GRADED BIALGEBRAS AND DELETION-CONTRACTION INVARIANTS

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partially based on joint work with
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Introduction ...

Plan of the lectures

1. MULTIPLICATIVE MINORS SYSTEMS AND BIALGEBRAS
2. GROTHENDIECK MONOID AND UNIVERSAL TUTTE CHARACTER
3. FIRST EXAMPLES: SETS, GRAPHS, MATROIDS
4. ARITHMETIC MATROIDS
5. KNOTS AND COLORED MATROIDS

1. Multiplicative Minors Systems and Bialgebras

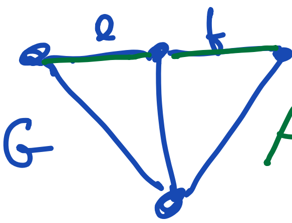
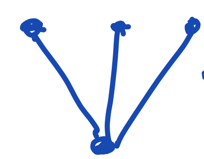
Example to keep in mind.

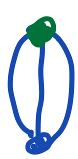
- S is the set of all the (isomorphism classes of) graphs
- $\forall G \in S$, $V(G)$ vertices and $E(G)$ edges
- $\forall G \in S$ $\forall A \subseteq E\,G$ two objects: <u>deletion</u> $G \setminus A$ with $V(G\setminus A) = V(G)$ and $E(G \setminus A) = E(G) \setminus A$ and <u>contraction</u> G/A with $V(G/A) = V(G)/\sim$ $E(G/A) = E(G) \setminus A$

$v \sim u$ if there is an edge $(v,u) \in A$

$\forall G_1, G_2 \in S$ $G_1 \oplus G_2$ the disjoint union

<u>Example</u>

G  $A = \{e, f\}$ $G \setminus A$ 

G/A 

<u>Def</u> A <u>multiplicative minors system</u> (MMS) is a set S with the following structure:

$\forall G \in S$ a finite set $E(G)$

$\forall G \in S$, $A \subseteq E(G)$ two objects $G \setminus A$

and G/A such that $E(G\backslash A) = E(G/A) = E(G)\backslash A$

$\forall G_1, G_2 \in S$, an object $G_1 \oplus G_2 \in S$ st

$E(G_1 \oplus G_2) = E(G_1) \sqcup E(G_2)$

with the following compatibilities.

1) $S\backslash\emptyset = S/\emptyset = S$

2) $\forall A, B \subseteq E(G)$ st $A \cap B = \emptyset$, $G\backslash(A \cup B) = (G\backslash A)\backslash B$

$G/(A\cup B) = (G/A)/B$ and $(G\backslash A)/B = (G/B)\backslash A$

3) $\oplus$ is associative and commutative, with 0_S

4) $\forall G_1, G_2 \in S$ $\forall A_1 \subseteq E(G_1)$, $A_2 \subseteq E(G_2)$,

$(G_1 \oplus G_2)\backslash(A_1 \sqcup A_2) = (G_1\backslash A_1) \oplus (G_2\backslash A_2)$ and same for $/$

A morphism of MMS is just a function $f: S \to S'$ compatible with E, $\backslash$, $/$, $\oplus$.

Let $\mathbb{K}$ be a commutative ring with 1 and $\mathbb{K}S$ be the free module on S. We will see that this is a bialgebra.

<u>Def:</u> An (associative) algebra: a $\mathbb{K}$-module A with $\mathbb{K}$ linear maps $\mu: A \otimes A \to A$ "product"

... linear maps $\mu: A\otimes A \to A$ ("product")

and $\eta: \mathbb{K} \to A$ ("unit") such that:

$$\begin{tikzcd} & A\otimes A \arrow[dr] & \\ A\otimes A\otimes A \arrow[ur, "\mu\otimes id"] \arrow[dr, "id\otimes\mu"'] & & A \\ & A\otimes A \arrow[ur] & \end{tikzcd} \qquad \begin{tikzcd} & \mathbb{K}\otimes A \arrow[r, "\eta\otimes id"] & A\otimes A \arrow[dr] & \\ A \arrow[ur, "a\mapsto 1\otimes a"] \arrow[dr] & & & A \\ & A\otimes \mathbb{K} \arrow[r, "id\otimes\eta"] & A\otimes A \arrow[ur] & \end{tikzcd}$$

commute

2) An (coassociative) coalgebra:
a $\mathbb{K}$-module A with $\mathbb{K}$ linear maps

$\Delta: A \to A\otimes A$ and $\varepsilon: A \to \mathbb{K}$ s.t.

$$\begin{tikzcd} & A\otimes A \arrow[dl, "\Delta\otimes id"'] & \\ A\otimes A\otimes A & & A \arrow[ul] \arrow[dl] \\ & A\otimes A \arrow[ul, "id\otimes\Delta"] & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & \mathbb{K}\otimes A \arrow[dl] & A\otimes A \arrow[l, "\varepsilon\otimes id"'] & \\ A & & & A \arrow[ul] \arrow[dl] \\ & A\otimes \mathbb{K} \arrow[ul] & A\otimes A \arrow[l, "id\otimes\varepsilon"] & \end{tikzcd}$$

commute

3) A bialgebra is a $\mathbb{K}$-module which is both an algebra and a coalgebra with the compatibilities:

μ and Δ:

$$\begin{tikzcd} A\otimes A\otimes A\otimes A \arrow[d, "(2,3)"] & A\otimes A \arrow[l, "\Delta\otimes\Delta"'] & \\ & & A \arrow[ul] \arrow[dl] \\ A\otimes A\otimes A\otimes A \arrow[r, "\mu\otimes\mu"] & A\otimes A & \end{tikzcd}$$

μ and ε

$$\begin{tikzcd} & \mathbb{K}\otimes\mathbb{K} \arrow[dr] & \\ A\otimes A \arrow[ur] \arrow[dr] & & \mathbb{K} \\ & A \arrow[ur] & \end{tikzcd}$$

commute

Δ and η

$$\begin{tikzcd} & \mathbb{K}\otimes\mathbb{K} \arrow[dl] & \\ A\otimes A & & \mathbb{K} \arrow[ul] \arrow[dl] \\ & A \arrow[ul] & \end{tikzcd}$$

ε and η :
$$\begin{tikzcd} \mathbb{k} \arrow[r, "\eta"] \arrow[rr, bend right, "\mathrm{id}"'] & A \arrow[r, "\varepsilon"] & \mathbb{k} \end{tikzcd}$$

<u>Def</u> Let $\mathbb{k}S$ be the free module on a MMS S. Naturally graded by

$$\mathbb{k}S = \bigoplus_{m \in \mathbb{N}} \mathbb{k}S_m, \text{ where } S_m = \{G \in S \mid |E(G)| = m\}$$

product: $\mu(G_1 \otimes G_2) = G_1 \oplus G_2$ with unit 0_S

coproduct: $\Delta(G) = \sum_{A \subseteq E(G)} G|_A \otimes G/A$ $\quad := G \setminus A^c$

counit: $\varepsilon(G) = \begin{cases} 1 & \text{if } G \in S_0 \\ 0 & \text{if } G \in S_n \end{cases}$

<u>Prop</u> $\mathbb{k}S$ is a graded bialgebra, and $S \mapsto \mathbb{k}S$ is a functor.

<u>Proof</u>: The operations respect the grading.
The coassociativity of Δ comes from axiom 2
The associativity of μ from axiom 3
compatibility of Δ, μ from axiom 4.
〃 〃 of Δ with η from axiom 1
and the other are trivial. Functoriality is also easy to verify. $\square$

Rem One can also define a "minors system" S, having similar axioms except that $G_1 \oplus G_2$ is defined only when one of the two is in S_0. This allows to deal with some classes, for example graphs with at most n edges.
Then $\mathbb{K}S$ is a not a bialgebra, but it is still a coalgebra.

Def A bialgebra A is called a Hopf algebra if $\exists\, a \in \mathrm{End}(A)$ ("the antipode") s.t.:

$$\begin{tikzcd}
 & A\otimes A \arrow[r, "a\otimes id"] & A\otimes A \arrow[dr] & \\
A \arrow[ur] \arrow[r, "\varepsilon"] \arrow[dr] & \mathbb{K} \arrow[rr, "\eta"] & & A \\
 & A\otimes A \arrow[r, "id\otimes a"] & A\otimes A \arrow[ur] &
\end{tikzcd}$$

commutes

Example: Let G be a group. The group algebra $\mathbb{K}[G]$ (basis t_g, $g\in G$ and $\mu(t_g \otimes t_h) = t_{gh}$, $\eta(1) = t_e$) is a bialgebra with $\Delta(t_g) = t_g \otimes t_g$, $\varepsilon(t_g) = 1$ $\forall g \in G$.
The antipode is $a(t_g) = t_{g^{-1}}$

Def A graded bialgebra A is connected if

$A_0 \simeq \mathbb{K}$

<u>Theor</u> If a bialgebra is graded and connected then it is Hopf, with antipode (Takeuchi):

$$a = \sum_{n\in\mathbb{N}} (-1)^n \mu^{n-1} \circ \pi^{\otimes n} \circ \Delta^{n-1}$$

where $\mu^{-1} = \eta$, $\Delta^{-1} = \varepsilon$, $\pi : S \to S_{>0}$ is natural proj.

<u>Def</u> A HMS is <u>connected</u> if $S_0 = \{0_S\}$ ($\Leftrightarrow \mathbb{K}S$ is connected)

<u>Example</u> $S =$ graphs. This is <u>not</u> connected

$S_0 = \{$graphs with no edges$\} = \{\bullet, \bullet\bullet, \bullet\bullet\bullet, \dots\}$
$\simeq \mathbb{N}$ $\Rightarrow$ $\mathbb{K}S_0$ is the algebra $\mathbb{K}[x]$ with coproduct $\Delta(x) = x \otimes x$. $\subseteq K[x, x^{-1}] \simeq \mathbb{K}[\mathbb{Z}]$

has no antipode (it should be $x \mapsto x^{-1}$)

So $\mathbb{K}S_0$ is not Hopf $\Rightarrow$ $\mathbb{K}S$ is not Hopf.

<u>Rem</u> We may obtain a Hopf algebra H by quotienting $\mathbb{K}S$ by the submodule spanned by $(U \oplus G) - G$, $\forall G \in S, U \in S_0$.

However, we will see that it is more convenient ... to not lose the information contained in S_0 and work with the whole bialgebra $\Bbbk S$.

A bit of history

'79 Joni–Rota: several combinatorial objects have a bialgebra structure

'94 Schmitt: many of them are Hopf (structure described, antipode computed)

'15 Krajewski–Moffatt–Tanasa: several polynomial invariants can be computed from the Hopf algebra

'17 Dupont–Fink–M: more invariants can be computed from the bialgebra/coalgebra $\Bbbk S$. Convolution (after coffee break!)

2. GROTHENDIECK MONOID AND TUTTE CHARACTERS

Def Let S be a (multiplicative)

minor system. A **norm** for S (with values in a commutative monoid X) is a function $S \to X$ satisfying:

1) $N(G) = N(G|_A)\, N(G/A)$ $\forall G \in S$, $A \subseteq E(G)$

$G|_A$ restriction $= G \setminus (E \setminus A)$.

2) $N(U \oplus G) = N(G)$ $\forall U \in S_0$, $G \in S$

3) $N(0_S) = 1$

__Rem__ If S is a MMS then any norm satisfies 2') $N(G_1 \oplus G_2) = N(G_1) \cdot N(G_2)$

In fact, using 1) with $A = E(G_1) \cup \phi$

$$N(G_1 \oplus G_2) = \underbrace{N(G_1 \oplus G_2|_\phi)}_{N(G_1)}\, \underbrace{N(G_1 / E(G_1) \oplus G_2)}_{N(G_2)}$$

by 2).

__Def__ The Grothendieck monoid $X(S)$ is the commutative monoid having generators $[G]$ for object $G \in S$ and relations:

1) $[G] = [G|_A][G/A] \quad \forall G \in S, A \subseteq E(G)$
2) $[U \oplus G] = [G] \quad \forall G \in S, U \in S_0$
3) $[0_S] = 1$

<u>Rem</u> Of course giving a morphism of monoids $X(S) \to X$ is equivalent to give a norm for X with values in X.
Then the map
$$S \to X(S)$$
$$G \mapsto [G]$$
is called the <u>universal norm</u>

<u>Theo</u> (DFM) $X(S)$ is generated by the classes $[G]$, $G \in S_1$ with relations:
1) $\forall G \in S_1, U \in S_0 \quad [U \oplus G] = [G]$
2) $\forall G \in S_2 \; (E(G) = \{e, f\})$
$$[G|_e][G/e] = [G|_f] \cdot [G/f].$$

Def If (C, Δ, ε) is a coalgebra and (A, μ, η) an algebra, the convolution of two linear maps $f, g : C \to A$ is $f * g = \mu \circ (f \otimes g) \circ \Delta$

$$A \leftarrow A \otimes A \longleftarrow C \otimes C \leftarrow C$$

Then $\mathrm{Hom}_{\Bbbk}(C, A)$ is an associative algebra with unit $u := \eta \circ \varepsilon$.

Def A <u>twist map</u> (with values in a monoid X) is a morphism of monoids $\tau : S_0 \longrightarrow X$

It extends linearly to $\Bbbk S_0$, then to $\Bbbk S$ by setting $\tau(\sigma) = 0 \;\; \forall\, \sigma \in S_n$, $n > 0$.

Let R be a commutative $\Bbbk$-algebra, R^* its multiplicative monoid, $N_1, N_2 : \Bbbk S \to R^* \subset R$ two norms, $\tau : S_0 \longrightarrow R^* \subset R$ a twist map

Def The Tutte character associated with the triple (N_1, τ, N_2) is the convolution product

$$T_{(N_1,\tau,N_2)} = \boxed{N_1 * \tau * N_2} : KS \to R.$$

Exercice 1 Show that

$$T_{(N_1,\tau,N_2)}(G) = \sum_{A \subseteq E(G)} N_1(G|_A)\,\tau\big((G|_A)/A\big)\,N_2(G/A).$$

Hint: use definition of coproduct and the fact that $\eta_{S>0} = 0$).

Rem "character"? morphism of algebras ($*$ of morphism)

"Tutte"

Exercice 2 Prove that $\forall G \in S$, $\forall e \in E(G)$

$e^c = E(G) \setminus \{e\}$

$$T_{(\dots)}(G) = N_1(G \setminus e^c)\, T_{(\dots)}(G/e) + N_2(G/e^c)\, T_{(\dots)}(G \setminus e)$$

Hint: write the formula from Ex 1 as

$$\sum_{A \subseteq E(G)} + \sum_{A \subseteq E(G)},$$ then use the fact

$e \in A$ $\quad e \notin A$ $\quad$ that N_1, N_2 are norms.

<u>Def</u> The universal Tutte character T^S of a MMS is the Tutte character associated to two copies of the universal norm and $\pi: \mathbb{K}S \to \mathbb{K}S_0$ the natural projection.

It is universal in the following sense:

<u>Prop</u> Let R be a commutative $\mathbb{K}$-algebra, $\phi: \mathbb{K}S \to R$ be a linear map satisfying

$$\phi(G) = N_1(G \setminus e^c)\,\phi(G/e) + N_2(G/e^c)\,\phi(G\setminus e)$$

for some norms N_1, N_2. Then there exist Φ such that

$$\begin{array}{ccc} \mathbb{K}[X(S)\times S_0\times X(S)] & \xrightarrow{\Phi} & R \\ {\scriptstyle T_S}\uparrow & \nearrow{\scriptstyle \phi} & \\ \mathbb{K}S & & \end{array}$$

Proof: Let us define Φ by

$$\Phi([G_1], U, [G_2]) = N_1(G_1)T(U)N_2(G_2).$$

By applying Φ to the deletion-contraction formula for T_S one gets

$$(\Phi \circ T_S)(G) = N_1(G \setminus e^c)(\Phi \circ T_S)(G/e) + N_2(G/e^c)(\Phi \circ T_S)(G \setminus e)$$

Then the result follows by induction on $|E(G)|$.

$U \in S_0$, $\Phi \circ T_S(U) = \Phi(U)$

$\square$

<u>Exercise 3</u> Compute $\chi(S)$ and T_S for $S =$ sets

<u>Lemma</u> If $N : \mathbb{K}S \to R$ is a norm, the $\overline{N} = (-1)^{|E(G)|} N(G)$ is the inverse for the condition:

$$N * \overline{N} = \overline{N} * N = u \; (= \eta \circ \varepsilon).$$

<u>Proof</u> if $U \in S_0$, $\Delta(U) = U \otimes U$ then

$$N * \overline{N}\,(U) = N(U)\,\overline{N}(U) = 1$$

If $G \in S_n$, $n > 0$

$$N * \overline{N}(G) = \sum_{A \subseteq E(G)} (-1)^{|A|}\, N(G/_A)\, N(G/A)$$

$$= \Big(\sum_{A \subseteq E(G)} (-1)^{|A|}\Big) N(G) = \overset{N(G)}{0 \cdot N(G)} = 0.$$

Convolution formula $\square$

<u>Theorem</u> (DFM) Let R be a commutative algebra, N_1, N_2, N be three norms, and τ_1, τ_2 be two twist maps. Then

$$T_{(N_1, \tau_1\tau_2, N_2)} = T_{(N_1, \tau_1, N)} * T_{(\overline{N}, \tau_2, N_2)}$$

<u>Proof</u> By definition this is

$$\underbrace{N_1 * \tau_1 * \underbrace{N * \overline{N}}_{u} * \tau_2}_{\tau_1 * \tau_2 \text{ the twist map } \tau_1\tau_2} * N_2$$

(pointwise product)

$\square$

In particular for $N_1, N_2, N =$ universal norm $S \longrightarrow X(S)$ and $T_1, T_2 =$ natural projection $\mathbb{K}S \to \mathbb{K}_{S_0}$ we get a "universal convolution formula"

3 FIRST EXAMPLES: SETS, GRAPHS, MATROIDS

<u>Example 0</u>: $S = \{\text{Sets}\}$ $\quad |S_n| = 1 \quad \forall n \in \mathbb{N}$

Grothendieck monoid $X(S) \simeq u^{\mathbb{N}}$

$$E \longmapsto u^{|E|}$$

T trivial $\Rightarrow$ universal Tutte character

$$N_1 * T * N_2 : \mathbb{K}S \longrightarrow \mathbb{K}[u_1, u_2]$$

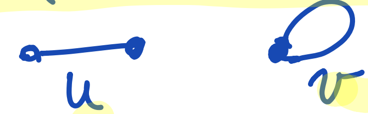
$$E \longmapsto \sum_{A \subseteq E} u_1^{|A|} u_2^{|E|-|A|} = (u_1 + u_2)^{|E|}$$

"doubling" of variables (because we took two copies of universal norm)

<u>Example 1</u> $S =$ graphs.

not connected $S_0 \simeq (\mathbb{N}, +) \simeq a^{\mathbb{N}}$

$X(S)$ generators:



relations are trivial in this case

$$X(S) \xrightarrow{\sim} u^{\mathbb{N}} v^{\mathbb{N}}$$

$$G \longmapsto u^{rk(G)} v^{cork(G)}$$

$k(G) = \#$ connected components

$rk(G) = |V(G)| - k(G)$

$cork(G) = |E(G)| - rk(G)$

Universal Tutte character

$$T^S : N_1 * T * N_2 : KS \longrightarrow K[\underbrace{u_1, v_1}_{N_1}, \underbrace{a}_{T}, \underbrace{u_2, v_2}_{N_2}]$$

$$G \longmapsto \sum_{A \subseteq E(G)} u_1^{rk(G|_A)} v_1^{cork(G|_A)} a^{k(G|_A / A)} u_2^{rk(G/A)} v_2^{cork(G/A)}$$

In order to understand this invariant, we will make a small digression

<u>Def</u> A <u>q-coloring</u> of a graph G is a map $c : V(G) \longrightarrow \mathbb{Z}/q$

c is <u>proper</u> if $c(i) \neq c(j)$ $\forall (i,j) \in E$

<u>Def</u> $\chi_G(q) = \#$ of proper q-colorings

$e \in E(G)$

<u>Prop</u> $\chi_G(q) = \chi_{G\setminus e}(q) - \chi_{G/e}(q)$

<u>Proof</u> # proper q-coloring =

# of colorings proper everywhere except eventually in e —

# of colorings proper everywhere except exactly in e

□

<u>Prop</u> $\chi_G(q)$ is a polynomial!

<u>Proof</u>: Induction on $|E(G)|$

"Chromatic"

Let's now choose an orientation of G

<u>Def</u> A <u>nowhere zero q-flow</u> is a map: $f: E(G) \to \mathbb{Z}^*/q$

such that $\forall v \in V(G)$, $\sum\limits_{e \to v} f(e) = \sum\limits_{v \overset{e}{\to}} f(e)$

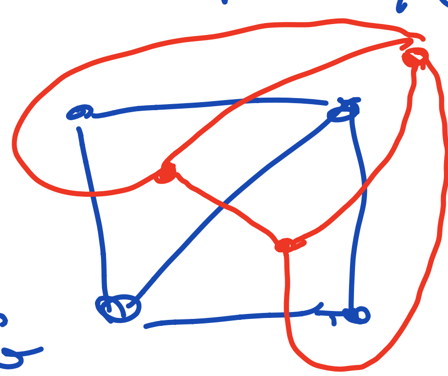
<u>Def</u> $\chi^*_G(q)$ = # nowhere zero q-flows

Exercise 4: prove that this does

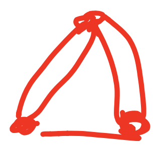
not depend on the orientation and that χ^*_G satisfies a del-contr recurrence $\Rightarrow$ "flow" polynomial

Fact If G is planar, let G^* be its dual graph. Then

$$\chi_{G^*} = q^{k(G^*)}\,\chi^*_G$$



G

$G^* =$ 

Tutte's idea: introduce a two-variable polynomial $D_G(x,y)$ "dichromate" s.t. $(-1)^{rk(G)} D_G(-q,0) = \chi_G(q)$ and $(-1)^{rk(G)} D_G(0,-q) = \chi^*_G(q)$ and $D_{G^*}(x,y) = D_G(y,x)$

$$D_G(x,y) = \sum_{A \subseteq E(G)} x^{|V(G)|}\, y^{cork(G)}.$$

A (more popular) variation is the following: "Tutte poly"

$$T_G(x,y) = (x-1)^{-k(G)} D_G(x-1, y-1) = \sum_{A \subseteq E(G)} (x-1)^{\operatorname{rk} E - \operatorname{rk} A} (y-1)^{|A| - \operatorname{rk} A}$$

$$T^S(1, y-1, 1, x-1, 1) = T_G(x,y)$$

$$T^S(1, y, x, 1, 1) = D_G(x,y)$$

universal deletion - contraction $\Rightarrow$ del/contr

univ convolution formula specializes to:

Theo: $T_G(x,y) = \sum_{A \subseteq E(G)} T_{G|_A}(0,y)\, T_{G/A}(x,0)$

Etienne las Vergnas in '98
Kook - Reiner - Stanton in '99

but we can get also new formulae for the dichromate, for instance

$$D(x_1 x_2, y) = \sum D_{G/A}(x_1, y)\, \chi_{G/A}(x_2)$$

<u>Example 2</u> S = matroids

<u>Def</u> A <u>matroid</u> M is a finite set $E(M)$ and a function $rk : 2^{E(M)} \to \mathbb{N}$ such that:

1) $rk(A) \leq |A|$
2) $A \subseteq B \Rightarrow rk(A) \leq rk(B)$
3) $rk(A \cup B) + rk(A \cap B) \leq rk(A) + rk(B)$

Example: list of vectors $(v_1, v_2, \dots, v_k)$

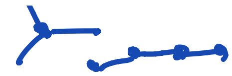
$rk(A) = \dim \langle v_i, i \in A \rangle$

$M|_A$ defined by $rk(M|_A) = (rk\, M)|_A$

M/A defined by $rk_{M/A}(B) = rk(B \cup A) - rk(A)$

<u>Exercise</u> The rank of a graph G defines a matroid $M(G)$

So we have a morphism of MMS

$\{\text{Graphs}\} \to \{\text{Matroids}\}$ 

that gives a morphism of bialgebras.

<u>Rem</u> $|S_0| = 1$, so $\mathbb{K}S$ is connected $\Rightarrow$ Hopf!

antipode: general Takeuchi formula

<u>Open problem</u>; give cancellation-free formula for the antipode

As before, $X(S) \xrightarrow{\sim} u^{\mathbb{N}} v^{\mathbb{N}}$

$$M \longmapsto u^{rk(M)} v^{cork(M)}$$

$$T^S(M) = \sum_{A \subseteq E(M)} u_1^{rk(M|A)} v_1^{cork\, M|A} u_2^{rk\, M/A} v_2^{cork\, M/A}$$

Our universal convolution formula here takes values in the ring $\mathbb{K}[\underbrace{u_1, v_1}_{N_1}, \underbrace{u, v}_{N}, \underbrace{u_2, v_2}_{N_2}]$ and says:

$$T^S(M)(u_1, v_1, u_2, v_2) = \sum_{A \subseteq E(M)} T^S(M|_A)(u_1, v_1, u, v).$$

$\cdot\, T^{S}(M/A)(-u, -v, u_2, v_2,$

$\Rightarrow$ EL, KRS formula for matroids

but also Kung's formula (2000)

$$T_M(1-ab, 1-cd) = \sum_{A \subseteq E(G)} a^{rk M - rk A}\, d^{|A| - rk A}\, T_{M|A}(1-a, 1-c)\, T_{M/A}(1-b, 1-d)$$

universal $\Rightarrow$ Kung $\Rightarrow$ EL-KRS

but also many others!

4 Recall

Matroid: $M=(E, rk)$, where:
E finite set, $rk: 2^E \to \mathbb{N}$ satisfying some axioms.

With the MMS $Mat = \{Matroids\}$ we associated:

- a bialgebra $\mathbb{K}Mat$ (connected, thus Hopf)
- the universal norm $S \to X(S) \simeq \mathbb{N}^2$

$$M \longmapsto u^{rk(M)} v^{cork(M)}$$

(where $u=[\bullet\!-\!\bullet]$, $v=[$  $]$)

- a universal Tutte character $T^{Mat} \in \mathbb{K}[u_1, v_1, u_2, v_2]$ that specializes to the Tutte polynomial

$$T_M(x,y) = \sum_{A \subseteq E(M)} (x-1)^{rk E - rk A} (y-1)^{|A| - rk A}$$

A matroid is <u>realizable</u>

when there is a matrix

with columns $\{v_i, i \in E\}$ such that

$rk(A) =$ rank of the submatrix with columns $\{v_i, i \in A\}$

In this case we can view every v_i

as a linear form on the dual space,

obtaining a hyperplane arrangement

$\{H_i := \ker v_i, i \in E\}$.

Then the Tutte polynomial specializes

to the Poincaré polynomial of the

complement of the arrangement:

$$P_{\mathbb{C}^d \setminus \cup H_i}(q) = \sum b_i q^i = q^d T_M\left(\frac{q+1}{q}, 0\right).$$

4. ARITHMETIC MATROIDS

If our matrix has integer coefficients, we can also view every column as a character of a torus:

$$v_i : \mathbb{C}^d/\mathbb{Z}^d \longrightarrow \mathbb{C}/\mathbb{Z} \simeq \mathbb{C}^* \quad \text{that is } e^{v_i} : (\mathbb{C}^*)^d \to \mathbb{C}^*$$

obtaining a toric arrangement

$$\{T_i = \ker e^{v_i},\ i \in E\}$$

<u>Exercise</u>: prove that $\forall A \subseteq E$, $|A| \le d$, $\bigcap_{i \in A} T_i$ has $m(A)$ connected components

where $m(A) = \left|\operatorname{tors}\left(\mathbb{Z}^d / \langle v_i \in A \rangle_{\mathbb{Z}}\right)\right| =$
$= \gcd(|A| \times |A|$ minors of the submatrix corresponding to $A)$.

It is then clear that the topology of the toric arrangement depends on these numbers

<u>Def</u> The <u>arithmetic Tutte polyn.</u>

$$T_{(M,m)} = \sum_{A \subseteq E} m(A)\,(x-1)^{\mathrm{rk}E - \mathrm{rk}A}\,(y-1)^{|A| - \mathrm{rk}A}$$

It specializes to the Poincaré polynomial of the toric arr:

$$P_{(\mathbb{C}^*)^d \setminus \cup T_i}(q) = q^d\, T_{(M,m)}\left(\frac{2q+1}{q}, 0\right)$$

Many other applications: integer points in polytopes, vpf, cellular complexes.....

<u>Def</u> Arithmetic matroid:
a couple $\mathcal{A} = (M, m)$ where M is a matroid and $m : 2^E \to \mathbb{N}$ a function satisfying some axioms.

<u>Def</u> $\mathcal{A}$ is realizable if it comes a "generalized" toric arrangement

<u>Theor</u> (Delucchi, M) If (M, m_1) and (M, m_2) are arithmetic matroids, then also $(M, m_1 \cdot m_2)$ is an aritmetic matroid

We have also restriction

$$m[\mathcal{A}|A) = m_{\mathcal{A}}\big|_A$$

$$m_{\mathcal{A}/A}(B) = m_{\mathcal{A}}(A \cup B)$$

$\oplus$ is given by product of multiplicities

Then $\mathrm{Ari} = \{$arithmetic matroids$\}$ is a MMS. ($\Longrightarrow$ bialgebra, ...)

Not connected: $m(\emptyset)$ can be any positive integer

$\mathrm{Ari}_0 \cong \{\mathbb{N} \setminus 0, \cdot\}$

Then the universal Tutte Character is

$$T^{\mathrm{Ari}}_{(M,m)} = \sum m(A)\, u_1^{rk(M|A)} v_1^{cork(M|A)} u_2^{rk(M/A)} v_2^{cork(M/A)}$$

which specializes to the arithmetic Tutte polynomial

<u>Question</u> How does the arith. Tutte polynomial of $(M, m_1 \cdot m_2)$ relates with the arith. Tutte polys of (M, m_1) and

(M, m_2) ?

We have morphisms

$$\mathrm{Mat} \to \mathrm{Ani} \qquad\qquad \mathrm{Ani} \to \mathrm{Mat}$$
$$M \mapsto (M,1) \qquad\qquad (M,m) \mapsto M$$

Let $\mathrm{Ani} \times_{\mathrm{Mat}} \mathrm{Ani}$

(objects: (M, m_1, m_2)).

Let's look at the universal convolution formula for $\mathrm{Ani} \times_{\mathrm{Mat}} \mathrm{Ani}$

It specializes to the following:

<u>Theor</u> (DFM) $T_{(M,m_1,m_2)} = T_{(M,m_1)} * T_{(M,m_2)}$

$$T_{M,m_1,m_2}(x,y) = \sum_{A \in E} T_{(M,m)/A}(0,y)\, T_{(M,m)/A}(x,0)$$

(generalizes a formula by Bachman-Lau?)

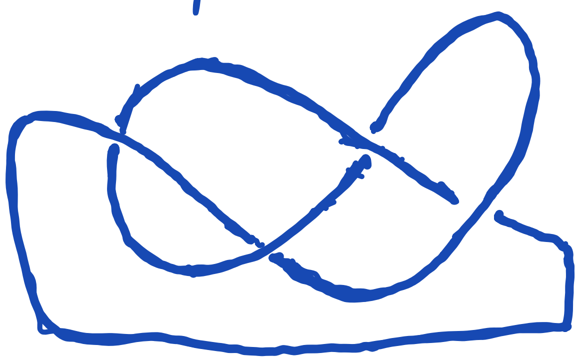
We also obtain more general

formulas, such as analogue
of Kung's formula.

5. KNOTS AND COLORED MATROIDS

<u>Def</u> A knot is an embedding of a circle S^1 in $\mathbb{R}^3$

$L \subseteq \mathbb{R}^3$ is a link if every connected component is a knot.



<u>Rem</u> Knots/links can be represented by a planar diagram, (decorated) planar quadrivalent graph Q.

<u>Def</u> Two knots K, K' (or links) are equivalent if there is an orientation-preserving homeomorphism $h: \mathbb{R}^3 \to \mathbb{R}^3$ s.t. $h(K) = K'$

In general it is difficult to say

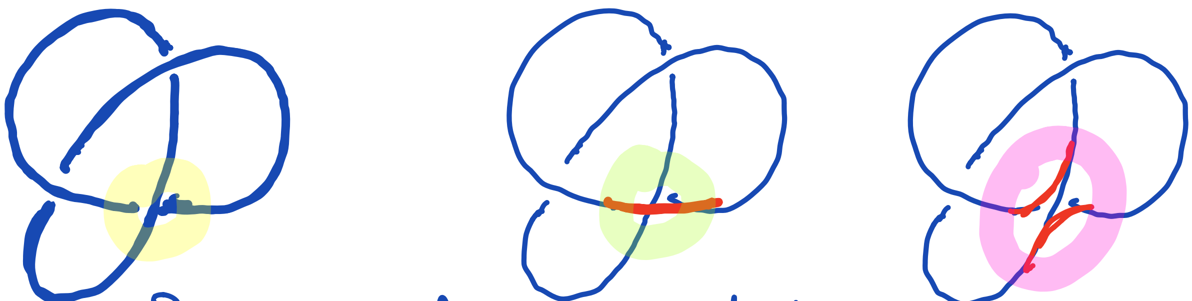
$\Rightarrow$ introduced many invariants to distinguish knots!

Def The Jones polynomial is the only $J(K) \in \mathbb{Z}[t^{\frac{1}{2}}, t^{-\frac{1}{2}}]$ s.t.

$J(\text{unknot}) = 1$ and

$(t^{\frac{1}{2}} - t^{-\frac{1}{2}}) J(L_0) = t^{-1} J(L_+) - t J(L_-)$ where

L_0, L_+, L_- differ only by a small region in which



Rem: morally, a sort of deletion-contraction!

(J arises from a representation of braid groups, or as the Euler characteristic of Khovanov homology).

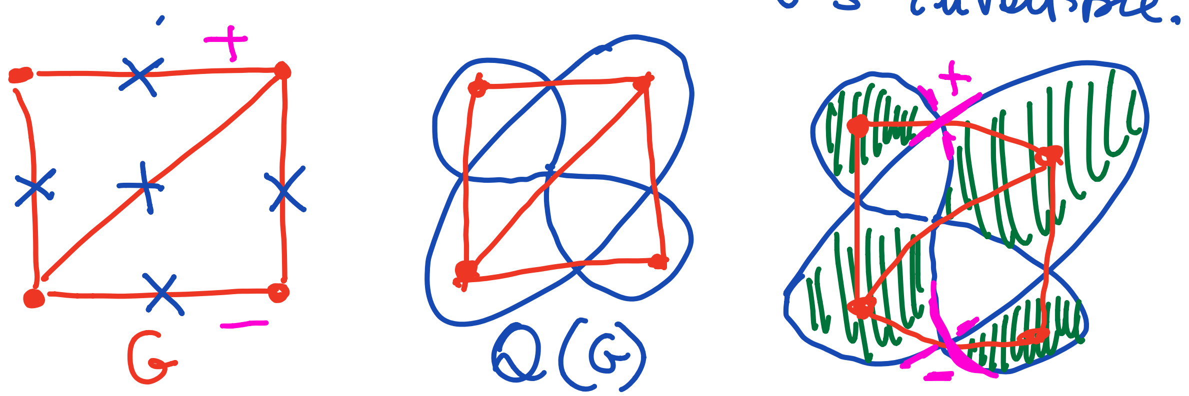
$t \mapsto t^{-1}$ gives J of mirror image

$t \mapsto e^x$ and expand $\Rightarrow$ Vassiliev invariant

How does this relate to graphs?

To a planar graph G we associate a quadrivalent planar graph $Q(G)$

this procedure is invertible.



Now if we put $+$ and $-$ at the crossings of $Q(G)$ we get the diagram of a knot. Via Q^{-1}, this is equivalent to put signs $+$ and $-$ on the edges of G

$$\{\text{connected planar link diagrams}\}$$
$$\updownarrow$$
$$\{\text{graphs with signed edges}\}$$

We need a "Tutte poly for signed graphs". More generally, we will define "Tutte polynomial of a colored matroid". (our case: two colors $+$ $-$ }.

<u>Def</u> A <u>colored matroid</u> is a couple (M, c) where $c : E(M) \to C$ ← finite set

<u>Rem</u> This is a MMS that we will call CMat

More generally, $\forall$ MMS S, we can define a new MMS CS.

$$\mathrm{CMat} \longrightarrow \mathrm{Mat}$$

Grothendieck monoid: a coloop and a loop of every color! $u_i = [\bullet \overset{i}{\text{———}} \bullet]$ $v_i = [\bigcirc^{i}]$ $i \in C$

and free (commutative) otherwise.

So $X(\mathrm{CMat}) \hookrightarrow X(\mathrm{Mat}) \times \{a_i,\ i \in C\}$

$$u_i \longmapsto u\, a_i$$
$$v_i \longmapsto v\, a_i$$

$$[M, c] \longrightarrow u^{rk(M)} v^{cork(M)} \prod_{e \in E(M)} a_{c(e)}$$

The universal Tutte character specialize to the "colored Tutte poly."

$$T_{(M,c)}(x,y) = \sum_{A \subseteq E(M)} \left(\prod_{e \in A} a_{c(e)}\right)(x-1)^{rkE - rkA}(y-1)^{|A| - rkA}$$

<u>Rem</u> is basically the same as Sokal "multivariate Tutte polynomial" arising in mechanical statistics.

<u>Theor</u> (Kauffman '89) For $C = \{+, -\}$ the Jones polynomial of a link L can be be computed from the colored Tutte of $Q^{-1}(L)$.
$\quad$ diagram

This theorem yied:

1) Efficient computations of Jones polynoms for some large knots
2) Proof that in general computing

the Jones polynomial of a knot is NP-hard.

Our universal convolution formula implies formulae for the Jones polynomials, whose applications we didn't understand yet.

Other applications:

- Graphs embedded into surfaces (Bollobás - Riordan polynomial, ...)
- delta - matroids, matroid perspectives...
- polymatroids
- ....

THANK YOU !