Asymptotic Analysis 40 (2004) 25–36
IOS Press

Global existence and asymptotic behaviour for a mildly degenerate dissipative hyperbolic equation of Kirchhoff type

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Abstract. We investigate the evolution problem

$$u'' + \delta u' + m(|A^{1/2}u|^2)Au = 0,$$
$$u(0) = u_0, \quad u'(0) = u_1,$$

where H is a Hilbert space, A is a self-adjoint non-negative operator on H with domain $D(A)$, $\delta > 0$ is a parameter, and $m : [0, +\infty[\to [0, +\infty[$ is a locally Lipschitz continuous function. We prove that this problem has a unique global solution for positive times, provided that the initial data $(u_0, u_1) \in D(A) \times D(A^{1/2})$ satisfy a suitable smallness assumption and the non-degeneracy condition $m(|A^{1/2}u_0|^2) > 0$. Moreover $(u(t), u'(t), u''(t)) \to (u_\infty, 0, 0)$ in $D(A) \times D(A^{1/2}) \times H$ as $t \to +\infty$, where $|A^{1/2}u_\infty| m(|A^{1/2}u_\infty|^2) = 0$. These results apply to degenerate hyperbolic PDEs with non-local non-linearities.

Keywords: hyperbolic equations, degenerate hyperbolic equations, dissipative equations, global existence, asymptotic behaviour, Kirchhoff equations

1. Introduction

Let H be a real Hilbert space, with norm $|\cdot|$ and scalar product $\langle\cdot,\cdot\rangle$. Let A be a self-adjoint linear non-negative operator on H with dense domain $D(A)$ (i.e., $\langle Au, u\rangle \geqslant 0$ for all $u \in D(A)$). We consider the Cauchy problem

$$\begin{cases} u''(t) + \delta u'(t) + m\big(|A^{1/2}u(t)|^2\big)Au(t) = 0, & t \geqslant 0, \\ u(0) = u_0, \quad u'(0) = u_1, \end{cases} \tag{1.1}$$

where $\delta > 0$, and $m : [0, +\infty[\to [0, +\infty[$ is a locally Lipschitz continuous function.

Problem (1.1) is an abstract setting of the initial-boundary value problem for the hyperbolic PDE with a non-local non-linearity of Kirchhoff type

$$u_{tt} + \delta u_t - m\left(\int_\Omega |\nabla u|^2 \, dx\right)\Delta u = 0, \quad \text{in } \Omega \times [0, +\infty[, \tag{1.2}$$

where $\Omega \subseteq \mathbb{R}^n$ is a (non-necessarily bounded) open set, ∇u is the gradient of u with respect to space variables, and Δ is the Laplace operator.

If Ω is an interval of the real line, this equation is a model for the damped small transversal vibrations of an elastic string with fixed endpoints (see, e.g., [4] where this concrete equation is considered in the case $m(r) = a + br$, with $a, b > 0$).

The case $\delta = 0$ (free vibrations) has long been studied: the interested reader can find appropriate references in the surveys [1,8].

The non-degenerate case (i.e., $m(r) \geqslant \nu > 0$ for all $r \geqslant 0$, which in the physical model corresponds to a pre-stressed string) with $\delta > 0$ was considered by [2,3,5,9]: they proved that for small initial data $(u_0, u_1) \in D(A) \times D(A^{1/2})$ there exists a unique global solution of (1.1) that decays exponentially as $t \to +\infty$.

Degenerate equations ($m(r) \geqslant 0$ for all $r \geqslant 0$) were considered by [6]. In the case where $m(r) = r^\gamma$ ($\gamma \geqslant 1$), and A is a coercive operator with a discrete spectrum, they proved existence and uniqueness of a global solution of (1.1) for small initial data $(u_0, u_1) \in D(A) \times D(A^{1/2})$ with $u_0 \neq 0$. They also proved that

$$(u(t), u'(t), u''(t)) \to (0,0,0) \quad \text{in } D(A^{1/2}) \times H \times H \tag{1.3}$$

with a polynomial rate as $t \to +\infty$, and that $|Au(t)|$ is uniformly bounded. Note that the functional space considered in (1.3) is *not* the natural space where the solution is defined.

The main argument of [6] relies on the smallness of some *ad hoc* energies, and seems to work only if $m(r) = r^\gamma$ or m behaves near zero like r^γ.

In this paper we consider problem (1.1) where m is any non-negative locally Lipschitz continuous function, and A is any non-negative operator. We prove that there exists a unique global solution provided that $(u_0, u_1) \in D(A) \times D(A^{1/2})$ satisfy a suitable smallness assumption (cf. Theorem 2.2) and the non-degeneracy condition $m(|A^{1/2}u_0|^2) > 0$. In the general case this solution may not decay to zero as $t \to +\infty$. However we prove that

$$(u(t), u'(t), u''(t)) \to (u_\infty, 0, 0) \quad \text{in } D(A) \times D(A^{1/2}) \times H \tag{1.4}$$

as $t \to +\infty$, and that $|A^{1/2}u_\infty| \cdot m(|A^{1/2}u_\infty|^2) = 0$ (cf. Theorem 2.3).

In particular, if the operator A is coercive and $m(r) > 0$ for every $r > 0$, then necessarily $u_\infty = 0$.

We point out that our asymptotic result (1.4) is stated in the natural space, and therefore it is stronger then (1.3).

The abstract results may be applied in a standard way to the concrete equation (1.2): for example for the Cauchy problem with homogeneous Dirichlet boundary conditions it is enough to take $H = L^2(\Omega)$, $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$, $D(A^{1/2}) = H_0^1(\Omega)$, and $A = -\Delta$.

2. Statement of the results

In this section we state the main results of this paper. For completeness shake, we recall the following local existence result, which may be proved by fixed point theorems (a sketch of the proof is included in Section 3 for the convenience of the reader).

Theorem 2.1 (Local existence). *Let $\delta \geqslant 0$, let $m : [0, +\infty[\to [0, +\infty[$ be a locally Lipschitz continuous function, and let $(u_0, u_1) \in D(A) \times D(A^{1/2})$ with $m(|A^{1/2}u_0|^2) > 0$.*

Then there exists $T > 0$ such that problem (1.1) has a unique solution

$$u \in C^2([0,T[; H) \cap C^1([0,T[; D(A^{1/2})) \cap C^0([0,T[; D(A)).$$

Moreover, u can be uniquely continued to a maximal solution defined in an interval $[0, T_*[$, and at least one of the following statements is valid:

(i) $T_* = +\infty$;
(ii) $\limsup_{t \to T_*^-} |A^{1/2}u'(t)|^2 + |Au(t)|^2 = +\infty$;
(iii) $\liminf_{t \to T_*^-} m(|A^{1/2}u(t)|^2) = 0$.

We remark that the dissipative term plays no role in this local existence theorem. On the contrary, it plays a crucial role in the following result.

Theorem 2.2 (Global existence). *Let $\delta > 0$, and let $m : [0, +\infty[\to [0, +\infty[$ be a locally Lipschitz continuous function. Let us assume that the initial data $(u_0, u_1) \in D(A) \times D(A^{1/2})$ satisfy the non-degeneracy condition*

$$m(|A^{1/2}u_0|^2) > 0, \tag{2.1}$$

and the smallness assumption

$$\|m'\|_{L^\infty([0,E(0)])} \max\left\{ \frac{|u_1|}{m(|A^{1/2}u_0|^2)}, \frac{2}{\delta}\sqrt{F(0)} \right\} \sqrt{F(0)} < \frac{\delta}{4}, \tag{2.2}$$

where

$$E(0) = \frac{|u_1|^2}{m(|A^{1/2}u_0|^2)} + |A^{1/2}u_0|^2, \qquad F(0) = \frac{|A^{1/2}u_1|^2}{m(|A^{1/2}u_0|^2)} + |Au_0|^2.$$

Then problem (1.1) admits a unique global solution

$$u \in C^2([0, +\infty[; H) \cap C^1([0, \infty[; D(A^{1/2})) \cap C^0([0, \infty[; D(A)).$$

As a consequence of Theorem 2.2 we have that, if the initial data $(u_0, u_1) \in D(A) \times D(A^{1/2})$ satisfy the non-degeneracy condition (2.1), then problem (1.1) has a global solution for every large enough δ.

Theorem 2.3 (Asymptotic behaviour). *Let us assume that all the hypotheses of Theorem 2.2 are satisfied.*

Then we have that:

(i) $m(|A^{1/2}u(t)|^2) > 0$ *for all* $t \geqslant 0$;

(ii) *there exists* $u_\infty \in D(A)$ *such that*

$$u(t) \to u_\infty \quad \text{in } D(A), \tag{2.3}$$

$$u'(t) \to 0 \quad \text{in } D(A^{1/2}), \tag{2.4}$$

$$u''(t) \to 0 \quad \text{in } H, \tag{2.5}$$

as $t \to +\infty$. *Moreover* $|A^{1/2}u_\infty| \cdot m(|A^{1/2}u_\infty|^2) = 0$.

The proof of Theorem 2.3 relies on a result about the asymptotic behaviour of solutions of the linearization of (1.1) (cf. Lemma 3.2 for the precise statement).

3. Proofs

Proof of Theorem 2.1. Since the argument is standard, we only sketch the main steps of the proof.

Step 1. Let us set

$$m_0 := m(|A^{1/2}u_0|^2), \qquad m_* := \max\left\{1, \frac{2}{m_0}\right\},$$

$$E_0 := |u_1|^2 + m_0|A^{1/2}u_0|^2, \qquad F_0 := |A^{1/2}u_1|^2 + m_0|Au_0|^2,$$

$$c := \|m'\|_{L^\infty([0,2m_*E_0])}.$$

Let us consider the functional space

$$X_{L,T} := \{a \in \mathrm{Lip}([0,T];\mathbb{R}):\ a(0) = m_0,\ \|a'\|_{L^\infty([0,T])} \leqslant L\},$$

with

$$L := 2cm_*(E_0 + F_0), \qquad T := \frac{m_0 \cdot \ln 2}{2L}. \tag{3.1}$$

Since $LT < m_0/2$ it follows that

$$a(t) > \frac{m_0}{2} \quad \forall a \in X_{L,T},\ \forall t \in [0,T]. \tag{3.2}$$

We can therefore define

$$[\Phi(a)](t) = m(|A^{1/2}u(t)|^2),$$

where $u \in C^2([0,T];H) \cap C^1([0,T];D(A^{1/2})) \cap C^0([0,T];D(A))$ is the unique solution (see, e.g., [8]) of the linear problem

$$\begin{cases} u''(t) + \delta u'(t) + a(t)Au(t) = 0, & t \geqslant 0, \\ u(0) = u_0, \quad u'(0) = u_1. \end{cases} \tag{3.3}$$

Step 2. We show that Φ maps $X_{L,T}$ into itself.

To this end, it is enough to show that for all $a \in X_{L,T}$ we have that

$$\left|[\Phi(a)]'(t)\right| \leqslant L \quad \text{for a.e. } t \in [0,T]. \tag{3.4}$$

Let us introduce the functions

$$E(t) := |u'(t)|^2 + a(t)|A^{1/2}u(t)|^2, \qquad F(t) := |A^{1/2}u'(t)|^2 + a(t)|Au(t)|^2.$$

In a standard way it is possible to prove that

$$E(t) \leqslant E_0\,\mathrm{e}^{2LT/m_0}, \qquad F(t) \leqslant F_0\,\mathrm{e}^{2LT/m_0},$$

hence by (3.2) and (3.1)

$$|u'(t)|^2 + |A^{1/2}u(t)|^2 \leqslant m_*E_0\,\mathrm{e}^{2LT/m_0} = 2m_*E_0,$$

$$|A^{1/2}u'(t)|^2 + |Au(t)|^2 \leqslant m_*F_0\,\mathrm{e}^{2LT/m_0} = 2m_*F_0.$$

Therefore

$$\left|[\Phi(a)]'(t)\right| = \left|2m'(|A^{1/2}u(t)|^2)\langle Au(t), u'(t)\rangle\right| \leqslant c(|Au(t)|^2 + |u'(t)|^2) \leqslant 2cm_*(E_0 + F_0) = L,$$

and (3.4) is proved.

Step 3. With a standard argument it is possible to prove that Φ is continuous with respect to the norm of the space $C^0([0,T];\mathbb{R})$. Since $X_{L,T}$ is a compact and convex subset of $C^0([0,T];\mathbb{R})$, by Schauder's fixed point theorem it follows that Φ has at least one fixed point a. The corresponding solution u of (3.3) is a solution of (1.1).

Since m is locally Lipschitz continuous, uniqueness of the solution follows in a standard way.

Step 4. Let us prove the last part of the statement.

Let $[0, T_*[$ be the maximal interval where the solution exists, and let us assume by contradiction that (i), (ii), and (iii) are false. Then there exist two constants ν, M such that $m(|A^{1/2}u(t)|^2) \geqslant \nu > 0$ in a left neighborhood of T_*, and $|A^{1/2}u'(t)|^2 + |Au(t)|^2 \leqslant M$ for every $t \in [0, T_*[$. From this inequality it follows that

$$|A^{1/2}u(t)| = \left|A^{1/2}u_0 + \int_0^t A^{1/2}u'(\tau)\,\mathrm{d}\tau\right| \leqslant |A^{1/2}u_0| + M^{1/2}T_*. \tag{3.5}$$

Moreover, since the function

$$H(t) = |u'(t)|^2 + \int_0^{|A^{1/2}u(t)|^2} m(s)\,\mathrm{d}s$$

is non-increasing, we have that $|u'(t)|^2 \leqslant H(0)$, hence by (3.5) there exists a constant $N > 0$ such that

$$|u'(t)|^2 + m(|A^{1/2}u(t)|^2)|A^{1/2}u(t)|^2 \leqslant N$$

for all $t \in [0, T_*[$. By (3.1) it follows that, for all S in a left neighborhood of T_*, the life span of the solution of

$$\begin{cases} w''(t) + \delta w'(t) + m(|A^{1/2}w(t)|^2)Aw(t) = 0, & t \geqslant S, \\ w(S) = u(S), \quad w'(S) = u'(S), \end{cases}$$

is larger then a strictly positive quantity independent of S. This contradicts the maximality of T_*. □

In the sequel we need the following comparison result for ODEs (the simple proof is omitted).

Lemma 3.1. *Let $T > 0$, and let $f \in C^1([0, T[; \mathbb{R})$. Let us assume that $f(t) \geqslant 0$ in $[0, T[$, and that there exist two constants $c_1 > 0$, $c_2 \geqslant 0$ such that*

$$f'(t) \leqslant -\sqrt{f(t)}(c_1\sqrt{f(t)} - c_2) \quad \forall t \in [0, T[.$$

Then

$$\sqrt{f(t)} \leqslant \max\left\{\sqrt{f(0)}, \frac{c_2}{c_1}\right\}$$

for all $t \in [0, T[$.

Proof of Theorem 2.2. *Step 1.* Let us assume that $m \in C^1([0, +\infty[; \mathbb{R})$, and let $[0, T_*[$ be the maximal interval where the solution exists. Let us set

$$c(t) := m(|A^{1/2}u(t)|^2),$$

and

$$T := \sup\left\{\tau \in [0, T_*[: \left|\frac{c'(t)}{c(t)}\right| \leqslant \frac{\delta}{2},\ c(t) > 0\ \forall t \in [0, \tau]\right\}.$$

Let us consider the functions

$$E(t) := \frac{|u'(t)|^2}{c(t)} + |A^{1/2}u(t)|^2, \qquad F(t) := \frac{|A^{1/2}u'(t)|^2}{c(t)} + |Au(t)|^2, \qquad G(t) := \frac{|u'(t)|}{c(t)}.$$

With simple computations it follows that

$$E'(t) = -\frac{1}{c(t)}\left(2\delta + \frac{c'(t)}{c(t)}\right)|u'(t)|^2, \tag{3.6}$$

$$F'(t) = -\frac{1}{c(t)}\left(2\delta + \frac{c'(t)}{c(t)}\right)|A^{1/2}u'(t)|^2, \tag{3.7}$$

$$(G^2)'(t) \leqslant -G(t)\left\{2\left(\delta + \frac{c'(t)}{c(t)}\right)G(t) - 2|Au(t)|\right\}, \tag{3.8}$$

for every $t \in [0, T[$.

Step 2. We show that $T = T_*$.

Let us assume by contradiction that $T < T_*$. Since $|c'(t)| \leqslant (\delta/2)c(t)$ in $[0, T[$ we have that

$$0 < c(0)\,\mathrm{e}^{-\delta T/2} \leqslant c(T) \leqslant c(0)\,\mathrm{e}^{\delta T/2}. \tag{3.9}$$

Since c and c' are continuous functions, by the maximality of T we have that necessarily

$$\left|\frac{c'(T)}{c(T)}\right| = \frac{\delta}{2}. \tag{3.10}$$

From (3.6) and (3.7) it follows that E and F are non-increasing functions, hence

$$|A^{1/2}u(t)|^2 \leqslant E(t) \leqslant E(0), \tag{3.11}$$

$$|Au(t)|^2 \leqslant F(t) \leqslant F(0). \tag{3.12}$$

Moreover by (3.8) we have that

$$(G^2)'(t) \leqslant -G(t)(\delta G(t) - 2\sqrt{F(0)})$$

hence, by Lemma 3.1 with $f = G^2$,

$$G(t) \leqslant \max\left\{G(0), \frac{2}{\delta}\sqrt{F(0)}\right\}, \quad \forall t \in [0, T]. \tag{3.13}$$

By (3.11)–(3.13), and the smallness assumption (2.2), we have that

$$\left|\frac{c'(T)}{c(T)}\right| = \left|\frac{2m'(|A^{1/2}u(T)|^2)\langle u'(T), Au(T)\rangle}{c(T)}\right| \leqslant 2\max_{0\leqslant r\leqslant E(0)}|m'(r)|\frac{|u'(T)|}{c(T)}|Au(T)|$$
$$\leqslant 2\max_{0\leqslant r\leqslant E(0)}|m'(r)|\max\left\{G(0), \frac{2}{\delta}\sqrt{F(0)}\right\}\sqrt{F(0)} < \frac{\delta}{2}.$$

This contradicts (3.10).

Step 3. Let us assume by contradiction that $T_* < +\infty$. By (3.9) and (3.12) it follows that

$$\liminf_{t\to T_*^-} m(|A^{1/2}u(t)|^2) \geqslant m(|A^{1/2}u_0|^2)\,\mathrm{e}^{-\delta T_*/2} > 0,$$

$$\limsup_{t\to T_*^-} |A^{1/2}u'(t)|^2 + |Au(t)|^2 \leqslant \max\{1, c(0)\,\mathrm{e}^{\delta T_*/2}\}F(0) < +\infty.$$

By the last statement of Theorem 2.1 this is a contradiction. This completes the proof if m' is continuous.

Step 4. If m is only locally Lipschitz continuous, thesis follows from a standard approximation argument. □

In order to study the asymptotic behaviour of the solutions of (1.1), we consider the linearized problem

$$\begin{cases} v''(t) + \delta v'(t) + c(t)Av(t) = 0, \quad t \geqslant 0, \\ v(0) = v_0, \quad v'(0) = v_1. \end{cases} \tag{3.14}$$

In the following lemma we examine the asymptotic behaviour of the solutions of (3.14).

Lemma 3.2. *Let $\delta > 0$, and let $c : [0, +\infty[\to]0, +\infty[$ be a Lipschitz continuous bounded function such that*

$$\left|\frac{c'(t)}{c(t)}\right| \leqslant \frac{\delta}{2} \quad \text{for a.e. } t \geqslant 0.$$

Let v be the unique global solution of (3.14) with $(v_0, v_1) \in D(A) \times D(A^{1/2})$.
Then there exists $v_\infty \in D(A)$ such that

$$v(t) \to v_\infty \quad \text{in } D(A), \tag{3.15}$$

$$v'(t) \to 0 \quad \text{in } D(A^{1/2}), \tag{3.16}$$

$$v''(t) \to 0 \quad \text{in } H, \tag{3.17}$$

as $t \to +\infty$. Furthermore, if $Av_\infty \neq 0$, then necessarily $c(t) \to 0$ as $t \to +\infty$.

Proof. We divide the proof in several steps, according to the following strategy. In Steps 1–6 we prove that

$$Av(t) \text{ has a limit } w_\infty \text{ in } H \text{ as } t \to +\infty, \tag{3.18}$$

$$A^{1/2}v'(t) \to 0 \text{ in } H \text{ as } t \to +\infty, \tag{3.19}$$

$$c(t)|Av(t)| \to 0 \text{ as } t \to +\infty. \tag{3.20}$$

If the operator A is coercive (i.e., there exists $\nu > 0$ such that $\langle Au, u\rangle \geqslant \nu |u|^2$ for all $u \in D(A)$), then Lemma 3.2 is proved.

If A is not coercive, it remains to prove (Step 7) that

$$v(t) \text{ has a limit } v_\infty \text{ in } H \text{ as } t \to +\infty, \tag{3.21}$$

$$v'(t) \to 0 \quad \text{in } H \text{ as } t \to +\infty. \tag{3.22}$$

Indeed, since A is closed, from (3.18) and (3.21) it follows that $v_\infty \in D(A)$, $w_\infty = Av_\infty$, and $v(t) \to v_\infty$ in $D(A)$.

Step 1. Let us consider the function

$$F(t) := \frac{|A^{1/2}v'(t)|^2}{c(t)} + |Av(t)|^2.$$

A simple computation shows that

$$F'(t) = -\left(2\delta + \frac{c'(t)}{c(t)}\right)\frac{|A^{1/2}v'(t)|^2}{c(t)} \leqslant -\frac{3}{2}\delta\frac{|A^{1/2}v'(t)|^2}{c(t)}. \tag{3.23}$$

In particular, the function $F(t)$ is non-increasing, hence there exists

$$F_\infty := \lim_{t\to+\infty} F(t).$$

If $F_\infty = 0$, then (3.18) holds true with $w_\infty = 0$, while (3.19) and (3.20) follow from the boundedness of c.

Now in Steps 2–6 we deal with the case $F_\infty > 0$ (in particular in Step 4 we show that in this case necessarily $c(t) \to 0$).

Step 2. We show that

$$\int_0^\infty |A^{1/2}v'(t)|^2 \, dt < +\infty. \tag{3.24}$$

Indeed, integrating (3.23) we have that

$$\int_0^\infty \frac{|A^{1/2}v'(t)|^2}{c(t)} \, dt \leqslant -\frac{2}{3\delta}\int_0^\infty F'(t) \, dt = \frac{2}{3\delta}(F(0) - F_\infty).$$

Since the function c is bounded, (3.24) follows from this inequality.

Step 3. We show that

$$\int_0^\infty c(t)|Av(t)|^2 \, dt < +\infty. \tag{3.25}$$

Indeed, taking the scalar product of the equation with Av, we obtain that

$$\begin{aligned} 0 &= \langle v''(t), Av(t)\rangle + \delta\langle v'(t), Av(t)\rangle + c(t)|Av(t)|^2 \\ &= \left(\langle v'(t), Av(t)\rangle + \frac{\delta}{2}|A^{1/2}v(t)|^2\right)' - |A^{1/2}v'(t)|^2 + c(t)|Av(t)|^2. \end{aligned}$$

Integrating in $[0, T]$ it follows that

$$\begin{aligned} &\int_0^T c(t)|Av(t)|^2 \, dt \\ &= \langle v_1, Av_0\rangle + \frac{\delta}{2}|A^{1/2}v_0|^2 - \langle v'(T), Av(T)\rangle - \frac{\delta}{2}|A^{1/2}v(T)|^2 + \int_0^T |A^{1/2}v'(t)|^2 \, dt \\ &\leqslant \langle v_1, Av_0\rangle + \frac{\delta}{2}|A^{1/2}v_0|^2 + \frac{1}{2\delta}|A^{1/2}v'(T)|^2 + \int_0^T |A^{1/2}v'(t)|^2 \, dt \\ &\leqslant \langle v_1, Av_0\rangle + \frac{\delta}{2}|A^{1/2}v_0|^2 + \frac{1}{2\delta}\|c\|_\infty F(0) + \int_0^T |A^{1/2}v'(t)|^2 \, dt. \end{aligned}$$

Passing to the limit as $T \to +\infty$, (3.25) follows from (3.24).

Step 4. From (3.24) and (3.25) it follows that

$$\int_0^\infty c(t)F(t)\,dt < +\infty.$$

Since $F(t) \geqslant F_\infty > 0$, then also

$$\int_0^\infty c(t)\,dt < +\infty. \tag{3.26}$$

Since c is Lipschitz continuous, it follows that $c(t) \to 0$ as $t \to +\infty$, hence (3.20) is proved. Since $|A^{1/2}v'(t)|^2 \leqslant c(t)F(0)$, then also (3.19) is proved.

Step 5. We show that (3.18) holds true with the additional assumption that $(v_0, v_1) \in D(A^2)\times D(A^{3/2})$. To this end, let us introduce the function

$$\widehat{F}(t) := \frac{|A^{3/2}v'(t)|^2}{c(t)} + |A^2 v(t)|^2.$$

As in Step 1, it is possible to prove that $\widehat{F}$ is non-increasing, hence

$$|A^2 v(t)|^2 \leqslant \widehat{F}(0)$$

for every $t \geqslant 0$. Now let us consider the function

$$\widehat{G}(t) := \frac{|Av'(t)|}{c(t)}.$$

Then we have that

$$(\widehat{G}^2)'(t) \leqslant -\widehat{G}(t)\left\{2\left(\delta + \frac{c'(t)}{c(t)}\right)\widehat{G}(t) - 2|A^2 v(t)|\right\} \leqslant -\widehat{G}(t)\left\{\delta\widehat{G}(t) - 2\sqrt{\widehat{F}(0)}\right\},$$

hence, by Lemma 3.1 with $f = \widehat{G}^2$, it follows that

$$\widehat{G}(t) \leqslant \max\left\{\widehat{G}(0), \frac{2}{\delta}\sqrt{\widehat{F}(0)}\right\}.$$

By (3.26), this implies that

$$\int_0^\infty |Av'(t)|\,dt < +\infty$$

and therefore $Av(t)$ has a limit as $t \to +\infty$.

Step 6. We show that (3.18) hold true for every initial data $(v_0, v_1) \in D(A) \times D(A^{1/2})$.

To this end, let us consider a sequence $\{(v_{0n}, v_{1n})\} \subseteq D(A^2) \times D(A^{3/2})$ converging to (v_0, v_1) in $D(A) \times D(A^{1/2})$. Let $\{v_n\}$ be the corresponding solutions of (3.14), and let us set $w_n := v - v_n$. Since w_n is a solution of (3.14), we have that

$$|Aw_n(t)|^2 \leqslant \frac{|A^{1/2}w_n'(t)|^2}{c(t)} + |Aw_n(t)|^2 \leqslant \frac{|A^{1/2}(v_{1,n} - v_1)|^2}{c(0)} + |A(v_{0,n} - v_0)|^2.$$

This proves that $\{Av_n\} \to Av$ uniformly in $[0, +\infty[$. Since $Av_n(t)$ has a limit as $t \to +\infty$ for every $n \in \mathbb{N}$ (see Step 5), then necessarily $Av(t)$ has a limit as $t \to +\infty$.

This completes the proof of (3.18).

Step 7. Let us prove (3.21) and (3.22) in the case where A is not coercive. Let us consider the function

$$E(t) := \frac{|v'(t)|^2}{c(t)} + |A^{1/2}v(t)|^2.$$

Arguing as in Step 1, it is possible to prove that, for every $T \geqslant 0$,

$$\int_0^T |v'(t)|^2\,dt \leqslant \frac{2}{3\delta}\|c\|_\infty E(0), \qquad |v'(T)|^2 \leqslant \|c\|_\infty E(0).$$

Moreover, taking the scalar product of (3.14) with v, and arguing as in Step 3, it follows that

$$\begin{aligned}\int_0^T c(t)|A^{1/2}v(t)|^2\,dt + \frac{\delta}{2}|v(T)|^2 &= \int_0^T |v'(t)|^2\,dt + \frac{\delta}{2}|v_0|^2 + \langle v_1, v_0\rangle - \langle v'(T), v(T)\rangle \\ &\leqslant \int_0^T |v'(t)|^2\,dt + \frac{\delta}{2}|v_0|^2 + \frac{\delta}{2}|v_0|^2 + \frac{1}{2\delta}|v_1|^2 + \frac{\delta}{4}|v(T)|^2 + \frac{1}{\delta}|v'(T)|^2,\end{aligned}$$

hence

$$|v(T)|^2 \leqslant \frac{4}{\delta}\left\{\frac{5\|c\|_\infty}{3\delta}\left(\frac{|v_1|^2}{c(0)} + |A^{1/2}v_0|^2\right) + \delta|v_0|^2 + \frac{1}{2\delta}|v_1|^2\right\}. \tag{3.27}$$

Now we use the spectral decomposition of A, following the notations of [7].

Let us fix $\sigma > 0$. Let E be the resolution of the identity associated with A, and let us consider the orthogonal decomposition

$$H = N_A \oplus N_B^\sigma \oplus N_C^\sigma,$$

where

$$N_A = \mathcal{R}(E(\{0\})), \qquad N_B^\sigma = \mathcal{R}(E(]0, \sigma[)), \qquad N_C^\sigma = \mathcal{R}(E([\sigma, +\infty[)).$$

Let $v_A(t)$, $v_B(t)$, $v_C(t)$ be the components of $v(t)$. Since N_A is the kernel of A, then v_A solves the ODE

$$v_A'' + \delta v_A' = 0,$$

hence by a direct computation we see that $v_A(t)$ has a limit as $t \to +\infty$.

Since the restriction of A to N_C^σ is coercive, by the results of Steps 1–6 it follows that also $v_C(t)$ has a limit as $t \to +\infty$.

Moreover, applying (3.27) to the function v_B, we have that

$$|v_B(t)|^2 \leqslant \frac{4}{\delta}\left\{\frac{5\|c\|_\infty}{3\delta}\left(\frac{|(v_1)_B|^2}{c(0)} + |A^{1/2}(v_0)_B|^2\right) + \delta|(v_0)_B|^2 + \frac{1}{2\delta}|(v_1)_B|^2\right\}$$

for every $t \geqslant 0$.

This proves that $|v_B(t)|$ is small, uniformly in t, provided that σ is small enough. Since v_A and v_C have a limit for every $\sigma > 0$, then (3.21) follows by a standard argument.

The proof of (3.22) is completely analogous. □

Proof of Theorem 2.3. We use the same notations as in the proof of Theorem 2.2. Let us first remark that u is the solution of (3.14) with

$$c(t) = m(|A^{1/2}u(t)|^2), \qquad (v_0, v_1) = (u_0, u_1).$$

In Step 2 of the proof of Theorem 2.2, we showed that $c(t) > 0$ for every $t \geqslant 0$ (this proves statement (i)), and

$$\left|\frac{c'(t)}{c(t)}\right| \leqslant \frac{\delta}{2} \quad \forall t \geqslant 0.$$

Since (3.11) holds true for every $t \geqslant 0$, it follows that c is bounded. Since m is locally Lipschitz continuous, and $|A^{1/2}u'(t)|^2 \leqslant F(t)c(t) \leqslant F(0)c(t)$ (see (3.12)), then it turns out that c is globally Lipschitz continuous.

By Lemma 3.2, there exists $u_\infty \in D(A)$ such that (2.3)–(2.5) are satisfied. If $A^{1/2}u_\infty \neq 0$, hence $Au_\infty \neq 0$, then by the last statement of Lemma 3.2 we have that $c(t) \to 0$ as $t \to +\infty$, and therefore

$$0 = \lim_{t\to+\infty} m(|A^{1/2}u(t)|^2) = m(|A^{1/2}u_\infty|^2). \qquad \square$$

References

[1] A. Arosio, Averaged evolution equations. The Kirchhoff string and its treatment in scales of Banach spaces, in: *Proceedings of the 2nd Workshop on Functional-Analytic Methods in Complex Analysis*, Trieste, 1993, World Scientific, Singapore.

[2] E.H. de Brito, The damped elastic stretched string equation generalized: existence, uniqueness, regularity and stability, *Appl. Anal.* **13** (1982), 219–233.

[3] E.H. de Brito, Decay estimates for the generalized damped extensible string and beam equation, *Nonlinear Anal.* **8** (1984), 1489–1496.

[4] R.W. Dickey, Infinite systems of nonlinear oscillation equations with linear damping, *SIAM J. Appl. Math.* **19** (1970), 208–214.

[5] K. Nishihara, Global existence and asymptotic behaviour of the solution of some quasilinear hyperbolic equation with linear damping, *Funkcial. Ekvac.* **32** (1989), 343–355.

[6] K. Nishihara and Y. Yamada, On global solutions of some degenerate quasilinear hyperbolic equations with dissipative terms, *Funkcial. Ekvac.* **33** (1990), 151–159.

[7] W. Rudin, *Functional Analysis*, McGraw-Hill, New York, 1973.

[8] S. Spagnolo, The Cauchy problem for the Kirchhoff equation, *Rend. Sem. Mat. Fis. Milano* **62** (1992), 17–51.

[9] Y. Yamada, On some quasilinear wave equations with dissipative terms, *Nagoya Math. J.* **87** (1982), 17–39.

Asymptotic Analysis 40 (2004) 37–49
IOS Press

The three divergence free matrix fields problem

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Abstract. We prove that for any connected open set $\Omega \subset \mathbb{R}^n$ and for any set of matrices $K = \{A_1, A_2, A_3\} \subset \mathbb{M}^{m\times n}$, with $m \geqslant n$ and $\operatorname{rank}(A_i - A_j) = n$ for $i \neq j$, there is no non-constant solution $B \in L^\infty(\Omega, \mathbb{M}^{m\times n})$, called exact solution, to the problem

$$\operatorname{Div} B = 0 \quad \text{in } \mathcal{D}'(\Omega, \mathbb{R}^m) \quad \text{and} \quad B(x) \in K \quad \text{a.e. in } \Omega.$$

In contrast, Garroni and Nesi [10] exhibited an example of set K for which the above problem admits the so-called approximate solutions. We give further examples of this type. We also prove non-existence of exact solutions when K is an arbitrary set of matrices satisfying a certain algebraic condition which is weaker than simultaneous diagonalizability.

Keywords: differential inclusions, phase transitions, homogenization

1. Introduction

The problem of characterizing solenoidal matrix fields which take values in a finite set of matrices, has been recently considered by Garroni and Nesi. This kind of problem is analogous to that on curl free matrix fields in which one asks whether a Lipschitz mapping using a finite number of gradients exists. Here the differential constraint of being the gradient of a mapping, and hence a curl free matrix field, is replaced by that of being a divergence free matrix field (i.e., a matrix valued function whose rows are divergence free in the distributional sense). To describe the problem we begin with some definitions.

Definition 1. Given two integers $m, n \geqslant 2$, a set of real $m \times n$ matrices $K \subset \mathbb{M}^{m\times n}$ and a bounded open set Ω in $\mathbb{R}^n$, we say that any $B \in L^\infty(\Omega, \mathbb{M}^{m\times n})$ satisfying

$$\begin{cases} \operatorname{Div} B = 0 & \text{in } \mathcal{D}'(\Omega, \mathbb{R}^m), \\ B(x) \in K & \text{a.e. in } \Omega, \\ B \text{ is non-constant,} & \end{cases} \tag{1.1}$$

is an exact solution of (1.1). We say that K is rigid for exact solutions if there is no solution to (1.1).

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