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Global solutions for dissipative Kirchhoff strings with non-Lipschitz nonlinear term

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Abstract

We investigate the evolution problem

$$\begin{cases} u'' + \delta u' + m\left(|A^{1/2}u|_H^2\right)Au = 0, \\ u(0) = u_0, \qquad u'(0) = u_1, \end{cases}$$

where H is a Hilbert space, A is a self-adjoint nonnegative operator on H with domain $D(A)$, $\delta > 0$ is a parameter, and $m(r)$ is a nonnegative function such that $m(0) = 0$ and m is nonnecessarily Lipschitz continuous in a neighborhood of 0.

We prove that this problem has a unique global solution for positive times, provided that the initial data $(u_0, u_1) \in D(A) \times D(A^{1/2})$ satisfy a suitable smallness assumption and the nondegeneracy condition $m(|A^{1/2}u_0|_H^2) > 0$. Moreover, we study the decay of the solution as $t \to +\infty$.

These results apply to degenerate hyperbolic PDEs with nonlocal nonlinearities.

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1. Introduction

Let H be a real Hilbert space, with norm $|\cdot|_H$ and scalar product $\langle\cdot,\cdot\rangle_H$. Let A be a self-adjoint linear nonnegative operator on H with dense domain $D(A)$ (i.e., $\langle Au, u\rangle_H \geqslant 0$ for all $u \in D(A)$). Let us consider the Cauchy problem

$$\begin{cases} u''(t) + \delta u'(t) + m\big(\big|A^{1/2}u(t)\big|_H^2\big)Au(t) = 0, & t \geqslant 0, \\ u(0) = u_0, \qquad u'(0) = u_1, \end{cases} \tag{1.1}$$

where δ is a nonnegative constant, and $m : [0, +\infty[\to [0, +\infty[$ is a continuous function.

Problem (1.1) is an abstract setting of the initial-boundary value problem for the equation

$$u_{tt} + \delta u_t + m\bigg(\int_\Omega |\nabla u|^2\,dx\bigg)\Delta u = 0, \quad \text{in } \Omega \times [0, +\infty[, \tag{1.2}$$

where $\Omega \subseteq \mathbb{R}^n$ is a (nonnecessarily bounded) open set. This last equation was introduced in the case $n = 1$ by G. Kirchhoff [10] as a model for the small transversal vibrations of an elastic string with fixed endpoints.

Equations (1.1)–(1.2) have long been studied under various conditions on the function m and on the regularity of the initial data: the interested reader can find appropriate references in the surveys of A. Arosio [1], S. Spagnolo [17] and L.A. Medeiros, J.L. Ferrel and S.-B. de Menezes [13].

In this context we will recall only some results on the existence of global solutions.

When the initial data are A-analytic, A. Arosio and S. Spagnolo [2] and later P. D'Ancona and S. Spagnolo [3] proved that (1.1) has a global solution. Various authors, beginning from the works of J.M. Greenberg and S.C. Hu [9] and P. D'Ancona and S. Spagnolo [4], proved the existence of global solutions of (1.2) for regular, nonanalytic, small initial data, when $\Omega = \mathbb{R}^n$, and $m \in C^1([0, +\infty[)$, with $m(r) \geqslant \nu > 0$, $\forall r \geqslant 0$ (see also T. Yamazaki [19,20], T. Matsuyama [12] when Ω are exterior domains). Moreover, some authors showed that, if $\Omega = \mathbb{R}^n$ or Ω is a bounded set, $m \in C^1([0, +\infty[)$ and $m(r) \geqslant \nu > 0$ for all $r \geqslant 0$, then there are global solutions for special classes of nonnecessarily regular or small initial data (see, for example, R. Manfrin [11]).

In the following let us consider (1.1) with $(u_0, u_1) \in D(A) \times D(A^{1/2})$ small initial data and $\delta > 0$ (i.e., we limit ourself to treat the dissipative case).

In the nondegenerate case (i.e., $m(r) \geqslant \nu > 0$ for all $r \geqslant 0$), when A is a coercive operator, and m is a C^1 function, E.H. de Brito, Y. Yamada, K. Nishihara [5,6,14,18] proved that there exists a unique global solution such that (u, u') decay with an exponential rate as $t \to +\infty$ in $D(A^{1/2}) \times H$. The same result, with a polynomial decay of the solution, was afterwards obtained by K. Nishihara and Y. Yamada [15] if $m(r) = r^\gamma$ ($\gamma \geqslant 1$), and $u_0 \neq 0$ (see also K. Ono [16]). When A is only a nonnegative operator, in [8] it was proved that, if m is a nonnegative C^1 function and $m(|A^{1/2}u_0|_H^2) > 0$, then there exists a global solution $u(t)$ and $(u(t), u'(t), u''(t)) \to (u_\infty, 0, 0)$ in $D(A) \times D(A^{1/2}) \times H$ as $t \to +\infty$, with $m(|A^{1/2}u_\infty|_H^2) = 0$.

In [7] it was then considered the case $m(r) = r^p$ ($0 < p < 1$) and $A^{1/2}u_0 \neq 0$, and proved the existence of global solutions for small regular (nonanalytic) initial data, where the regularity of the data depends on p. Moreover, in [7] it was proved that the solutions decay with a polynomial rate.

The purpose of this paper is to extend the result in [7] for the unique global existence of a solution to a more general class of functions $m(r)$ nonnecessarily Lipschitz continuous near the origin. Moreover, we only assume that the initial data are in $D(A) \times D(A^{1/2})$ (see Theorem 2.1). Then we study the asymptotic behaviour of the solutions when $m(x) \to 0$ (as $x \to 0^+$) slowly (see Theorem 2.2).

2. Preliminaries and statements

To begin with, let us state the conditions on the function m we need in the following:

(1) $m \in C^1(]0, a]) \cap C^0([0, a])$ for some $a > 0$;
(2) m is a nonnegative function, nonidentically zero;
(3) there exists a constant C such that

$$\left|xm'(x)\right| \leqslant C, \quad 0 < x \leqslant a; \tag{2.1}$$

(4) $m(x) \neq 0$ for $x \neq 0$ and for all $\alpha > 0$ we have

$$\lim_{x \to 0^+} x^\alpha m(x)^{-1} = 0.$$

Let us discuss shortly the assumptions on the function m.

- The condition (3) is verified by large classes of functions: for example, $m(r)$ can be any C^1 function or can behave near the origin as r^p $(p > 0)$ or as $|\log(r)|^{-\alpha}$ $(\alpha > 0)$.
- A sufficient condition in order that m verifies (4) is that m satisfies (1)–(2) and

$$\lim_{x \to 0^+} \frac{|m'(x)x|}{m(x)} = 0.$$

Now let us set for $|A^{1/2}u_0|_H > 0$ and $m(|A^{1/2}u_0|_H^2) > 0$:

$$E(0) = \frac{|u_1|_H^2}{m(|A^{1/2}u_0|_H^2)} + \left|A^{1/2}u_0\right|_H^2, \qquad G(0) = \frac{|u_1|_H^2}{|A^{1/2}u_0|_H^2 m^2(|A^{1/2}u_0|_H^2)},$$
$$F(0) = \frac{|A^{1/2}u_1|_H^2}{m(|A^{1/2}u_0|_H^2)|A^{1/2}u_0|_H^2} + \frac{|Au_0|_H^2}{|A^{1/2}u_0|_H^2} - \frac{\langle Au_0, u_1\rangle_H^2}{|A^{1/2}u_0|_H^4 m(|A^{1/2}u_0|_H^2)}.$$

Moreover, for $0 < b \leqslant a$ let us define

$$\gamma_b := \sup_{0 \leqslant x \leqslant b} m(x) + \sup_{0 < x \leqslant b} x\left|m'(x)\right|.$$

Let us observe that, if m verifies (1)–(3), then the quantity γ_b is always well defined.

We shall prove the following result:

Theorem 2.1. *Let us assume $\delta > 0$, and that m verifies* (1)–(3). *Let us suppose that* $(u_0, u_1) \in D(A) \times D(A^{1/2})$, *with* $|A^{1/2}u_0|_H^2 \neq 0$, $m(|A^{1/2}u_0|_H^2) \neq 0$ *and*

$$E(0) < a_1 \leqslant a, \quad \gamma_{a_1} \frac{|\langle Au_0, u_1\rangle_H|}{m(|A^{1/2}u_0|_H^2)|A^{1/2}u_0|_H^2} < \frac{\delta}{4}, \tag{2.2}$$

$$\gamma_{a_1}\sqrt{F(0)}\max\left\{\sqrt{G(0)}, \frac{4}{\delta}\sqrt{F(0)}\right\} < \frac{\delta}{4}. \tag{2.3}$$

Then there exists a unique global solution u of (1.1) *such that $|A^{1/2}u(t)|_H^2 > 0$ and $m(|A^{1/2}u(t)|_H^2) > 0$ for all $t \geqslant 0$ and*

$$u \in C^2([0, +\infty[; H) \cap C^1([0, \infty[; D(A^{1/2})) \cap C^0([0, \infty[; D(A)).$$

Let us remark that:

- Theorem 2.1 can be enunciated also as: let us fix $(u_0, u_1) \in D(A) \times D(A^{1/2})$, with $|A^{1/2}u_0|_H^2 \neq 0$, $m(|A^{1/2}u_0|_H^2) \neq 0$ and $E(0) < a$, then, if δ is sufficiently big, (1.1) has a unique global solution;
- if $m(0) = 0$ and $xm'(x) \to 0$ as $x \to 0$, then, of course, we can find $a_1 > 0$ and initial data such that inequalities in (2.2)–(2.3) are verified, otherwise it could be necessary take δ sufficiently big (note that, if A is a coercive operator, we get $F(0) \geqslant c_0 > 0$).

Since the asymptotic behaviour when m is a C^1 function or $m(r) = r^p$ has been considered in the previous works, we limit ourself to study the asymptotic behaviour of the solutions of (1.1) when $m(x) \to 0$ (as $x \to 0^+$) slowly (i.e., m satisfies (4)).

Theorem 2.2. *Let us assume that all the conditions of Theorem* 2.1 *are satisfied, that A is a coercive operator and that m verifies also* (4)*. Let u be the solution of* (1.1).

Then, for all $k > 0$ and for all $0 < \alpha < 1$, there exists $C_{k,\alpha} > 0$ such that for all $t \geqslant 0$,

$$(1+t)^k\left(\frac{|A^{1/2}u'(t)|_H^2}{|A^{1/2}u(t)|_H^{2\alpha}} + \frac{m(|A^{1/2}u(t)|_H^2)|Au(t)|_H^2}{|A^{1/2}u(t)|_H^{2\alpha}}\right) \leqslant C_{k,\alpha}. \tag{2.4}$$

Since the operator A is coercive, (2.4) imply that the same estimates hold true for $|u'|_H$, $|A^{1/2}u|_H$, $|u|_H$. Moreover, let us remark that, thanks to (4), the estimate (2.4) implies that $(1+t)^k|Au(t)|_H^2$ is bounded for every k.

3. Proofs

Let us enunciate, first of all, a result of existence of local solutions for (1.1). The proof of this theorem can be obtained by a simple adaptation of the one of [8, Theorem 2.1], then it is omitted.

Theorem 3.1 *(Local existence). Let us assume $\delta \geqslant 0$, that m verifies* (1)–(3)*, and $(u_0, u_1) \in D(A) \times D(A^{1/2})$ with $|A^{1/2}u_0|_H^2 > 0$ and $m(|A^{1/2}u_0|_H^2) > 0$.*

Then there exists $T > 0$ such that problem (1.1) *has a unique solution u with $|A^{1/2}u(t)|_H^2 > 0$ and $m(|A^{1/2}u(t)|_H^2) > 0$ in $[0, T[$ and*

$$u \in C^2([0, T[; H) \cap C^1([0, T[; D(A^{1/2})) \cap C^0([0, T[; D(A)).$$

Moreover, u can be uniquely continued to a maximal solution defined in an interval $[0, T_*[$, *and at least one of the following statements is valid*:

(i) $T_* = +\infty$;
(ii) $\limsup_{t\to T_*^-} |A^{1/2}u(t)|_H^2 = a$;
(iii) $\limsup_{t\to T_*^-} |A^{1/2}u'(t)|^2 + |Au(t)|^2 = +\infty$;
(iv) $\liminf_{t\to T_*^-} |A^{1/2}u(t)|_H^2 = 0$;
(v) $\liminf_{t\to T_*^-} m(|A^{1/2}u(t)|_H^2) = 0$.

Now we can prove Theorem 2.1.

Proof of Theorem 2.1. Let $[0, T_*[$ be the maximal interval where the solution exists.
Let us set

$$c(t) := m\big(\big|A^{1/2}u(t)\big|_H^2\big), \qquad h(t) = \frac{\langle Au(t), u'(t)\rangle_H}{|A^{1/2}u(t)|_H^2},$$
$$T := \sup\left\{\tau \in [0, T_*[:\ 0 < \big|A^{1/2}u(t)\big|_H^2 \leqslant a_1,\ c(t) > 0,\ \gamma_{a_1}\left|\frac{h(t)}{c(t)}\right| \leqslant \frac{\delta}{4},\ \forall t \in [0, \tau]\right\}.$$

To begin with let us remark that $T > 0$ and that in $[0, T[$ we have $0 < c(t) < \gamma_{a_1}$ and

$$\frac{|(|A^{1/2}u(t)|_H^2)'|}{|A^{1/2}u(t)|_H^2} = 2\big|h(t)\big| \leqslant 2\frac{\delta}{4\gamma_{a_1}}c(t) \leqslant \frac{\delta}{2}, \tag{3.1}$$

that implies for $t \in [0, T[$,

$$0 < \big|A^{1/2}u_0\big|_H^2 e^{-\delta t/2} \leqslant \big|A^{1/2}u(t)\big|_H^2 \leqslant \big|A^{1/2}u_0\big|_H^2 e^{\delta t/2}. \tag{3.2}$$

Moreover, in $[0, T[$ it holds true that $c(t) > 0$ and

$$\left|\frac{c'(t)}{c(t)}\right| = 2\big|m'\big(\big|A^{1/2}u(t)\big|_H^2\big)\big|\frac{|h(t)|}{c(t)}\big|A^{1/2}u(t)\big|_H^2 \leqslant 2\gamma_{a_1}\frac{|h(t)|}{c(t)} \leqslant \frac{\delta}{2}, \tag{3.3}$$

that implies for $t \in [0, T[$,

$$0 < c(0)e^{-\delta t/2} \leqslant c(t) \leqslant c(0)e^{\delta t/2}. \tag{3.4}$$

Let us now consider for $t \in [0, T[$ the following function:

$$E(t) := \frac{|u'(t)|_H^2}{c(t)} + \big|A^{1/2}u(t)\big|_H^2.$$

Since it holds true that

$$E'(t) = -\frac{1}{c(t)}\left(2\delta + \frac{c'(t)}{c(t)}\right)\big|u'(t)\big|_H^2 \leqslant -\frac{\delta\,|u'(t)|_H^2}{c(t)}, \tag{3.5}$$

hence we get

$$E(t) \leqslant E(0) < a_1, \quad \forall t \in [0, T[. \tag{3.6}$$

In the same way we find

$$E_1(t) := \frac{|A^{1/2}u'(t)|_H^2}{c(t)} + \left|Au(t)\right|_H^2 \leqslant E_1(0). \tag{3.7}$$

Our purpose is to prove that $T = T_* = +\infty$. Indeed in this case we have a global solution, and by (3.2)–(3.4), we get $|A^{1/2}u(t)|_H^2 > 0$ and $m(|A^{1/2}u(t)|_H^2) > 0$ for all $t \geqslant 0$, hence this solution is also unique.

Now let us assume that $T = T_* < +\infty$; then by (3.6), (3.7), (3.2) and (3.4) we obtain that (ii), (iii), (iv) and (v), respectively, of Theorem 3.1 are false, and this contradicts Theorem 3.1.

Then we must only prove that $T = T_*$.

Let us assume by contradiction that $T < T_*$. Hence by (3.6), (3.2) and (3.4) and by the maximality of T we have that necessarily

$$\frac{\gamma_{a_1}|h(T)|}{c(T)} = \frac{\delta}{4}. \tag{3.8}$$

Let us consider for $0 \leqslant t \leqslant T$,

$$F(t) = \frac{|A^{1/2}u'(t)|_H^2}{c(t)|A^{1/2}u(t)|_H^2} + \frac{|Au(t)|_H^2}{|A^{1/2}u(t)|_H^2} - \frac{h^2(t)}{c(t)}.$$

We have

$$\begin{aligned}
F'(t) = {} & -2\delta \frac{|A^{1/2}u'(t)|_H^2}{c(t)|A^{1/2}u(t)|_H^2} - \frac{c'(t)}{c(t)} \frac{|A^{1/2}u'(t)|_H^2}{c(t)|A^{1/2}u(t)|_H^2} \\
& - 2\frac{\langle Au(t), u'(t)\rangle_H}{|A^{1/2}u(t)|_H^4} \frac{|A^{1/2}u'(t)|_H^2}{c(t)} - 2\left|Au(t)\right|_H^2 \frac{\langle Au(t), u'(t)\rangle_H}{|A^{1/2}u(t)|_H^4} \\
& - 2\frac{h(t)}{c(t)} \frac{\langle Au(t), -c(t)Au(t) - \delta u'(t)\rangle_H + |A^{1/2}u'(t)|_H^2}{|A^{1/2}u(t)|_H^2} \\
& + 4\frac{h(t)}{c(t)} \frac{\langle Au(t), u'(t)\rangle_H^2}{|A^{1/2}u(t)|_H^4} + \frac{h^2(t)c'(t)}{c^2(t)} \\
= {} & 2\left[-\frac{|A^{1/2}u'(t)|_H^2}{c(t)|A^{1/2}u(t)|_H^2} + \frac{h^2(t)}{c(t)}\right]\left[\delta + 2h(t) + \frac{c'(t)}{2c(t)}\right].
\end{aligned}$$

Since it holds true that

$$-\frac{|A^{1/2}u'(t)|_H^2}{c(t)|A^{1/2}u(t)|_H^2} + \frac{h^2(t)}{c(t)} = -\frac{|A^{1/2}u'(t)|_H^2}{c(t)|A^{1/2}u(t)|_H^2} + \frac{\langle Au(t), u'(t)\rangle_H^2}{c(t)|A^{1/2}u(t)|_H^4} \leqslant 0$$

and (see (3.1), (3.3))

$$\delta + 2h(t) + \frac{c'(t)}{2c(t)} \geqslant \delta - \frac{\delta}{2} - \frac{\delta}{4} > 0,$$

then we have $F' \leqslant 0$, hence we get $F(t) \leqslant F(0)$ for all $0 \leqslant t \leqslant T$ and in particular

$$\frac{|Au(t)|_H^2}{|A^{1/2}u(t)|_H^2} \leqslant F(0), \quad 0 \leqslant t \leqslant T. \tag{3.9}$$

Let now us set

$$G(t) = \frac{|u'(t)|_H^2}{c^2(t)|A^{1/2}u(t)|_H^2}.$$

Using (3.9) we get

$$\begin{aligned}
G'(t) &= 2\frac{\langle -c(t)Au(t) - \delta u'(t), u'(t)\rangle_H}{c^2(t)|A^{1/2}u(t)|_H^2} - 2\frac{|u'(t)|_H^2}{c^2(t)|A^{1/2}u(t)|_H^4}\langle Au(t), u'(t)\rangle_H \\
&\quad - 2\frac{c'(t)}{c(t)}\frac{|u'(t)|_H^2}{c^2(t)|A^{1/2}u(t)|_H^2} \\
&= -2G(t)\left[\delta + h(t) + \frac{c'(t)}{c(t)}\right] - 2\frac{\langle Au(t), u'(t)\rangle_H}{c(t)|A^{1/2}u(t)|_H^2} \\
&\leqslant -\frac{\delta}{2}G(t) + 2\sqrt{G(t)}\sqrt{F(0)}.
\end{aligned}$$

Using standard techniques for differential inequalities, it is easy hence prove that, for all $0 \leqslant t \leqslant T$,

$$G(t) \leqslant \max\left\{G(0), \frac{16}{\delta^2}F(0)\right\} := G_0. \tag{3.10}$$

Therefore, using (3.9) and (3.10), by (2.3) we obtain

$$\frac{\gamma_{a_1}|h(T)|}{c(T)} = \gamma_{a_1}\left|\frac{\langle Au(T), u'(T)\rangle_H}{c(T)|A^{1/2}u(T)|_H^2}\right| \leqslant \gamma_{a_1}\sqrt{F(0)}\sqrt{G_0} < \frac{\delta}{4},$$

that contradicts (3.8). □

Now we show Theorem 2.2.

Proof of Theorem 2.2. We use the same notations as in Theorem 2.1.

First of all let us remark that the estimates (3.1)–(3.3)–(3.6) hold true for all $t \geqslant 0$, that is

$$2\left|\frac{\langle Au(t), u'(t)\rangle_H}{|A^{1/2}u(t)|_H^2}\right| \leqslant \frac{\delta}{2}, \qquad \left|\frac{c'(t)}{c(t)}\right| \leqslant \frac{\delta}{2}, \qquad \left|A^{1/2}u(t)\right|_H^2 \leqslant E(0). \tag{3.11}$$

Now let us remark that to prove Theorem 2.2 it is enough to show that, for all $0 < \alpha < 1$ and $k \geqslant 0$, the estimate (2.4) holds true for

$$t \geqslant t_{k,\alpha} := \max\left\{\frac{2k}{\delta(1-\alpha)} - 1, 0\right\}.$$

Let us set

$$P_{k,\alpha}(t) := (1+t)^k\left(\frac{|A^{1/2}u'(t)|_H^2}{|A^{1/2}u(t)|_H^{2\alpha}} + \frac{c(t)|Au(t)|_H^2}{|A^{1/2}u(t)|_H^{2\alpha}}\right) + \frac{\delta(1-\alpha)}{2}\int_{t_{k,\alpha}}^{t}(1+s)^k\frac{|A^{1/2}u'(s)|_H^2}{|A^{1/2}u(s)|_H^{2\alpha}}\,ds$$

and

$$H_{k,\alpha}(t) := \int_{t_{k,\alpha}}^{t} c(s)\left|Au(s)\right|_H^2\left|A^{1/2}u(s)\right|_H^{-2\alpha}(1+s)^k\,ds.$$

We have only to prove by induction that $P_{k,\alpha}$ and $H_{k,\alpha}$ are bounded functions for $t \geqslant t_{k,\alpha}$.

In the following we denote by $c^j_{k,\alpha}$ all constants (independent on t) we use in each step.

Let us consider $k = 0$. By (3.11) we get for $t \geqslant 0$,

$$\begin{aligned}
P'_{0,\alpha}(t) &= -2\delta\frac{|A^{1/2}u'(t)|_H^2}{|A^{1/2}u(t)|_H^{2\alpha}} - 2\alpha\frac{\langle Au(t), u'(t)\rangle_H}{|A^{1/2}u(t)|_H^2}\frac{|A^{1/2}u'(t)|_H^2}{|A^{1/2}u(t)|_H^{2\alpha}} \\
&\quad + \left[\frac{c'(t)}{c(t)} - 2\alpha\frac{\langle Au(t), u'(t)\rangle_H}{|A^{1/2}u(t)|_H^2}\right]c(t)\frac{|Au(t)|_H^2}{|A^{1/2}u(t)|_H^{2\alpha}} + \frac{\delta}{2}(1-\alpha)\frac{|A^{1/2}u'(t)|_H^2}{|A^{1/2}u(t)|_H^{2\alpha}} \\
&\leqslant -\frac{|A^{1/2}u'(t)|_H^2}{|A^{1/2}u(t)|_H^{2\alpha}}\left(2\delta - \frac{\alpha\delta}{2} - \frac{\delta}{2}(1-\alpha)\right) + \frac{\delta}{2}(1+\alpha)c(t)\frac{|Au(t)|_H^2}{|A^{1/2}u(t)|_H^{2\alpha}} \\
&= -\frac{3\delta}{2}\frac{|A^{1/2}u'(t)|_H^2}{|A^{1/2}u(t)|_H^{2\alpha}} + \frac{\delta}{2}(1+\alpha)c(t)\frac{|Au(t)|_H^2}{|A^{1/2}u(t)|_H^{2\alpha}}.
\end{aligned}$$

Now a straightforward computation gives

$$\begin{aligned}
-\left(\frac{\langle Au(t), u'(t)\rangle_H}{|A^{1/2}u(t)|_H^{2\alpha}}\right)' &= -\frac{|A^{1/2}u'(t)|_H^2}{|A^{1/2}u(t)|_H^{2\alpha}} + 2\alpha\frac{\langle Au(t), u'(t)\rangle_H^2}{|A^{1/2}u(t)|_H^{2(\alpha+1)}} \\
&\quad + c(t)\frac{|Au(t)|_H^2}{|A^{1/2}u(t)|_H^{2\alpha}} + \delta\frac{\langle Au(t), u'(t)\rangle_H}{|A^{1/2}u(t)|_H^{2\alpha}}.
\end{aligned}$$

Hence we obtain

$$\begin{aligned}
c(t)\frac{|Au(t)|_H^2}{|A^{1/2}u(t)|_H^{2\alpha}} &\leqslant -\left[\frac{\langle Au(t), u'(t)\rangle_H}{|A^{1/2}u(t)|_H^{2\alpha}} + \frac{\delta}{2}\frac{|A^{1/2}u(t)|_H^{2(1-\alpha)}}{1-\alpha}\right]' \\
&\quad + \frac{|A^{1/2}u'(t)|_H^2}{|A^{1/2}u(t)|_H^{2\alpha}},
\end{aligned} \tag{3.12}$$

that implies

$$\begin{aligned}P'_{0,\alpha}(t) \leqslant{}& \left(-\frac{3\delta}{2}+\frac{\delta}{2}(1+\alpha)\right)\frac{|A^{1/2}u'(t)|_H^2}{|A^{1/2}u(t)|_H^{2\alpha}} \\ &- \frac{\delta}{2}(1+\alpha)\left[\frac{\langle Au(t),u'(t)\rangle_H}{|A^{1/2}u(t)|_H^{2\alpha}}+\frac{\delta}{2}\frac{|A^{1/2}u(t)|_H^{2(1-\alpha)}}{1-\alpha}\right]'.\end{aligned}$$

Therefore integrating this last inequality over $[0,t]$ and using once more (3.11), we have:

$$\begin{aligned}P_{0,\alpha}(t) &\leqslant P_{0,\alpha}(0)+\frac{\delta}{2}(1+\alpha)\left[\frac{\langle Au(0),u'(0)\rangle_H}{|A^{1/2}u(0)|_H^{2\alpha}}+\frac{\delta}{2}\frac{|A^{1/2}u(0)|_H^{2(1-\alpha)}}{1-\alpha}\right] \\ &\quad - \frac{\delta}{2}(1+\alpha)\left[\frac{\langle Au(t),u'(t)\rangle_H}{|A^{1/2}u(t)|_H^{2\alpha}}+\frac{\delta}{2}\frac{|A^{1/2}u(t)|_H^{2(1-\alpha)}}{1-\alpha}\right] \\ &\leqslant c_{0,\alpha}^1+\frac{\delta}{2}(1+\alpha)\left|\frac{\langle Au(t),u'(t)\rangle_H}{|A^{1/2}u(t)|_H^2}\right|\left|A^{1/2}u(t)\right|_H^{2(1-\alpha)} \leqslant c_{0,\alpha}^2.\end{aligned} \tag{3.13}$$

Since $P_{0,\alpha}$ is bounded, integrating (3.12) over $[0,t]$ we get also, for all $t \geqslant 0$,

$$H_{0,\alpha}(t) \leqslant c_{0,\alpha}^3 - \frac{\langle Au(t),u'(t)\rangle_H}{|A^{1/2}u(t)|_H^{2\alpha}}+\int_0^t \frac{|A^{1/2}u'(s)|_H^2}{|A^{1/2}u(s)|_H^{2\alpha}}\,ds \leqslant c_{0,\alpha}^4. \tag{3.14}$$

Now let us assume that $P_{k,\alpha}$ and $H_{k,\alpha}$ are bounded for all $0<\alpha<1$ and we prove that $P_{k+1,\alpha}$ and $H_{k+1,\alpha}$ are bounded for all α. We have for $t \geqslant t_{k+1,\alpha}$, using once more (3.11),

$$\begin{aligned}P'_{k+1,\alpha}(t) &= -(1+t)^{k+1}\frac{|A^{1/2}u'(t)|_H^2}{|A^{1/2}u(t)|_H^{2\alpha}}\left(2\delta+2\alpha\frac{\langle Au(t),u'(t)\rangle_H}{|A^{1/2}u(t)|_H^2}-\frac{\delta}{2}(1-\alpha)-\frac{k+1}{1+t}\right) \\ &\quad + (1+t)^{k+1}c(t)\frac{|Au(t)|_H^2}{|A^{1/2}u(t)|_H^{2\alpha}}\left(\frac{c'(t)}{c(t)}-2\alpha\frac{\langle Au(t),u'(t)\rangle_H}{|A^{1/2}u(t)|_H^2}+\frac{k+1}{1+t}\right) \\ &\leqslant -(1+t)^{k+1}\frac{|A^{1/2}u'(t)|_H^2}{|A^{1/2}u(t)|_H^{2\alpha}}\left(2\delta-\frac{\alpha\delta}{2}-\frac{\delta}{2}(1-\alpha)-\frac{\delta}{2}(1-\alpha)\right) \\ &\quad + (1+t)^{k+1}c(t)\frac{|Au(t)|_H^2}{|A^{1/2}u(t)|_H^{2\alpha}}\left(\frac{\delta}{2}+\frac{\alpha\delta}{2}+\frac{\delta}{2}(1-\alpha)\right) \\ &\leqslant -(1+t)^{k+1}\delta\left(1+\frac{\alpha}{2}\right)\frac{|A^{1/2}u'(t)|_H^2}{|A^{1/2}u(t)|_H^{2\alpha}}+\delta(1+t)^{k+1}c(t)\frac{|Au(t)|_H^2}{|A^{1/2}u(t)|_H^{2\alpha}}.\end{aligned} \tag{3.15}$$

Moreover, we easily see that

$$
\begin{aligned}
&-\left((1+t)^{k+1}\frac{\langle Au(t),u'(t)\rangle_H}{|A^{1/2}u(t)|_H^{2\alpha}}\right)'\\
&\quad=-(1+t)^{k+1}\frac{|A^{1/2}u'(t)|_H^2}{|A^{1/2}u(t)|_H^{2\alpha}}+2\alpha(1+t)^{k+1}\frac{\langle Au(t),u'(t)\rangle_H^2}{|A^{1/2}u(t)|_H^{2(\alpha+1)}}\\
&\qquad-(k+1)(1+t)^k\frac{\langle Au(t),u'(t)\rangle_H}{|A^{1/2}u(t)|_H^{2\alpha}}+(1+t)^{k+1}c(t)\frac{|Au(t)|_H^2}{|A^{1/2}u(t)|_H^{2\alpha}}\\
&\qquad+\delta(1+t)^{k+1}\frac{\langle Au(t),u'(t)\rangle_H}{|A^{1/2}u(t)|_H^{2\alpha}}.
\end{aligned}
$$

Hence we get

$$
\begin{aligned}
&(1+t)^{k+1}c(t)\frac{|Au(t)|_H^2}{|A^{1/2}u(t)|_H^{2\alpha}} &(3.16)\\
&\quad\leqslant(1+t)^{k+1}\frac{|A^{1/2}u'(t)|_H^2}{|A^{1/2}u(t)|_H^{2\alpha}}-\left[(1+t)^{k+1}\frac{\langle Au(t),u'(t)\rangle_H}{|A^{1/2}u(t)|_H^{2\alpha}}\right]'\\
&\qquad+\left[(k+1)(1+t)^k\frac{|A^{1/2}u(t)|_H^{2(1-\alpha)}}{2(1-\alpha)}\right]'-k(k+1)(1+t)^{k-1}\frac{|A^{1/2}u(t)|_H^{2(1-\alpha)}}{2(1-\alpha)}\\
&\qquad-\delta\left[(1+t)^{k+1}\frac{|A^{1/2}u(t)|_H^{2(1-\alpha)}}{2(1-\alpha)}\right]'+\delta(k+1)(1+t)^k\frac{|A^{1/2}u(t)|_H^{2(1-\alpha)}}{2(1-\alpha)}. &(3.17)
\end{aligned}
$$

Putting this last inequality in (3.15) we have

$$
\begin{aligned}
P'_{k+1,\alpha}(t)\leqslant{}&-\frac{\alpha}{2}(1+t)^{k+1}\frac{|A^{1/2}u'(t)|_H^2}{|A^{1/2}u(t)|_H^{2\alpha}}+\frac{\delta^2}{2}(k+1)(1+t)^k\frac{|A^{1/2}u(t)|_H^{2(1-\alpha)}}{(1-\alpha)}\\
&-\delta\left[(1+t)^{k+1}\frac{\langle Au(t),u'(t)\rangle_H}{|A^{1/2}u(t)|_H^{2\alpha}}+\delta(1+t)^{k+1}\frac{|A^{1/2}u(t)|_H^{2(1-\alpha)}}{2(1-\alpha)}\right]'\\
&+\delta\left[(k+1)(1+t)^k\frac{|A^{1/2}u(t)|_H^{2(1-\alpha)}}{2(1-\alpha)}\right]'.
\end{aligned}
$$

Integrating over $[t_{k+1,\alpha},t]$ we get, using also Holder inequality,

$$
\begin{aligned}
P_{k+1,\alpha}(t)\leqslant{}&c^1_{k+1,\alpha}+\frac{\delta^2(k+1)}{2(1-\alpha)}\int_{t_{k+1,\alpha}}^t(1+s)^k\left|A^{1/2}u(s)\right|_H^{2(1-\alpha)}ds\\
&-\delta(1+t)^{k+1}\frac{\langle Au(t),u'(t)\rangle_H}{|A^{1/2}u(t)|_H^{2\alpha}}\\
&-\frac{\delta^2}{2(1-\alpha)}(1+t)^{k+1}\left|A^{1/2}u(t)\right|_H^{2(1-\alpha)}\left(1-\frac{k+1}{\delta(1+t)}\right)
\end{aligned}
$$

$$\leqslant c^1_{k+1,\alpha} + \frac{\delta^2(k+1)}{2(1-\alpha)} \int_{t_{k+1,\alpha}}^{t} (1+s)^k \left|A^{1/2}u(s)\right|_H^{2(1-\alpha)} ds$$
$$+ \frac{\delta^2}{4}\left|A^{1/2}u(t)\right|_H^{2(1-\alpha)} \frac{1+\alpha}{1-\alpha}(1+t)^{k+1} + \frac{1-\alpha}{1+\alpha}(1+t)^{k+1} \frac{|A^{1/2}u'(t)|_H^2}{|A^{1/2}u(t)|_H^{2\alpha}}$$
$$- \frac{\delta^2}{4}\left|A^{1/2}u(t)\right|_H^{2(1-\alpha)} \frac{1+\alpha}{1-\alpha}(1+t)^{k+1}.$$

Let us consider $\alpha < \beta < 1$ such that $t_{k,\beta} \leqslant t_{k+1,\alpha}$ and let us remark that, since A is a coercive operator and $c(t) = m(|A^{1/2}u(t)|_H^2)$ verifies (4), then there exists a constant $\lambda_{\alpha,\beta}$ such that

$$\left|A^{1/2}u(t)\right|_H^{2(1-\alpha)} = \frac{|A^{1/2}u(t)|_H^2}{|A^{1/2}u(t)|_H^{2\alpha}} \leqslant \lambda_{\alpha,\beta} c(t) \frac{|Au(t)|_H^2}{|A^{1/2}u(t)|_H^{2\beta}}. \tag{3.18}$$

Hence recalling the inductive hypothesis on $H_{k,\beta}$ we get

$$\int_{t_{k+1,\alpha}}^{+\infty} (1+s)^k \left|A^{1/2}u(s)\right|_H^{2(1-\alpha)} ds < +\infty.$$

Using this last inequality we obtain

$$\frac{2\alpha}{1+\alpha} P_{k+1,\alpha}(t) \leqslant c^2_{k+1,\alpha}.$$

Since $P_{k+1,\alpha}$ is bounded for all α, integrating (3.17) and using (3.18) we get

$$H_{k+1,\alpha}(t) \leqslant c^3_{k+1,\alpha} + \int_{t_{k+1,\alpha}}^{t} (1+s)^{k+1} \frac{|A^{1/2}u'(s)|_H^2}{|A^{1/2}u(s)|_H^{2\alpha}} ds - (1+t)^{k+1} \frac{\langle Au(t), u'(t)\rangle_H}{|A^{1/2}u(t)|_H^{2\alpha}}$$
$$+ (k+1)(1+t)^k \frac{|A^{1/2}u(t)|_H^{2(1-\alpha)}}{2(1-\alpha)} - \delta(1+t)^{k+1} \frac{|A^{1/2}u(t)|_H^{2(1-\alpha)}}{2(1-\alpha)}$$
$$+ \delta(k+1) \int_{t_{k+1,\alpha}}^{t} (1+s)^k \frac{|A^{1/2}u(s)|_H^{2(1-\alpha)}}{2(1-\alpha)} ds$$
$$\leqslant c^4_{k+1,\alpha} + \delta(k+1)\lambda_{\alpha,\beta} \int_{t_{k+1,\alpha}}^{+\infty} (1+s)^k c(s) \frac{|Au(s)|_H^2}{|A^{1/2}u(s)|_H^{2\beta}} ds < +\infty. \qquad \square$$

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