{G}_\varepsilon(t)\le 4\gamma_{10}^2\cdot\varepsilon^{2/3}+\varepsilon\int_0^{+\infty}|u''(s)|^2\,ds \qquad \forall t\ge 0.$$

Since the integral is finite (see (3.11)), inequality (2.7) is proved.

The same argument works if we replace the fixed initial data (u_0,u_1) with a family of initial data $(u_{0\varepsilon},u_{1\varepsilon})$ satisfying (2.8), (2.9), and (2.10). □

4. **Open problems.**

It is well known that the main open problem in the theory of Kirchhoff equations is the existence of global solutions without smallness assumptions on ε, both for the dissipative and for the non-dissipative case (see [1, 11, 12, 16]).

In this section we present some "minor" open problems related to the singular perturbation topic.

The first one concerns once again the existence of global solutions to (1.1), (1.2). The classical local existence results in the nondissipative case hold true for initial data $(u_0,u_1)\in D(A^{3/4})\times D(A^{1/4})$. The global existence results for the dissipative case can be easily extended to initial data $(u_0,u_1)\in D(A^{3/4})\times D(A^{1/4})$ provided that the equation is nondegenerate or $m(\sigma)=\sigma^\gamma$ with $\gamma\ge 2$.

On the contrary, the proof given in [5] for a general locally Lipschitz continuous non-linearity $m(\sigma)\ge 0$ seems to require in an essential way that $(u_0,u_1)\in D(A)\times D(A^{1/2})$. So the first open problem is the following.

Open problem 4.1. *Let $m:[0,+\infty)\to[0,+\infty)$ be a locally Lipschitz continuous function. Let us assume that $(u_0,u_1)\in D(A^{3/4})\times D(A^{1/4})$ satisfy the nondegeneracy condition (1.6).*

Does problem (1.1), (1.2) admit a global solution for every small enough ε?

The convergence estimates for the singular perturbation can probably be improved in several directions. For instance, it could be interesting to understand the role of the monotonicity assumption in Theorem 2.1.

Open problem 4.2. *Are the conclusions of Theorem 2.1 true without the assumption that $\sigma m(\sigma^2)$ is nondecreasing in a right neighborhood of σ_1?*

Concerning the converge rate, it could be interesting to replace $\sqrt{\varepsilon}$ with ε in (2.3), or ε with ε^2 in (2.4). We recall that ε is the convergence rate for $|A^{1/2}r_\varepsilon|$ which appears both in the local-in-time estimates for the mildly degenerate case, and in the global-in-time estimates for the nondegenerate case.

Open problem 4.3. *Let* $(u_0, u_1) \in D(A^{3/2}) \times D(A^{1/2})$. *Is it true that*

$$|A^{1/2} r_\varepsilon(t)| \leq C\varepsilon \qquad \forall t \geq 0,$$

for a suitable constant C *independent on* ε *and* t*?*

Finally, it could be interesting to mix singular perturbation and decay estimates, as it was done in [10] in the non-degenerate case. We state a possible question in this direction.

Open problem 4.4. *Let* $(u_0, u_1) \in D(A^{3/2}) \times D(A^{1/2})$. *Is it true that*

$$|A^{1/2} r_\varepsilon(t)| \leq \varepsilon\gamma(t) \qquad \forall t \geq 0,$$

for a suitable function $\gamma(t)$, *independent on* ε, *such that* $\gamma(t) \to 0$ *as* $t \to +\infty$*?*

Concerning the choice of $\gamma(t)$, we suspect that in many cases $|A^{1/2} r_\varepsilon(t)|$, as $t \to +\infty$, may decay faster than $|A^{1/2} u_\varepsilon(t)|$ and $|A^{1/2} u(t)|$ separately (see [10] for the nondegenerate case).

Appendix A. Local-in-time convergence.

Proposition A.1. *Let* A *be a nonnegative operator, and let* $m : [0, +\infty) \to [0, +\infty)$ *be a function of class* C^1. *Let us assume that the initial data* $(u_0, u_1) \in D(A^{3/2}) \times D(A^{1/2})$ *satisfy the nondegeneracy condition (1.6), and let* ε_1 *be as in Lemma 3.4.*

Then for every $T > 0$ *there exists a constant* C_T *such that estimates (1.13), (1.14), and (1.15) hold true for every* $\varepsilon \in (0, \varepsilon_1)$.

Proof. Let $\varepsilon \in (0, \varepsilon_1)$. In the following $\alpha_1, \ldots, \alpha_{13}$ denote some constants, depending on T and on the initial data, but independent on ε. From (3.25) we have that

$$c_\varepsilon(t) \geq \alpha_1 > 0 \qquad \forall t \in [0, T]. \tag{A.1}$$

As in the proof of Theorem 2.1, due to the explicit expression (3.38), it is equivalent to prove estimates (1.13) and (1.14) with r_ε or with $\rho_\varepsilon := r_\varepsilon + \theta_\varepsilon$. Let us consider the function (note that in the definition we use both r_ε and ρ_ε)

$$\mathcal{F}_\varepsilon := \varepsilon \frac{|r_\varepsilon'(t)|^2}{c_\varepsilon} + |A^{1/2} \rho_\varepsilon|^2.$$

Then we have that

$$\begin{aligned} \mathcal{F}_\varepsilon' &= -\frac{|r_\varepsilon'|^2}{c_\varepsilon} \left(2 + \varepsilon \frac{c_\varepsilon'}{c_\varepsilon} \right) + 2\langle \theta_\varepsilon', A\rho_\varepsilon \rangle + \frac{2(c - c_\varepsilon)}{c_\varepsilon} \langle r_\varepsilon', Au \rangle - \frac{2\varepsilon}{c_\varepsilon} \langle r_\varepsilon', u'' \rangle \\ &=: I_1 + I_2 + I_3 + I_4. \end{aligned} \tag{A.2}$$

Let us estimate the four summands. From (3.26) we have that

$$I_1 \leq -\frac{|r_\varepsilon'|^2}{c_\varepsilon}.$$

From (3.38) we have that

$$I_2 = 2\langle A^{1/2} \theta_\varepsilon', A^{1/2} \rho_\varepsilon \rangle \leq 2|A^{1/2} w_0| e^{-t/\varepsilon} \cdot \sup_{s \in [0,T]} |A^{1/2} \rho_\varepsilon(s)|.$$

From (3.47) and (A.1) we have that

$$\begin{aligned} I_3 &\leq 2 \cdot \frac{|r_\varepsilon'|}{\sqrt{c_\varepsilon}} \cdot \frac{|c - c_\varepsilon|}{\sqrt{c_\varepsilon}} |Au| \leq \frac{1}{4} \frac{|r_\varepsilon'|^2}{c_\varepsilon} + 4 \frac{|c - c_\varepsilon|^2}{c_\varepsilon} |Au|^2 \\ &\leq \frac{1}{4} \frac{|r_\varepsilon'|^2}{c_\varepsilon} + \alpha_2 |A^{1/2} \rho_\varepsilon|^2 \leq \frac{1}{4} \frac{|r_\varepsilon'|^2}{c_\varepsilon} + \alpha_2 \mathcal{F}_\varepsilon. \end{aligned}$$

Finally using (A.1) once more we have that

$$I_4 \leq 2 \cdot \frac{|r_\varepsilon'|}{\sqrt{c_\varepsilon}} \cdot \frac{\varepsilon|u''|}{\sqrt{c_\varepsilon}} \leq \frac{1}{4}\frac{|r_\varepsilon'|^2}{c_\varepsilon} + 4\varepsilon^2\frac{|u''|^2}{c_\varepsilon} \leq \frac{1}{4}\frac{|r_\varepsilon'|^2}{c_\varepsilon} + \alpha_3\varepsilon^2|u''|^2.$$

Replacing all these estimates in (A.2) we obtain that

$$\mathcal{F}_\varepsilon' \leq -\frac{|r_\varepsilon'|^2}{2c_\varepsilon} + \alpha_2\mathcal{F}_\varepsilon + \alpha_3\varepsilon^2|u''|^2 + 2e^{-t/\varepsilon}|A^{1/2}w_0| \cdot \sup_{s\in[0,T]}|A^{1/2}\rho_\varepsilon(s)|.$$

This is a differential inequality satisfied by the function $\mathcal{F}_\varepsilon$. Integrating it, and recalling that $\mathcal{F}_\varepsilon(0) = 0$, we obtain that

$$\begin{aligned}
\mathcal{F}_\varepsilon(t) + e^{\alpha_2 t}\int_0^t \frac{|r_\varepsilon'(s)|^2}{2c_\varepsilon(s)}e^{-\alpha_2 s}\,ds \;&\leq\; \alpha_3\varepsilon^2 e^{\alpha_2 t}\int_0^t e^{-\alpha_2 s}|u''(s)|^2 ds + \\
&\quad +2|A^{1/2}w_0| \cdot \sup_{s\in[0,T]}|A^{1/2}\rho_\varepsilon(s)| \cdot e^{\alpha_2 t}\int_0^t e^{-\alpha_2 s - s/\varepsilon}ds \\
&=: \;\; J_1 + J_2.
\end{aligned}$$

By (3.11) and the boundedness of t we have that

$$J_1 \leq \alpha_4\varepsilon^2, \quad J_2 \leq \alpha_5\varepsilon \cdot \sup_{s\in[0,T]}|A^{1/2}\rho_\varepsilon(s)| \leq \alpha_6\varepsilon^2 + \frac{1}{2}\sup_{s\in[0,T]}|A^{1/2}\rho_\varepsilon(s)|^2.$$

We have thus proved that

$$\begin{aligned}
|A^{1/2}\rho_\varepsilon(t)|^2 + \frac{1}{2\mu_2}\int_0^t |r_\varepsilon'(s)|^2\,ds \quad &\leq \quad \mathcal{F}_\varepsilon(t) + e^{\alpha_2 t}\int_0^t \frac{|r_\varepsilon'(s)|^2}{2c_\varepsilon(s)}e^{-\alpha_2 s}\,ds \\
&\leq \quad \alpha_7\varepsilon^2 + \frac{1}{2}\sup_{s\in[0,T]}|A^{1/2}\rho_\varepsilon(s)|^2. \qquad \text{(A.3)}
\end{aligned}$$

In particular

$$\sup_{s\in[0,T]}|A^{1/2}\rho_\varepsilon(s)|^2 \leq \alpha_7\varepsilon^2 + \frac{1}{2}\sup_{s\in[0,T]}|A^{1/2}\rho_\varepsilon(s)|^2,$$

hence $|A^{1/2}\rho_\varepsilon(t)|^2 \leq \alpha_8\varepsilon^2$ for every $t \in [0,T]$, which proves (1.14). Coming back to (A.3) this proves also (1.15).

It remains to prove (1.13). To this end we define $\mathcal{D}_\varepsilon$ as in (3.49), and we estimate the right hand side of (3.50). Using (3.47) and (1.14) we have that

$$|\langle\rho_\varepsilon, c_\varepsilon Au_\varepsilon - cAu\rangle| = \left|\langle A^{1/2}\rho_\varepsilon, (c_\varepsilon - c)A^{1/2}u + c_\varepsilon A^{1/2}\rho_\varepsilon\rangle\right| \leq \alpha_9\varepsilon^2.$$

Moreover

$$-\varepsilon\langle\rho_\varepsilon, u''\rangle \leq \varepsilon|u''(t)| \cdot \sup_{s\in[0,T]}|\rho_\varepsilon(s)|.$$

Integrating (3.50) in $[0,t]$ we therefore obtain that

$$\mathcal{D}_\varepsilon(t) \leq \varepsilon\int_0^t |\rho_\varepsilon'(s)|^2\,ds + \alpha_{10}\varepsilon^2 + \varepsilon \cdot \sup_{s\in[0,T]}|\rho_\varepsilon(s)| \cdot \int_0^T |u''(s)|\,ds. \qquad \text{(A.4)}$$

By (1.15) and (3.38) we have that

$$\varepsilon\int_0^t |\rho_\varepsilon'(s)|^2\,ds \leq 2\varepsilon\int_0^t |r_\varepsilon'(s)|^2\,ds + 2\varepsilon\int_0^t |\theta_\varepsilon'(s)|^2\,ds \leq \alpha_{11}\varepsilon^2,$$

and by (3.12) we have that

$$\sup_{s\in[0,T]} |\rho_\varepsilon(s)| \cdot \varepsilon \int_0^T |u''(s)|\, ds \leq \frac{1}{8} \sup_{s\in[0,T]} |\rho_\varepsilon(s)|^2 + \alpha_{12}\varepsilon^2.$$

Coming back to (A.4) we have thus proved that

$$\begin{aligned} \frac{|\rho_\varepsilon(t)|^2}{2} &\leq -\varepsilon\langle \rho_\varepsilon(t), \rho_\varepsilon'(t)\rangle + \alpha_{13}\varepsilon^2 + \frac{1}{8}\sup_{s\in[0,T]} |\rho_\varepsilon(s)|^2 \\ &\leq \frac{1}{8}|\rho_\varepsilon(t)|^2 + 2\varepsilon^2 |\rho_\varepsilon'(t)|^2 + \alpha_{13}\varepsilon^2 + \frac{1}{8}\sup_{s\in[0,T]} |\rho_\varepsilon(s)|^2. \end{aligned}$$

Taking the supremum in $[0,T]$ of both sides and estimating $|\rho_\varepsilon'(t)|$ as in (3.53), we finally obtain (1.13). □

Remark A.2. The proof of Proposition A.1 is an example of what in the introduction we called a "linear argument". The advantage of this approach is that it can be extended word-by-word to more regular data. For instance, if we assume that $(u_0, u_1) \in D(A^{\alpha+1}) \times D(A^\alpha)$ for some $\alpha \geq 1/2$, then the same argument proves also that

$$\begin{aligned} |A^\alpha r_\varepsilon(t)| &\leq C_T\varepsilon \qquad \forall t \in [0,T], \\ \int_0^T |A^{(2\alpha-1)/2} r_\varepsilon'(s)|^2\, ds &\leq C_T\varepsilon^2. \end{aligned}$$

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E-mail address: ghisi@dm.unipi.it
E-mail address: m.gobbino@dma.unipi.it