Moreover, in some classes of spaces our pseudo solutions are also strong C-solutions (in separable Banach spaces for instance).

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A NOTE ON THE ABSTRACT CAUCHY–KOVALEVSKAYA THEOREM

Abstract. We give a version of the abstract Cauchy–Kovalevskaya Theorem for the Cauchy problem

$$u' = A(t,u), \quad u(0) = u_0$$

when A is not necessarily a Lipschitz continuous operator. The operator $A(t,u) = F(t,u,u)$ verifies

1) $F : I \times B_{r_1,R} \times B_{r,R} \to X_s$ for $s < r < r_0$ ($r_1 < r_0$ is fixed), $F(t,u,\cdot)$ is Lipschitz continuous, and $F(t,\cdot,\cdot)$ is α-Lipschitz continuous

or

2) $F : I \times B_{r_1,R} \times X_r \to X_s$ for $s < r < r_0$ ($r_1 < r_0$ is fixed), and $F(t,\cdot,\cdot)$ is α-Lipschitz continuous,

where $B_{r,R}$ denotes the ball of radius R in X_r.

We prove the result by using Tonelli approximations and fixed point theorems.

1. Introduction

Let us consider the Cauchy problem

$$(1.1)\qquad \begin{cases} u' = A(t,u), & t \in I, \\ u(0) = u_0 \in X_{r_0} \end{cases}$$

where $A(t,\cdot)$ is a continuous (but non necessarily Lipschitz continuous) operator in a scale of Banach spaces $(X_r)_{0<r\leq r_0}$ (see Definition 2.1).

Many authors considered this problem under a Lipschitz condition; i.e., there exist [1] $C, M > 0$ such that for $s < r$

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[1] $B_{r,R}(u_0)$ is the ball in X_r of radius R centered in u_0

$$(1.2)\quad \begin{cases} \|A(t,u)-A(t,w)\|_s \le \dfrac{C\|u-w\|_r}{r-s}, & \forall u,w\in B_{r,R}(u_0)\\[2ex] \|A(t,0)\|_s \le \dfrac{M}{r_0-s} \end{cases}$$

(see for example [3], [8], [10], [11], [12], for a bibliography we refer to [7]).

Moreover some authors proved that (1.1) has a local solution under non-Lipschitz assumptions.

K. Deimling [6] assumed that

$$A(t,u) = B(t)u + f(t,u),$$

where $B(t): X_r \to X_s$ $(s<r)$ is a continuous linear operator for every $t\in I$ and $f: I\times B_{r,R} \to X_{r_0}$ $(r<r_0)$, is a uniformly continuous regularizing function such that for some constant K there is

$$\alpha_{r_0}(f(t,B)) \le K\alpha_r(B) \quad (r<r_0), \quad \forall B\subseteq B_{r,R}(u_0)$$

where α_r is the Hausdorff noncompactness measure in X_r (see Definition 2.2).

Furthermore some authors ([4], [9], [14]) treated (1.1), by assuming conditions of "compact type", i.e., they supposed that the imbeddings $X_r \hookrightarrow X_s$ $(s<r)$ are compact, and that A is a continuous function, verifying:

$$\|A(t,u)\|_s \le \frac{M}{r-s} \quad \text{if} \quad \|u\|_r \le \frac{R}{r-s} \quad \text{H. Begehr [4];}$$

$$(1.3)\quad \|A(t,u)\|_s \le \frac{M+C\|u\|_r}{r-s} \qquad \forall t\in I, \quad u\in X_r, \quad \text{V. I. Nazarov [9];}$$

for some $0<\varepsilon<1$ and $u_0=0$:

$$\|A(t,u)\|_s \le C\frac{\|u\|_r}{r-s} + \frac{M}{(r_0-s)^\varepsilon} \qquad \forall t\in I, \quad u\in B_{r,R}(0)$$

in H. Reissig [14] (he treated some more general systems) Later on in [7] it was considered the case in which the imbeddings $X_r \hookrightarrow X_s$ $(s<r)$ are not necessary compact and $A: I\times X_r \to X_s$ $(s<r)$ is a Carathéodory (weakly Carathéodory) operator [2] such that:

1. A verifies (1.3);
2. there exists a constant K such that for every bounded set $U\subset X_r$

$$\alpha_s(A(I\times U)) \le K\frac{\alpha_r(U)}{r-s}.$$

[2] A is Carathéodory (weakly Carathéodory) if $A(t,\cdot)$ is continuous (weakly continuous) and $A(\cdot,u)$ is measurable (weakly measurable).

In this paper we give a generalization of the results of [6] and [7].

THEOREM 1.1. (see [6]) *Let $(X_r)_{0<r\le r_0}$ be a scale of Banach spaces and $0<r_1<r_0$. Let us assume that $A(t,u)=B(t,u,u)$ and that:*

(i) $B: I\times B_{r_1,R}(u_0)\times B_{r,R}(u_0) \to X_s$ $(s<r)$;
(ii) *$B(\cdot,v,u)$ is measurable and $B(t,\cdot,\cdot)$ is continuous;*
(iii) *$B(t,v,\cdot)$ is uniformly Lipschitz continuous (i.e. verifies (1.2) independently from v);*
(iv) *there exists a constant K such that* [3] *for every bounded set $U\subseteq B_{r,R}(u_0)$, and $V\subseteq B_{r_1,R}(u_0)$:*

$$\limsup_{|J|\to 0} \alpha_s(B(J,V,U)) \le K\Big(\frac{\alpha_r(U)}{r-s} + \alpha_{r_1}(V)\Big) \qquad (s<r).$$

Then problem (1.1) *has at least a local solution.*

The second one generalize [7].

THEOREM 1.2. *Let $(X_r)_{0<r\le r_0}$ be a scale of Banach spaces. Let $0<r_1<r_0$. Let us suppose that $A(t,u)=B(t,u,u)$ and that:*

(i) $B: I\times B_{r_1,R}(u_0)\times X_r \to X_s$ $(s<r)$,
(ii) *$B(\cdot,v,u)$ is measurable and $B(t,\cdot,\cdot)$ is continuous,*
(iii) *there exist $C,M>0$ such that*

$$\|B(t,v,u)\|_s \le \frac{C\|u\|_r+M}{r-s} \qquad (u\in X_r \quad v\in B_{r_1,R}(u_0)),$$

(iv) *there exists a constant K such that for every bounded set $U\subseteq B_{r,R}(u_0)$, and $V\subseteq B_{r_1,R}(u_0)$*

$$\limsup_{|J|\to 0} \alpha_s(B(J,V,U)) \le K\Big(\frac{\alpha_r(U)}{r-s} + \alpha_{r_1}(V)\Big) \qquad (s<r).$$

Then problem (1.1) *has at least a local solution.*

Let us remark that in the case of a single Banach space the assumption (iv) of Theorems 1.1, 1.2 was introduced by G. Pianigiani [13].

These results may be applied to prove the existence of local analytic solutions of systems of PDE. For example by using Theorem 1.1, setting [4]

$$B(t,V,U) = A(\|v_1\|_2^2,\dots,\|v_n\|_2^2)U_x,$$

we can prove that the problem

$$(1.4)\qquad U_t = A(\|u_1\|_2^2,\dots,\|u_n\|_2^2)U_x, \quad U(0,x)=U_0(x)$$

[3] $|J|$ denote the Lebesgue measure of the interval J

[4] $\|u\|_2$ is the L^2 norm of u

has a local L^2-analytic solution, where $U = (u_1, \dots, u_n)$ is a n-dimensional vector, $A(\cdot) : \Re^n \to \Re$ is a continuous $n \times n$ matrix and U_0 is a L^2-analytic vector, that is there exists $r_0 > 0$ such that

$$\sum_{j\in N} \left\| \frac{d^j U_0}{dx^j} \right\|_2^2 \frac{r_0^{2j}}{(j!)^2} < \infty.$$

This result improves the one well known in the particular case of the Kirchhoff equation $u_{tt} = m(\|u_x\|_2^2)u_{xx}$, where $m \geq 0$ is a continuous real function.

2. Preliminaries

DEFINITION 2.1. A scale of Banach spaces is a family of Banach spaces $(X_r)_r$ (where $0 < r \leq r_0$) with norm $\|\cdot\|_r$ such that

$$X_r \hookrightarrow X_s \quad \text{and} \quad \|\cdot\|_s \leq \|\cdot\|_r \qquad (s < r).$$

DEFINITION 2.2. Let X be a Banach space, C a bounded subset of X. The Hausdorff noncompactness measure of C is

$\alpha_X(C)$

$= \inf\{\varepsilon > 0 : C$ can be covered by a finite number of balls of radius $\varepsilon\}$.

When the Banach space X is unambiguosly determined by the context, we denote α_X only by α.

Let C be a subset of X. Let us indicate by $cl(C)$ the closure of C and by $co(C)$ its convex hull.

PROPOSITION 2.3. (see [6], p. 19). *Let A, B be bounded subsets of X, then:*

1. $\alpha(co(A)) = \alpha(A)$;
2. $\alpha(cl(A)) = \alpha(A)$;
3. $\alpha(A \cup B) \leq \alpha(A) \vee \alpha(B)$;
4. $\alpha(A) = 0$ *if and only if A is relatively compact;*
5. $\alpha(\lambda A) = |\lambda|\alpha(A)$;
6. $\alpha(A + B) \leq \alpha(A) + \alpha(B)$;
7. $\alpha(A) \leq \alpha(B)$ *if* $A \subset B$.

We recall now some well known results that we use in the sequel.

THEOREM 2.4. (see [6] p. 25). *Let C be a closed, bounded convex subset of X. Let $F : C \to C$ be continuous and F be α-condensing, that is there exists $K < 1$ such that: $\alpha(F(C)) \leq K\alpha(C)$. Then F has at least one fixed point.*

PROPOSITION 2.5. (see [1], Lemma 2.1). *Let $I := [0, a]$, $L \subseteq C^0(I, X) = H$. Let us suppose that there exists $K > 0$ such that*

$$v \in L \Rightarrow \|v(t) - v(s)\| \leq K|t - s| \quad t, s \in I.$$

Then $\alpha_H(L) = \sup_{0\leq t\leq a} \alpha_X(L(t))$, where $L(t) := \{x \in X : \ \exists u \in L \ \ x = u(t)\}$.

PROPOSITION 2.6. (see [6] p. 25). *Let $(X_n)_n$ be a nonincreasing sequence of nonempty bounded closed subsets of X such that $\lim_{n\to+\infty} \alpha(X_n) = 0$. Then $X_\infty = \bigcap X_n$ is a compact nonempty set.*

DEFINITION 2.7. A set valued map F is upper semicontinuous at x if for every open set $N \supseteq F(x)$ there exists a neighborhood M of x such that $F(M) \subseteq N$.

Let us remark that if $F : X \to 2^X$, where X is a complete metric space and $F(x)$ is relatively compact, then F is upper semicontinuous at x if and only if

(2.1) $$\forall\varepsilon > 0 \ \ \exists\eta > 0 : v \in B(x, \eta) \Rightarrow F(v) \subseteq B(F(x), \varepsilon).$$

For the set valued maps the following fixed point theorem holds true.

THEOREM 2.8. (see [2]). *Let K be a convex compact subset of a Banach space X. Let $F : K \to 2^K$ be an upper semicontinuous map. Then F has a fixed point, i.e., there exists $x \in K$ such that $x \in F(x)$.*

We shall use also the known existence result for (1.1) in the case of Lipschitz continuous mappings.

THEOREM 2.9. *Let us suppose that $A : I \times B_{r,R}(u_0) \to X_s \quad (s < r)$ and that $A(\cdot, u)$ is measurable and $A(t, \cdot)$ is Lipschitz continuous* (see (1.2)).

Then the problem (1.1) *has a unique local solution u and for every $\varepsilon > 0$ there exists $S_\varepsilon = S_\varepsilon(R, M, C)$ such that $u(t) \in B_{r_\varepsilon(t),R}(u_0)$, where $r_\varepsilon(t) = 1 - \varepsilon - S_\varepsilon t$ for $t \leq \frac{1-\varepsilon}{S_\varepsilon}$.*

3. Proofs of the result

First of all let us remark that we can assume $u_0 = 0$, and $r_0 = 1$ without loss of generality; moreover in the following we denote $B_{r,R}(0)$ only by $B_{r,R}$. We divide the proofs in some parts.

3.1. An "auxiliary" problem

Let us set

$$L = C^0([0, a]; B_{r_1,R}),$$
$$L_\beta = \{v \in L : \|v(t) - v(s)\|_{r_1} \leq \beta|t - s| \quad \forall t, s \in [0, a]\}$$

for $a > 0$, $\beta > 0$. Let v be in L. In this section we discuss the problem of existence of local solutions for

(3.1) $$u' = B(t, v, u), \quad u(0) = 0.$$

The following lemma is a straightforward consequence of Theorem 2.9.

LEMMA 3.1. *Let us suppose that the hypotheses of theorem 1.1 are verified. Then there exist $a > 0$, $\beta > 0$ such that for every $v \in L$ the problem (3.1) has a solution u defined on $[0,a]$ such that $u(t) \in B_{r(t)+\beta,R}$, where $r(t) = 1 - St - \beta$ for some $S > 0$ and $r(a) \geq r_1$.*

Now let us set

$$(3.2)\quad S = 4\max\left\{\frac{M}{\nu R} + C, 2K\right\} \quad a = \frac{1}{4}S^{-1}(1-r_1), \quad \nu = \frac{1}{12}(1-r_1).$$

LEMMA 3.2. *Let us suppose that the hypotheses of Theorem 1.2 are verified. Then for every $v \in L$ the problem (3.1) has a solution u defined on $[0,a]$ such that $u(t) \in B_{r(t)+\beta,R}$ where a, S are defined by (3.2), $r(t) = 1 - St - \frac{3}{4}(1-r_1)$ and $\beta = \frac{1}{2}(1-r_1)$.*

Proof. We can use the same method as in [7]; so we give only the outline of the proof. We introduce for $n \in \mathbb{N}$ the following approximate problems of Tonelli type (see [15])

$$(3.3)\quad \begin{cases} u_n(t) = 0, & t \leq 0 \\ u_n(t) = \int_0^t B(\tau, v(\tau), u_n(\tau - \frac{a}{n}))\,d\tau, & 0 \leq t \leq a. \end{cases}$$

It is easy to see that for every $n \in \mathbb{N}$ the problem (3.3) has a solution defined on $[0,a]$ and that $u_n(t) \in X_r$ for every $r < 1$ and $t \in [0,a]$.

Step 1 (some well-known estimate). Let us define:

$$\|v\|_{s,t} = \|v\|_s(1 - St - s).$$

It is easy to see that (see [3] and [7]):

$$(3.4)\quad \|u_n(t)\|_{s,t} \leq \nu R \quad \forall t \leq a, \quad s < 1 - St.$$

If we then set $\rho(t) = 1 - St - \nu$, we have $\|u_n(t)\|_{\rho(t)} \leq R$.

Step 2 (equicontinuity). Now let us define $\rho_1(t) = 1 - St - 2\nu$. For every $t \leq a$ the functions $u_n : [0,t] \to B_{\rho_1(t),R}$ are equi-Lipchitz continuous. Indeed if $\sigma, \tau \in [0,t]$, $\sigma < \tau$, then

$$\|u_n(\sigma) - u_n(\tau)\|_{\rho_1(t)} \leq \int_\sigma^\tau \left\|B\left(x, v(x), u_n\left(x - \frac{a}{n}\right)\right)\right\|_{\rho_1(t)} dx$$
$$\leq \nu^{-1}(CR + M)|\tau - \sigma|.$$

Step 3 (compactness). For a subset U of X_s let us denote by $cl_s(U)$ the closure of U in X_s and by $co(U)$ its convex hull. Let us set

$$\Omega(t) = \{u_n(t) : n \in \mathbb{N}\}, \qquad \Omega(\sigma, t) = \bigcup_{\sigma \leq \tau \leq t} \Omega(\tau),$$

$$\alpha(\Omega((0,a))) = \sup_{t \leq a}\ \sup_{s < \rho_1(t)} (\rho_1(t) - s)\alpha_s(\Omega(t)).$$

Let $\varepsilon > 0$, using step 2, we can find $\delta > 0$ such that

$$0 < t - \tau \leq \delta \Longrightarrow \Omega(\tau, t) \subseteq \Omega(t) + \varepsilon B_{\rho_1(t),1}.$$

Now let $t \in [0,a]$, and $(\lambda_i), i = 0, \dots, 2^m - 1$ be a subdivision of $[0,t]$ in 2^m equal parts, with $2^{-m} \leq \delta$. Let $k \geq m$ and (t_j^k) be a finite partition of $[0,t]$ in 2^k equal parts. Let $s < \rho_1(t)$. Then, as in [7], for all $n_0 \geq 1$ we get

$$\Omega_{n_0}(t) \subseteq \sum_{j=0}^{2^k-1} (t_{j+1}^k - t_j^k) cl_s co(B([t_j^k, t_{j+1}^k], V, \Omega(t_j^k - \frac{1}{n_0}, t_{j+1}^k)) \cup \{0\}),$$

where $V = \{v(t) : t \in [0,a]\}$, $\Omega_{n_0}(t) = \{u_n(t) : n \geq n_0\}$.

We denote by $i(j,k)$, the index i such that $t_j^k \in [\lambda_{i(j,k)}, \lambda_{i(j,k)+1}]$, $j = 0, \dots, 2^k - 1$, $k \geq m$. Then, if $\frac{1}{n_0} \leq \delta$, we find

$$\Omega(t_j^k - \frac{1}{n_0}, t_{j+1}^k) \subseteq \Omega(\lambda_{i(j,k)+1}) + 2\varepsilon B_{\rho_1(\lambda_{i(j,k)+1}),1};$$

hence

$$\Omega_{n_0}(t) \subseteq \sum_{j=0}^{2^k-1} (t_{j+1}^k - t_j^k) cl_s co(B([t_j^k, t_{j+1}^k], V, \Omega(\lambda_{i(j,k)+1})$$
$$+ 2\,\varepsilon B_{\rho_1(\lambda_{i(j,k)+1}),1}) \cup \{0\}).$$

Furthermore let us set

$$\phi_s(\tau) = \begin{cases} 1 & \text{if } \tau \leq 0, \\ \frac{1}{2}(\rho_1(\tau) + s) & \text{otherwise;} \end{cases}$$

and

$$c_{k,j} = B([t_j^k, t_{j+1}^k], V, \Omega(\lambda_{i(j,k)+1}) + 2\,\varepsilon B_{\rho_1(\lambda_{i(j,k)+1}),1}).$$

By (iv) there exists $\overline{k}$ such that for $k \geq \overline{k}$ and for every $j = 0, \dots, 2^k - 1$, $i = 0, \dots, 2^m - 1$

$$\alpha_s(c_{kj}) \leq K\left(\frac{\alpha_{\phi_s(\lambda_{i(j,k)+1})}(\Omega(\lambda_{i(j,k)+1})) + 2\,\varepsilon}{\phi_s(\lambda_{i(j,k)+1}) - s} + \alpha_{r_1}(V)\right) + \varepsilon.$$

Since V is compact in X_{r_1}, and $\Omega(t) \setminus \Omega_{n_0}(t)$ is a finite set, we obtain

$$\alpha_s(\Omega(t)) \leq \varepsilon + K \sum_{j=0}^{2^k-1} (t_{j+1}^k - t_j^k)\left(\frac{\alpha_{\phi_s(\lambda_{i(j,k)+1})}(\Omega(\lambda_{i(j,k)+1})) + 2\,\varepsilon}{\phi_s(\lambda_{i(j,k)+1}) - s}\right)$$

$$\le \varepsilon\Big(1+2K\sum_{j=0}^{2^k-1}\frac{t_{j+1}^k-t_j^k}{\phi_s(\lambda_{i(j,k)+1})-s}\Big)$$
$$+K\alpha(\Omega(0,a))\sum_{j=0}^{2^k-1}\frac{t_{j+1}^k-t_j^k}{(\phi_s(\lambda_{i(j,k)+1})-s)^2}.$$

Moreover, by the choice of λ_i and t_j^k, we get

$$\sum_{j=0}^{2^k-1}\frac{t_{j+1}^k-t_j^k}{(\phi_s(\lambda_{i(j,k)+1})-s)^q}=\sum_{i=0}^{2^m-1}\frac{\lambda_{i+1}-\lambda_i}{(\phi_s(\lambda_{i+1})-s)^q},\quad q=1,2;$$

hence, by taking first $m\to+\infty$ and second $\varepsilon\to 0$, we have

$$\alpha_s(\Omega(t))\le K\alpha(\Omega(0,a))\int_0^t\frac{1}{(\phi_s(\tau)-s)^2}\,d\tau$$
$$\le+\frac{4K}{S}\alpha(\Omega(0,a))\frac{1}{(\rho_1(t)-s)}.$$

Therefore

$$\alpha(\Omega(0,a))=0. \tag{3.5}$$

Now let us set $r_1(t)=1-St-3\nu$; thanks to (3.5) for every $t\in[0,a]$, the set $\Omega(t)$ is compact in $B_{r_1(t),R}$.

Step 4 (final step) By the Ascoli-Arzelà theorem and by a diagonal argument (see [14] and [7]) we can prove that there exists $u_{n_k}\to u$, where u is solution of (3.1) $u:[0,t]\to B_{r(t)+\beta,R}$, for $t\le a$: where $r(t)=r_1(t)-\beta$, $\beta=\frac{1}{2}(1-r_1)$, $r_1(a)\ge r_1+\beta$. ∎

3.2. Properties of solutions

Let us denote by u_v the solutions of (3.1) as in Lemmas 3.1, 3.2.

We have the following result.

LEMMA 3.3. *The functions u_v are equi-Lipschitz continuous, i.e. for every $t\in[0,a]$ there is*

$$\|u_v(\tau)-u_v(\sigma)\|_{r(t)+\frac{\beta}{2}}\le\frac{2}{\beta}(CR+M)|\tau-\sigma|,\quad \tau,\sigma\in[0,t].$$

Proof. It is enough to use the method of step 2 in the proof of Lemma 3.2, by remarking that in both the cases

$$\|B(t,v,u)\|_s\le\frac{M+CR}{r-s}\quad(u\in B_{r,R},\quad v\in B_{r_1,R},\quad s<r).\ ∎$$

Let us set

$$\gamma=2\frac{CR+M}{\beta}. \tag{3.6}$$

REMARK 3.4. Let a,β be as in Lemmas 3.1, 3.2 and $v\in L_\gamma$. Let u_v be a solution of (3.1) as in Lemmas 3.1–3.2. Then, thanks to Lemma 3.3, $u_v\in L_\gamma$.

If we define

$$V=\bigcup_{0\le t\le a}\{v(t):v\in L_\gamma\},\quad \Omega(t)=\{u_v(t):v\in L_\gamma\},$$
$$\alpha(\Omega(0,a))=\sup_{0\le t\le a}\sup_{s<r(t)+\frac{\beta}{2}}\Big(r(t)+\frac{1}{2}\beta-s\Big)\alpha_s(\Omega(t));$$

we can also prove the following lemma, using the method of step 3 in the proof of Lemma 3.2.

LEMMA 3.5. *It holds true*

$$\alpha(\Omega(0,a))\le\frac{4K}{S}\alpha(\Omega(0,a))+K\,a\,\alpha_{r_1}(V).\ ∎$$

3.3. Proof of Theorem 1

Let us remark that in Lemma 3.1 we can suppose $S\ge 5K$ and replace a by a positive number a_1 such that

$$r(a_1)-5Ca_1\ge r_1\quad\text{and}\quad \lambda=\frac{10\,K\,a_1}{\beta}<1.$$

Let us consider the map $F:L_\gamma\to L_\gamma$ (where γ is as in (3.6)) defined by

$$F(v)=u_v$$

where u_v is the solution of (3.1). Thanks to Remark 3.4, the map F is well defined.

If we prove that

1. F is continuous;
2. F is α-condensing,

then, by Theorem 2.4, the map F has at least one fixed point.

Ad 1. Let $v_n\to v$ in L_γ. Define $\rho(t)=r(a_1)-5Ct+\frac{\beta}{2}$. Let $t\in[0,a_1]$, $s<\rho(t)$ and

$$h_s(\tau)=\begin{cases}1 & \text{if } \tau\le 0\\ \dfrac{\rho(\tau)+s}{2} & \text{otherwise.}\end{cases}$$

Hence

$$\|u_{v_n}(t)-u_v(t)\|_s\le\int_0^t\|B(\tau,v_n(\tau),u_{v_n}(\tau))-B(\tau,v(\tau),u_v(\tau))\|_s\,d\tau$$

$$\leq \int_0^t \frac{C\|u_{v_n}(\tau)-u_v(\tau)\|_{h_s(\tau)}}{h_s(\tau)-s}d\tau + \int_0^t \|B(\tau,v_n(\tau),u_v(\tau))-B(\tau,v(\tau),u_v(\tau))\|_s\,d\tau.$$

Let us set

$$|\!|\!|u_{v_n}-u_v|\!|\!| = \sup_{0\leq t\leq a_1}\sup_{s<\rho(t)}(\rho(t)-s)\|u_{v_n}(t)-u_v(t)\|_s.$$

Therefore

$$\|u_{v_n}(t)-u_v(t)\|_s \leq \tfrac{4}{5}|\!|\!|u_{v_n}-u_v|\!|\!|\frac{1}{\rho(t)-s} + \int_0^t \|B(\tau,v_n(\tau),u_v(\tau))-B(\tau,v(\tau),u_v(\tau))\|_s d\tau.$$

Since $s<r(a_1)+\frac{\beta}{2}$, we have

$$|\!|\!|u_{v_n}-u_v|\!|\!| \leq 5\int_0^{a_1}\|B(\tau,v_n(\tau),u_v(\tau))-B(\tau,v(\tau),u_v(\tau))\|_{r(a_1)+\frac{\beta}{2}}\,d\tau;$$

hence, for some $\delta_1>0$ one gets

$$\sup_{0\leq t\leq a_1}\|u_{v_n}(t)-u_v(t)\|_{r_1} \leq \delta_1\int_0^{a_1}\|B(\tau,v_n(\tau),u_v(\tau))-B(\tau,v(\tau),u_v(\tau))\|_{r(a_1)+\frac{\beta}{2}}\,d\tau.$$

Finally, by the Lebesgue theorem, for the dominate convergence $u_{v_n}\to u_v$ in L; so F defined by (3.6), is continuous.

Ad 2. By Lemma 3.5, we get $\sup_t \alpha_{r_1}(\Omega(t))\leq\lambda\,\alpha_{r_1}(V)$. Then, by Proposition 2.5, we have $\alpha_H(F(L_\gamma))\leq\lambda\;\alpha_H(L_\gamma)$, where α_H is the Hausdorff noncompactness measure in $H=C^\circ([0,a_1],X_{r_1})$. Since $\lambda<1$, then F is α-condensing. ∎

3.4. Proof of Theorem 2

We can suppose that

$$\lambda=\frac{10\,K\,a}{\beta}<1.$$

Let us consider the map $F: L_\gamma\to 2^{L_\gamma}$ (where γ is as in (3.6)) defined as follows:

$u_v\in F(v)$ if u_v is a solution of (3.1), that for every $t\in[0,a]$ and $s<r(t)+\frac{\beta}{2}$ verifies

$$u_v(t)\in B_{s,R}\quad\text{and}\quad\|u_v(\sigma)-u_v(\tau)\|_s\leq\gamma|\tau-\sigma|\quad\text{for}\quad\sigma,\tau\in[0,t].$$

Thanks to Lemmas 3.2, 3.3 and Remark 3.4, the map F is well defined, since for every v the set $F(v)$ is not empty. If we prove that:

1. there exists a compact convex set $N\subseteq L_\gamma$ such that $F:N\to 2^N$;
2. $F|_N$ is upper semicontinuous,

then, by Theorem 2.8 the map F has at least one fixed point, solution of (1.1).

Ad 1. By Lemma 3.5 and Proposition 2.5, we get $\sup_{0\leq t\leq a}\alpha_{r_1}(\Omega(t))\leq\lambda\alpha_{r_1}(V)$ and

$$\alpha_H(F(L_\gamma))\leq\lambda\alpha_H(L_\gamma) \tag{3.7}$$

where α_H is the Hausdorff noncompactness measure in $H=C^\circ([0,a],X_{r_1})$.

Define

$$Y_n=cl_{r_1}co\Big(\bigcup_{v\in L_\gamma}F^n(v)\Big),$$

where $cl_{r_1}(U)$, denotes the closure of U and $co(U)$ its convex hull in X_{r_1}.

Let us remark that Y_n is a nonincreasing sequence of nonempty closed convex bounded sets and, by (3.7) $\lim_{n\to+\infty}\alpha_H(Y_n)=0$. By Proposition 2.6, $N=\bigcap Y_n$ is a nonempty convex compact set; moreover $F:N\to 2^N$.

Ad 2. For every $v\in N$, the set $F(v)$ is relatively compact. Now let us assume, by contradiction, that there exists v_0 such that F is not upper semicontinuous at v_0. Then (see (2.1)), there exist $\varepsilon>0$, $v_n\in N$ and $u_n\in F(v_n)$ such that [5]

$$\|v_0-v_n\|_H\leq\frac{1}{n}\quad\text{and}\quad\|u_n-w\|_H\geq\varepsilon\quad\forall w\in F(v_0). \tag{3.8}$$

Since $(u_n)_n\subseteq N$, there exists a subsequence $u_{n_k}\to u$ in H. Therefore by the Lebesgue theorem for the dominate convergence,

$$u(t)=\lim_{k\to+\infty}u_{n_k}(t)=\int_0^t B(\tau,v_0(\tau),u(\tau))\,d\tau.$$

Now let us prove that $u\in F(v_0)$. Remark that

$$\alpha_{r_1}\Big(\bigcup_{0\leq t\leq a}\{v_n(t):n\in\mathbf{N}\}\Big)=0.$$

Let us set

$$\Omega_1(t)=\{u_n(t):n\in\mathbf{N}\},$$

$$\alpha(\Omega_1(0,a))=\sup_{0\leq t\leq a}\sup_{s<r(t)+\frac{\beta}{2}}\Big(r(t)+\frac{\beta}{2}-s\Big)\alpha_s(\Omega_1(t)).$$

[5] $\|\cdot\|_H$ is the norm in H

As in Lemma 3.5 we have $\alpha(\Omega_1(0,a)) = 0$. Therefore, by Proposition 2.5, for every $t \in [0,a]$ the sequence u_{n_k} is relatively compact in $C^0([0,t], B_{s,R})$ for every $s < r(t) + \frac{\beta}{2}$. By this fact, $u(t) \in B_{s,R}$ for every $s < r(t) + \frac{\beta}{2}$, and

$$\|u(\tau) - u(\sigma)\|_s \leq \gamma|\tau - \sigma| \quad \sigma, \tau \in [0,t].$$

Hence $u \in F(v_0)$, but, by (3.8); $\|u - w\|_{r_1} \geq \varepsilon \ \forall w \in F(v_0)$. By this contradiction, F must be upper semicontinuous at v_0. ■

Final remarks

It is also possible to prove Theorems 1.1, 1.2 directly (i.e., without introducing the auxiliary problem (3.1)), using Tonelli approximations instead of fixed point theorems. By using this latter method, one can prove also the analogous of Theorems 1.1, 1.2 with respect to the weak topology, *i.e.* these results are still true if we replace: measurable by weakly measurable, continuous by weakly continuous and the noncompactness measure by the weak noncompactness measure [6].

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[6] If U is a bounded subset of a Banach space X, the weak noncompactness measure of U is (see [5]):

$$\alpha_w(U) = \inf\{\varepsilon > 0 : \exists K_\varepsilon, \text{weakly compact, such that} \quad U \subset K_\varepsilon + \varepsilon B_1\}.$$

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