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Mildly degenerate Kirchhoff equations with weak dissipation: Global existence and time decay

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ABSTRACT

We consider the hyperbolic–parabolic singular perturbation problem for a *degenerate* quasilinear Kirchhoff equation with *weak* dissipation. This means that the coefficient of the dissipative term tends to zero when $t \to +\infty$.

We prove that the hyperbolic problem has a unique global solution for suitable values of the parameters. We also prove that the solution decays to zero, as $t \to +\infty$, with the same rate of the solution of the limit problem of parabolic type.

1. Introduction

Let H be a real Hilbert space. For every x and y in H, $|x|$ denotes the norm of x, and $\langle x, y\rangle$ denotes the scalar product of x and y. Let A be a self-adjoint linear operator on H with dense domain $D(A)$. We assume that A is nonnegative, namely $\langle Ax, x\rangle \geqslant 0$ for every $x \in D(A)$, so that for every $\alpha \geqslant 0$ the power $A^{\alpha}x$ is defined provided that x lies in a suitable domain $D(A^{\alpha})$.

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We consider the Cauchy problem

$$\varepsilon u_\varepsilon''(t) + \frac{1}{(1+t)^p} u_\varepsilon'(t) + \left|A^{1/2} u_\varepsilon(t)\right|^{2\gamma} A u_\varepsilon(t) = 0 \quad \forall t \geqslant 0, \tag{1.1}$$

$$u_\varepsilon(0) = u_0, \qquad u_\varepsilon'(0) = u_1, \tag{1.2}$$

where $\varepsilon > 0$, $p \geqslant 0$, and $\gamma > 0$. Eq. (1.1) is the prototype of all degenerate Kirchhoff equations with weak dissipation

$$\varepsilon u_\varepsilon''(t) + b(t) u_\varepsilon'(t) + m\left(\left|A^{1/2} u_\varepsilon(t)\right|^2\right) A u_\varepsilon(t) = 0 \quad \forall t \geqslant 0, \tag{1.3}$$

where $b : [0, +\infty) \to (0, +\infty)$ and $m : [0, +\infty) \to [0, +\infty)$ are given functions which are always assumed to be of class C^1 (or at least locally Lipschitz continuous), unless otherwise stated. It is well known that (1.1) is the abstract setting of a quasilinear nonlocal partial differential equation of hyperbolic type which was proposed as a model for small vibrations of strings and membranes.

Eq. (1.3) is called nondegenerate (or strictly hyperbolic) when

$$\mu := \inf_{\sigma \geqslant 0} m(\sigma) > 0,$$

and mildly degenerate when $\mu = 0$ but $m(|A^{1/2} u_0|^2) \neq 0$. In the special case of Eq. (1.1) this assumption reduces to

$$A^{1/2} u_0 \neq 0. \tag{1.4}$$

Concerning the dissipation term $b(t) u_\varepsilon'(t)$, we have constant dissipation when $b(t) \equiv \delta > 0$ is a positive constant, and weak dissipation when $b(t) \to 0$ as $t \to +\infty$. Finally, the operator A is called coercive when

$$\nu := \inf\left\{ \frac{\langle Ax, x\rangle}{|x|^2} : x \in D(A),\ x \neq 0 \right\} > 0, \tag{1.5}$$

and noncoercive when $\nu = 0$.

The singular perturbation problem in its generality consists in proving the convergence of solutions of (1.3), (1.2) to solutions of the first order problem

$$b(t) u_\varepsilon'(t) + m\left(\left|A^{1/2} u_\varepsilon(t)\right|^2\right) A u_\varepsilon(t) = 0, \qquad u(0) = u_0, \tag{1.6}$$

obtained setting formally $\varepsilon = 0$ in (1.3), and omitting the second initial condition in (1.2).

The singular perturbation problem gives rise to several subproblems. The first step is of course the existence of global solutions for the limit problem (1.6). This has been established in [10] under very general assumptions. The second step is the existence of a global solution for the hyperbolic problem (1.3), (1.2). The third step is the convergence of solutions $u_\varepsilon(t)$ of the hyperbolic problem to the solution $u(t)$ of the parabolic problem. The final goal are the so-called error-decay estimates which prove in the same time that the difference $u_\varepsilon(t) - u(t)$ decays to 0 as $t \to +\infty$ (with the same rate of $u(t)$), and tends to 0 as $\varepsilon \to 0^+$.

The second and third steps have generated a considerable literature, which we sum up below.

Nondegenerate Kirchhoff equations with constant dissipation. The first results where obtained in the eighties by E.H. de Brito [2] and Y. Yamada [21]. They independently proved the global solvability of the hyperbolic problem with initial data $(u_0, u_1) \in D(A) \times D(A^{1/2})$ under a suitable assumption involving ε, the initial data, and the constant dissipation δ. Once that δ and the initial data are fixed, this condition holds true provided that ε is small enough. The key step in their proofs, as well as in all the subsequent literature, is to show that solutions satisfy an a priori estimate such as

$$\varepsilon \frac{|m'(|A^{1/2}u_\varepsilon(t)|^2)|}{m(|A^{1/2}u_\varepsilon(t)|^2)} \cdot \left|u'_\varepsilon(t)\right| \cdot \left|Au_\varepsilon(t)\right| \leqslant b(t), \tag{1.7}$$

which is clearly more likely to be true when ε is small enough. Existence of global solutions without the smallness assumption on ε remains a challenging open problem, as well as the nondissipative case $b(t) \equiv 0$.

More recently H. Hashimoto and T. Yamazaki [12] obtained optimal error-decay estimates for the singular perturbation problem. Thanks to these estimates, which improve or extend all previous works (see [3,11]), this case can be considered quite well understood.

Degenerate Kirchhoff equations with constant dissipation. The case where $m(\sigma) = \sigma^\gamma$ (with $\gamma \geqslant 1$) has been studied in the nineties by K. Nishihara and Y. Yamada [16] (see also [20]). The result is the existence of a unique global solution for the mildly degenerate equation provided that ε is small enough. Later on this existence result was extended by the authors [6] to arbitrary locally Lipschitz continuous nonlinearities $m(\sigma) \geqslant 0$, and by the first author [4,5] to non-Lipschitz nonlinearities of the form $m(\sigma) = \sigma^\gamma$ with $\gamma \in (0, 1)$.

All the quoted papers considered also the asymptotic behavior of solutions, but the estimates proved therein where in general far from being optimal. In the meanwhile sharp decay estimates were the subject of a series of papers by T. Mizumachi [13,14] and K. Ono [17,18], in which however only the special case $m(\sigma) = \sigma$ was considered. More recently the authors [7] obtained optimal and ε-independent decay estimates for the general case. As expected the result is that solutions of the hyperbolic problem always decay as the corresponding solutions of the limit problem.

As for the singular perturbation problem, in [8] the authors proved that $u_\varepsilon(t) \to u(t)$ uniformly in time, but without sharp error-decay estimates, which in this case remain an open problem.

From the technical point of view, the difficulty is that in the degenerate case the denominator in (1.7) may vanish. This cannot happen for $t = 0$ due to the mild nondegeneracy assumption, but it does happen in the limit as $t \to +\infty$ due to the decay of solutions.

Nondegenerate Kirchhoff equations with weak dissipation. Let us come to the nondegenerate case with $b(t) = (1 + t)^{-p}$. What complicates things is the competition between the smallness of ε and the smallness of $b(t)$. In particular it is no more enough to prove that the left-hand side of (1.7) is bounded, but it is necessary to prove that it decays with an a priori fixed rate.

This is the reason why this problem has been solved only in recent years in some papers by M. Nakao and J. Bae [15], by T. Yamazaki [22,23], and by the authors [9]. The result is that for every $p \in [0, 1]$, and every $(u_0, u_1) \in D(A) \times D(A^{1/2})$, the hyperbolic problem has a unique global solution provided that ε is small enough. Moreover the solution decays to 0 as $t \to +\infty$ as the solution of the limit problem, and optimal error-decay estimates for the singular perturbation have been proved (see [9,22,23]). When $p > 1$ the existence of global solutions for the hyperbolic problem is still an open problem, but in any case solutions cannot decay to 0 as $t \to +\infty$. On the other hand, solutions of the limit parabolic problem decay to zero also for $p > 1$, faster and faster as p grows.

This means that a threshold appears. When $p \in [0, 1]$ the smallness of ε is dominant over the smallness of $b(t)$, and (1.3) behaves like a parabolic equation. When $p > 1$ the smallness of $b(t)$ is dominant over the smallness of ε, and (1.3) behaves like a nondissipative hyperbolic equation.

Degenerate Kirchhoff equations with weak dissipation. Let us finally come to Eq. (1.1), which is the object of this paper. Now in (1.7) the smallness of ε has to compete both with the decay of $b(t)$, and with the vanishing of the denominator. So one needs a priori decay estimates for terms whose denominator vanishes in the limit.

To our knowledge the only previous result for this equation was obtained at the end of the nineties by K. Ono [19], who proved global existence for the mildly degenerate case when $\gamma = 1$, $p \in [0, 1/3]$, and of course ε is small enough. We recall that $m(\sigma) = \sigma$ is the only case where sharp decay estimates were already available in those years.

In this paper we consider the global solvability of (1.1), (1.2) with more general values of the parameters, and initial data $(u_0, u_1) \in D(A) \times D(A^{1/2})$ satisfying (1.4).

Our first result (Theorem 2.1) concerns the coercive case. Under this assumption we prove that a unique global solution exists provided that $\gamma > 0$, $p \in [0, 1]$ and ε is small enough. We also prove sharp decay estimates as $t \to +\infty$.

Our second result (Theorem 2.2) concerns the noncoercive case. In this case we prove that a unique global solution exists provided that $\gamma \geqslant 1$, $p \in [0, (\gamma^2 + 1)/(\gamma^2 + 2\gamma - 1)]$, and ε is small enough. Note that the supremum of this interval is 1 both when $\gamma = 1$ and when $\gamma \to +\infty$, but it is strictly smaller than 1 for $\gamma > 1$. We also provide decay estimates for solutions.

Finally in both cases we show (Theorem 2.3) that for $p > 1$ solutions of (1.1) (provided that they exist, which remains an open problem) do not decay to 0 as $t \to +\infty$.

From the point of view of global solvability and decay properties these results show that in the coercive case Eq. (1.1) behaves like the nondegenerate one, exhibiting nondissipative hyperbolic behavior for $p > 1$, and parabolic behavior for $p \in [0, 1]$. We point out that this is true also in the non-Lipschitz case $\gamma \in (0, 1)$. In the noncoercive case (with $\gamma \geqslant 1$) we have once again hyperbolic behavior for $p > 1$, and parabolic behavior for $p \in [0, (\gamma^2 + 1)/(\gamma^2 + 2\gamma - 1)]$.

Proofs rely on the techniques introduced in [7] in order to prove sharp decay estimates. When the operator is coercive the decay rate depends only on p and γ. When the operator is noncoercive the decay rate belongs to a range depending on p and γ, but within this range it seems to depend on the initial conditions. The existence of a range of possible decay rates is what in the noncoercive case creates the no-man's land between $(\gamma^2 + 1)/(\gamma^2 + 2\gamma - 1)$ and 1. What happens when p lies in this interval is not clear yet.

In this paper we do not consider the behavior of solutions as $\varepsilon \to 0^+$, even if all our ε-independent estimates are for sure a first step in this direction. We just mention that a simple adaptation of the arguments of [11] and [8] should be enough to prove two types of result: that $u_\varepsilon \to u$ uniformly in time (without estimates of the convergence rate), and that $u_\varepsilon \to u$ in every interval $[0, T]$ with an estimate of the error depending on T. On the other hand, obtaining error-decay estimates analogous to the nondegenerate case seems to be a much more difficult task. Apart from the partial results of [8] this problem is still open in the degenerate case, both with constant and with weak dissipation.

2. Statements

Our first result concerns the global solvability of the hyperbolic problem and decay properties of solutions in the case of coercive operators.

Theorem 2.1. *Let H be a Hilbert space, and let A be a nonnegative self-adjoint* (*unbounded*) *operator with dense domain. Let us assume that A satisfies the coerciveness condition* (1.5)*. Let $\gamma > 0$, and let $p \in [0, 1]$. Let us assume that $(u_0, u_1) \in D(A) \times D(A^{1/2})$ satisfy* (1.4)*.*

Then there exists $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_0)$ problem (1.1), (1.2) *has a unique global solution*

$$u_\varepsilon \in C^2\big([0, +\infty); H\big) \cap C^1\big([0, +\infty); D\big(A^{1/2}\big)\big) \cap C^0\big([0, +\infty); D(A)\big). \tag{2.1}$$

Moreover there exist positive constants C_1 and C_2 such that

$$\frac{C_1}{(1+t)^{(p+1)/\gamma}} \leqslant \big|A^{1/2} u_\varepsilon(t)\big|^2 \leqslant \frac{C_2}{(1+t)^{(p+1)/\gamma}} \quad \forall t \geqslant 0; \tag{2.2}$$

$$\frac{C_1}{(1+t)^{(p+1)/\gamma}} \leqslant \left|Au_\varepsilon(t)\right|^2 \leqslant \frac{C_2}{(1+t)^{(p+1)/\gamma}} \quad \forall t \geqslant 0; \tag{2.3}$$

$$\left|u'_\varepsilon(t)\right|^2 \leqslant \frac{C_2}{(1+t)^{2+(p+1)/\gamma}} \quad \forall t \geqslant 0. \tag{2.4}$$

Our second result is the counterpart of Theorem 2.1 in the case of noncoercive operators.

Theorem 2.2. *Let H be a Hilbert space, and let A be a nonnegative self-adjoint (unbounded) operator with dense domain. Let $\gamma \geqslant 1$, and let*

$$0 \leqslant p \leqslant \frac{\gamma^2+1}{\gamma^2+2\gamma-1}. \tag{2.5}$$

Let us assume that $(u_0, u_1) \in D(A) \times D(A^{1/2})$ satisfy (1.4).

Then there exists $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_0)$ problem (1.1), (1.2) has a unique global solution satisfying (2.1).

Moreover there exist constants C_1 and C_2 such that

$$\frac{C_1}{(1+t)^{(p+1)/\gamma}} \leqslant \left|A^{1/2}u_\varepsilon(t)\right|^2 \leqslant \frac{C_2}{(1+t)^{(p+1)/(\gamma+1)}} \quad \forall t \geqslant 0; \tag{2.6}$$

$$\left|Au_\varepsilon(t)\right|^2 \leqslant \frac{C_2}{(1+t)^{(p+1)/\gamma}} \quad \forall t \geqslant 0; \tag{2.7}$$

$$\left|u'_\varepsilon(t)\right|^2 \leqslant \frac{C_2}{(1+t)^{[2\gamma^2+(1-p)\gamma+p+1]/(\gamma^2+\gamma)}} \quad \forall t \geqslant 0. \tag{2.8}$$

The last result of this paper concerns the case $p > 1$. An analogous result holds true for nondegenerate equations (see [9, Theorem 2.3]).

Theorem 2.3. *Let H and A be as in Theorem 2.2. Let $m : [0, +\infty) \to [0, +\infty)$ be a continuous function. Let $b : [0, +\infty) \to (0, +\infty)$ be a continuous function such that*

$$\int_0^{+\infty} b(s)\,ds < +\infty. \tag{2.9}$$

Let $(u_0, u_1) \in D(A) \times D(A^{1/2})$ be such that

$$|u_1|^2 + \int_0^{|A^{1/2}u_0|^2} m(\sigma)\,d\sigma > 0. \tag{2.10}$$

Let us assume that for some $\varepsilon > 0$ problem (1.3), (1.2) has a global solution u_ε satisfying (2.1).

Then

$$\liminf_{t\to+\infty}\left(\left|u'_\varepsilon(t)\right|^2 + \left|A^{1/2}u_\varepsilon(t)\right|^2\right) > 0. \tag{2.11}$$

Remark 2.4. The constants ε_0, C_1, C_2 given in Theorem 2.1 above may be taken as continuous functions of ν, γ, p, $|u_0|$, $|A^{1/2}u_0|$, $|A^{1/2}u_0|^{-1}$, $|Au_0|$, $|u_1|$, $|A^{1/2}u_1|$. The same is true for the constants ε_0, C_1, C_2 given in Theorem 2.2, apart from the fact that in this case there is no dependence on ν.

Remark 2.5. Our results can be easily extended to more general Kirchhoff equations. For example with the same technique we can deal with nonlinearities $m : [0, +\infty) \to [0, +\infty)$ of class C^1 such that

$$c_1\sigma^\gamma \leqslant m(\sigma) \leqslant c_2\sigma^\gamma, \qquad c_1\sigma^{\gamma-1} \leqslant m'(\sigma) \leqslant c_2\sigma^{\gamma-1}$$

in a right-hand neighborhood of $\sigma = 0$ for suitable positive constants c_1 and c_2. However this generality only complicates proofs without introducing any new idea.

Remark 2.6. In Theorem 2.1 we assume that $\gamma > 0$, while in Theorem 2.2 we assume that $\gamma \geqslant 1$. Some weaker results can be obtained with similar techniques also when the operator is noncoercive and $\gamma > 0$. For example for every $\gamma > 0$ one can prove the global solvability for every $p \in [0, \gamma/(\gamma + 2)]$ (and of course ε small enough). The solution also satisfies (2.6). We sketch the argument in Remark 3.4. Note that when $\gamma \geqslant 1$ the upper bound $\gamma/(\gamma + 2)$ is always less than the upper bound in (2.5).

3. Proofs

Proofs are organized as follows. First of all in 3.1 we state and prove two simple comparison results for ordinary differential equations, which we need several times in the sequel. Then we prove Theorems 2.1 and 2.2. Their proofs have a common part, which we concentrate in 3.2 in the form of an a priori estimate (Proposition 3.3). Then in 3.3 we conclude the proof of Theorem 2.1, and in 3.4 we conclude the proof of Theorem 2.2. Finally, in 3.5 we prove Theorem 2.3.

3.1. Comparison results for ODEs

Numerous variants of the following comparison result have already been used in [4–9].

Lemma 3.1. *Let $T > 0$, let $p \geqslant 0$, and let $f : [0, T] \to [0, +\infty)$ be a function of class C^1. Let us assume that there exist two constants $c_1 > 0$, $c_2 \geqslant 0$ such that*

$$f'(t) \leqslant -\frac{c_1}{(1+t)^p} f(t) + c_2\sqrt{f(t)} \quad \forall t \in [0, T]. \tag{3.1}$$

Then we have that

$$f(t) \leqslant f(0) + \left(\frac{c_2}{c_1}\right)^2 (1+t)^{2p} \quad \forall t \in [0, T]. \tag{3.2}$$

Proof. From (3.1) if follows that

$$f'(t) \leqslant -\frac{c_1}{2(1+t)^p} f(t) + \frac{c_2^2}{2c_1}(1+t)^p,$$

which is equivalent to say that $f(t)$ is a subsolution of the differential equation

$$y'(t) = -\frac{c_1}{2(1+t)^p} y(t) + \frac{c_2^2}{2c_1}(1+t)^p. \tag{3.3}$$

Let $g(t)$ denote the right-hand side of (3.2). Then

$$-\frac{c_1}{2(1+t)^p} g(t) + \frac{c_2^2}{2c_1}(1+t)^p \leqslant 0 \leqslant g'(t),$$

which is equivalent to say that $g(t)$ is a supersolution of (3.3). Since $f(0) \leqslant g(0)$ the conclusion follows from the standard comparison principle between subsolutions and supersolutions. □

The next comparison result looks quite technical, but the basic idea is the following. When $f(t) \equiv 0$ the differential inequalities (3.5) and (3.7) can be explicitly integrated, thus providing estimates for $w(t)$. When $f(t)$ is small enough according to (3.4) estimates of the same order can still be proved. The interested reader is referred to [7, Lemma 4.2] for a similar comparison result.

Lemma 3.2. *Let $p \geqslant 0$, $\gamma > 0$, $\alpha > 0$, and $T > 0$ be real numbers. Let $w : [0, T] \to [0, +\infty)$ be a function of class C^1 with $w(0) > 0$, and let $f : [0, T] \to \mathbb{R}$ be a continuous function.*

Let us assume that

$$\left| \int_0^t (1+s)^p f(s)\, ds \right| \leqslant \min\left\{ \frac{1}{4\gamma[w(0)]^\gamma}, \frac{\alpha}{2(p+1)} \right\} (1+t)^{p+1} \quad \forall t \in [0, T]. \tag{3.4}$$

Then we have the following implications.

(1) *If w satisfies the differential inequality*

$$w'(t) \leqslant -2(1+t)^p \big[w(t)\big]^{1+\gamma} \big(\alpha + f(t)\big) \quad \forall t \in [0, T], \tag{3.5}$$

then we have the following estimate

$$w(t) \leqslant w(0) \left[\max\left\{ 2, \frac{p+1}{\alpha\gamma[w(0)]^\gamma} \right\} \right]^{1/\gamma} \cdot \frac{1}{(1+t)^{(p+1)/\gamma}} \quad \forall t \in [0, T]. \tag{3.6}$$

(2) *If w satisfies the differential inequality*

$$w'(t) \geqslant -2(1+t)^p \big[w(t)\big]^{1+\gamma} \big(\alpha + f(t)\big) \quad \forall t \in [0, T], \tag{3.7}$$

then we have the following estimate

$$w(t) \geqslant w(0) \left[1 + \frac{3\alpha\gamma[w(0)]^\gamma}{p+1} \right]^{-1/\gamma} \cdot \frac{1}{(1+t)^{(p+1)/\gamma}} \quad \forall t \in [0, T]. \tag{3.8}$$

Proof. Let $y(t)$ be the solution of the Cauchy problem

$$y'(t) = -2\big[y(t)\big]^{\gamma+1}, \qquad y(0) = w(0).$$

It is easy to see that

$$y(t) = w(0)\big(1 + 2\gamma\big[w(0)\big]^\gamma t\big)^{-1/\gamma} \quad \forall t > -\frac{1}{2\gamma[w(0)]^\gamma}.$$

For every $t \in [0, T]$ let us set

$$\Phi(t) := \frac{\alpha}{p+1}\big[(1+t)^{p+1} - 1\big] + \int_0^t (1+s)^p f(s)\, ds, \qquad z(t) := y\big(\Phi(t)\big).$$

First of all we have to prove that $z(t)$ is well defined. This is true because if we set

$$C := \min\left\{\frac{1}{4\gamma[w(0)]^{\gamma}}, \frac{\alpha}{2(p+1)}\right\},$$

then from (3.4) we have that

$$\Phi(t) \geqslant \left(\frac{\alpha}{p+1} - C\right)\left[(1+t)^{p+1} - 1\right] - C \geqslant -C > -\frac{1}{2\gamma[w(0)]^{\gamma}}.$$

Moreover a simple calculation shows that $z(t)$ is a solution of the Cauchy problem

$$z'(t) = -2(1+t)^{p}\left[z(t)\right]^{\gamma+1}\left(\alpha + f(t)\right) \quad \forall t \in [0, T], \tag{3.9}$$

$$z(0) = w(0). \tag{3.10}$$

Proof of statement (1). Assumption (3.5) is equivalent to say that $w(t)$ is a subsolution of the Cauchy problem (3.9), (3.10). The usual comparison principle implies that $w(t) \leqslant z(t)$. Now we have to estimate $z(t)$. From (3.4) it follows that

$$\begin{aligned}
1 + 2\gamma\left[w(0)\right]^{\gamma}\Phi(t) &\geqslant 1 + 2\gamma\left[w(0)\right]^{\gamma}\left(\left(\frac{\alpha}{p+1} - C\right)\left[(1+t)^{p+1} - 1\right] - C\right) \\
&\geqslant 1 + 2\gamma\left[w(0)\right]^{\gamma}\left(\frac{\alpha}{2(p+1)}\left[(1+t)^{p+1} - 1\right] - \frac{1}{4\gamma[w(0)]^{\gamma}}\right) \\
&= \frac{1}{2} + \frac{\alpha\gamma[w(0)]^{\gamma}}{p+1}\left[(1+t)^{p+1} - 1\right] \\
&\geqslant \min\left\{\frac{1}{2}, \frac{\alpha\gamma[w(0)]^{\gamma}}{p+1}\right\}(1+t)^{p+1},
\end{aligned}$$

where in the last step we exploited the elementary inequality

$$A + B(x-1) \geqslant \min\{A, B\}x \quad \forall A \geqslant 0,\ \forall B \geqslant 0,\ \forall x \geqslant 1.$$

It follows that

$$\begin{aligned}
w(t) \leqslant z(t) &= w(0)\left[1 + 2\gamma\left[w(0)\right]^{\gamma}\Phi(t)\right]^{-1/\gamma} \\
&\leqslant w(0)\left[\max\left\{2, \frac{p+1}{\alpha\gamma[w(0)]^{\gamma}}\right\}\right]^{1/\gamma} \cdot \frac{1}{(1+t)^{(p+1)/\gamma}},
\end{aligned}$$

which is exactly (3.6).

Proof of statement (2). Assumption (3.7) is equivalent to say that $w(t)$ is a supersolution of the Cauchy problem (3.9), (3.10), hence $w(t) \geqslant z(t)$ for every $t \in [0, T]$.

Since

$$\begin{aligned}
1 + 2\gamma\left[w(0)\right]^{\gamma}\Phi(t) &\leqslant (1+t)^{p+1} + 2\gamma\left[w(0)\right]^{\gamma}\left(\frac{\alpha}{p+1} + C\right)(1+t)^{p+1} \\
&\leqslant \left(1 + \frac{3\alpha\gamma[w(0)]^{\gamma}}{p+1}\right)(1+t)^{p+1},
\end{aligned}$$

the conclusion follows as in the previous case. □

3.2. Basic energy estimates

In this section we prove some energy estimates and a lower bound for $|A^{1/2}u_\varepsilon(t)|$. Such estimates do not require the coerciveness of the operator, and they are fundamental both in the proof of Theorem 2.1 and in the proof of Theorem 2.2. They extend to the weakly dissipative equation the estimates stated in [7, Section 3.4] in the case of constant dissipation.

The estimates involve the following energies

$$F_\varepsilon(t) := \varepsilon \frac{|A^{1/2}u_\varepsilon'(t)|^2}{|A^{1/2}u_\varepsilon(t)|^{2\gamma}} + \bigl|Au_\varepsilon(t)\bigr|^2, \tag{3.11}$$

$$P_\varepsilon(t) := \varepsilon \frac{|A^{1/2}u_\varepsilon(t)|^2 |A^{1/2}u_\varepsilon'(t)|^2 - \langle Au_\varepsilon(t), u_\varepsilon'(t)\rangle^2}{|A^{1/2}u_\varepsilon(t)|^{2\gamma+4}} + \frac{|Au_\varepsilon(t)|^2}{|A^{1/2}u_\varepsilon(t)|^2}, \tag{3.12}$$

$$Q_\varepsilon(t) := \frac{|u_\varepsilon'(t)|^2}{|A^{1/2}u_\varepsilon(t)|^{4\gamma+2}}, \tag{3.13}$$

$$R_\varepsilon(t) := \varepsilon \frac{|A^{1/2}u_\varepsilon'(t)|^2}{|A^{1/2}u_\varepsilon(t)|^{2\gamma+2}} + \frac{|Au_\varepsilon(t)|^2}{|A^{1/2}u_\varepsilon(t)|^2}. \tag{3.14}$$

We point out that the first summand in the definition of $P_\varepsilon(t)$ is nonnegative due to the Cauchy–Schwarz inequality.

We state the result in the form of an a priori estimate. We assume that in some interval $[0, S)$ there exists a solution of the hyperbolic problem satisfying a given estimate (see (3.15) below), and we deduce that this solution satisfies several energy inequalities in the same interval. We point out that all constants do not depend on S.

Proposition 3.3. *Let H and A be as in Theorem 2.2. Let $\gamma > 0$, $p \in [0, 1]$, $K > 0$ be real numbers, and let $(u_0, u_1) \in D(A) \times D(A^{1/2})$ satisfy (1.4).*

Then there exist positive constants ε_0, σ_0, σ_1 with the following property. If $\varepsilon \in (0, \varepsilon_0)$, $S > 0$, and

$$u_\varepsilon \in C^2\bigl([0, S); H\bigr) \cap C^1\bigl([0, S); D\bigl(A^{1/2}\bigr)\bigr) \cap C^0\bigl([0, S); D(A)\bigr)$$

is a solution of (1.1), (1.2) such that

$$A^{1/2}u_\varepsilon(t) \neq 0 \quad \textit{and} \quad \frac{|\langle Au_\varepsilon(t), u_\varepsilon'(t)\rangle|}{|A^{1/2}u_\varepsilon(t)|^2} \leqslant \frac{K}{(1+t)^p} \quad \forall t \in [0, S), \tag{3.15}$$

then for every $t \in [0, S)$ we have that

$$F_\varepsilon(t) + \int_0^t \frac{1}{(1+s)^p} \frac{|A^{1/2}u_\varepsilon'(s)|^2}{|A^{1/2}u_\varepsilon(s)|^{2\gamma}}\, ds \leqslant F_\varepsilon(0); \tag{3.16}$$

$$P_\varepsilon(t) \leqslant P_\varepsilon(0); \tag{3.17}$$

$$Q_\varepsilon(t) \leqslant Q_\varepsilon(0) + 4P_\varepsilon(0)(1+t)^{2p}; \tag{3.18}$$

$$(1+t)^{2p} R_\varepsilon(t) + \int_0^t (1+s)^p \frac{|A^{1/2}u_\varepsilon'(s)|^2}{|A^{1/2}u_\varepsilon(s)|^{2\gamma+2}}\, ds$$

$$\leqslant \bigl[R_\varepsilon(0) + 2(K+1)P_\varepsilon(0)\bigr](1+t)^{p+1}; \tag{3.19}$$

$$\left|\int_0^t (1+s)^p \frac{\langle u_\varepsilon''(s), Au_\varepsilon(s)\rangle}{|A^{1/2}u_\varepsilon(s)|^{2\gamma+2}}\,ds\right| \leqslant \sigma_0(1+t)^{p+1}; \tag{3.20}$$

$$\left|A^{1/2}u_\varepsilon(t)\right|^2 \geqslant \frac{\sigma_1}{(1+t)^{(p+1)/\gamma}}. \tag{3.21}$$

Proof. Let us set

$$\sigma_0 := \frac{|\langle u_1, Au_0\rangle|}{|A^{1/2}u_0|^{2\gamma+2}} + \frac{3}{2}\left(\sqrt{P_1(0)Q_1(0)} + 2P_1(0)\right) + (2\gamma+3)\left(R_1(0) + 2(K+1)P_1(0)\right),$$

$$\sigma_1 := \left|A^{1/2}u_0\right|^2\left(1 + \frac{3\gamma P_1(0)|A^{1/2}u_0|^{2\gamma}}{p+1}\right)^{-1/\gamma}. \tag{3.22}$$

Let us choose ε_0 in such a way that

$$4\varepsilon_0 \leqslant 1, \qquad 4\varepsilon_0 K(\gamma+1) \leqslant 1, \qquad \sigma_0\varepsilon_0 \leqslant \min\left\{\frac{1}{4\gamma|A^{1/2}u_0|^{2\gamma}}, \frac{P_1(0)}{2(p+1)}\right\}. \tag{3.23}$$

Proof of (3.16) *through* (3.19). Let us compute the time derivative of the energies (3.11) through (3.14). After some computations we find that

$$F_\varepsilon' = -2\left(\frac{1}{(1+t)^p} + \gamma\varepsilon\frac{\langle u_\varepsilon', Au_\varepsilon\rangle}{|A^{1/2}u_\varepsilon|^2}\right)\frac{|A^{1/2}u_\varepsilon'|^2}{|A^{1/2}u_\varepsilon|^{2\gamma}}, \tag{3.24}$$

$$P_\varepsilon' = -2\left(\frac{1}{(1+t)^p} + (\gamma+2)\varepsilon\frac{\langle u_\varepsilon', Au_\varepsilon\rangle}{|A^{1/2}u_\varepsilon|^2}\right)\frac{|A^{1/2}u_\varepsilon|^2|A^{1/2}u_\varepsilon'|^2 - \langle Au_\varepsilon, u_\varepsilon'\rangle^2}{|A^{1/2}u_\varepsilon|^{2\gamma+4}}, \tag{3.25}$$

$$Q_\varepsilon' = -\frac{2}{\varepsilon}\left(\frac{1}{(1+t)^p} + (2\gamma+1)\varepsilon\frac{\langle u_\varepsilon', Au_\varepsilon\rangle}{|A^{1/2}u_\varepsilon|^2}\right)Q_\varepsilon - \frac{2}{\varepsilon}\frac{\langle u_\varepsilon', Au_\varepsilon\rangle}{|A^{1/2}u_\varepsilon|^{2\gamma+2}}, \tag{3.26}$$

$$R_\varepsilon' = -2\left(\frac{1}{(1+t)^p} + (\gamma+1)\varepsilon\frac{\langle u_\varepsilon', Au_\varepsilon\rangle}{|A^{1/2}u_\varepsilon|^2}\right)\frac{|A^{1/2}u_\varepsilon'|^2}{|A^{1/2}u_\varepsilon|^{2\gamma+2}} - 2\frac{\langle u_\varepsilon', Au_\varepsilon\rangle|Au_\varepsilon|^2}{|A^{1/2}u_\varepsilon|^4}. \tag{3.27}$$

Thanks to assumption (3.15) and the second inequality in (3.23) we have that

$$F_\varepsilon'(t) \leqslant -\frac{1}{(1+t)^p}\frac{|A^{1/2}u_\varepsilon'(t)|^2}{|A^{1/2}u_\varepsilon(t)|^{2\gamma}}; \tag{3.28}$$

$$P_\varepsilon'(t) \leqslant 0; \tag{3.29}$$

$$Q_\varepsilon'(t) \leqslant -\frac{1}{\varepsilon}\frac{1}{(1+t)^p}Q_\varepsilon(t) - \frac{2}{\varepsilon}\frac{\langle u_\varepsilon'(t), Au_\varepsilon(t)\rangle}{|A^{1/2}u_\varepsilon(t)|^{2\gamma+2}}; \tag{3.30}$$

$$R_\varepsilon'(t) \leqslant -\frac{3}{2}\frac{1}{(1+t)^p}\frac{|A^{1/2}u_\varepsilon'(t)|^2}{|A^{1/2}u_\varepsilon(t)|^{2\gamma+2}} - 2\frac{\langle u_\varepsilon'(t), Au_\varepsilon(t)\rangle|Au_\varepsilon(t)|^2}{|A^{1/2}u_\varepsilon(t)|^4}. \tag{3.31}$$

Integrating (3.28) in $[0,t]$ we obtain (3.16).
Conclusion (3.17) trivially follows from (3.29).
From (3.30) we deduce that

$$\begin{aligned} Q_\varepsilon'(t) &\leqslant -\frac{1}{\varepsilon}\frac{1}{(1+t)^p}Q_\varepsilon(t)+\frac{2}{\varepsilon}\frac{|u_\varepsilon'(t)|\cdot|Au_\varepsilon(t)|}{|A^{1/2}u_\varepsilon(t)|^{2\gamma+2}} \\ &\leqslant -\frac{1}{\varepsilon}\frac{1}{(1+t)^p}Q_\varepsilon(t)+\frac{2}{\varepsilon}\sqrt{P_\varepsilon(0)}\sqrt{Q_\varepsilon(t)}. \end{aligned}$$

Therefore applying Lemma 3.1 we obtain (3.18).
From (3.31) we have that

$$\begin{aligned} \left[(1+t)^{2p}R_\varepsilon(t)\right]' &= 2p(1+t)^{2p-1}R_\varepsilon(t)+(1+t)^{2p}R_\varepsilon'(t) \\ &\leqslant 2p(1+t)^{2p-1}\varepsilon\frac{|A^{1/2}u_\varepsilon'|^2}{|A^{1/2}u_\varepsilon|^{2\gamma+2}}+2p(1+t)^{2p-1}\frac{|Au_\varepsilon|^2}{|A^{1/2}u_\varepsilon|^2} \\ &\quad -\frac{3}{2}(1+t)^p\frac{|A^{1/2}u_\varepsilon'|^2}{|A^{1/2}u_\varepsilon|^{2\gamma+2}}-2(1+t)^{2p}\frac{|Au_\varepsilon|^2}{|A^{1/2}u_\varepsilon|^2}\frac{\langle u_\varepsilon', Au_\varepsilon\rangle}{|A^{1/2}u_\varepsilon|^2} \\ &=: I_1(t)+I_2(t)+I_3(t)+I_4(t). \end{aligned}$$

Since $2p-1\leqslant p$ and $2p\varepsilon\leqslant 2\varepsilon_0\leqslant 1/2$, we have that

$$I_1(t)+I_3(t)\leqslant\left(2p\varepsilon-\frac{3}{2}\right)(1+t)^p\frac{|A^{1/2}u_\varepsilon'(t)|^2}{|A^{1/2}u_\varepsilon(t)|^{2\gamma+2}}\leqslant-(1+t)^p\frac{|A^{1/2}u_\varepsilon'(t)|^2}{|A^{1/2}u_\varepsilon(t)|^{2\gamma+2}}.$$

From (3.15), (3.17), and the fact that $2p-1\leqslant p$ we have that

$$I_2(t)+I_4(t)\leqslant 2(K+p)(1+t)^p\frac{|Au_\varepsilon(t)|^2}{|A^{1/2}u_\varepsilon(t)|^2}\leqslant 2(K+1)(1+t)^pP_\varepsilon(0).$$

It follows that

$$\left[(1+t)^{2p}R_\varepsilon(t)\right]'\leqslant-(1+t)^p\frac{|A^{1/2}u_\varepsilon'(t)|^2}{|A^{1/2}u_\varepsilon(t)|^{2\gamma+2}}+2(K+1)P_\varepsilon(0)(1+t)^p.$$

Integrating in $[0,t]$ we obtain (3.19).

Proof of (3.20). Let us consider the following identity

$$\begin{aligned} (1+t)^p\frac{\langle u_\varepsilon'', Au_\varepsilon\rangle}{|A^{1/2}u_\varepsilon|^{2\gamma+2}} &= \left[(1+t)^p\frac{\langle u_\varepsilon', Au_\varepsilon\rangle}{|A^{1/2}u_\varepsilon|^{2\gamma+2}}\right]'-(1+t)^p\frac{|A^{1/2}u_\varepsilon'|^2}{|A^{1/2}u_\varepsilon|^{2\gamma+2}} \\ &\quad +(2\gamma+2)(1+t)^p\frac{\langle u_\varepsilon', Au_\varepsilon\rangle^2}{|A^{1/2}u_\varepsilon|^{2\gamma+4}}-p(1+t)^{p-1}\frac{\langle u_\varepsilon', Au_\varepsilon\rangle}{|A^{1/2}u_\varepsilon|^{2\gamma+2}} \\ &=: J_1(t)+J_2(t)+J_3(t)+J_4(t). \end{aligned} \tag{3.32}$$

In order to estimate the integral of the left-hand side, we estimate the integrals of the four terms in the right-hand side. By (3.17) and (3.18) we have that

$$\begin{aligned} \frac{|\langle u_\varepsilon'(t), Au_\varepsilon(t)\rangle|}{|A^{1/2}u_\varepsilon(t)|^{2\gamma+2}} &\leqslant \frac{|Au_\varepsilon(t)|}{|A^{1/2}u_\varepsilon(t)|}\cdot\frac{|u_\varepsilon'(t)|}{|A^{1/2}u_\varepsilon(t)|^{2\gamma+1}} \\ &\leqslant \sqrt{P_\varepsilon(0)}\sqrt{Q_\varepsilon(0)+4P_\varepsilon(0)(1+t)^{2p}} \\ &\leqslant \left(\sqrt{P_\varepsilon(0)Q_\varepsilon(0)}+2P_\varepsilon(0)\right)(1+t)^p, \end{aligned} \tag{3.33}$$

hence

$$\left|\int_0^t J_1(s)\,ds\right| \leqslant \frac{|\langle u_1, Au_0\rangle|}{|A^{1/2}u_0|^{2\gamma+2}} + (1+t)^{2p}\left(\sqrt{P_\varepsilon(0)Q_\varepsilon(0)} + 2P_\varepsilon(0)\right).$$

The integral of $J_2(t)$ can be easily estimated using (3.19).

As for $J_3(t)$, by Cauchy–Schwarz inequality we have that

$$\frac{\langle u_\varepsilon'(t), Au_\varepsilon(t)\rangle^2}{|A^{1/2}u_\varepsilon(t)|^{2\gamma+4}} \leqslant \frac{|A^{1/2}u_\varepsilon'(t)|^2|A^{1/2}u_\varepsilon(t)|^2}{|A^{1/2}u_\varepsilon(t)|^{2\gamma+4}} = \frac{|A^{1/2}u_\varepsilon'(t)|^2}{|A^{1/2}u_\varepsilon(t)|^{2\gamma+2}},$$

and therefore we reduce once again to (3.19).

Finally, from (3.33) we obtain that

$$\left|\int_0^t J_4(s)\,ds\right| \leqslant \frac{1}{2}\left(\sqrt{P_\varepsilon(0)Q_\varepsilon(0)} + 2P_\varepsilon(0)\right)(1+t)^{2p}.$$

Plugging all these estimates in (3.32), and recalling once again that $1 \leqslant (1+t)^{2p} \leqslant (1+t)^{p+1}$ for every $t \geqslant 0$, we obtain (3.20).

Proof of (3.21). Let us set $w_\varepsilon(t) := |A^{1/2}u_\varepsilon(t)|^2$. Then

$$w_\varepsilon'(t) = -2(1+t)^p\left[w_\varepsilon(t)\right]^{\gamma+1}\left(\frac{|Au_\varepsilon(t)|^2}{|A^{1/2}u_\varepsilon(t)|^2} + \varepsilon\frac{\langle u_\varepsilon''(t), Au_\varepsilon(t)\rangle}{|A^{1/2}u_\varepsilon(t)|^{2\gamma+2}}\right), \tag{3.34}$$

hence by (3.17)

$$w_\varepsilon'(t) \geqslant -2(1+t)^p\left[w_\varepsilon(t)\right]^{\gamma+1}\left(P_1(0) + \varepsilon\frac{\langle u_\varepsilon''(t), Au_\varepsilon(t)\rangle}{|A^{1/2}u_\varepsilon(t)|^{2\gamma+2}}\right).$$

This means that w_ε satisfies a differential inequality of the form (3.7) with

$$\alpha := P_1(0), \qquad f(t) := \varepsilon\frac{\langle u_\varepsilon''(s), Au_\varepsilon(s)\rangle}{|A^{1/2}u_\varepsilon(s)|^{2\gamma+2}}. \tag{3.35}$$

Thanks to (3.20) and the last inequality in (3.23) we have that

$$\left|\int_0^t (1+s)^p f(s)\,ds\right| \leqslant \varepsilon\sigma_0(1+t)^{p+1} \leqslant \min\left\{\frac{1}{4\gamma|A^{1/2}u_0|^{2\gamma}}, \frac{P_1(0)}{2(p+1)}\right\}(1+t)^{p+1},$$

and therefore the function $f(t)$ satisfies assumption (3.4) of Lemma 3.2. From statement (2) of that lemma we obtain (3.21). □

3.3. Proof in the coercive case

Local maximal solutions. Problem (1.1), (1.2) admits a unique local-in-time solution, and this solution can be continued to a solution defined in a maximal interval $[0, T)$, where either $T = +\infty$, or

$$\limsup_{t\to T^-}\left(\left|A^{1/2}u'_\varepsilon(t)\right|^2 + \left|Au_\varepsilon(t)\right|^2\right) = +\infty, \tag{3.36}$$

or

$$\liminf_{t\to T^-}\left|A^{1/2}u_\varepsilon(t)\right|^2 = 0. \tag{3.37}$$

We omit the proof of these standard results. The interested reader is referred to [6] (see also [1]).

Preliminaries and notations. Let ν satisfy (1.5), and let

$$\sigma_2 := \left|A^{1/2}u_0\right|^2\left[\max\left\{2, \frac{p+1}{\nu\gamma|A^{1/2}u_0|^{2\gamma}}\right\}\right]^{1/\gamma}.$$

Let K be such that

$$K > \frac{|\langle Au_0, u_1\rangle|}{|A^{1/2}u_0|^2}, \qquad K > \left(\sqrt{P_1(0)Q_1(0)} + 2P_1(0)\right)\sigma_2^\gamma. \tag{3.38}$$

Starting with this value of K let us define σ_0 and σ_1 as in the proof of Proposition 3.3, and let us choose ε_0 satisfying (3.23), and the further requirement

$$\sigma_0\varepsilon_0 \leqslant \min\left\{\frac{\nu}{2(p+1)}, \frac{1}{4\gamma|A^{1/2}u_0|^{2\gamma}}\right\}. \tag{3.39}$$

Let us finally set

$$S := \sup\left\{\tau \in [0, T):\ A^{1/2}u_\varepsilon(t) \neq 0 \text{ and } \frac{|\langle Au_\varepsilon(t), u'_\varepsilon(t)\rangle|}{|A^{1/2}u_\varepsilon(t)|^2} \leqslant \frac{K}{(1+t)^p}\ \forall t \in [0, \tau]\right\}.$$

From the mild nondegeneracy assumption (1.4) and the first inequality in (3.38) it is easy to see that $S > 0$. Moreover in the interval $[0, S)$ all the conclusions of Proposition 3.3 hold true.

Estimate from above for $|A^{1/2}u_\varepsilon(t)|$. Let us set $w_\varepsilon(t) := |A^{1/2}u_\varepsilon(t)|^2$ as in the proof of Proposition 3.3. Once again $w_\varepsilon(t)$ is a solution of (3.34). Since we are in the coercive case we have that $|Au_\varepsilon(t)|^2 \geqslant \nu|A^{1/2}u_\varepsilon(t)|^2$. Therefore from (3.34) it follows that

$$w'_\varepsilon(t) \leqslant -2(1+t)^p\left[w_\varepsilon(t)\right]^{\gamma+1}\left(\nu + \varepsilon\frac{\langle u''_\varepsilon(t), Au_\varepsilon(t)\rangle}{|A^{1/2}u_\varepsilon(t)|^{2\gamma+2}}\right),$$

which means that w_ε satisfies an inequality of the form (3.5) with $\alpha := \nu$, and $f(t)$ defined as in (3.35). Thanks to (3.20) and (3.39) the function $f(t)$ satisfies assumption (3.4) of Lemma 3.2. From statement (1) of that lemma we obtain that

$$\left|A^{1/2}u_\varepsilon(t)\right|^2 \leqslant \frac{\sigma_2}{(1+t)^{(p+1)/\gamma}} \quad \forall t \in [0, S). \tag{3.40}$$

Global existence. We prove that $S = T = +\infty$. Let us assume by contradiction that $S < T$. By definition of S this means that

$$\text{either} \quad \left|A^{1/2}u_\varepsilon(S)\right|^2 = 0 \quad \text{or} \quad \frac{|\langle Au_\varepsilon(S), u'_\varepsilon(S)\rangle|}{|A^{1/2}u_\varepsilon(S)|^2} = \frac{K}{(1+S)^p}. \tag{3.41}$$

By continuity all the estimates proved so far hold true also for $t = S$. In particular (3.21) rules out the first possibility in (3.41).

From (3.33), (3.40), and the second inequality in (3.38), we have that

$$\begin{aligned}\frac{|\langle Au_\varepsilon(S), u'_\varepsilon(S)\rangle|}{|A^{1/2}u_\varepsilon(S)|^2} &\leqslant \frac{|Au_\varepsilon(S)|}{|A^{1/2}u_\varepsilon(S)|}\cdot\frac{|u'_\varepsilon(S)|}{|A^{1/2}u_\varepsilon(S)|^{2\gamma+1}}\cdot\left|A^{1/2}u_\varepsilon(S)\right|^{2\gamma}\\ &\leqslant \left(\sqrt{P_1(0)Q_1(0)} + 2P_1(0)\right)(1+S)^p\cdot\frac{\sigma_2^\gamma}{(1+S)^{p+1}} < \frac{K}{1+S} \leqslant \frac{K}{(1+S)^p},\end{aligned}$$

which rules out the second possibility in (3.41).

It remains to prove that $T = +\infty$. Let us assume by contradiction that $T < +\infty$. Then the quoted local existence result says that either (3.36) or (3.37) holds true.

On the other hand now we know that (3.21) is satisfied for every $t \in [0, T)$, which rules (3.37) out. Moreover from (3.40) we have that $|A^{1/2}u_\varepsilon(t)|^2$ is uniformly bounded from above in $[0, T)$, hence by (3.16) it follows that also $|A^{1/2}u'_\varepsilon(t)|$ and $|Au_\varepsilon(t)|$ are uniformly bounded from above in $[0, T)$. This rules (3.36) out.

Decay estimates. Let us prove estimates (2.2), (2.3), and (2.4). Now we know that the solution is global, and that all the estimates proved so far hold true for every $t \geqslant 0$.

Therefore (2.2) follows from (3.21) and (3.40). Moreover from (3.17) and the coerciveness assumption (1.5) we have that

$$\nu \leqslant \frac{|Au_\varepsilon(t)|^2}{|A^{1/2}u_\varepsilon(t)|^2} \leqslant P_1(0) \quad \forall t \geqslant 0,$$

hence (2.3) follows from (2.2). Finally, (2.4) follows from (3.18) and (3.40). □

3.4. *Proof in the noncoercive case*

Local maximal solutions. As in the coercive case there exists a unique local-in-time solution which can be continued to a solution defined in a maximal interval $[0, T)$, where either $T = +\infty$, or (3.36) holds true, or (3.37) holds true.

Preliminaries and notations. Let σ_1 be the constant defined in (3.22), let

$$\sigma_3 := 16(\gamma+1)\left(|u_1|^2 + \left|A^{1/2}u_0\right|^{2\gamma+2} + 2|u_0|^2\right),$$

$$\sigma_4 := 2\frac{|A^{1/2}u_1|^2}{|A^{1/2}u_0|^{2\gamma}} + 2|Au_0|^2 + \frac{1}{2}\frac{|\langle Au_0, u_1\rangle|}{|A^{1/2}u_0|^{2\gamma}} + 36\sigma_1^{1-\gamma},$$

and let K be such that

$$K > \frac{|\langle Au_0, u_1\rangle|}{|A^{1/2}u_0|^2}, \qquad K > \left[(1+\gamma)\sigma_3\right]^{(\gamma-1)/(\gamma+1)}\left(\frac{|u_1|}{|A^{1/2}u_0|^{2\gamma}}\sqrt{\sigma_4} + 4\sigma_4\right). \tag{3.42}$$

Starting with this value of K let us define σ_0 as in the proof of Proposition 3.3, and let us choose ε_0 satisfying (3.23) and the further condition

$$16\varepsilon_0 \leqslant 1.$$

As in the coercive case let us finally set

$$S := \sup\left\{\tau \in [0,T):\ A^{1/2}u_\varepsilon(t) \neq 0 \text{ and } \frac{|\langle Au_\varepsilon(t), u'_\varepsilon(t)\rangle|}{|A^{1/2}u_\varepsilon(t)|^2} \leqslant \frac{K}{(1+t)^p}\ \forall t \in [0,\tau]\right\}.$$

From the mild nondegeneracy assumption (1.4), and the first inequality in (3.42), it is easy to see that $S > 0$. Moreover in the interval $[0,S)$ all the conclusions of Proposition 3.3 hold true.

In the following we set

$$\beta = \frac{p+1}{\gamma},$$

and we prove estimates involving the following energies

$$H_\varepsilon(t) := \varepsilon\left|u'_\varepsilon(t)\right|^2 + \frac{1}{\gamma+1}\left|A^{1/2}u_\varepsilon(t)\right|^{2\gamma+2}; \tag{3.43}$$

$$D_\varepsilon(t) := \varepsilon(1+t)^p\left\langle u'_\varepsilon(t), u_\varepsilon(t)\right\rangle + \frac{1}{2}\left(1 - \frac{\varepsilon p}{(1+t)^{1-p}}\right)\left|u_\varepsilon(t)\right|^2; \tag{3.44}$$

$$\widehat{D}_\varepsilon(t) := \varepsilon(1+t)^{2\beta-1}\frac{\langle u'_\varepsilon(t), Au_\varepsilon(t)\rangle}{|A^{1/2}u_\varepsilon(t)|^{2\gamma}}; \tag{3.45}$$

$$G_\varepsilon(t) := (1+t)^\beta\frac{|u'_\varepsilon(t)|^2}{|A^{1/2}u_\varepsilon(t)|^{4\gamma}}. \tag{3.46}$$

All the estimates we present are first claimed in the interval $[0,S)$. At the end of the proof we show that $S = T = +\infty$, thus obtaining that all the estimates actually hold true for every $t \geqslant 0$.

First order estimate. In this section of the proof we show that

$$(1+t)^{p+1}H_\varepsilon(t) + \left|u_\varepsilon(t)\right|^2 + \int_0^t (1+s)\left|u'_\varepsilon(s)\right|^2 ds \leqslant \sigma_3 \quad \forall t \in [0,S). \tag{3.47}$$

To this end we begin by taking the time derivative of (3.44):

$$D'_\varepsilon(t) = -(1+t)^p\left|A^{1/2}u_\varepsilon(t)\right|^{2\gamma+2} + \varepsilon(1+t)^p\left|u'_\varepsilon(t)\right|^2 + \frac{\varepsilon p(1-p)}{2}\frac{|u_\varepsilon(t)|^2}{(1+t)^{2-p}}.$$

Integrating in $[0,t]$ we obtain that

$$\int_0^t (1+s)^p\left|A^{1/2}u_\varepsilon(s)\right|^{2\gamma+2} ds$$

$$= D_\varepsilon(0) - D_\varepsilon(t) + \varepsilon\int_0^t (1+s)^p\left|u'_\varepsilon(s)\right|^2 ds + \frac{\varepsilon p(1-p)}{2}\int_0^t \frac{|u_\varepsilon(s)|^2}{(1+s)^{2-p}} ds. \tag{3.48}$$

From our assumptions on ε and p we have that $2\varepsilon < 1/4$, $2p \leqslant p+1$, $\varepsilon p \leqslant 1/2$. Therefore

$$-D_\varepsilon(t) \leqslant 2\varepsilon^2(1+t)^{2p}\left|u'_\varepsilon(t)\right|^2 + \frac{1}{8}\left|u_\varepsilon(t)\right|^2 + \frac{\varepsilon p}{2(1+t)^{1-p}}\left|u_\varepsilon(t)\right|^2 - \frac{1}{2}\left|u_\varepsilon(t)\right|^2$$
$$\leqslant \frac{1}{4}\varepsilon(1+t)^{p+1}\left|u'_\varepsilon(t)\right|^2 - \frac{1}{8}\left|u_\varepsilon(t)\right|^2.$$

Plugging this estimate in (3.48) we obtain that

$$\frac{1}{8}\left|u_\varepsilon(t)\right|^2 + \int_0^t (1+s)^p\left|A^{1/2}u_\varepsilon(s)\right|^{2\gamma+2}ds$$
$$\leqslant D_\varepsilon(0) + \frac{1}{4}\varepsilon(1+t)^{p+1}\left|u'_\varepsilon(t)\right|^2 + \varepsilon\int_0^t (1+s)\left|u'_\varepsilon(s)\right|^2 ds + \frac{\varepsilon p(1-p)}{2}\int_0^t \frac{|u_\varepsilon(s)|^2}{(1+s)^{2-p}}ds. \tag{3.49}$$

Let us consider now the energy defined in (3.43). A simple calculation gives that

$$\left[(1+t)^{p+1}H_\varepsilon\right]' = -(1+t)\left(2 - \frac{\varepsilon(p+1)}{(1+t)^{1-p}}\right)\left|u'_\varepsilon\right|^2 + \frac{p+1}{\gamma+1}(1+t)^p\left|A^{1/2}u_\varepsilon\right|^{2\gamma+2}.$$

Let us integrate in $[0,t]$. Using (3.49) and rearranging the terms we obtain that

$$(1+t)^{p+1}\left(1 - \frac{1}{4}\frac{p+1}{\gamma+1}\right)\varepsilon\left|u'_\varepsilon(t)\right|^2 + \frac{(1+t)^{p+1}}{\gamma+1}\left|A^{1/2}u_\varepsilon(t)\right|^{2\gamma+2}$$
$$\leqslant H_\varepsilon(0) - \left(2 - \varepsilon(p+1) - \varepsilon\frac{p+1}{\gamma+1}\right)\int_0^t (1+s)\left|u'_\varepsilon(s)\right|^2 ds$$
$$+ \frac{p+1}{\gamma+1}\left(D_\varepsilon(0) - \frac{1}{8}\left|u_\varepsilon(t)\right|^2 + \frac{\varepsilon p(1-p)}{2}\int_0^t \frac{|u_\varepsilon(s)|^2}{(1+s)^{2-p}}ds\right).$$

From the smallness assumptions on ε, and the fact that $(p+1)/(\gamma+1) \leqslant 2$, it follows that

$$\frac{1}{2}(1+t)^{p+1}H_\varepsilon(t) + \int_0^t (1+s)\left|u'_\varepsilon(s)\right|^2 ds + \frac{1}{8}\frac{p+1}{\gamma+1}\left|u_\varepsilon(t)\right|^2$$
$$\leqslant \left(H_\varepsilon(0) + 2\left|D_\varepsilon(0)\right|\right) + \frac{p+1}{\gamma+1}\frac{\varepsilon p(1-p)}{2}\int_0^t \frac{|u_\varepsilon(s)|^2}{(1+s)^{2-p}}ds. \tag{3.50}$$

In particular we have that

$$\left|u_\varepsilon(t)\right|^2 \leqslant \frac{8(\gamma+1)}{p+1}\left(H_\varepsilon(0) + 2\left|D_\varepsilon(0)\right|\right) + 4\varepsilon(1-p)\int_0^t \frac{|u_\varepsilon(s)|^2}{(1+s)^{2-p}}ds,$$

hence by Gronwall's lemma

$$\begin{aligned}\left|u_\varepsilon(t)\right|^2 &\leqslant \frac{8(\gamma+1)}{p+1}\left(H_\varepsilon(0)+2\left|D_\varepsilon(0)\right|\right)\exp\left(4\varepsilon(1-p)\int_0^t \frac{1}{(1+s)^{2-p}}\,ds\right)\\ &\leqslant \frac{8(\gamma+1)}{p+1}\left(H_\varepsilon(0)+2\left|D_\varepsilon(0)\right|\right)\exp(4\varepsilon)\\ &\leqslant \frac{16(\gamma+1)}{p+1}\left(H_\varepsilon(0)+2\left|D_\varepsilon(0)\right|\right).\end{aligned}$$

Integrating in $[0,t]$ we obtain that

$$(1-p)\int_0^t \frac{|u_\varepsilon(s)|^2}{(1+s)^{2-p}}\,ds \leqslant 16\frac{\gamma+1}{p+1}\left(H_\varepsilon(0)+2\left|D_\varepsilon(0)\right|\right).$$

Coming back to (3.50) we have therefore that

$$\begin{aligned}&\frac{1}{2}(1+t)^{p+1}H_\varepsilon(t)+\int_0^t (1+s)\left|u'_\varepsilon(s)\right|^2 ds+\frac{1}{8}\frac{p+1}{\gamma+1}\left|u_\varepsilon(t)\right|^2\\ &\quad\leqslant (1+8p\varepsilon)\left(H_\varepsilon(0)+2\left|D_\varepsilon(0)\right|\right).\end{aligned}\tag{3.51}$$

It remains to estimate the right-hand side. This can be easily done because $8p\varepsilon \leqslant 1$, and

$$\begin{aligned}H_\varepsilon(0)+2\left|D_\varepsilon(0)\right| &\leqslant \varepsilon|u_1|^2+\frac{1}{\gamma+1}\left|A^{1/2}u_0\right|^{2\gamma+2}+2\varepsilon\left|\langle u_1,u_0\rangle\right|+(1-\varepsilon p)|u_0|^2\\ &\leqslant |u_1|^2+\left|A^{1/2}u_0\right|^{2\gamma+2}+2|u_0|^2.\end{aligned}$$

Plugging this estimate in (3.51), and multiplying by $8(\gamma+1)$, we obtain (3.47).

Second order estimate. In this section of the proof we show that

$$(1+t)^\beta F_\varepsilon(t)+\frac{1}{2}\frac{1}{(1+t)^\beta}\int_0^t (1+s)^{2\beta-p}\frac{|A^{1/2}u'_\varepsilon(s)|^2}{|A^{1/2}u_\varepsilon(s)|^{2\gamma}}\,ds \leqslant \sigma_4 \quad \forall t\in[0,S).\tag{3.52}$$

To this end we begin by computing the time derivative of (3.45):

$$\begin{aligned}\widehat{D}'_\varepsilon(t) &= -(1+t)^{2\beta-1}\left|Au_\varepsilon(t)\right|^2+\varepsilon(1+t)^{2\beta-1}\frac{|A^{1/2}u'_\varepsilon(t)|^2}{|Au_\varepsilon(t)|^{2\gamma}}\\ &\quad -2\gamma\varepsilon(1+t)^{2\beta-1}\frac{\langle u'_\varepsilon(t),Au_\varepsilon(t)\rangle^2}{|A^{1/2}u_\varepsilon(t)|^{2\gamma+2}}\\ &\quad -(1+t)^{2\beta-p-1}\left(1-\varepsilon\frac{2\beta-1}{(1+t)^{1-p}}\right)\frac{\langle u'_\varepsilon(t),Au_\varepsilon(t)\rangle}{|A^{1/2}u_\varepsilon(t)|^{2\gamma}}\\ &=: I_1(t)+I_2(t)+I_3(t)+I_4(t).\end{aligned}\tag{3.53}$$

Let us estimate this derivative from above. To this end in $I_2(t)$ we replace the exponent $2\beta - 1$ with the bigger exponent $2\beta - p$. The term $I_3(t)$ is nonpositive and can be neglected. In order to estimate $I_4(t)$ we remark that $0 < \beta \leqslant 2$, hence $0 \leqslant |2\beta - 1| \leqslant 3$. Due to the smallness of ε we have therefore that

$$\left|1 - \varepsilon \frac{2\beta - 1}{(1+t)^{1-p}}\right| \leqslant 1 + \frac{|2\beta - 1|\varepsilon}{(1+t)^{1-p}} \leqslant 1 + 3\varepsilon \leqslant 2,$$

and thus

$$\begin{aligned}\left|I_4(t)\right| &\leqslant 2(1+t)^{2\beta-p-1} \frac{|A^{1/2}u'_\varepsilon(t)|}{|A^{1/2}u_\varepsilon(t)|^{2\gamma-1}} \\ &\leqslant \frac{1}{4\beta}(1+t)^{2\beta-p} \frac{|A^{1/2}u'_\varepsilon(t)|^2}{|A^{1/2}u_\varepsilon(t)|^{2\gamma}} + 4\beta(1+t)^{2\beta-p-2} \frac{1}{|A^{1/2}u_\varepsilon(t)|^{2\gamma-2}}.\end{aligned}$$

Since $\gamma \geqslant 1$ we can estimate the last term using (3.21). After some calculations with the exponents we obtain that

$$\frac{1}{|A^{1/2}u_\varepsilon(t)|^{2\gamma-2}} \leqslant \sigma_1^{1-\gamma}(1+t)^{p+1-\beta}, \tag{3.54}$$

hence

$$\left|I_4(t)\right| \leqslant \frac{1}{4\beta}(1+t)^{2\beta-p} \frac{|A^{1/2}u'_\varepsilon(t)|^2}{|A^{1/2}u_\varepsilon(t)|^{2\gamma}} + 4\beta\sigma_1^{1-\gamma}(1+t)^{\beta-1}.$$

Plugging these estimates in (3.53) we have proved that

$$\begin{aligned}\widehat{D}'_\varepsilon(t) \leqslant{}& -(1+t)^{2\beta-1}\left|Au_\varepsilon(t)\right|^2 + (1+t)^{2\beta-p}\left(\varepsilon + \frac{1}{4\beta}\right)\frac{|A^{1/2}u'_\varepsilon(t)|^2}{|A^{1/2}u_\varepsilon(t)|^{2\gamma}} \\ &+ 4\beta\sigma_1^{1-\gamma}(1+t)^{\beta-1}.\end{aligned}$$

Integrating in $[0, t]$ we obtain that

$$\begin{aligned}\int_0^t (1+s)^{2\beta-1}\left|Au_\varepsilon(s)\right|^2 ds \leqslant{}& \left(\varepsilon + \frac{1}{4\beta}\right)\int_0^t (1+s)^{2\beta-p} \frac{|A^{1/2}u'_\varepsilon(s)|^2}{|A^{1/2}u_\varepsilon(s)|^{2\gamma}}\, ds \\ &+ \widehat{D}_\varepsilon(0) - \widehat{D}_\varepsilon(t) + 4\sigma_1^{1-\gamma}(1+t)^{\beta}.\end{aligned} \tag{3.55}$$

Using (3.54) once more we have that

$$\begin{aligned}-\widehat{D}_\varepsilon(t) &\leqslant \frac{\varepsilon^2}{2}(1+t)^{2\beta} \frac{|A^{1/2}u'_\varepsilon(t)|^2}{|A^{1/2}u_\varepsilon(t)|^{2\gamma}} + \frac{1}{2}(1+t)^{2\beta-2} \frac{1}{|A^{1/2}u_\varepsilon(t)|^{2\gamma-2}} \\ &\leqslant \frac{\varepsilon^2}{2}(1+t)^{2\beta} \frac{|A^{1/2}u'_\varepsilon(t)|^2}{|A^{1/2}u_\varepsilon(t)|^{2\gamma}} + \frac{1}{2}\sigma_1^{1-\gamma}(1+t)^{\beta+p-1}.\end{aligned}$$

Since $\beta + p - 1 \leqslant \beta$, plugging this estimate in (3.55) we obtain that

$$\int_0^t (1+s)^{2\beta-1}\big|Au_\varepsilon(s)\big|^2\,ds \leqslant \widehat{D}_\varepsilon(0) + \frac{\varepsilon^2}{2}(1+t)^{2\beta}\frac{|A^{1/2}u_\varepsilon'(t)|^2}{|A^{1/2}u_\varepsilon(t)|^{2\gamma}} + \frac{9}{2}\sigma_1^{1-\gamma}(1+t)^\beta + \left(\varepsilon + \frac{1}{4\beta}\right)\int_0^t (1+s)^{2\beta-p}\frac{|A^{1/2}u_\varepsilon'(s)|^2}{|A^{1/2}u_\varepsilon(s)|^{2\gamma}}\,ds. \tag{3.56}$$

Let us consider now the energy defined in (3.11). A simple calculation gives that

$$\big[(1+t)^{2\beta}F_\varepsilon\big]' = -(1+t)^{2\beta}\left(\frac{2}{(1+t)^p} + 2\varepsilon\gamma\frac{\langle Au_\varepsilon, u_\varepsilon'\rangle}{|A^{1/2}u_\varepsilon|^2} - \frac{2\beta\varepsilon}{1+t}\right)\frac{|A^{1/2}u_\varepsilon'|^2}{|A^{1/2}u_\varepsilon|^{2\gamma}} + 2\beta(1+t)^{2\beta-1}|Au_\varepsilon|^2.$$

By definition of S and the second inequality in (3.23), we have that

$$2\varepsilon\gamma\frac{|\langle Au_\varepsilon(t), u_\varepsilon'(t)\rangle|}{|A^{1/2}u_\varepsilon(t)|^2} + \frac{2\beta\varepsilon}{1+t} \leqslant (2\varepsilon\gamma K + 2\beta\varepsilon)\frac{1}{(1+t)^p} \leqslant \frac{1}{(1+t)^p}, \tag{3.57}$$

hence

$$\big[(1+t)^{2\beta}F_\varepsilon(t)\big]' \leqslant -(1+t)^{2\beta-p}\frac{|A^{1/2}u_\varepsilon'(t)|^2}{|A^{1/2}u_\varepsilon(t)|^{2\gamma}} + 2\beta(1+t)^{2\beta-1}\big|Au_\varepsilon(t)\big|^2.$$

Let us integrate in $[0,t]$. Using (3.56) and rearranging the terms we obtain that

$$(1+t)^{2\beta}(1-\beta\varepsilon)\varepsilon\frac{|A^{1/2}u_\varepsilon'(t)|^2}{|A^{1/2}u_\varepsilon(t)|^{2\gamma}} + (1+t)^{2\beta}\big|Au_\varepsilon(t)\big|^2 + \left(\frac{1}{2} - 2\beta\varepsilon\right)\int_0^t (1+s)^{2\beta-p}\frac{|A^{1/2}u_\varepsilon'(s)|^2}{|A^{1/2}u_\varepsilon(s)|^{2\gamma}}\,ds \leqslant F_\varepsilon(0) + 2\beta\widehat{D}_\varepsilon(0) + 9\beta\sigma_1^{1-\gamma}(1+t)^\beta.$$

Since $\beta \leqslant 2$, and $2\beta\varepsilon \leqslant 4\varepsilon \leqslant 1/4$, it follows that

$$\begin{aligned}\frac{1}{2}(1+t)^{2\beta}F_\varepsilon(t) + \frac{1}{4}\int_0^t (1+s)^{2\beta-p}\frac{|A^{1/2}u_\varepsilon'(s)|^2}{|A^{1/2}u_\varepsilon(s)|^{2\gamma}}\,ds &\leqslant F_\varepsilon(0) + 2\beta\widehat{D}_\varepsilon(0) + 9\beta\sigma_1^{1-\gamma}(1+t)^\beta \\ &\leqslant \frac{|A^{1/2}u_1|^2}{|A^{1/2}u_0|^{2\gamma}} + |Au_0|^2 + \frac{1}{4}\frac{|\langle Au_0, u_1\rangle|}{|A^{1/2}u_0|^{2\gamma}} + 18\sigma_1^{1-\gamma}(1+t)^\beta.\end{aligned}$$

Dividing by $(1+t)^\beta/2$ we obtain (3.52).

Estimate on the derivative. Let us consider the energy defined in (3.46). Its time derivative is given by

$$G'_\varepsilon(t) = -\frac{1}{\varepsilon}(1+t)^\beta\left(\frac{2}{(1+t)^p} + 4\varepsilon\gamma\frac{\langle u'_\varepsilon(t), Au_\varepsilon(t)\rangle}{|A^{1/2}u_\varepsilon(t)|^2} - \frac{\beta\varepsilon}{1+t}\right)\frac{|u'_\varepsilon(t)|^2}{|A^{1/2}u_\varepsilon(t)|^{4\gamma}} - \frac{2}{\varepsilon}(1+t)^\beta\frac{\langle u'_\varepsilon(t), Au_\varepsilon(t)\rangle}{|A^{1/2}u_\varepsilon(t)|^{2\gamma}}.$$

Arguing as in (3.57) we find that

$$4\varepsilon\gamma\frac{|\langle u'_\varepsilon(t), Au_\varepsilon(t)\rangle|}{|A^{1/2}u_\varepsilon(t)|^2} + \frac{\beta\varepsilon}{1+t} \leqslant \frac{3}{2}\frac{1}{(1+t)^p},$$

hence

$$G'_\varepsilon(t) \leqslant -\frac{1}{2\varepsilon}\frac{1}{(1+t)^p}G_\varepsilon(t) + \frac{2}{\varepsilon}(1+t)^{\beta/2}\left|Au_\varepsilon(t)\right| \cdot \sqrt{G_\varepsilon(t)}.$$

Thanks to (3.52) we have therefore that

$$G'_\varepsilon(t) \leqslant -\frac{1}{2\varepsilon}\frac{1}{(1+t)^p}G_\varepsilon(t) + \frac{2}{\varepsilon}\sqrt{\sigma_4} \cdot \sqrt{G_\varepsilon(t)},$$

hence by Lemma 3.1

$$G_\varepsilon(t) \leqslant G_\varepsilon(0) + 16\sigma_4(1+t)^{2p} \quad \forall t \in [0, S). \tag{3.58}$$

Global existence. We prove that $S = T = +\infty$. Let us assume by contradiction that $S < T$. Then by continuity all the estimates proved so far hold true also for $t = S$. Moreover by definition of S we have the alternative (3.41).

The first possibility can be ruled out using (3.21) exactly as in the coercive case.

In order to rule out the second possibility we consider the inequality

$$\frac{|\langle Au_\varepsilon(S), u'_\varepsilon(S)\rangle|}{|A^{1/2}u_\varepsilon(S)|^2} \leqslant \left|Au_\varepsilon(S)\right| \cdot \frac{|u'_\varepsilon(S)|}{|A^{1/2}u_\varepsilon(S)|^{2\gamma}} \cdot \left|A^{1/2}u_\varepsilon(S)\right|^{2\gamma-2}. \tag{3.59}$$

Let us estimate the three factors. From (3.52) we have that

$$\left|Au_\varepsilon(S)\right| \leqslant \frac{\sqrt{\sigma_4}}{(1+S)^{\beta/2}}.$$

From (3.58) we have that

$$\frac{|u'_\varepsilon(S)|}{|A^{1/2}u_\varepsilon(S)|^{2\gamma}} \leqslant \frac{\sqrt{G_\varepsilon(0) + 16\sigma_4(1+S)^{2p}}}{(1+S)^{\beta/2}} \leqslant \left(\frac{|u_1|}{|A^{1/2}u_0|^{2\gamma}} + 4\sqrt{\sigma_4}\right)\frac{1}{(1+S)^{\beta/2-p}}.$$

Since $\gamma \geqslant 1$ the last factor in (3.59) can be estimated using (3.47). We obtain that

$$\left|A^{1/2}u_\varepsilon(S)\right|^{2\gamma-2} \leqslant \left[(\gamma+1)\sigma_3\right]^{(\gamma-1)/(\gamma+1)}\frac{1}{(1+S)^{(p+1)(\gamma-1)/(\gamma+1)}}.$$

Plugging all these estimates in (3.59), and recalling (3.42), we obtain that

$$\frac{|\langle Au_\varepsilon(S), u'_\varepsilon(S)\rangle|}{|A^{1/2}u_\varepsilon(S)|^2} < \frac{K}{(1+S)^p}(1+S)^{2p-\beta-(p+1)(\gamma-1)/(\gamma+1)}.$$

If p satisfies (2.5), then the last exponent is less than or equal to zero, and this is enough to rule out the second possibility in (3.41).

It remains to prove that $T = +\infty$. So let us assume by contradiction that $T < +\infty$. Then the quoted local existence result says that either (3.36) or (3.37) holds true.

On the other hand as in the coercive case we have that (3.21) is satisfied for every $t \in [0, T)$, which rules (3.37) out. Moreover from (3.47) we have that $|A^{1/2}u_\varepsilon(t)|^2$ is uniformly bounded from above in $[0, T)$, hence by (3.16) it follows that also $|A^{1/2}u'_\varepsilon(t)|$ and $|Au_\varepsilon(t)|$ are uniformly bounded from above in $[0, T)$. This rules (3.36) out.

Decay estimates. Let us prove estimates (2.6), (2.7), and (2.8). Now we know that the solution is global, and that all the estimates proved so far hold true for every $t \geqslant 0$.

Therefore the lower bound in (2.6) follows from (3.21), while the upper bound follows from (3.47). Moreover (2.7) follows from (3.52). Finally, (2.8) follows from (3.58) and (2.6).

Remark 3.4. A careful inspection of the proofs reveals that (3.47) was proved without using the assumption $\gamma \geqslant 1$. At this point one can modify (3.59) as follows:

$$\frac{|\langle Au_\varepsilon(S), u'_\varepsilon(S)\rangle|}{|A^{1/2}u_\varepsilon(S)|^2} \leqslant \frac{|Au_\varepsilon(S)|}{|A^{1/2}u_\varepsilon(S)|} \cdot \frac{|u'_\varepsilon(S)|}{|A^{1/2}u_\varepsilon(S)|^{2\gamma+1}} \cdot \left|A^{1/2}u_\varepsilon(S)\right|^{2\gamma}.$$

Now we can estimate the first and second factor using (3.33) as we did in the coercive case, and then estimate the last factor using (3.47). All these inequalities require neither the coerciveness of the operator, nor $\gamma \geqslant 1$.

We end up with an estimate such as

$$\frac{|\langle Au_\varepsilon(S), u'_\varepsilon(S)\rangle|}{|A^{1/2}u_\varepsilon(S)|^2} \leqslant \frac{K_1}{(1+S)^p}(1+S)^{2p-\gamma(p+1)/(\gamma+1)}$$

for a suitable constant K_1. The last exponent is less than or equal to zero provided that $p \leqslant \gamma/(\gamma+2)$. This is the key point of the proof of global solvability for $\gamma > 0$ and $p \in [0, \gamma/(\gamma+2)]$ without coerciveness assumptions. We leave the details to the interested reader.

3.5. *Proof of Theorem 2.3*

In analogy with (3.43) let us set

$$H_\varepsilon(t) := \varepsilon\left|u'_\varepsilon(t)\right|^2 + \int_0^{|A^{1/2}u_\varepsilon(t)|^2} m(\sigma)\, d\sigma.$$

Assumption (2.10) is equivalent to say that $H_\varepsilon(0) > 0$. Moreover we have that

$$H'_\varepsilon(t) = -2b(t)\left|u'_\varepsilon(t)\right|^2 \geqslant -\frac{2}{\varepsilon}b(t)H_\varepsilon(t) \quad \forall t \geqslant 0,$$

hence

$$H_\varepsilon(t) \geqslant H_\varepsilon(0) \exp\left(-\frac{2}{\varepsilon} \int_0^t b(s)\, ds\right) \quad \forall t \geqslant 0.$$

The right-hand side is greater than a positive constant independent of t because of (2.9) and the fact that $H_\varepsilon(0) > 0$. This implies (2.11).

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