1. Introduction

Let $E$ be a set of finite perimeter in $\mathbb{R}^n$. The isoperimetric deficit of $E$ is defined as

$$D(E) := \frac{P(E) - P(B_r)}{P(B_r)}$$

where $B_r$ is the ball having the same volume of $B$, that is $|E| = \omega_n r^n$. For a general set of finite perimeter the Fraenkel asymmetry of $E$, measuring in a natural way how far is $E$ from being a ball of the same volume, is defined as

$$\lambda(E) = \min \left\{ \frac{|E \triangle B_r(x)|}{r^n} : x \in \mathbb{R}^n, \ |E| = \omega_n r^n \right\}.$$

We recall the following quantitative version of the classical isoperimetric inequality proved in [2] see also [?]

$$\lambda(E) \leq c \sqrt{D(E)}$$

where $c$ is a constant depending only on the dimension.

The deviation from spherical shape of a set $K \in \mathcal{K}$ is defined by

$$\lambda_H(K) = \min_{x \in \mathbb{R}^n} \left\{ \frac{d_H(K, B_r(x))}{r} : |K| = \omega_n r^n \right\}.$$

The main result of the paper is the following:

**Theorem 1.1.** Let $0 < \gamma < 1$. There exists $0 < \delta_\gamma < 1$ such that for any $K \in \mathcal{C}_\gamma$ with $D(K) < \delta_\gamma$ then

$$\lambda_H(K) \leq C \begin{cases} D(K)^{\frac{1}{2}} & \text{for } n = 2 \\ D(K)^{\frac{1}{2}} \left( \log \frac{1}{D(K)} \right)^{\frac{1}{2}} & \text{for } n = 3 \\ D(K)^{\frac{n-1}{n-2}} & \text{for } n \geq 4 \end{cases}$$

where $C$ is a constant depending only on $\gamma$ and $n$. 

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Remark 1.2. (provv.) We point out that, up to a suitable rescaling in the definition of $\lambda_H$, equality (2) coincides with the one proved in Theorem 1.2 in [1] in the particular case of nearly spherical domains. This inequality is known to be optimal in case $n = 2$ or $n = 3$ see Example 3.1 in [1].

2. Preliminaries

- Richiami sugli insiemi di perimetro finito funzioni BV
- Formule di area e coarea
- Formule per il calcolo della superficie e del volume di un dominio nearly spherical

Throughout we will denote by $B_r(x)$ the closed ball centered in $x$ with radius $r$ and we will write simply $B_r$ if the center is at the origin. The volume of the unit ball will be denoted by $\omega_n$.

We denote by $K$ the family of all compact subsets of $\mathbb{R}^n$, and recall that if $H, K \in K$ the Hausdorff distance between $K$ and $H$ is defined as

$$d_H(K, H) = \inf\{\varepsilon > 0 : H \subset K + B_{\varepsilon}, K \subset H + B_{\varepsilon}\}.$$ 

Let us now introduce a class of sets of finite perimeter satisfying a mild regularity property, namely an interior cone property with aperture $\frac{\pi}{2}$. To this aim, given $x_0 \in \mathbb{R}^n$, $\gamma > 0$ and $\nu \in S^{n-1}$ the spherical sector with vertex in $x_0$, axis of symmetry parallel to $\nu$ and height $\gamma$ is defined as

$$S_{x_0, \gamma, \nu} := \left\{ x \in \mathbb{R}^n : |x - x_0| \leq \gamma, \langle x - x_0, \nu \rangle \geq \frac{\sqrt{2}}{2} |x - x_0| \right\}.$$ 

Finally for $\gamma > 0$ we set

$$C_\gamma = \left\{ E \in C : |E| < \infty \text{ and } \forall x \in \partial E \; \exists \nu \in S^{n-1} \; \text{with} \; S_{x, \gamma', \nu} \subset E, \; \gamma' = \gamma |E|^{\frac{n}{n-2}} \omega_n^{\frac{2}{n}} \right\}.$$ 

Remark 2.1. We point out that the interior cone property required for sets in $C_\gamma$ is a very mild regularity assumption. Indeed one can construct a compact set of finite perimeter $K \subset \mathbb{R}^2$ satisfying a uniform interior sphere condition such that $H^1(\partial E \setminus \partial^* E) > 0$ as shown by the next example.

Example 2.2. This example is inspired to Example 4.1 in ADD (Colombo-Marigoand, see also Mantegazza -Mennucci pag. 10). Let $C \subset S^1$ be a compact set with $H^1(C) > 0$ and empty interior. Set

$$K = B_1 \cup (B_4 \setminus \text{int}B_2) \cup \left( \bigcup_{x \in C} B_1(x) \right).$$ 

Since $C$ is closed it is easily check that $K$ is compact (recall that with $B_r(x)$ we denote a closed ball). Let $A = S^1 \setminus C$ then $A = \bigcup_{i=1}^{\infty} \Gamma_i$, where each $\Gamma_i$ is a connected open arc such that $\Gamma_i \cap \Gamma_j = \emptyset$ if $i \neq j$. By adding to $C$ finitely many points if needed we may assume w.l.o.g. that $B_1 \subset \text{int}K$. Thus $\partial(K) \subset \partial B_1 \cup \partial B_2 \cup \partial \bigcup_{x \in C} B_1(x)$. We claim that $K$ is a set of finite perimeter. To prove this, by theorem ADD (Federer) it is enough to show that $H^1(\partial^M K) < \infty$ which in turn is true if we prove that $H^1(\partial \bigcup_{x \in C} B_1(x) \cap \text{int}B_2 < \infty$. To this
Lemma 3.2. \( \text{continuity} \)

Let us denote by \( a_i, b_i \) the end points of \( \Gamma_i \) and by \( S_i, T_i \) the open arcs of \( B_1(a_i), B_1(b_i) \), respectively, whose projection on \( S^1 \) coincides with \( \Gamma_i \). Clearly

\[
\mathcal{H}^1(S_i \cap T_i) \leq c \mathcal{H}^1(\Gamma_i).
\]

(3)

Proof. Let \( \chi \) and this implies the contradiction \( |E \Delta B_1(x)| > \varepsilon \).

Moreover by compactness properties of the Hausdorff distance in \( Q \), there exists \( \bar{y}_n \) such that \( \pi(\bar{x}_n) = x_n \) and \( y_n \in intB_2 \setminus (B_1(a_{n_0}) \cap B_1(b_{n_0})) \).

By construction each \( y_n \notin K \) and the sequence \( y_n \) converges to \( 2x \). On the other hand since \( B_1(x) \subset K \) is tangent to \( \partial K \) in \( 2x \) and \( B_1(3x) \subset K \) is also tangent to \( \partial K \) in \( 2x \) we get that the \( \Theta_n(2x, K) = 1 \) thus proving that \( 2x \in \partial K \setminus \partial M K \).

Therefore \( \mathcal{H}^1(\partial K \setminus \partial M K) \geq 2 \mathcal{H}^1(C) > 0. \)

3. PROOF OF THE MAIN RESULT

We observe that since all the quantities considered are scaling invariant, it is not restrictive to work in the class

\[ C^1_\gamma = \{ E \in C_\gamma : |E| = \omega_n \}. \]

In the following, when the dependence on \( x_0 \) and \( \nu \) is not relevant, we will use the notation \( S_\gamma \) to denote a generic spherical sector \( S_{x_0, \gamma, \nu} \).

Lemma 3.1. There exist \( \delta_0 \) and \( L > 0 \) such that for any \( E \in C^1_\gamma \) with \( D(E) < \delta_0 \) we have \( \text{diam}(E) < L \).

Proof. Let \( \varepsilon = \frac{|S_\gamma|}{2} \) and \( \delta_0 = \varepsilon^2 c \), where \( c \) is the constant appearing in (1). Then \( L := 2 + 2 \text{diam}(S_\gamma) \) and \( \delta_0 \) satisfy the required property. Indeed, by contradiction, assume that \( \text{diam}(E) \geq L \); then there exists \( y \in \partial E \) with \( \text{dist}(y, B_1(x_0)) > \text{diam}(S_\gamma) \), where \( B_1(x_0) \) is such that \( \lambda(E) = |E \Delta B_1(x_0)| \). Since \( E \in C^1_\gamma \) there exists \( S_{y, \gamma, \nu} \subset E \) with \( S_{\gamma} \subset E \setminus B_1(x_0) \) and this implies the contradiction \( |E \Delta B_1(x_0)| > 2 \varepsilon \).

From the previous result, w.l.o.g. we may suppose that all the sets are contained in \( Q_L = [-L, L]^n \).

Lemma 3.2. For any \( \varepsilon > 0 \) there exists \( \delta_1 > 0 \) such that for any \( E \in C^1_\gamma \) with \( D(E) < \delta_1 \) we have \( \lambda(E) < \varepsilon \).

Proof. Arguing by contradiction there exist \( \varepsilon_0 > 0 \) and a sequence \( \{E_j\} \subset C^1_\gamma \) such that \( \lim_{j \to \infty} D(E_j) = 0 \) and \( \lambda(E_j) \geq \varepsilon_0 \). Since \( \chi_{E_j} \) is precompact in \( BV(Q_L) \), then, up to a subsequence not relabeled, we may assume \( \chi_{E_j} \to \chi_F \) in \( L^1(Q_L) \) for a suitable set \( F \). Note that \( |F| = \omega_n \) and, by lower semicontinuity of the perimeter, \( D(F) \leq \lim \inf_{j \to \infty} D(E_j) = 0. \)

The isoperimetric inequality yields at ones that \( F \) coincides a.e. with a unit ball, say \( B_1 \). Moreover by compactness properties of the Hausdorff distance in \( Q_L \), we may assume also \( E_j \to E_\infty \) in \( d_H \). We claim that \( E_\infty = B_1 \). Indeed, the inclusion \( B_1 \subset E_\infty \) is straightforward, since a.e. \( x \in B_1 \) is limit of a sequence \( \{x_j\} \) with \( x_j \in E_j \). For the opposite inequality if \( E_\infty \not\subset B_1 \) then there exists \( \bar{x} \in \partial E_\infty \setminus B_1 \), and in particular \( \text{dist}(\bar{x}, B_1) \geq r_0 > 0. \) By Hausdorff convergence \( \bar{x} \) is limit of points \( \bar{x}_j \in \partial E_j \). Letting \( S_{\bar{x}_j, \gamma, \nu} \subset E_j \) be the interior cones relative to \( \bar{x}_j \), we easily get that \( |S_{\bar{x}_j, \gamma, \nu} \setminus B_1| > 0 \) for \( j \) large enough (so that...
Lemma 3.3. There exists $\varepsilon_0 = \varepsilon_0(\gamma)$ such that for any $0 < \varepsilon \leq \varepsilon_0$ there exists $1 - \varepsilon < s_\varepsilon < 1$ with the property that for any $s_\varepsilon < r < 1$, $y \in \partial B_r$ and $z \in K_{y,s_\varepsilon,r}$, with

$$K_{y,s_\varepsilon,r} := \left\{ z \in B_1 \setminus B_r \mid \langle y, \frac{z - y}{|y - z|} \rangle \geq s_\varepsilon \right\}$$

it holds

$$|S_{z,\gamma,\nu} \setminus B_1| \geq \varepsilon \quad \forall \nu \in S^{n-1} \text{ with } \langle \nu, \frac{y - z}{|y - z|} \rangle \leq \frac{1}{\sqrt{2}}$$

Proof. Let $\varepsilon_0 = \varepsilon_0(\gamma) = |S_{\varepsilon_0,\gamma,\nu} \setminus B_1|$ with $\nu = \frac{\varepsilon_1 + \varepsilon_2}{\sqrt{2}}$. Arguing by contradiction there exist $0 < \tilde{\varepsilon} < \varepsilon_0$ and sequences $\{r_j\}$, with $1 - \frac{1}{j} < r_j < 1$, $\{y_j\} \subset \partial B_{r_j}$, $\{z_j\} \subset B_1 \setminus B_{r_j}$ and $\{\nu_j\} \subset S^{n-1}$ satisfying

$$\langle y_j, \frac{y_j - z_j}{|y_j - z_j|} \rangle \geq 1 - \frac{1}{j} \quad \text{and} \quad \langle \nu_j, \frac{y_j - z_j}{|y_j - z_j|} \rangle \leq \frac{1}{\sqrt{2}}$$

and

$$|S_{z_j,\gamma,\nu_j} \setminus B_1| < \tilde{\varepsilon} < \varepsilon_0.$$ 

By a compactness argument up to subsequences we have $\nu_j \to \nu_0$, $z_j \to z_0$, $y_j \to y_0$ and $\frac{y_j - z_j}{|y_j - z_j|} \to \zeta_0$. Taking into account that $|y_j - z_j| \leq \varepsilon/j$ we easily get that $z_0 = y_0$. Moreover since

$$r_j \geq \langle y_j, \frac{y_j - z_j}{|y_j - z_j|} \rangle \geq 1 - \frac{1}{j}$$

passing to the limit we deduce $y_0 = \zeta_0$. Similarly from (4) we get $\langle y_0, \nu_0 \rangle \leq \frac{1}{\sqrt{2}}$. Finally, since $S_{z_j,\gamma,\nu_j} \to S_{z_0,\gamma,\nu_0}$ in the Hausdorff topology, passing to the limit in (5) we infer

$$|S_{z_0,\gamma,\nu_0} \setminus B_1| < \varepsilon < \varepsilon_0.$$ 

The last inequality contradicts the definition of $\varepsilon_0$. 

Proposition 3.4. For all $0 < \varepsilon < \varepsilon_0$ there exist $\delta, r_1, r_2 > 0$ with $1 > r_1 > r_2$ such that for any $E \in C^1$ with $D(E) < \delta$, we have

$$B_{r_1}(x_0) \setminus B_{r_2}(x_0) \subset E,$$

where $B_1(x_0)$ realizes $\lambda(E) = |E \Delta B_1(x_0)|$.

Proof. Let $s_\varepsilon$ as in Lemma 3.3 and define $r_2 = s_\varepsilon \vee 1 - \varepsilon$. In order to define $r_1$, we consider, for any $s_\varepsilon < r < 1$, $k_r = |K_{y,s_\varepsilon,r}|$. By continuity of $k_r$ we may define $r_1$ such that $k_{r_1} = \frac{k_r}{2}$. Finally choose $\delta$ such that

$$D(E) \leq \delta \Rightarrow \lambda(E) < \varepsilon \wedge \frac{k_{r_2}}{2}.$$ 

By contradiction assume that there exists $y \in \partial B_{r_1}(x_0) \setminus E$ for a suitable $r \in (r_2, r_1]$. Since $k_r \geq k_{r_1}$ we claim that there exists $z \in K_{y,s_\varepsilon,r} \cap \partial E$. Indeed if $K_{y,s_\varepsilon,r} \cap \partial E = \emptyset$ we would have $\lambda(E) \geq |K_{y,s_\varepsilon,r}| \geq k_r \geq \frac{k_{r_2}}{2}$. Let $S_{z,\gamma,\nu} \subset E$ be the internal cone relative to $z$; since $y \notin E$ we have $\langle \nu, \frac{y - z}{|y - z|} \rangle < \frac{1}{\sqrt{2}}$. Then by Lemma 3.3 we get the contradiction $\lambda(E) \geq |S_{z,\gamma,\nu} \setminus B_{r_1}(x_0)| \geq \varepsilon$. 

□
Lemma 3.5. There exists $\delta > 0$ such that for any $E \in C^1_\gamma$ with $D(E) < \delta$ one of the following holds true

(i) $\lambda_H(E) \leq D(E)^{\frac{n-1}{n}}$;
(ii) there exists $\tilde{E} \in C^1_{\gamma/2}$ with $B_{r_1}(x_0) \subset \tilde{E}$ satisfying

$$\lambda_H(E) \leq \lambda_H(\tilde{E}) + c\sqrt{D(E)}$$
and
$$D(\tilde{E}) \leq D(E).$$

Proof. ADD (chi delta). Let $E \in C^1_\gamma$ with $D(E) \leq \delta$ be fixed. We define $H = B_{r_1}(x_0) \setminus E$, $F = E \cup H$, $\tilde{E} = \alpha F$ with $\alpha = \left(\frac{\omega_n}{\omega_n + |H|}\right)^\frac{1}{n}$ and

$$h = \sup\{r > 0 : B_r(x) \subset H \text{ for some } x \in H\}.$$

Case I: Assume that $h \geq \frac{1}{2} \lambda_H(E)$.

By Proposition 3.4 we have

$$D(E) = \frac{P(E) - n\omega_n}{n\omega_n} = \frac{P(F) + P(H) - n\omega_n}{n\omega_n}$$

$$\geq \frac{P(F) + P(B_h) - n\omega_n}{n\omega_n}$$

$$= \frac{P(F) + n\omega_nh^{n-1} - n\omega_n}{n\omega_n}.$$  \hspace{1cm} (6)

We observe that $P(F) - n\omega_n \geq 0$: indeed, by the classical isoperimetric inequality, since $\alpha < 1$ and $|E| = \omega_n$, it holds

$$P(F) - n\omega_n = \frac{1}{\alpha^{n-1}}P(\tilde{E}) - n\omega_n \geq \frac{1}{\alpha^{n-1}}P(B_1) - n\omega_n$$

$$\geq n\omega_n \left(\frac{1}{\alpha^{n-1}} - 1\right) > 0.$$  

Taking into account the last inequality in (6) we thus infer

$$D(E) \geq h^{n-1} \geq \frac{1}{2^{n-1}} \lambda_H(E)^{n-1}.$$  

Hence in Case I property (i) holds.

Case II: Assume that $h < \frac{1}{2} \lambda_H(E)$.

Note that, being $B^\infty$ a ball realizing the infimum in $\lambda_H(E)$, it is easy to show that

$$\lambda_H(E) = d_H(E, B^\infty) = d_H(F, B^\infty).$$

Moreover we have

$$\lambda_H(\tilde{E}) = \lambda_H(\alpha F) = d_H(\alpha F, B^0)$$

$$\geq d_H(F, B^0) - d_H(F, \alpha F) \geq d_H(F, B^\infty) - 2(1 - \alpha).$$

An easy computation leads to

$$1 - \alpha = 1 - \left(\frac{\omega_n}{\omega_n + |H|}\right)^\frac{1}{n} \leq c|H| \leq c\lambda(E) \leq cD(E)^{\frac{1}{2}}.$$  

Thus Case II is proved to hold. \hfill $\square$
Lemma 3.6. There exist $\varepsilon > 0$ and $\delta > 0$ such that if $E \in \mathcal{C}^1$ with $B_{1-\varepsilon} \subset E$ and $D(E) < \delta$, then for any $\xi \in \mathbb{S}^{n-1}$ there exists a unique $t > 0$ such that $x_0 + tv \in \partial(E)$.

Proof. Let $\varepsilon := \gamma/4 \land 1 - s_{\varepsilon_0}$ being $s_{\varepsilon_0}$ as in Lemma 3.3. By using Lemma 3.2 we can choose $\delta$ such that $\lambda(H(E) < \varepsilon$ and $\lambda(E) < \varepsilon_0$. This implies that

\[
\text{eq:7} \quad d_H(E, B_{1-\varepsilon}(x_0)) \leq 4\varepsilon < \gamma.
\]

Indeed, denoted by $B_1(x_\infty)$ the unit ball realizing $\lambda(E) = d_H(E, B_1(x_\infty))$, we have

\[
d_H(E, B_{1-\varepsilon}(x_0)) \leq d_H(E, B_1(x_\infty)) + d_H(B_1(x_\infty), B_{1-\varepsilon}(x_0)) \leq \varepsilon + \varepsilon + |x_0 - x_\infty| \leq 4\varepsilon.
\]

Assume by contradiction that there exist $\xi \in \mathbb{S}^{n-1}$, $0 < t_1 < t_2$ such that $z_i = x_0 + t_i\xi \in \partial E$ for $i = 1, 2$. According to (7) we have that the interior cone related to $z_2$, $S_{zz,\gamma,\nu} \subset E$, is such that $\langle \nu, \frac{z_i - z_2}{|z_i - z_2|} \rangle \leq \frac{1}{\sqrt{2}}$; indeed if it is not the case, $z_1$ would lie in \emph{interno}E. It remains to apply Lemma 3.3 to infer the contradiction $\lambda(E) \geq \varepsilon_0$.

\[\square\]

Remark 3.7. As a direct consequence of Lemma 3.6, $\partial E$ can be represented in spherical coordinates as the graph of a suitable function $\rho : \mathbb{S}^{n-1} \to \mathbb{R}$. The regularity properties of such a function will be investigated in the sequel.

In the following we will assume that all the hypotheses required for $\partial E$ being a graph are satisfied. In particular we assume that $D(E)$ is sufficiently small to ensure that $d_H(E, B_1(x_0)) \leq s_\varepsilon$ with $s_\varepsilon$ defined as in Lemma 3.3.

Lemma 3.8. The function $\rho$ is a $W^{1,1} (\mathbb{S}^{n-1})$ function.

Proof. We start by proving that $\rho \in BV (\mathbb{S}^{n-1})$. We will argue locally using the spherical coordinates $\Phi$. Let $J \in \mathbb{S}^{n-1}$ be a neighborhood of $e_n$ and set $V = \{ x \in \mathbb{R}^n : \frac{x}{|x|} \in J, \ 1 - \eta < |x| < 1 + \eta \}$. Set $I \times (1 - \eta, 1 + \eta) = \Phi^{-1}(V)$. Since $\Phi$ is a diffeomorphism and $E \cap V$ is a set of finite perimeter, so is $U = \Phi^{-1}(E \cap V)$. Define $\sigma : I \to \mathbb{R}$ as $\sigma(v) = \rho(\Phi(v, 0))$ and note that $U$ is the subgraph of $\sigma$. Applying Theorem B in [?] we get that $\sigma \in BV(I)$. The claim will follows once we show that $\sigma \in W^{1,1}(I)$. Let $\Gamma_\sigma^*$ be the extended graph ?? of $\sigma$, i.e.

\[
\text{eq:8} \quad \Gamma_\sigma^* = \{ (v, t) \in I \times (1 - \eta, 1 + \eta) : \sigma^-(v) \leq t \leq \sigma^+(v) \}.
\]

It is enough to show that for $\mathcal{H}^{n-1}$-a.e. $z \in \Gamma_\sigma^*$ it holds $\langle \nu_{\mathcal{G}_\sigma^*}(z), e_n \rangle \neq 0$, where $\nu_{\mathcal{G}_\sigma^*}(z)$ is the measure theoretic unit normal to $\Gamma_\sigma^*$.

Assume by contradiction that there exists $z \in \Gamma_\sigma^*$ such that $\langle \nu_{\mathcal{G}_\sigma^*}(z), e_n \rangle = 0$, since by Federer theorem ADD $\Gamma_\sigma^*$ coincides $\mathcal{H}^{n-1}$-a.e. with $\partial * (U)$ we may assume that $z \in \partial * (U)$. Hence, denoted by $B_r(z, \nu) = \{ x \in B_r(z) : \langle x - z, \nu \rangle \leq 0 \},$

\[
\text{eq:9} \quad \lim_{r \to 0^+} \frac{|U \cap B_{r}^-(z, \nu_{\mathcal{G}_\sigma^*}(z))|}{|r^n|} = 0.
\]

By the area formula

\[
\text{eq:9} \quad |U \cap B_{r}^-(z, \nu_{\mathcal{G}_\sigma^*}(z))| = \int_{E \cap V \cap \Phi(B_r^-(z, \nu_{\mathcal{G}_\sigma^*}(z)))} J(\Phi^{-1}) dy.
\]

Moreover, by the uniform bound on $J(\Phi)$ we have that

\[
\Phi(B_r^-(z, \nu_{\mathcal{G}_\sigma^*}(z))) \supset B_{r/2}(\Phi(z), \xi(z)),
\]

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with

\[ \langle \xi(z), \Phi(z) \rangle = 0. \tag{10} \]

Without loss of generality we may assume \( \Phi(z) \in \partial^*(E) \), and, by (8) and (9), we get

\[ \lim_{r \to 0^+} \frac{|E \cap V \cap B_{r/2}^-(\Phi(z), \xi(z))|}{|(r/2)^n|} = 0. \tag{11} \]

From (11) we immediately infer that \( \xi(z) \) is the measure theoretical inner normal to \( \partial^*(E) \) in \( \Phi(x) \). By Lemma 3.3, the interior cone property and (10) we get a contradiction. \( \square \)

**Lemma 3.9.** The function \( \rho \) is a \( W^{1,\infty}(S^{n-1}) \) function.

**Proof.** As usual we will work locally and by rotation invariance it is sufficient to prove the statement in a neighborhood of \( e_n \). Let \( y_0 \) such that \( \Phi(y_0) = e_n \). We note that we have \( \frac{\partial \Phi(e_n)}{\partial y_i} = e_i \) for \( i \in \{1, \ldots, n-1\} \). Moreover \( y \to \Phi(y)\rho(\Phi(y)) \) is a parametrization of \( \partial E \) in a neighborhood of \( P = \rho(e_n)e_n \) and the tangent space is spanned by the the vectors

\[ \frac{\partial \Phi(y)\rho(\Phi(y))}{\partial y_i} \big|_{y_0} = \rho(\Phi(y_0))e_i + \langle \nabla \rho(\Phi(y_0)), e_i \rangle e_n. \]

Hence the outer normal to \( \partial E \) in \( P \) is given by

\[ \nu = \frac{e_n - \frac{1}{\rho(e_n)} \sum_{i=1}^{n-1} \langle \nabla \rho(e_n), e_i \rangle e_i}{\sqrt{1 + \frac{1}{\rho^2(e_n)}|\nabla \rho(e_n)|^2}} \]

where \( \nabla \tau \rho \) is the tangential gradient of \( \rho \). Let \( \theta > 0 \) be the angle between \( \nu \) and \( e_n \). An easy computation shows that

\[ \cos(\theta) = |\langle \nu, e_n \rangle| = \frac{1}{\sqrt{1 + \frac{1}{\rho^2(e_n)}|\nabla \tau \rho(e_n)|^2}}. \tag{12} \]

Let \( M > 0 \) be fixed. If \( |\nabla \tau \rho(e_n)| > M \) then

\[ \cos(\theta) \leq \frac{1}{\sqrt{1 + \frac{1}{4}M^2}}. \tag{13} \]

By Lemma ADD (geometrico) it also holds that \( \theta \in [0, \frac{\pi}{2} - \beta] \) for a given \( \beta > 0 \) which, combined with (11) provides a uniform bound on \( M \). This concludes the proof. \( \square \)

In the following we will denote by \( u(x) = \rho(x) - 1 \).

**Lemma 3.10.** There exists a constant \( c = c(n, c_0, M) \) such that for any \( u \in W^{1,\infty}(S^{n-1}) \) with

\[ |u|_\infty \leq c_0 < 1, \quad |\nabla u|_\infty \leq M \]

it holds

\[ |u|_{n-1, \infty} \leq c \lambda_H(E) \]

where \( E = \{ x \in \mathbb{R}^n : |x| \leq u(\frac{x}{|x|}) + 1 \} \).
References


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