1. COHOMOLOGY OF GROUPS

Let $G$ be a group (which has to be thought as endowed with the discrete topology). We begin by recalling the usual definition of the cohomology of $G$ with coefficients in a $G$-module $V$. All the results described here are classical, and date back to the works of Eilenberg and Mac Lane, Hopf, Eckmann, and Freudenthal in the 1940s, and to the contributions of Cartan and Eilenberg to the birth of the theory of homological algebra in the early 1950s.

For the whole course, unless otherwise stated, group actions will always be on the left, and modules over (possibly non-commutative) rings will always be left modules. If $R$ is any commutative ring (in fact, we will be interested only in the cases $R = \mathbb{Z}$, $R = \mathbb{R}$), we denote by $R[G]$ the group ring associated to $R$ and $G$, i.e. the set of finite linear combinations of elements of $G$ with coefficients in $R$, endowed with the operations

\[
\left( \sum_{g \in G} a_g g \right) + \left( \sum_{g \in G} b_g g \right) = \sum_{g \in G} (a_g + b_g)g,
\]

\[
\left( \sum_{g \in G} a_g g \right) \cdot \left( \sum_{g \in G} b_g g \right) = \sum_{g \in G} \sum_{h \in G} a_{gh} b_{g^{-1}h} g,
\]

(when the sums on the left-hand sides of these equalities are finite, then the same is true for the corresponding right-hand sides). Observe that an $R[G]$-module $V$ is just an $R$-module $V$ endowed with an action of $G$ by $R$-linear maps, and that an $R[G]$-map between $R[G]$-modules is just an $R$-linear map which commutes with the actions of $G$. When $R$ is understood, we will often refer to $R[G]$-maps as to $G$-maps. If $V$ is an $R[G]$-module, then we denote by $V^G$ the subspace of $G$-invariants of $V$, i.e. the set

\[
V^G = \{ v \in V \mid g \cdot v = v \text{ for every } g \in G \}.
\]

We are now ready to describe the complex of cochains which defines the cohomology of $G$ with coefficients in $V$. For every $n \in \mathbb{N}$ we set

\[
C^n(G, V) = \{ f : G^{n+1} \to V \},
\]

and we define $\delta^n : C^n(G, V) \to C^{n+1}(G, V)$ as follows:

\[
\delta^n f(g_0, \ldots, g_{n+1}) = \sum_{i=0}^{n+1} (-1)^i f(g_0, \ldots, \hat{g}_i, \ldots, g_{n+1}).
\]

It is immediate to check that $\delta^{n+1} \circ \delta^n = 0$ for every $n \in \mathbb{N}$, so the pair $(C^\bullet(G, V), \delta^\bullet)$ is indeed a complex, which is usually known as the homogeneous complex associated to the pair $(G, V)$. The formula

\[(g \cdot f)(g_0, \ldots, g_n) = g(f(g^{-1}g_0, \ldots, g^{-1}g_n))\]

endows $C^n(G, V)$ with an action of $G$, whence with the structure of an $R[G]$-module. It is immediate to check that $\delta^n$ is a $G$-map, so the $G$-invariants
$C^\bullet(G, V)^G$ provide a subcomplex of $C^\bullet(G, V)$, whose homology is by definition the cohomology of $G$ with coefficients in $V$. More precisely, if

$$Z^n(G, V) = C^n(G, V)^G \cap \ker \delta^n, \quad B^n(G, V) = \delta^{n-1}(C^{n-1}(G, V)^G)$$

(where we understand that $B^0(G, V) = 0$), then $B^n(G, V) \subseteq Z^n(G, V)$, and

$$H^n(G, V) = Z^n(G, V)/B^n(G, V).$$

**Definition 1.1.** The $R$-module $H^n(G, V)$ is the $n$-th cohomology module of $G$ with coefficients in $V$.

By definition, we have $H^0(G, V) = V^G$ for every $R[G]$-module $V$. In this course we will be mainly concerned with the case when $R = V = \mathbb{R}$ (or, less frequently, when $R = V = \mathbb{Z}$), and the action of $G$ on $V$ is trivial.

1.1. **Functoriality.** Group cohomology provides a binary functor. If $G$ is a group and $\alpha: V_1 \to V_2$ is an $R[G]$-map, then $f$ induces an obvious change of coefficients map $\alpha^*: C^\bullet(G, V_1) \to C^\bullet(G, V_2)$ obtained by composing any cochain with values in $V_1$ with $\alpha$. It is not difficult to show that, if

$$0 \to V_1 \xrightarrow{\alpha} V_2 \xrightarrow{\beta} V_3 \to 0$$

is an exact sequence of $R[G]$-modules, then the induced sequence of complexes

$$0 \to C^\bullet(G, V_1)^G \xrightarrow{\alpha^*} C^\bullet(G, V_2)^G \xrightarrow{\beta^*} C^\bullet(G, V_3)^G \to 0$$

is also exact, so there is a long exact sequence

$$0 \to H^0(V_1, G) \to H^0(V_2, G) \to H^0(V_3, G) \to H^1(G, V_1) \to H^1(G, V_2) \to \ldots$$

in cohomology.

Let us now consider the functoriality of cohomology with respect to the first variable. Let $\psi: G_1 \to G_2$ be a group homomorphism, and let $V$ be an $R[G_2]$-module. Then $G_1$ acts on $V$ via $\psi$, so $V$ is endowed with a natural structure of $R[G_2]$-module. If we denote this structure by $\psi^{-1}V$, then the maps

$$\psi^n: C^n(G_2, V) \to C^n(G_1, \psi^{-1}V), \quad \psi^n(f)(g_0, \ldots, g_n) = f(\psi(g_0), \ldots, \psi(g_n))$$

provide a chain map such that $\psi^n(C^n(G_2, V)^{G_2}) \subseteq C^n(G_1, \psi^{-1}V)^{G_1}$. As a consequence, we get a well-defined map

$$H^n(\psi^n): H^n(G_2, V) \to H^n(G_1, \psi^{-1}V)$$

in cohomology. We will consider this map mainly in the case when $V$ is the trivial module $R$. In that context, the discussion above shows that every homomorphism $\psi: G_1 \to G_2$ induces a map

$$H^n(\psi^n): H^n(G_2, R) \to H^n(G_1, R)$$

in cohomology.
1.2. The topological interpretation of group cohomology. Let us recall the well-known topological interpretation of group cohomology. We restrict our attention to the case when $V = \mathbb{R}$ is a trivial $G$-module. If $X$ is any topological space, then we denote by $C_\bullet(X, \mathbb{R})$ (resp. $C^\bullet(X, \mathbb{R})$) the complex of singular chains (resp. cochains) with coefficients in $\mathbb{R}$, and by $H_\bullet(X, \mathbb{R})$ (resp. $H^\bullet(X, \mathbb{R})$) the corresponding singular homology module.

Suppose now that $X$ is any path-connected topological space satisfying the following properties:

1. the fundamental group of $X$ is isomorphic to $G$,
2. the space $X$ admits a universal covering $\tilde{X}$, and
3. $\tilde{X}$ is $\mathbb{R}$-acyclic, i.e. $H_n(\tilde{X}, \mathbb{R}) = 0$ for every $n \geq 1$.

Then $H^\bullet(G, \mathbb{R})$ is canonically isomorphic to $H^\bullet(X, \mathbb{R})$ (see Subsection 7.4).

If $X$ is a CW-complex, then condition (2) is automatically satisfied, and Whitehead Theorem implies that condition (3) may be replaced by one of the following equivalent conditions:

3' $\tilde{X}$ is contractible, or
3'' $\pi_n(X) = 0$ for every $n \geq 2$.

If a CW-complex satisfies conditions (1), (2) and (3) (or (3'), or (3'')), then one usually says that $X$ is a $K(G, 1)$, or an Eilenberg-MacLane space. Whitehead Theorem implies that the homotopy type of a $K(G, 1)$ only depends on $G$, so if $X$ is any $K(G, 1)$, then it makes sense to define $H^i(G, \mathbb{R})$ by setting $H^i(G, \mathbb{R}) = H^i(X, \mathbb{R})$. It is not difficult to show that this definition agrees with the definition of group cohomology given above. In fact, associated to $G$ there is the $\Delta$-complex $BG$, having one $n$-simplex for every ordered $(n + 1)$-tuple of elements of $G$, and obvious face operators (see e.g. [Hat02]). When endowed with the weak topology, $BG$ is clearly contractible. Moreover, the group $G$ acts freely and simplicially on $BG$, so the quotient $X_G$ of $BG$ by the action of $G$ inherits the structure of a $\Delta$-complex, and is such that $\pi_1(X_G) = G$. In other words, $X_G$ is a $K(G, 1)$. By definition, the space of the simplicial $n$-cochains on $BG$ coincides with the module $C^n(G, \mathbb{R})$ introduced above, and the simplicial homology of $X_G$ is isomorphic to the homology of the $G$-invariant simplicial cochains on $BG$. This allows us to conclude that $H^\bullet(X_G, \mathbb{R})$ is canonically isomorphic to $H^\bullet(G, \mathbb{R})$.

2. Bounded cohomology of groups

Let us now shift our attention to bounded cohomology of groups. In order to do so we first need to define the notion of normed $G$-module. Let $R$ and $G$ be as above. For the sake of simplicity, we assume that $R = \mathbb{Z}$ or $R = \mathbb{R}$, and we denote by $| \cdot |$ the usual absolute value on $R$. A normed $R[G]$-module $V$ is an $R[G]$-module endowed with an invariant norm, i.e. a map $\| \cdot \|: V \to \mathbb{R}$ such that:

- $\|v\| = 0$ if and only if $v = 0$,
- $\|r \cdot v\| = |r| \cdot \|v\|$ for every $r \in R$, $v \in V$,
2.1. Functoriality.

A $G$-map between normed $R[G]$-modules is a $G$-map between the underlying $R[G]$-modules, which is bounded with respect to the norms.

Let $V$ be a normed $R[G]$-module, and recall that $C^n(G,V)$ is endowed with the structure of an $R[G]$-module. For every $f \in C^n(G,V)$ one may consider the $\ell^\infty$-norm

$$\|f\|_\infty = \sup\{\|f(g_0, \ldots, g_n)\| : (g_0, \ldots, g_n) \in G^{n+1}\} \in [0, +\infty].$$

We set

$$C^n_b(G,V) = \{ f \in C^n(G,V) \mid \|f\|_\infty < \infty \}$$

and we observe that $C^n_b(G,V)$ is a $R[G]$-submodule of $C^n(G,V)$. Therefore, $C^n_b(G,V)$ is a normed $R[G]$-module. The differential $\delta^n : C^n(G,V) \to C^{n+1}(G,V)$ restricts to a $G$-map of normed $R[G]$-modules $\delta^n : C^n_b(G,V) \to C^{n+1}_b(G,V)$, so one may define as usual

$$Z^n_b(G,V) = \ker \delta^n \cap C^n_b(G,V)^G, \quad B^n_b(G,V) = \delta^n^{-1}(C^{n-1}_b(G,V)^G)$$

(where we understand that $B^0_b(G,V) = \{0\}$), and set

$$H^n_b(G,V) = Z^n_b(G,V) / B^n_b(G,V).$$

The $\ell^\infty$-norm on $C^n_b(G,V)$ restricts to a norm on $Z^n_b(G,V)$, which descends to a seminorm on $H^n_b(G,V)$ by taking the infimum over all the representatives of a coclass: namely, for every $\alpha \in H^n_b(G,V)$ one sets

$$\|\alpha\|_\infty = \inf\{\|f\|_\infty : f \in Z^n_b(G,V), [f] = \alpha\}.$$

**Definition 2.1.** The $R$-module $H^n_b(G,V)$ is the $n$-th bounded cohomology module of $G$ with coefficients in $V$. The seminorm $\|\cdot\|_\infty : H^n_b(G,V) \to \mathbb{R}$ is called the canonical seminorm of $H^n_b(G,V)$.

The canonical seminorm on $H^n_b(G,V)$ is a norm if and only if the subspace $B^n_b(G,V)$ is closed in $Z^n_b(G,V)$ (whence in $C^n_b(G,V)$). However, this is not always the case: for example, if $G$ is the fundamental group of a closed hyperbolic surface, then it is proved in [Som97, Som98] that $H^2_b(G,\mathbb{R})$ contains non-trivial elements with null seminorm. On the other hand, it was proved independently by Ivanov [Iva90] and Matsumoto and Morita [MM85] that $H^2_b(G,\mathbb{R})$ is a Banach space for every group $G$ (see Theorem 11.1).

2.1. **Functoriality.** Just as in the case of ordinary cohomology, also bounded cohomology provides a binary functor. The discussion carried out in Subsection 1.1 applies word by word in the bounded case. Namely, every exact sequence of normed $R[G]$-modules

$$0 \to V_1 \xrightarrow{\alpha} V_2 \xrightarrow{\beta} V_3 \to 0$$

induces a long exact sequence

$$0 \to H^0_b(V_1,G) \to H^0_b(V_2,G) \to H^0_b(V_3,G) \to H^1_b(G,V_1) \to H^1_b(G,V_2) \to \ldots.$$
in cohomology. Moreover, if $\psi: G_1 \to G_2$ is a group homomorphism, and $V$ is a normed $R[G_2]$-module, then $V$ admits a natural structure of normed $R[G_1]$-module, which is denoted by $\psi^{-1}V$. Then, the homomorphism $\psi$ induces a well-defined map

$$H^n_b(\psi^n): H^n_b(G_2, V) \to H^n_b(G_1, \psi^{-1}V)$$

in bounded cohomology. In particular, in the case of trivial coefficients we get a map

$$H^n(\psi^n): H^n_b(G_2, R) \to H^n_b(G_1, R).$$

2.2. The comparison map. The inclusion $C^\bullet_b(G, V) \to C^\bullet(G, V)$ induces a map

$$c: H^\bullet_b(G, V) \to H^\bullet(G, V)$$

called the comparison map. We will see soon that the comparison map is neither injective nor surjective in general. One could approach the study of $H^n_b(G, V)$ by looking separately at the kernel and at the image of the comparison map. This strategy is described in Section 4 for bounded cohomology groups in low degree.

2.3. Looking for a topological interpretation of bounded cohomology of groups. Let us now restrict to the case when $V = R$ is a trivial normed $G$-module. We would like to compare the bounded cohomology of $G$ with a suitably defined singular bounded cohomology of a suitable topological model for $G$. There is a straightforward notion of boundedness for singular cochains, so the notion of bounded singular cohomology of a topological space is easily defined (see Section 10). Then, it is still true that the bounded cohomology of a group $G$ is canonically isomorphic to the bounded singular cohomology of a $K(G, 1)$ (in fact, even more is true: in the case of real coefficients, the bounded cohomology of $G$ is isometrically isomorphic to the bounded singular cohomology of any countable CW-complex $X$ such that $\pi_1(X) = G$ – see Theorem 10.8). However, unfortunately the proof described in Subsection 1.2 for classical cohomology does not carry over to the bounded context. It is still true that homotopically equivalent spaces have isometrically isomorphic bounded cohomology, but, if $X_G$ is the $K(G, 1)$ defined in Subsection 1.2, then there is no clear reason why the bounded simplicial cohomology of $X_G$ (which coincides with the bounded cohomology of $G$) should be isomorphic to the bounded singular cohomology of $X_G$.

For example, if $X$ is any finite $\Delta$-complex, then every simplicial cochain on $X$ is obviously bounded, so the cohomology of the bounded simplicial cochains on $X$ coincides with the classical singular cohomology of $X$, which in general is very different from the bounded singular cohomology of $X$. 
3. The bar resolution

We have defined the cohomology (resp. the bounded cohomology) of $G$ as the cohomology of the complex $C^\bullet(G,V)^G$ (resp. $C^\bullet_b(G,V)^G$). Of course, an element $f \in C^\bullet(G,V)^G$ is completely determined by the values it takes on $(n + 1)$-tuples having 1 as first entry. More precisely, if we denote by

$$C^0(G,V) = V, \quad C^1(G,V) = C^{n-1}(G,V) = \{ f : G^n \to V \},$$

and we consider $C^n(G,V)$ simply as an $R$-module for every $n \in \mathbb{N}$, then we have $R$-isomorphisms

$$C^n(G,V)^G \to C^n(G,V) \quad \varphi \mapsto ((g_1, \ldots, g_n) \mapsto \varphi(1, g_1g_2, \ldots, g_1 \cdot g_n)).$$

Under these isomorphisms, the differential $\delta^* : C^\bullet(G,V)^G \to C^{\bullet+1}(G,V)^G$ translates into the differential $\delta^* : C^\bullet(G,V) \to C^{\bullet+1}(G,V)$ such that

$$\delta^0(v)(g) = g \cdot v - v \quad v \in V = C^0(G,V), \quad g \in G,$$

and

$$\delta^1(f)(g_1, \ldots, g_{n+1}) = g_1 \cdot f(g_2, \ldots, g_{n+1})$$

$$+ \sum_{i=1}^{n} (-1)^i f(g_1, \ldots, g_i g_{i+1}, \ldots, g_{n+1})$$

$$+ (-1)^{n+1} f(g_1, \ldots, g_n),$$

for $n \geq 1$. The complex

$$0 \to C^0(G,V) \xrightarrow{\delta^0} C^1(G,V) \xrightarrow{\delta^1} \ldots \xrightarrow{\delta^{n-1}} C^n(G,V) \xrightarrow{\delta^n} \ldots$$

is usually known as the inhomogeneous complex associated to the pair $(G,V)$. By construction, the cohomology of this complex is canonically isomorphic to $H^\bullet(G,V)$.

Just as we did for the homogeneous complex, if $V$ is a normed $R[G]$-module, then we may define the submodule $C^\bullet_b(G,V)$ of bounded elements of $C^\bullet(G,V)$. For every $n \in \mathbb{N}$, the isomorphism $C^n(G,V)^G \cong C^n(G,V)$ restricts to an isometric isomorphism $C^n_b(G,V)^G \cong C^n_b(G,V)$, so $C^\bullet_b(G,V)$ is a subcomplex of $C^\bullet(G,V)$, whose cohomology is canonically isometrically isomorphic to $H^\bullet_b(G,V)$.

4. (Bounded) cohomology of groups in low degree

In this section we analyze the (bounded) cohomology modules of a group $G$ in degree 0, 1, 2. We restrict our attention to the case when $V = R$ is equal either to $\mathbb{Z}$, or to $\mathbb{R}$, both endowed with the structure of trivial $R[G]$-module. In order to simplify the computations, it will be convenient to work with the inhomogeneous complexes of (bounded) cochains.
By the very definitions, we have \( C^0(G, R) = C^0_b(G, R) = R \), and \( \delta^0 = 0 \), so
\[
H^0(G, R) = H^0_b(G, R) = R.
\]

4.1. **(Bounded) group cohomology in degree one.** Let us now describe what happens in degree one. By definition, for every \( \varphi \in C^1(G, R) \) we have
\[
\delta(f)(g_1, g_2) = f(g_1) + f(g_2) - f(g_1 g_2)
\]
(recall that we are assuming that the action of \( G \) on \( R \) is trivial). In other words, if we denote by \( Z^1(G, R) \) and \( B^1(G, R) \) the spaces of cocycles and coboundaries of the inhomogeneous complex \( C^\ast(G, R) \), then we have
\[
H^1(G, R) = Z^1(G, R) = \text{Hom}(G, R).
\]
Moreover, every bounded homomorphism with values in \( \mathbb{Z} \) or in \( \mathbb{R} \) is obviously trivial, so
\[
H^1_b(G, R) = Z^1_b(G, R) = 0
\]
(here and henceforth, we denote by \( Z^\ast_b(G, R) \) and \( B^\ast_b(G, R) \) the spaces of cocycles and coboundaries of the bounded inhomogeneous complex \( C^\ast_b(G, R) \)).

4.2. **Group cohomology in degree two.** The situation in degree two is more interesting. A central extension of \( G \) by \( R \) is an exact sequence
\[
1 \rightarrow R \rightarrow G' \rightarrow G \rightarrow 1
\]
such that \( \iota(R) \) is contained in the center of \( G \). In what follows, in this situation we will identify \( R \) with \( \iota(R) \in G \) via \( \iota \). Two such extensions are equivalent if they may be put into a commutative diagram as follows:
\[
\begin{array}{ccc}
1 & \rightarrow & R \\
\downarrow \text{Id} & & \downarrow f \\
1 & \rightarrow & G_1 \rightarrow G \rightarrow 1
\end{array}
\]
(by the commutativity of the diagram, the map \( f \) is necessarily an isomorphism). Associated to an exact sequence
\[
1 \rightarrow R \rightarrow G' \rightarrow G \rightarrow 1
\]
there is a cocyle \( \varphi \in C^2(G, R) \) which is defined as follows: let \( s: G \rightarrow G' \) be any map such that \( \pi \circ s = \text{Id}_G \). Then we set
\[
\varphi(g_1, g_2) = s(g_1 g_2)^{-1} s(g_1) s(g_2).
\]
By construction we have \( \varphi(g_1, g_2) \in \ker \pi = R \), so \( \varphi \) is indeed an element in \( \overline{C}^2(G, R) \). It is easy to check that \( \overline{\delta}(\varphi) = 0 \), so \( \varphi \in \overline{Z}^2(G, R) \). Moreover, different choices for the section \( s \) give rise to cocycles which differ one from the other by a coboundary, so any central extension of \( G \) by \( R \) defines an element in \( H^2(G, R) \). It is not difficult to reverse this construction to show that every element in \( H^2(G, R) \) is represented by a central extension, and
two central extensions are equivalent if and only if they define the same element in $H^2(G, R)$ (see e.g. [Bro82] for full details). Therefore:

**Proposition 4.1.** The group $H^2(G, R)$ is in natural bijection with the set of equivalence classes of central extensions of $G$ by $R$.

**Remark 4.2.** Proposition 4.1 may be easily generalized to the case of arbitrary extensions of $G$ by $R$. Once such an extension is given, if $G'$ is the middle term of the extension, then the action of $G'$ on $R$ by conjugacy descends to a well-defined action of $G$ on $R$, thus defining on $R$ the structure of a (possibly non-trivial) $G$-module (this structure is trivial precisely when $R$ is central in $G'$). Then, one may associate to the extension an element of the cohomology group $H^2(G, R)$ with coefficients in the (possibly non-trivial) $R[G']$-module $R$.

4.3. **Bounded group cohomology in degree two: quasimorphisms.** As mentioned above, the study of $H^2_b(G, R)$ may be reduced to the study of the kernel and of the image of the comparison map

$$c : H^2_b(G, R) \to H^2(G, R).$$

We will describe in Section 11 a characterization of groups with injective comparison map due to Matsumoto and Morita [MM85]. In this subsection we describe the relationship between the kernel of the comparison map and the space of quasimorphisms on $G$.

**Definition 4.3.** A map $f : G \to R$ is a quasimorphism if there exists a constant $D \geq 0$ such that

$$|f(g_1) + f(g_2) - f(g_1g_2)| \leq D$$

for every $g_1, g_2 \in G$. The least $D \geq 0$ for which the above inequality is satisfied is the defect of $f$, and it is denoted by $D(f)$. The space of quasimorphisms is an $R$-module, and it is denoted by $Q(G, R)$, or simply by $Q(G)$ when $R$ is understood.

By the very definition, a quasimorphism is an element of $\overline{C}^1(G, R)$ having bounded differential. Of course, both bounded functions (i.e. elements in $\overline{C}^1_b(G, R)$) and homomorphisms (i.e. elements in $\overline{Z}^1(G, R) = \text{Hom}(G, R)$) are quasimorphisms.

If every quasimorphism could be obtained just by adding a bounded function to a homomorphism, the notion of quasimorphism would not really introduce something new. Observe that every bounded homomorphism with values in $R$ is necessarily trivial, so $\overline{C}^1_b(G, R) \cap \text{Hom}(G, R) = \{0\}$, and we may think of $\overline{C}^1_b(G, R) \oplus \text{Hom}(G, R)$ as of the space of “trivial” quasimorphisms on $G$.

The following result is an immediate consequence of the definitions, and shows that the existence of “non-trivial” quasimorphisms is equivalent to the non-injectivity of the comparison map $c : H^2_b(G, R) \to H^2(G, R)$. 
Proposition 4.4. The differential $\overline{\delta}: Q(G, R) \rightarrow \overline{C}^2_b(G, R)$ induces an isomorphism

$$Q(G, R) / \left( \overline{C}^1_b(G, R) \oplus \text{Hom}_\mathbb{Z}(G, R) \right) \cong \ker c.$$ 

Therefore, in order to show that $H^2_b(G, R)$ is non-trivial it is sufficient to construct quasimorphisms which do not stay at finite distance from a homomorphism.

4.4. Homogeneous quasimorphisms. Let us introduce the following:

Definition 4.5. A quasimorphism $f: G \rightarrow R$ is homogeneous if $f(g^n) = n \cdot f(g)$ for every $g \in G$, $n \in \mathbb{Z}$. The space of homogeneous quasimorphisms is a submodule of $Q(G, R)$, and it is denoted by $Q^h(G, R)$.

Of course, there are no non-trivial bounded homogeneous quasimorphisms. In particular, for every quasimorphism $f$ there exists at most one homogeneous quasimorphism $f^h$ such that $\|f^h - f\|_\infty < +\infty$, so a homogeneous quasimorphism which is not a homomorphism cannot stay at finite distance from a homomorphism. When $R = \mathbb{Z}$, it may happen that homogeneous quasimorphisms are quite sparse in $Q(G, \mathbb{Z})$: for example, for every $\alpha \in \mathbb{R} \setminus \mathbb{Z}$, if we denote by $\lfloor x \rfloor$ the largest integer which is not bigger than $x$, then the quasimorphism $f_\alpha: \mathbb{Z} \rightarrow \mathbb{Z}$ defined by

$$f_\alpha(n) = \lfloor \alpha n \rfloor$$

is not at finite distance from any element in $Q^h(\mathbb{Z}, \mathbb{Z}) = \text{Hom}(\mathbb{Z}, \mathbb{Z})$. When $R = \mathbb{R}$ homogeneous quasimorphisms play a much more important role, due to the following:

Proposition 4.6. Let $f \in Q(G, \mathbb{R})$ be a quasimorphism. Then, there exists a unique element $\overline{f} \in Q^b(G, \mathbb{R})$ that stays at a finite distance from $f$. Moreover, we have

$$\|f - \overline{f}\|_\infty \leq D(f), \quad D(\overline{f}) \leq 4D(f).$$

Proof. For every $g \in G$, $m, n \in \mathbb{N}$ we have

$$|f(g^{mn} - nf(g^m))| \leq (n - 1)D(f),$$

so

$$\left| \frac{f(g^n)}{n} - \frac{f(g^m)}{m} \right| \leq \left| \frac{f(g^n)}{n} - \frac{f(g^{mn})}{mn} \right| + \left| \frac{f(g^{mn})}{mn} - \frac{f(g^m)}{m} \right| \leq \left( \frac{1}{n} + \frac{1}{m} \right) D(f).$$

Therefore, the sequence $f(g^n)/n$ is a Cauchy sequence, and the same is true for the sequence $f(g^{-n})/(-n)$. Since $f(g^n) + f(g^{-n}) \leq f(0) + D(f)$ for every $n$, we may conclude that the limit

$$\overline{f}(g) = \lim_{n \to \infty} \frac{f(g^n)}{n}$$
exists for every \( g \in G \). Moreover, the inequality \(|f(g^n) - nf(g)| \leq (n-1)D(f)\) implies that
\[
\left| f(g) - \frac{f(g^n)}{n} \right| \leq D(f)
\]
so by passing to the limit we obtain that \( \|f \|_\infty \leq D(f) \). This immediately implies that \( f \) is a quasimorphism such that \( D(f) \leq 4D(f) \). Finally, the fact that \( f \) is homogeneous is obvious. \( \square \)

In fact, the stronger inequality \( D(f) \leq 2D(f) \) holds (see e.g. [Cal09] for a proof). Propositions 4.4 and 4.6 imply the following:

**Corollary 4.7.** The space \( Q(G, \mathbb{R}) \) decomposes as a direct sum
\[
Q(G, \mathbb{R}) = Q^h(G, \mathbb{R}) \oplus \overline{C}_b^1(G, \mathbb{R}) .
\]
Moreover, the restriction of \( \overline{\delta} \) to \( Q^h(G, \mathbb{R}) \) induces an isomorphism
\[
Q^h(G, \mathbb{R})/\text{Hom}(G, \mathbb{R}) \cong \ker c .
\]

4.5. **Quasimorphisms on abelian groups.** Suppose now that \( G \) is abelian.
Then, for every \( g_1, g_2 \in G \), every element \( f \in Q^h(G, \mathbb{R}) \), and every \( n \in \mathbb{N} \) we have
\[
|nf(g_1g_2) - nf(g_1) - nf(g_2)| = |f((g_1g_2)^n) - f(g_1^n) - f(g_2^n)|
\]
\[
= |f(g_1^n g_2^n) - f(g_1^n) - f(g_2^n)| \leq D(f) .
\]
Dividing by \( n \) this inequality and passing to the limit for \( n \to \infty \) we get that \( f(g_1g_2) = f(g_1) + f(g_2) \), i.e. \( f \) is a homomorphism. Therefore, every homogeneous quasimorphism on \( G \) is a homomorphism. Putting together Proposition 4.4 and Corollary 4.7 we obtain the following:

**Corollary 4.8.** If \( G \) is abelian, then the comparison map
\[
c: H^2_b(G, \mathbb{R}) \to H^2(G, \mathbb{R})
\]
is injective.

In fact, a much stronger result holds: if \( G \) is abelian, then it is amenable (see Definition 6.1), so \( H^p_b(G, \mathbb{R}) = 0 \) for every \( n \geq 1 \) (see Corollary 6.7). We stress that Corollary 4.8 does not hold in the case with integer coefficients. For example, we have the following:

**Proposition 4.9.** We have \( H^2_b(\mathbb{Z}, \mathbb{Z}) = \mathbb{R}/\mathbb{Z} \).

**Proof.** The short exact sequence
\[
0 \to \mathbb{Z} \to \mathbb{R} \to \mathbb{R}/\mathbb{Z} \to 0
\]
induces a short exact sequence of complexes
\[
0 \to \overline{C}_b^1(\mathbb{Z}, \mathbb{Z}) \to \overline{C}_b^1(\mathbb{Z}, \mathbb{R}) \to \overline{C}^* (\mathbb{Z}, \mathbb{R}/\mathbb{Z}) \to 0 .
\]
Let us consider the following portion of the long exact sequence induced in cohomology:
\[
\ldots \to H^1_b(\mathbb{Z}, \mathbb{R}) \to H^1(\mathbb{Z}, \mathbb{R}/\mathbb{Z}) \to H^2_b(\mathbb{Z}, \mathbb{Z}) \to H^2_b(\mathbb{Z}, \mathbb{R}) \to H^2(\mathbb{Z}, \mathbb{R}/\mathbb{Z}) \to \ldots
\]
Recall from Subsection 4.1 that $H^1_b(\mathbb{Z}, \mathbb{R}) = 0$. Moreover, we have $H^2(\mathbb{Z}, \mathbb{R}) = H^2(S^1, \mathbb{R}) = 0$, so $H^2_b(\mathbb{Z}, \mathbb{R})$ is equal to the kernel of the comparison map $c: H^2_b(\mathbb{Z}, \mathbb{R}) \to H^2(\mathbb{Z}, \mathbb{R})$, which vanishes by Corollary 4.8. Therefore, we have

$$H^2_b(\mathbb{Z}, \mathbb{Z}) \cong H^1(\mathbb{Z}, \mathbb{R}/\mathbb{Z}) = \text{Hom}(\mathbb{Z}, \mathbb{R}/\mathbb{Z}) \cong \mathbb{R}/\mathbb{Z}.$$ 

\[\square\]

It follows from the proof of the previous proposition that the isomorphism $\mathbb{R}/\mathbb{Z} \to H^2_b(\mathbb{Z}, \mathbb{Z})$ is given by the map

$$\mathbb{R}/\mathbb{Z} \ni [\alpha] \mapsto [\delta f_\alpha] \in H^2_b(\mathbb{Z}, \mathbb{Z}),$$

where $f_\alpha$ is the quasimorphism defined in (1). It is interesting to notice that the module $H^2_b(\mathbb{Z}, \mathbb{Z})$ is not finitely generated over $\mathbb{Z}$. This fact already shows that bounded cohomology may be very different from classical cohomology. The same phenomenon may occur in the case of real coefficients: in the following section we show that, if $F_2$ is the free group on 2 elements, then $H^2_b(F_2, \mathbb{R})$ is an infinite dimensional vector space.

4.6. The image of the comparison map. In Subsection 4.2 we have interpreted elements in $H^2(G, \mathbb{R})$ as equivalence classes of central extensions of $G$ by $R$. Of course, one may wonder whether elements of $H^2(G, \mathbb{R})$ admitting bounded representatives represent peculiar central extensions. It turns out that this is indeed the case: bounded classes represent central extensions which are quasi-isometrically trivial (i.e. quasi-isometrically equivalent to product extension). Of course, in order to give a sense to this statement we need to restrict our attention to the case when $G$ is finitely generated, and $R = \mathbb{Z}$ (or, more in general, $R$ is a finitely generated abelian group). Under these assumptions, let us consider the central extension

$$1 \to \mathbb{Z} \to G' \to G \to 1,$$

and the following condition:

(*) there exists a quasi-isometry $q: G' \to G \times \mathbb{Z}$ which makes the following diagram commute:

$$\begin{array}{ccc}
G' & \xrightarrow{\pi} & G \\
\downarrow q & & \downarrow \text{Id} \\
G \times \mathbb{Z} & \longrightarrow & G
\end{array}$$

where the horizontal arrow on the bottom represents the obvious projection.

Condition (*) is equivalent to the existence of a Lipschitz section $s: G \to G'$ such that $\pi \circ s = \text{Id}_G$ (see e.g. [KL01, Proposition 8.2]). Gersten proved that a sufficient condition for a central extension to satisfy condition (*) is that its coclass in $H^2(G, \mathbb{Z})$ admits a bounded representative (see [Ger92, Theorem 3.1]). Therefore, the image of the comparison map $H^2_b(G, \mathbb{Z}) \to H^2(G, \mathbb{Z})$
determines extensions which satisfy condition (*). As far as the author knows, it is not known whether the converse implication holds, i.e. if every central extension satisfying (*) is represented by a cohomology class lying in the image of the comparison map.

Let us now consider the case with real coefficients, and list some results about the surjectivity of the comparison map:

1. If $G$ is the fundamental group of a closed locally symmetric space of non-compact type, then the comparison map $c: H^n_b(G, \mathbb{R}) \to H^n(G, \mathbb{R}) \cong \mathbb{R}$ is surjective [LS06].

2. If $G$ is word hyperbolic, then the comparison map $c: H^n_b(G, V) \to H^n(G, V)$ is surjective for every $n \geq 2$ when $V$ is any normed $\mathbb{R}[G]$ module or any finitely generated group (considered as a trivial $\mathbb{Z}[G]$-module) [Min01].

3. If $G$ is finitely presented and the comparison map $c: H^2_b(G, V) \to H^2(G, V)$ is surjective for every normed $\mathbb{R}[G]$-module, then $G$ is word hyperbolic [Min02].

4. The set of closed 3-manifolds $M$ for which the comparison map $H^\bullet_b(\pi_1(M), \mathbb{R}) \to H^\bullet(\pi_1(M), \mathbb{R})$ is surjective in every degree is completely characterized in [FS02].

Observe that points (2) and (3) provide a characterization of word hyperbolicity in terms of bounded cohomology. Moreover, putting together point (2) with the discussion above, we see that every extension of a word hyperbolic group by $\mathbb{Z}$ is quasi-isometrically trivial.

5. The second bounded cohomology group of free groups

Let $F_2$ be the free group on two generators. Since $H^2(F_2, \mathbb{R}) = H^2(S^1 \lor S^1, \mathbb{R}) = 0$, the computation of $H^2_b(F_2, \mathbb{R})$ is reduced to the analysis of the space $Q(G, \mathbb{R})$ of real quasimorphisms on $F_2$. There are several constructions of elements in $Q(G, \mathbb{R})$ available in the literature. The first such example is probably due to Johnson [Joh72], who proved that $H^2_b(F_2, \mathbb{R}) \neq 0$. Afterwards, Brooks produced an infinite family of quasimorphisms inducing linearly independent elements of $H^2_b(F_2, \mathbb{R})$ [Bro81]. Since then, other constructions have been provided by many authors for more general classes of groups (see Remark 5.3 below). We describe here a family of quasimorphism which is due to Rolli [Rol].

Let $s_1, s_2$ be the generators of $F_2$, and let

$$\ell^\infty_{\text{odd}}(\mathbb{Z}) = \{ \alpha: \mathbb{Z} \to \mathbb{R} \mid \alpha(n) = -\alpha(-n) \text{ for every } n \in \mathbb{Z} \}.$$ 

For every $\alpha \in \ell^\infty_{\text{odd}}(\mathbb{Z})$ we consider the map

$$f_\alpha: F_2 \to \mathbb{R}$$

defined by

$$f_\alpha(s_1^{n_1} \cdots s_k^{n_k}) = \sum_{j=1}^k \alpha(n_k),$$
where we are identifying every element of $F_2$ with the unique reduced word representing it. It is elementary to check that $f_\alpha$ is a quasimorphism (see [Rol, Proposition 2.1]). Moreover, we have the following:

**Proposition 5.1** ([Rol]). The map
\[
\ell_{\text{odd}}(\mathbb{Z}) \to H^2_b(F_2, \mathbb{R}) \quad \alpha \mapsto [\delta(f_\alpha)]
\]
is injective.

**Proof.** Suppose that $[\delta(f_\alpha)] = 0$. By Proposition 4.4, this implies that $f_\alpha = h + b$, where $h$ is a homomorphism and $b$ is bounded. Then, for $i = 1, 2$, $k \in \mathbb{N}$, we have $k \cdot h(s_i) = h(s_i^k) = f_\alpha(s_i^k) - b(s_i^k) = \alpha(k) - b(s_i^k)$. By dividing by $k$ and letting $k$ going to $\infty$ we get $h(s_i) = 0$, so $h = 0$, and $f_\alpha = b$ is bounded.

Observe now that for every $k, l \in \mathbb{Z}$ we have $f_\alpha((s_1 s_2)^k) = 2k \cdot \alpha(l)$. Since $f_\alpha$ is bounded, this implies that $\alpha = 0$, whence the conclusion. \[\square\]

**Corollary 5.2.** Let $G$ be a group admitting an epimorphism
\[
\varphi : G \to F_2.
\]
Then the vector space $H^2_b(G, \mathbb{R})$ is infinite-dimensional.

**Proof.** Since $F_2$ is free, the epimorphism admits a right inverse $\psi$. The composition
\[
H^2_b(F_2, \mathbb{R}) \xrightarrow{\varphi^*} H^2_b(G, \mathbb{R}) \xrightarrow{\psi^*} H^2_b(F_2, \mathbb{R})
\]
of the induced maps in bounded cohomology is the identity, and Proposition 5.1 ensures that $H^2_b(F_2, \mathbb{R})$ is infinite-dimensional. The conclusion follows. \[\square\]

**Remark 5.3.** The previous corollary implies that a large class of groups has infinite-dimensional second bounded cohomology module. In fact, this holds true for every group belonging to one of the following families (groups as in (1) also satisfy condition (2), and groups as in (2) also satisfy (3): the cited papers generalized one the result of the other):

1. non-elementary Gromov hyperbolic groups [EF97];
2. groups acting geometrically on Gromov hyperbolic spaces with limit set consisting of at least three points [Fuj98];
3. groups admitting a non-elementary *weakly proper discontinuous action* on a Gromov hyperbolic space [BF02];
4. groups having infinitely many ends [Fuj00].

An important application of point (3) is that every subgroup of a the mapping class group of a compact surface either is virtually abelian, or has infinite-dimensional second bounded cohomology. A nice geometric interpretation of (homogeneous) quasimorphisms is given in [Man05].
6. Amenability

The original definition of amenable group is due to von Neumann [vN29], who introduced the class of amenable groups while studying the Banach-Tarski paradox. As usual, we will restrict our attention to the definition of amenability in the context of discrete groups, referring the reader e.g. to [Pie84, Pat88] for a thorough account on amenability in the wider context of locally compact groups.

Let $G$ be a group. We denote by $\ell^\infty(G) = C^0_b(G, \mathbb{R})$ the space of bounded real functions on $G$, endowed with the usual structure of normed $\mathbb{R}[G]$-module (in fact, of Banach $G$-module). A mean on $G$ is a map $m: \ell^\infty(G) \to \mathbb{R}$ satisfying the following properties:

1. $m$ is linear;
2. if $1_G$ denotes the map taking every element of $G$ to $1 \in \mathbb{R}$, then $m(1_G) = 1$;
3. $m(f) \geq 0$ for every non-negative $f \in \ell^\infty(G)$.

If conditions (1) and (2) are satisfied, then condition (3) is equivalent to

$$(3') \inf_{g \in G} f(g) \leq m(f) \leq \sup_{g \in G} f(g)$$

for every $f \in \ell^\infty(G)$.

In particular, the dual norm of a mean on $G$, when considered as a functional on $\ell^\infty(G)$, is equal to 1. A mean $m$ is left invariant (or simply invariant) if it satisfies the following additional condition:

4. $m(g \cdot f) = m(f)$ for every $g \in G, f \in \ell^\infty(G)$.

**Definition 6.1.** A group $G$ is amenable if it admits an invariant mean.

The following lemma describes some equivalent definitions of amenability that will prove useful later.

**Lemma 6.2.** Let $G$ be a group. Then, the following conditions are equivalent:

1. $G$ is amenable;
2. there exists a non-trivial left invariant continuous functional $\varphi \in \ell^\infty(G)^*$;
3. $G$ admits a left invariant finitely additive probability measure.

**Proof.** If $A$ is a subset of $G$, we denote by $\chi_A$ the characteristic function of $A$.

(1) $\Rightarrow$ (2): If $m$ is an invariant mean on $\ell^\infty(G)$, then the map $f \mapsto m(f)$ defines the desired functional on $\ell^\infty(G)$.

(2) $\Rightarrow$ (3): Let $\varphi: \ell^\infty(G) \to \mathbb{R}$ be a non-trivial continuous functional. For every $A \subseteq G$ we define a non-negative real number $\mu(A)$ as follows. For every partition $P$ of $A$ into a finite number of subsets $A_1, \ldots, A_n$, we set

$$\mu_P(A) = |\varphi(\chi_{A_1})| + \ldots + |\varphi(\chi_{A_n})|.$$ 

Observe that, if $\varepsilon_i$ is the sign of $\varphi(\chi_{A_i})$, then $\mu_P(A) = \varphi(\sum_{i} \varepsilon_i \chi_{A_i}) \leq \|\varphi\|$, so

$$\mu(A) = \sup_P \mu_P(A)$$
is a finite non-negative number for every $A \subseteq G$. By the linearity of $\varphi$, if $P'$ is a refinement of $P$, then $\mu_{P'}(A) \leq \mu_P(A)$, and this easily implies that $\mu$ is a finitely additive measure on $G$. Recall now that the set of characteristic functions generates a dense subspace of $\ell^\infty(G)$. As a consequence, since $\varphi$ is non-trivial, there exists a subset $A \subseteq G$ such that $\mu(A) \geq |\varphi(\chi_A)| > 0$. In particular, we have $\mu(G) > 0$, so after rescaling we may assume that $\mu$ is a probability measure on $G$. The fact that $\mu$ is $G$-invariant is now a consequence of the $G$-invariance of $\varphi$.

(3) $\Rightarrow$ (1): Let $\mu$ be a left invariant finitely additive measure on $G$, and let $Z \subseteq \ell^\infty(G)$ be the subspace generated by the functions $\chi_A$, $A \subseteq G$. Using the finite additivity of $\mu$, it is not difficult to show that there exists a linear functional $m_Z : Z \to \mathbb{R}$ such that $m(\chi_A) = \mu(A)$ for every $A \subseteq G$. Moreover, $m_Z$ has operator norm equal to one, so it is continuous. Since $Z$ is dense in $\ell^\infty(G)$, the functional $m_Z$ uniquely extends to a continuous functional $m \in \ell^\infty(G)'$. The fact that $m$ is a mean is an immediate consequence of the fact that every non-negative function in $\ell^\infty(G)$ may be approximated by linear combinations of characteristic functions with positive coefficients. Finally, the $G$-invariance of $\mu$ implies that $m_Z$, whence $m$, is also invariant. \qed

The action of $G$ on itself by right translations endows $\ell^\infty(G)$ also with the structure of a right Banach $G$-module, and it is well-known that the existence of a left-invariant mean on $G$ implies the existence of a bi-invariant mean on $G$: in other words, $G$ is amenable if and only if it admits a bi-invariant mean. However, we won’t use this fact in the sequel.

Finite groups are obviously amenable: an invariant mean $m$ is obtained by setting $m(f) = (1/|G|) \sum_{g \in G} f(g)$ for every $f \in \ell^\infty(G)$. Of course, we are interested in finding less obvious examples of amenable groups.

6.1. **Abelian groups are amenable.** A key result in the theory of amenable groups is the following theorem, which is originally due to von Neumann:

**Theorem 6.3** ([vN29]). *Every abelian group is amenable.*

**Proof.** We follow here the proof given in [Pat88], which is based on the Markov-Kakutani Fixed Point Theorem.

Let $G$ be an abelian group, and let $\ell^\infty(G)'$ be the topological dual of $\ell^\infty(G)$, endowed with the weak* topology. We consider the subset $K \subseteq \ell^\infty(G)'$ given by (not necessarily invariant) means on $G$, i.e. we set

$$K = \{ \varphi \in \ell^\infty(G)' | \varphi(1_G) = 1, \varphi(f) \geq 0 \text{ for every } f \geq 0 \} .$$

The set $K$ is non-empty (it contains the map $\varphi$ sending every element of $\ell^\infty(G)$ to the value it takes on the identity of $G$). Moreover, it is closed, convex, and contained in the closed ball of radius one in $\ell^\infty(G)'$. Therefore, $K$ is compact by the Banach-Alaouglu Theorem.

For every $g \in G$ let us now consider the map $L_g : \ell^\infty(G)' \to \ell^\infty(G)'$ defined by

$$L_g(\varphi)(f) = \varphi(g \cdot f) .$$
We want to show that the $L_g$ admit a common fixed point in $K$. To this aim, one may apply the Markov-Kakutani Theorem cited above, which we prove here (in the case we are interested in) for completeness.

We first show that, if $H$ is any non-empty compact convex subset of $K$ which is left invariant by a single $L_g$, then $L_g$ admits a fixed point in $H$. Let us fix $\varphi \in H$. For every $n \in \mathbb{N}$ we set

$$\varphi_n = \frac{1}{n+1} \sum_{i=0}^{n} L_g^i(\varphi).$$

By convexity, $\varphi_n \in H$ for every $n$, so there exists a subsequence $\varphi_{n_i}$ tending to $\varphi \in H$. Recall that $\|y\| = 1$ for every $y \in K$, so for every $f \in \ell^\infty(G)$ we have

$$|\varphi_{n_i}(g \cdot f) - \varphi_{n_i}(f)| = |(L_g(\varphi_{n_i}) - \varphi_{n_i})(f)| = \frac{|(L_g^{n+1}(\varphi) - \varphi)(f)|}{n_i + 1} \leq \frac{2\|f\|}{n_i + 1}.$$

By passing to the limit for $n_i$ tending to $\infty$, we obtain that $\varphi(g \cdot f) = \varphi(f)$ for every $f \in \ell^\infty(G)$, so $L_g$ has a fixed point in $H$.

Let us now observe that $K$ is $L_g$-invariant for every $g \in G$. Therefore, the set $K_g$ of points of $K$ which are fixed by $L_g$ is non-empty. Moreover, it is easily seen that $K_g$ is closed (whence compact) and convex. Since $G$ is abelian, if $1 \leq g_1, g_2$ are elements of $G$, then the maps $L_{g_1}$ and $L_{g_2}$ commute with each other, so $K_{g_1}$ is $L_{g_2}$-invariant. The argument above shows that $L_{g_2}$ has a fixed point in $K_{g_1}$, so $K_{g_1} \cap K_{g_2} \neq \emptyset$. One may iterate this argument to show that every finite intersection $K_{g_1} \cap \ldots \cap K_{g_m}$ is non-empty. In other words, the family $K_g, g \in G$ satisfies the finite intersection property, so $K_G = \bigcap_{g \in G} K_g$ is non-empty. But $K_G$ is precisely the set of invariant means on $G$. \hfill $\Box$

6.2. Other amenable groups. Starting from abelian groups, it is possible to construct larger classes of amenable groups:

**Proposition 6.4.** Let $G, H$ be amenable groups. Then:

1. Every subgroup of $G$ is amenable.
2. If the sequence

$$1 \to H \to G' \to G \to 1$$

is exact, then $G'$ is amenable (in particular, the direct product of a finite number of amenable groups is amenable).

3. Every direct union of amenable groups is amenable.

**Proof.** (1): Let $K$ be a subgroup of $G$. Let $S$ be a set of representatives in $G$ for the set $K \setminus G$ of right lateral classes of $K$ in $G$. For every $f \in \ell^\infty(K)$ we define $\hat{f} \in \ell^\infty(G)$ by setting $\hat{f}(g) = f(k)$, where $g = ks, s \in S, k \in K$. If $m$ is an invariant mean on $G$, then we set $m_H(f) = m(\hat{f})$. It is easy to check that $m_H$ is an invariant mean on $K$, so $K$ is amenable.

(2): Let $m_H, m_G$ be invariant means on $H, G$ respectively. For every $f \in \ell^\infty(G')$ we construct a map $f_G \in \ell^\infty(G)$ as follows. For $g \in G$, we
take $g'$ in the preimage of $g$ in $G'$, and we define $f_{g'} \in \ell^\infty(H)$ by setting
$f_{g'}(h) = f(g'h)$. Then, we set $f_G(g) = m_H(f_{g'})$. Using that $m_H$ is $H$-
invariant one may show that $f_G(g)$ does not depend on the choice of $g'$, so
$f_G$ is well-defined. We obtain the desired mean on $G$ by setting $m(f) = m_G(f_G)$.

(3): Let $G$ be the direct union of the amenable groups $G_i$, $i \in I$. For
every $i$ we consider the set
$K_i = \{ \varphi \in \ell^\infty(G) : \varphi(1_G) = 1, \varphi(f) \geq 0 \text{ for every } f \geq 0, \varphi \text{ is } G_i\text{-invariant} \}$.

If $m_i$ is an invariant mean on $G_i$, then the map $f \mapsto m_i(f|_{G_i})$ defines an
element of $K_i$. Therefore, each $K_i$ is non-empty. Moreover, the fact that the
union of the $G_i$ is direct implies that the family $K_i$, $i \in I$ satisfies the finite
intersection property. Finally, each $K_i$ is closed and compact in the weak*
topology on $\ell^\infty(G)$, so the intersection $K = \bigcap_{i \in I} K_i$ is non-empty. But $K$
is precisely the set of $G$-invariants means on $G$, whence the conclusion. □

The class of elementary amenable groups is the smallest class of groups
which contains all finite and all abelian groups, and is closed under the
operations of taking subgroups, forming quotients, forming extensions, and
taking direct unions [Day57]. For example, virtually solvable groups are
elementary amenable. Proposition 6.4 (together with Theorem 6.3) shows
that every elementary amenable group is indeed amenable. However, any
group of intermediate growth is amenable [Pat88] but is not elementary
amenable [Cho80] In particular, there exist amenable groups which are not
virtually solvable.

Remark 6.5. We have already noticed that, if $G_1, G_2$ are amenable groups,
then so is $G_1 \times G_2$. More precisely, in the proof of Proposition 6.4-(2) we
have shown how to construct an invariant mean $m$ on $G_1 \times G_2$ starting
from invariant means $m_i$ on $G_i$, $i = 1, 2$. Under the identification of means
with finitely additive probability measures, the mean $m$ corresponds to the
product measure $m_1 \times m_2$, so it makes sense to say that $m$ is the product
of $m_1$ and $m_2$. If $G_1, \ldots, G_n$ are amenable groups with invariant means
$m_1, \ldots, m_n$, then we denote by $m_1 \times \ldots \times m_n$ the invariant mean on $G$
inductively defined by the formula $m_1 \times \ldots \times m_n = m_1 \times (m_2 \times \ldots \times m_n)$.

6.3. Amenability and bounded cohomology. In this subsection we show
that amenable groups are somewhat invisible to bounded cohomology with
coefficients in normed $\mathbb{R}[G]$-modules (things are quite different in the case
of $\mathbb{Z}[G]$-modules). We say that an $\mathbb{R}[G]$-module $V$ is a dual normed $\mathbb{R}[G]$-
module if $V$ is isomorphic (as a normed $\mathbb{R}[G]$-module) to the topological
dual of some normed $\mathbb{R}[G]$-module $W$. In other words, $V$ is the space of
bounded linear functionals on the normed $\mathbb{R}[G]$-module $W$, endowed with
the action defined by $g \cdot f(w) = f(g^{-1}w)$, $g \in G$, $w \in W$.

Theorem 6.6. Let $G$ be an amenable group, and let $V$ be a dual normed
$\mathbb{R}[G]$-module. Then $H^n_b(G, V) = 0$ for every $n \geq 1$. 
Proof. Recall that the bounded cohomology $H^*_b(G,V)$ is defined as the cohomology of the $G$-invariants of the homogeneous complex

$$0 \longrightarrow C^0(G,V) \xrightarrow{\delta^0} C^1(G,V) \xrightarrow{\delta^1} \ldots \longrightarrow C^n(G,V) \longrightarrow \ldots$$

Of course, the cohomology of the complex $C^*_b(G,V)$ of possibly non-invariant cochains vanishes in positive degree, since the maps

$$k^{n+1}: C^{n+1}_b(G,V) \rightarrow C^n_b(G,V), \quad k^{n+1}(f)(g_0, \ldots, g_n) = f(1, g_0, \ldots, g_n), \quad n \geq 0$$

provide a (partial) homotopy between the identity and the zero map of $C^*_b(G,V)$ in positive degree. In order to prove the theorem, we are going to show that, under the assumption that $G$ is amenable, a similar homotopy can be defined on the complex of invariant cochains. Roughly speaking, while $k^*$ is obtained by coning over the identity of $G$, we will define a $G$-invariant homotopy $j^*$ by averaging the cone operator over all the elements of $G$. Let us fix an invariant mean $m$ on $G$, and let $f$ be an element of $C^{n+1}_b(G,V)$. Recall that $V = W'$ for some $\mathbb{R}[G]$-module $W$, so $f(g, g_0, \ldots, g_n)$ is a bounded functional on $W$ for every $(g, g_0, \ldots, g_n) \in G^{n+2}$. For every $(g_0, \ldots, g_n) \in G^{n+1}$, $w \in W$, we consider the function

$$f_w: G \rightarrow \mathbb{R}, \quad f_w(g) = f(g, g_0, \ldots, g_n)(w).$$

It follows from the definitions that $f_w$ is an element of $\ell^\infty(G)$, so we may set $(j^{n+1}(f))(g_0, \ldots, g_n)(w) = m(f_w)$. It is immediate to check that this formula defines a continuous functional on $W$, whose norm is bounded in terms of the norm of $f$. In other words, $j^{n+1}(f)(g_0, \ldots, g_n)$ is indeed an element of $V$, and the map $j^{n+1}: C^{n+1}_b(G,V) \rightarrow C^n_b(G,V)$ is bounded. The $G$-invariance of the mean $m$ implies that $j^{n+1}$ is $G$-equivariant. The collection of maps $j^{n+1}$, $n \in \mathbb{N}$ provides the required partial $G$-equivariant homotopy between the identity and the zero map of $C^n_b(G,V)$, $n \geq 1$. □

Corollary 6.7. Let $G$ be an amenable group. Then $H^n_b(G,\mathbb{R}) = 0$ for every $n \geq 1$.

Recall from Section 4.3 that the second bounded cohomology group with trivial real coefficients is strictly related to the space of real quasimorphisms. Putting together Corollary 6.7 with Proposition 4.4 and Corollary 4.7 we get the following result, which extends to amenable groups the characterization of real quasimorphisms on abelian groups we described in Subsection 4.5:

Corollary 6.8. Let $G$ be an amenable group. Then every real quasimorphism on $G$ is at bounded distance from a homomorphism. Equivalently, every homogeneous real quasimorphism on $G$ is a homomorphism.

From Corollary 5.2 and Corollary 6.7 we deduce that there exist many groups which are not amenable:

Corollary 6.9. Suppose that the group $G$ contains a non-abelian free subgroup. Then $G$ is not amenable.
Proof. We know from Corollary 5.2 that non-abelian free groups have non-trivial second bounded cohomology group with real coefficients, so they cannot be amenable. The conclusion follows from the fact that the subgroup of an amenable group is amenable. □

It was a long-standing problem to understand whether the previous corollary could be sharpened into a characterization of amenable groups as those groups which do not contain any non-abelian free subgroup. This question is usually attributed to von Neumann, and appeared first in [Day57]. Ol’shanskii answered von Neumann’s question in the negative in [Ol’80]. The first examples of finitely presented groups which are not amenable but do not contain any non-abelian free group are due to Ol’shanskii and Sapir [OS02].

6.4. Johnson’s characterization of amenability. We have just seen that the bounded cohomology of any amenable group with values in any dual co-

We keep notation from the preceding paragraph. If \([J] = 0\), then \(J = \delta \psi\) for some \(\psi \in C_b^1(G, (\ell^\infty(G)/\mathbb{R}))'\). For every \(g \in G\) we denote by \(\hat{\psi}(g) \in \ell^\infty(G)\) the pullback of \(\psi(g)\) via the projection map \(\ell^\infty(G) \to \ell^\infty(G)/\mathbb{R}\). We consider the element \(\varphi \in \ell^\infty(G)\) defined by \(\varphi = \delta_1 - \hat{\psi}(1)\). Since \(\hat{\psi}(1)\) vanishes on constant functions, we have \(\varphi(1_G) = 1\), so \(\varphi\) is non-trivial, and we are left to show that \(\varphi\) is \(G\)-invariant.

\[\delta J = \delta g \cdot \delta(g^{-1} \cdot f) = f(gg_0) = \delta g g_0(f) .\]

Let us consider the element

\[ J \in C_b^1(G, (\ell^\infty(G)/\mathbb{R}'))', \quad J(g_0, g_1) = \delta_{g_1} - \delta_{g_0} .\]

Of course we have \(\delta J = 0\). Moreover, for every \(g, g_0, g_1 \in G\) we have

\[(g \cdot J)(g_0, g_1) = g(J(g^{-1}g_0, g^{-1}g_1)) = g(\delta_{g^{-1}g_0} - \delta_{g^{-1}g_1}) = \delta_{g_0} - \delta_{g_1} = J(g_0, g_1)\]

so \(J\) is \(G\)-invariant, and defines an element \([J] \in H_b^1(G, (\ell^\infty(G)/\mathbb{R}))'\), called the Johnson class of \(G\).

Theorem 6.10. Suppose that the Johnson class of \(G\) vanishes. Then \(G\) is amenable.

Proof. By Lemma 6.2, it is sufficient to show that the topological dual \(\ell^\infty(G)'\) contains a non-trivial \(G\)-invariant element.

We keep notation from the preceding paragraph. If \([J] = 0\), then \(J = \delta \psi\) for some \(\psi \in C_b^1(G, (\ell^\infty(G)/\mathbb{R}))'\). For every \(g \in G\) we denote by \(\hat{\psi}(g) \in \ell^\infty(G)\) the pullback of \(\psi(g)\) via the projection map \(\ell^\infty(G) \to \ell^\infty(G)/\mathbb{R}\). We consider the element \(\varphi \in \ell^\infty(G)\) defined by \(\varphi = \delta_1 - \hat{\psi}(1)\). Since \(\hat{\psi}(1)\) vanishes on constant functions, we have \(\varphi(1_G) = 1\), so \(\varphi\) is non-trivial, and we are left to show that \(\varphi\) is \(G\)-invariant.
The $G$-invariance of $\psi$ implies the $G$-invariance of $\hat{\psi}$, so
\begin{equation}
\hat{\psi}(g) = (g \cdot \hat{\psi})(g) = g \cdot (\hat{\psi}(1)) \quad \text{for every } g \in G.
\end{equation}
From $J = \delta \psi$ we deduce that
\[ \delta g_1 - \delta g_0 = \hat{\psi}(g_1) - \hat{\psi}(g_0) \quad \text{for every } g_0, g_1 \in G. \]
In particular, we have
\begin{equation}
\delta_g - \hat{\psi}(g) = \delta_1 - \hat{\psi}(1) \quad \text{for every } g \in G.
\end{equation}
Therefore, using Equations (2), (3), for every $g \in G$ we get
\[ g \cdot \varphi = g \cdot (\delta_1 - \hat{\psi}(1)) = (g \cdot \delta_1) - g \cdot (\hat{\psi}(1)) = \delta_g - \hat{\psi}(g) = \delta_1 - \hat{\psi}(1) = \varphi, \]
and this concludes the proof. \hfill \Box

Putting together Theorems 6.6 and 6.10 we get the following result, which characterizes amenability in terms of bounded cohomology:

**Corollary 6.11.** Let $G$ be a group. Then $G$ is amenable if and only if $H^n_{\text{b}}(G, V) = 0$ for every dual normed $\mathbb{R}[G]$-module $V$ and every $n \geq 1$. \hfill \Box

### 7. Group cohomology via resolutions

Computing group cohomology by means of its very definition is usually very hard. The topological interpretation of group cohomology already provides a powerful tool for computations: for example, one may estimate the cohomological dimension of a group $G$ (i.e. the maximal $n \in \mathbb{N}$ such that $H^n(G, R) \neq 0$) in terms of the dimension of a $K(G, 1)$ as a CW-complex. We have already mentioned that a topological interpretation for the bounded cohomology of a group is still available, but in order to prove this fact more machinery has to be introduced. Before going into the bounded case, we describe some well-known results which hold in the classical case: namely, we will show that the cohomology of $G$ may be computed by looking at several complexes of cochains. This crucial fact was already observed in the pioneering works on group cohomology of the 1940s and the 1950s. There are several ways to define group cohomology in terms of resolutions. We privilege here an approach that better extends to the case of bounded cohomology. We briefly compare our approach to more traditional ones in Subsection 7.3.

#### 7.1. Relatively injective modules.
We begin by introducing the notions of relatively injective $R[G]$-module and of strong $G$-resolution of an $R[G]$-module. The counterpart of these notions in the context of normed $R[G]$-modules will play an important role in the theory of bounded cohomology of groups. The importance of these notions is due to the fact that the cohomology of $G$ may be computed by looking at any strong $G$-resolution of the coefficient module by relatively injective modules (see Corollary 7.5 below).
A $G$-map $\iota: A \to B$ between $R[G]$-modules is \textit{strongly injective} if there is an $R$-linear map $\sigma: B \to A$ such that $\sigma \circ \iota = \operatorname{Id}_A$ (in particular, $\iota$ is injective). We emphasize that, even if $A$ and $B$ are $G$-modules, the map $\sigma$ is \textit{not} required to be $G$-equivariant.

**Definition 7.1.** An $R[G]$-module $U$ is \textit{relatively injective} (over $R[G]$) if the following holds: whenever $A, B$ are $G$-modules, $\iota: A \to B$ is a strongly injective $G$-map and $\alpha: A \to U$ is a $G$-map, there exists a $G$-map $\beta: B \to U$ such that $\beta \circ \iota = \alpha$.

\[
\begin{array}{ccc}
0 & \longrightarrow & A \\
& & \ \downarrow \iota \\
& \sigma & \hookrightarrow \ B \\
& & \ \downarrow \alpha \\
& & \ \downarrow \beta \\
& & U
\end{array}
\]

In the case when $R = \mathbb{R}$, a map is strongly injective if and only if it is injective, so in the context of $\mathbb{R}[G]$-modules the notions of relative injectivity and the traditional notion of injectivity coincide. However, relative injectivity is rather weaker than injectivity in general. For example, if $R = \mathbb{Z}$ and $G = \{1\}$, then $C^n(G, R)$ is isomorphic to the trivial $G$-module $\mathbb{Z}$, which of course is not injective over $\mathbb{Z}[G] = \mathbb{Z}$. On the other hand, we have the following:

**Lemma 7.2.** For every $R[G]$-module $V$ and every $n \in \mathbb{N}$, the $R[G]$-module $C^n(G, V)$ is relatively injective.

**Proof.** Let us consider the extension problem described in Definition 7.1, with $U = C^n(G, V)$. Then we define $\beta$ as follows:

\[
\beta(b)(g_0, \ldots, g_n) = \alpha(g_0 \sigma(g_0^{-1}b))(g_0, \ldots, g_n).
\]

The fact that $\beta$ is $R$-linear and that $\beta \circ \iota = \alpha$ is straightforward, so we just need to show that $\beta$ is $G$-equivariant. But for every $g \in G$, $b \in B$ and $(g_0, \ldots, g_n) \in C^{n+1}$ we have

\[
g(\beta(b))(g_0, \ldots, g_n) = g(\beta(b)(g^{-1}g_0, \ldots, g^{-1}g_n))
\]

\[
= g(\alpha(g^{-1}g_0 \sigma(g_0^{-1}gb))(g^{-1}g_0, \ldots, g^{-1}g_n))
\]

\[
= g((g^{-1}(\alpha(g_0 \sigma(g_0^{-1}gb)))(g^{-1}g_0, \ldots, g^{-1}g_n))
\]

\[
= \alpha(g_0 \sigma(g_0^{-1}gb))(g_0, \ldots, g_n)
\]

\[
= \beta(gb)(g_0, \ldots, g_n)
\]

whence the conclusion. \(\square\)

### 7.2. Complexes and resolutions

An $R[G]$-\textit{complex} (or simply a $G$-\textit{complex} or a \textit{complex}) is a sequence of $R[G]$-modules $E^i$ and $G$-maps $\delta^i: E^i \to E^{i+1}$, $i \in \mathbb{N}$, such that $\delta^{i+1} \circ \delta^i = 0$ for every $i$:

\[
0 \longrightarrow E^0 \overset{\delta^0}{\longrightarrow} E^1 \overset{\delta^1}{\longrightarrow} \ldots \overset{\delta^n}{\longrightarrow} E^{n+1} \overset{\delta^{n+1}}{\longrightarrow} \ldots
\]
Such a sequence will be denoted by \((E^\bullet, \delta^\bullet)\). Moreover, we set \(Z^n(E^\bullet) = \ker \delta^n \cap (E^n)^G\), \(B^n(E^\bullet) = \delta^{n-1}(E^{n-1})^G\) (where again we understand that \(B^0(E^\bullet) = 0\)), and we define the cohomology of the complex \(E^\bullet\) by setting
\[
H^n(E^\bullet) = Z^n(E^\bullet)/B^n(E^\bullet).
\]

A chain map between \(G\)-complexes \((E^\bullet, \delta^\bullet_E)\) and \((F^\bullet, \delta^\bullet_F)\) is a sequence of \(G\)-maps \(\{\alpha^i: E^i \to F^i \mid i \in \mathbb{N}\}\) such that \(\delta^i_F \circ \alpha^i = \alpha^{i+1} \circ \delta^i_E\) for every \(i \in \mathbb{N}\). If \(\alpha^\bullet, \beta^\bullet\) are chain maps between \((E^\bullet, \delta^\bullet_E)\) and \((F^\bullet, \delta^\bullet_F)\), a \(G\)-homotopy between \(\alpha^\bullet\) and \(\beta^\bullet\) is a sequence of \(G\)-maps \(\{T^i: E^i \to F^{i-1} \mid i \geq 0\}\) such that \(T^i \circ \delta^0_E = \alpha^0 - \beta^0\) and \(\delta^i_E \circ T^i + T^{i+1} \circ \delta^i_E = \alpha^i - \beta^i\) for every \(i \geq 1\). Every chain map induces a well-defined map in cohomology, and \(G\)-homotopic chain maps induce the same map in cohomology.

If \(E\) is an \(R[G]\)-module, an augmented \(G\)-complex \((E, E^\bullet, \delta^\bullet)\) with augmentation map \(\varepsilon: E \to E^0\) is a complex
\[
0 \longrightarrow E^0 \xrightarrow{\varepsilon} E^0 \xrightarrow{\delta^0} \cdots \xrightarrow{\delta^n} E^{n+1} \xrightarrow{\delta^{n+1}} \cdots
\]
A resolution of \(E\) (over \(G\)) is an exact augmented complex \((E, E^\bullet, \delta^\bullet)\) (over \(G\)). A resolution \((E, E^\bullet, \delta^\bullet)\) is relatively injective if \(E^n\) is relatively injective for every \(n \geq 0\). It is well-known that any map between modules extends to a chain map between injective resolutions of the modules. Unfortunately, the same result for relatively injective resolutions does not hold. The point is that relative injectivity guarantees the needed extension property only for strongly injective maps. Therefore, we need to introduce the notion of strong resolution.

A contracting homotopy for a resolution \((E, E^\bullet, \delta^\bullet)\) is a sequence of \(R\)-linear maps \(k^i: E^i \to E^{i-1}\) such that \(\delta^{i-1} \circ k^i + k^{i+1} \circ \delta^i = \text{Id}_{E^i}\) if \(i \geq 0\), and \(k^0 \circ \delta^i = \text{Id}_{E^i}\):
\[
0 \longrightarrow E \xrightarrow{k^0} E^0 \xrightarrow{k^1} E^1 \xrightarrow{k^2} \cdots \xrightarrow{k^n} E^{n+1} \xrightarrow{k^{n+1}} \cdots
\]
Note however that it is not required that \(k^i\) be \(G\)-equivariant. A resolution is strong if it admits a contracting homotopy.

The following proposition shows that the chain complex \(C^\bullet(G, V)\) provides a relatively injective strong resolution of \(V\):

**Proposition 7.3.** Suppose that \(R = \mathbb{R}\). Let \(\varepsilon: V \to C^0(G, V)\) be defined by \(\varepsilon(v)(g) = v\) for every \(v \in V\), \(g \in G\). Then the augmented complex
\[
0 \longrightarrow V \xrightarrow{\varepsilon} C^0(G, V) \xrightarrow{\delta^0} C^1(G, V) \longrightarrow \cdots \longrightarrow C^n(G, V) \longrightarrow \cdots
\]
provides a relatively injective strong resolution of \(V\).

**Proof.** We already know that each \(C^i(G, V)\) is relatively injective. In order to show that the augmented complex \((V, C^\bullet(G, V), \delta^\bullet)\) is a strong resolution it is sufficient to observe that the maps
\[
k^{n+1}: C^{n+1}(G, V) \to C^n(G, V) \quad k^{n+1}(f)(g_0, \ldots, g_n) = f(1, g_0, \ldots, g_n)
\]
provide the required contracting homotopy.

The resolution of $V$ described in the previous proposition is obtained just by augmenting the homogeneous complex associated to $(G, V)$, and it is usually called the standard resolution of $V$ (over $G$).

The following result may be proved by means of standard homological algebra arguments (see e.g. [Iva87], [Mon01, Lemmas 7.2.4 and 7.2.6] for a detailed proof in the bounded case – the argument there extends to this context just by forgetting any reference to the norms). It implies that any relatively injective strong resolution of a $G$-module $V$ may be used to compute the cohomology modules $H^\bullet(G, V)$ (see Corollary 7.5).

**Theorem 7.4.** Let $\alpha: E \to F$ be a $G$-map between $R[G]$-modules, let $(E, E^\bullet, \delta^\bullet_E)$ be a strong resolution of $E$, and suppose that $(F, F^\bullet, \delta^\bullet_F)$ is an augmented complex such that $F^i$ is relatively injective for every $i \geq 0$. Then $\alpha$ extends to a chain map $\alpha^\bullet$, and any two extensions of $\alpha$ to chain maps are $G$-homotopic.

**Corollary 7.5.** Let $(V, V^\bullet, \delta^\bullet_V)$ be a relatively injective strong resolution of $V$. Then for every $n \in \mathbb{N}$ there is a canonical isomorphism

$$H^n(G, V) \cong H^n(V^\bullet).$$

**Proof.** By Proposition 7.3, both $(V, V^\bullet, \delta^\bullet_V)$ and the standard resolution of $V$ are relatively injective strong resolutions of $V$ over $G$. Therefore, Theorem 7.4 provides chain maps between $C^\bullet(G, V)$ and $V^\bullet$, which are one the $G$-homotopy inverse of the other. Therefore, these chain maps induce isomorphisms in cohomology.

7.3. **The classical approach to group cohomology via resolutions.** In this subsection we describe the relationship between the description of group cohomology via resolutions given above and more traditional approaches to the subject (see e.g. [Bro82]). The reader who is not interested can safely skip the section, since the results cited below will not be used elsewhere in the course.

The category of abelian groups endowed with a left action by $G$ obviously coincides with the category of left $\mathbb{Z}[G]$-modules. If $V, W$ are two such modules, then the space $\text{Hom}_{\mathbb{Z}}(V, W)$ is endowed with the structure of a $\mathbb{Z}[G]$-module by setting $(g \cdot f)(v) = g(f(g^{-1}v))$ for every $g \in G$, $f \in \text{Hom}_{\mathbb{Z}}(V, W)$, $v \in V$. Then, the cohomology $H^\bullet(G, V)$ is often defined in the following equivalent ways (we refer e.g. to [Bro82] for the definition of projective module (over a ring $R$); for our discussion, it is sufficient to know that any free $R$-module is projective over $R$, so any free resolution is projective):

1. **(Via injective resolutions of $V$):** Let $(V, V^\bullet, \delta^\bullet_V)$ be an injective resolution of $V$ over $\mathbb{Z}[G]$, and take the complex $W^\bullet = \text{Hom}_{\mathbb{Z}}(V^\bullet, \mathbb{Z})$, endowed with the action $g \cdot f(v) = f(g^{-1}v)$. Then $(W^\bullet)^G = \text{Hom}_{\mathbb{Z}[G]}(V^\bullet, \mathbb{Z})$ (where $\mathbb{Z}$ is endowed with the structure of trivial $G$-module), and one may define $H^\bullet(G, V)$ as the homology of the $G$-invariants of $W^\bullet$. 
(2) (Via projective resolutions of $\mathbb{Z}$): Let $(\mathbb{Z}, P_\bullet, d_\bullet)$ be a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}[G]$, and take the complex $Z^\bullet = \text{Hom}_\mathbb{Z}(P_\bullet, V)$. We have again that $(Z^\bullet)^G = \text{Hom}_{\mathbb{Z}[G]}(P_\bullet, V)$, and again we may define $H^\bullet(G, V)$ as the homology of the $G$-invariants of the complex $Z^\bullet$.

The fact that these two definitions are indeed equivalent is proved e.g. in [Bro82].

We have already observed that, if $V$ is a $\mathbb{Z}[G]$-module, the module $C^n(G, V)$ is not injective in general. However, this is not really a problem, since the complex $C^\bullet(G, V)$ may be recovered from a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}[G]$. Namely, let $C_n(G, \mathbb{Z})$ be the free $\mathbb{Z}$-module admitting the set $G^{n+1}$ as a basis. The diagonal action of $G$ onto $G^{n+1}$ endows $C_n(G, \mathbb{Z})$ with the structure of a $\mathbb{Z}[G]$-module. The modules $C_n(G, \mathbb{Z})$ may be arranged into a resolution

$$0 \leftarrow \mathbb{Z} \xrightarrow{\varepsilon} C_0(G, \mathbb{Z}) \xrightarrow{d_1} C_1(G, \mathbb{Z}) \xrightarrow{d_2} \ldots \xrightarrow{d_n} C_n(G, \mathbb{Z}) \xrightarrow{d_{n+1}} \ldots$$

of the trivial $\mathbb{Z}[G]$-module $\mathbb{Z}$ over $\mathbb{Z}[G]$ (the homology of the $G$-coinvariants of this resolution is by definition the homology of $G$, we refer to Section 11 for more details). Now, it is easy to check that $C_n(G, \mathbb{Z})$ is free, whence projective, as a $\mathbb{Z}[G]$-module. Moreover, the module $\text{Hom}_{\mathbb{Z}}(C_n(G, \mathbb{Z}), V)$ is $\mathbb{Z}[G]$-isomorphic to $C^n(G, V)$. This shows that, in the context of the traditional definition of group cohomology via resolutions, the complex $C^\bullet(G, V)$ arises from a projective resolution of $\mathbb{Z}$, rather than from an injective resolution of $V$.

### 7.4. The topological interpretation of group cohomology revisited.

Let us show how Corollary 7.5 may be exploited to prove that the cohomology of $G$ is isomorphic to the singular cohomology of any path-connected topological space $X$ satisfying conditions (1), (2) and (3) described in Subsection 1.2. We fix an identification between $G$ and the group of the covering automorphisms of the universal covering $\tilde{X}$ of $X$. The action of $G$ on $\tilde{X}$ induces an action of $G$ on $C_\bullet(\tilde{X}, R)$, whence an action of $G$ on $C^\bullet(\tilde{X}, R)$, which is defined by $(g \cdot \varphi)(c) = \varphi(g^{-1}c)$ for every $c \in C_s(\tilde{X}, R)$, $\varphi \in C^\bullet(\tilde{X}, R)$, $g \in G$. Therefore, for $n \in \mathbb{N}$ both $C_n(\tilde{X}, R)$ and $C^n(\tilde{X}, R)$ are endowed with the structure of $R[G]$-modules. We have a natural identification $C^\bullet(\tilde{X}, R)^G \cong C^\bullet(X, R)$.

**Lemma 7.6.** For every $n \in \mathbb{N}$, the singular cochain module $C^n(\tilde{X}, R)$ is relatively injective.

**Proof.** For every topological space $Y$, let us denote by $S_n(Y)$ the set of singular simplices with values in $Y$.

We denote by $L_n(\tilde{X})$ a set of representatives for the action of $G$ on $S_n(\tilde{X})$ (for example, if $F$ is a set of representatives for the action of $G$ on $\tilde{X}$, we may define $L_n(\tilde{X})$ as the set of singular $n$-simplices whose first vertex lies in $F$). Then, for every $n$-simplex $s \in S_n(\tilde{X})$, there exist a unique $g_s \in G$, and
a unique $\overline{s} \in L_n(\tilde{X})$ such that $g_s \cdot \overline{s} = s$. Let us now consider the extension problem:

\[
\begin{array}{c}
0 \rightarrow A \xrightarrow{\alpha} B \\
\downarrow \beta \hspace{1cm} \downarrow \psi \hspace{1cm} \\
C^n(\tilde{X}, R)
\end{array}
\]

We define the desired extension $\beta$ by setting

$$\beta(b)(s) = \alpha(\sigma(g_s^{-1} \cdot b))(\overline{s})$$

for every $s \in S_n(\tilde{X})$. It is easy to verify that the map $\beta$ is an $R[G]$-map, and that $\alpha = \beta \circ \iota$.

An alternative proof (providing the same solution to the extension problem) is the following. Using that the action of $G$ on $\tilde{X}$ is free, it is easy to show that $L_n(\tilde{X})$, when considered as a subset of $C_n(\tilde{X}, R)$, is a free basis of $C_n(\tilde{X}, R)$ over $R[G]$. In other words, every $c \in C_n(\tilde{X}, R)$ may be expressed uniquely as a sum of the form $c = \sum_{i=1}^{k} a_i g_i s_i$, $a_i \in R$, $g_i \in G$, $s_i \in L_n(\tilde{X})$. Therefore, the map

$$\psi: C^0(G, C^0(L_n(\tilde{X}), R)) \rightarrow C^n(\tilde{X}, R), \quad \psi(f) \left(\sum_{i=1}^{k} a_i g_i s_i \right) = \sum_{i=1}^{k} a_i f(g_i)(s_i)$$

is well-defined. If we endow $C^0(L_n(\tilde{X}), R)$ with the structure of trivial $G$-module, then a straightforward computation shows that $\psi$ is in fact a $G$-isomorphism, whence the conclusion by Lemma 7.2.

**Proposition 7.7.** Let $\varepsilon: R \rightarrow C^0(\tilde{X}, R)$ be defined by $\varepsilon(t)(s) = t$ for every singular $0$-simplex $s$ in $\tilde{X}$. Suppose that $H_i(\tilde{X}, R) = 0$ for every $i \geq 1$. Then the augmented complex

$$0 \rightarrow R \xrightarrow{\varepsilon} C^0(\tilde{X}, R) \xrightarrow{\delta^1} C^1(\tilde{X}, R) \rightarrow \cdots \xrightarrow{\delta^{n-1}} C^n(\tilde{X}, R) \xrightarrow{\delta^n} C^{n+1}(\tilde{X}, R)$$

is a relative injective strong resolution of the trivial $R[G]$-module $R$.

**Proof.** Since $\tilde{X}$ is path-connected, we have that $\operatorname{Im} \varepsilon = \ker \delta^0$. Observe now that $C_n(\tilde{X}, R)$ is $R$-free for every $n \in \mathbb{N}$. As a consequence, the obvious augmented complex associated to $C_n(\tilde{X}, R)$, being acyclic, is homotopically trivial over $R$. Since $C^n(\tilde{X}, R) \cong \operatorname{Hom}_R(C_n(\tilde{X}, R), R)$, we may conclude that the augmented complex described in the statement is a strong resolution of $R$ over $R[G]$. Then the conclusion follows from Lemma 7.6.

Putting together Propositions 7.3, 7.7 and Corollary 7.5 we may provide the following topological characterization of $H^n(G, R)$:

**Corollary 7.8.** Let $X$ be a path-connected space admitting a universal covering $\tilde{X}$, and suppose that $H_i(\tilde{X}, R) = 0$ for every $i \geq 1$. Then $H^i(X, R)$ is canonically isomorphic to $H^i(\pi_1(X), R)$ for every $i \in \mathbb{N}$. 

8. BOUNDED COHOMOLOGY VIA RESOLUTIONS

Just as in the case of classical cohomology, it is often useful to have alternative ways for computing the bounded cohomology of a group. This section is devoted to an approach to bounded cohomology which closely follows the traditional approach to classical cohomology via homological algebra. The circle of ideas we are going to describe first appeared (in the case with trivial real coefficients) in a paper by Brooks [Bro81], where it was exploited to give an independent proof of Gromov’s result that the isomorphism type of the bounded cohomology of a space (with real coefficients) only depends on its fundamental group [Gro82]. Brooks’ theory was then developed by Ivanov in his foundational paper [Iva87]. Ivanov gave a new proof of the vanishing of the bounded cohomology (with real coefficients) of simply connected space (this result played an important role in Brooks’ argument, and was originally due to Gromov [Gro82]), and managed to incorporate the seminorm into the homological algebra approach to bounded cohomology, thus proving that the bounded cohomology of a space is isometrically isomorphic to the bounded cohomology of its fundamental group (in the case with real coefficients). Ivanov’s theory was further developed by Monod [Mon01], who paid a particular attention to the continuous bounded cohomology of topological groups.

Both Ivanov’s and Monod’s theory are concerned with Banach $G$-modules, which are in particular $\mathbb{R}[G]$-modules. For the moment, we prefer to consider also the (quite different) case with integral coefficients. Therefore, we let $R$ be either $\mathbb{Z}$ or $\mathbb{R}$, and we concentrate our attention on the category of normed $R[G]$-modules introduced in Section 2. In the next subsections we will see that relative injective modules and strong resolutions may be defined in this context just by adapting to normed $R[G]$-modules the analogous definitions for generic $R[G]$-modules.

8.1. Relative injectivity. Throughout the whole section, unless otherwise stated, we will deal only with normed $R[G]$-modules. Therefore, $G$-morphisms will be always assumed to be bounded.

The following definitions are taken from [Iva87] (where only the case when $R = \mathbb{R}$ and $V$ is Banach is dealt with). A bounded linear map $\iota: A \to B$ of normed $R$-modules is strongly injective if there is a linear map $\sigma: B \to A$ with $\|\sigma\| \leq 1$ and $\sigma \circ \iota = \text{Id}_A$ (in particular, $\iota$ is injective). We emphasize that, even when $A$ and $B$ are $R[G]$-modules, the map $\sigma$ is not required to be $G$-equivariant.

**Definition 8.1.** A normed $R[G]$-module $E$ is relatively injective if for every strongly injective $G$-morphism $\iota: A \to B$ of normed $R[G]$-modules and every $G$-morphism $\alpha: A \to E$ there is a $G$-morphism $\beta: B \to E$ satisfying $\beta \circ \iota = \alpha$. 
and $||\beta|| \leq ||\alpha||$.

\[
\begin{array}{c}
0 \rightarrow A \xrightarrow{\sigma} B \\
\alpha \downarrow \hspace{2cm} \downarrow \beta \\
E
\end{array}
\]

**Remark 8.2.** Let $E$ be a normed $R[G]$-module, and let $\hat{E}$ be the underlying $R[G]$-module. Then no obvious implication exists between the fact that $E$ is relatively injective (in the category of normed $R[G]$-modules, i.e. according to Definition 8.1), and the fact that $\hat{E}$ is (in the category of $R[G]$-modules, i.e. according to Definition 7.1). This could suggest that the use of the same name for these different notions could indeed be an abuse. However, unless otherwise stated, henceforth we will deal with relatively injective modules only in the context of normed $R[G]$-modules, so the reader may safely take Definition 8.1 as the only definition of relative injectivity.

The following result is due to Ivanov [Iva87] (see also Remark 8.5), and shows that the modules involved in the definition of bounded cohomology are relatively injective.

**Lemma 8.3.** Let $V$ be a normed $R[G]$-module. Then the normed $R[G]$-module $C^n_b(G,V)$ is relatively injective.

**Proof.** Let us consider the extension problem described in Definition 8.1, with $E = C^n_b(G,V)$. Then we define $\beta$ as follows:

$$
\beta(b)(g_0,\ldots,g_n) = \alpha(g_0\sigma(g_0^{-1}b))(g_0,\ldots,g_n).
$$

It is immediate to check that $\beta \circ \iota = \alpha$. Moreover, since $||\sigma|| \leq 1$, we have $||\beta|| \leq ||\alpha||$. Finally, the fact that $\beta$ commutes with the actions of $G$ may be proved by the very same computation given in the proof of Lemma 7.2. \hfill $\square$

8.2. **Resolutions of normed $R[G]$-modules.** A normed $R[G]$-complex is an $R[G]$-complex whose modules are normed $R[G]$-spaces, and whose differential is a bounded $R[G]$-map in every degree. A chain map between $(E^\bullet,\delta^\bullet) \xrightarrow{\iota_E} (F^\bullet,\delta^\bullet_F)$ is a chain map between the underlying $R[G]$-complexes which is bounded in every degree, and a $G$-homotopy between two such chain maps is just a $G$-homotopy between the underlying maps of $R[G]$-modules, which is bounded in every degree. The cohomology $H^*_G(E^\bullet)$ of the normed $G$-complex $(E^\bullet,\delta^\bullet_E)$ is defined as usual by taking the cohomology of the subcomplex of $G$-invariants. The norm on $E^n$ restricts to a norm on $G$-invariant cocycles, which induces in turn a seminorm on $H^n(E^\bullet)$ for every $n \in \mathbb{N}$.

An augmented normed $G$-complex $(E,E^\bullet,\delta^\bullet)$ with augmentation map $\varepsilon: E \rightarrow E^0$ is a $G$-complex

$$
0 \rightarrow E \xrightarrow{\varepsilon} E^0 \xrightarrow{\delta^0} E^1 \xrightarrow{\delta^1} \cdots \xrightarrow{\delta^n} E^{n+1} \xrightarrow{\delta^{n+1}} \cdots
$$
(in particular, \(\varepsilon\) is bounded). A resolution of \(E\) (as a normed \(R[G]\)-complex) is an exact augmented normed complex \((E, E^*, \delta^*)\). It is \textit{relatively injective} if \(E^n\) is relatively injective for every \(n \geq 0\). From now on, we will call simply \textit{complex} any normed complex.

Let \((E, E^*, \delta^*_E)\) be an augmented complex, and suppose that \((F, F^*, \delta^*_F)\) is a relatively injective resolution of \(F\). We would like to be able to extend any \(G\)-map \(E \rightarrow F\) to a chain map between \(E^*\) and \(F^*\). As observed in the preceding section, the assumption that \((F, F^*, \delta^*_F)\) is relatively injective is not sufficient to ensure that any \(G\)-map between \(E\) and \(F\) extends to a chain map between \((E, E^*, \delta^*_E)\) and \((F, F^*, \delta^*_F)\). The point is that relative injectivity guarantees the needed extension property only for \textit{strongly} injective maps, so a corresponding notion of \textit{strong} resolution is needed.

A \textit{contracting homotopy} for a resolution \((E, E^*, \delta^*)\) is a sequence of linear maps \(k^i: E^i \rightarrow E^{i-1}\) such that \(\|k^i\| \leq 1\) for every \(i \in \mathbb{N}\), \(\delta^{i-1} \circ k^i + k^{i+1} \circ \delta^i = \text{Id}_{E^i}\) if \(i \geq 0\), and \(k^0 \circ \delta^1 = \text{Id}_{E^1}\):

\[
\begin{array}{ccccccc}
0 & \rightarrow & E & \xrightarrow{k^0} & E^0 & \xrightarrow{k^1} & E^1 & \cdots & \xrightarrow{k^n} & E^n & \xrightarrow{k^{n+1}} & \cdots \\
\end{array}
\]

Note however that it is not required that \(k^i\) be \(G\)-equivariant. A resolution is \textit{strong} if it admits a contracting homotopy.

**Proposition 8.4.** Let \(V\) be a normed \(R[G]\)-space, and let \(\varepsilon: V \rightarrow C^0_b(G, V)\) be defined by \(\varepsilon(v)(g) = v\) for every \(v \in V\), \(g \in G\). Then the augmented complex

\[
\begin{array}{ccccccc}
0 & \rightarrow & V & \xrightarrow{\varepsilon} & C^0_b(G, V) & \xrightarrow{k^0} & C^1_b(G, V) & \rightarrow & \cdots & \rightarrow & C^n_b(G, V) & \rightarrow & \cdots \\
\end{array}
\]

provides a relatively injective \textit{strong} resolution of \(V\).

**Proof.** We already know that each \(C^i_b(G, V)\) is relatively injective, so in order to conclude it is sufficient to observe that the map

\[
k^{n+1}: C^{n+1}_b(G, V) \rightarrow C^n_b(G, V) \quad k^{n+1}(f)(g_0, \ldots, g_n) = f(1, g_0, \ldots, g_n)
\]

provides a contracting homotopy for the resolution \((V, C^*_b(G, V), \delta^*)\). \(\square\)

The resolution described in Proposition 8.4 is the \textit{standard resolution} of \(V\) as a normed \(R[G]\)-module.

**Remark 8.5.** Let us briefly compare our notion of standard resolution with Ivanov’s and Monod’s ones. In [Iva87], for every \(n \in \mathbb{N}\) the set \(C^n_b(G, \mathbb{R})\) is denoted by \(B(G^{n+1})\), and it is endowed with the structure of a \textit{right} Banach \(G\)-module by the action \(g \cdot f(g_0, \ldots, g_n) = f(g_0, \ldots, g_n \cdot g)\). Moreover, the sequence of modules \(B(G^n)\), \(n \in \mathbb{N}\), is equipped with a structure of \(G\)-complex

\[
\begin{array}{ccccccc}
0 & \rightarrow & \mathbb{R} & \xrightarrow{d_{-1}} & B(G) & \xrightarrow{d_0} & B(G^2) & \xrightarrow{d_1} & \cdots & \xrightarrow{d_n} & B(G^{n+2}) & \xrightarrow{d_{n+1}} & \cdots,
\end{array}
\]
where $d_{-1}(t)(g) = t$ and
\[
d_n(f)(g_0, \ldots, g_{n+1}) = (-1)^{n+1} f(g_1, \ldots, g_{n+1}) + \sum_{i=0}^{n} (-1)^{n-i} f(g_0, \ldots, g_i g_i, \ldots, g_{n+1})
\]
for every $n \geq 0$ (here we are using Ivanov’s notation also for the differential). Now, it is readily seen that Ivanov’s resolution is isomorphic to our standard resolution via the isometric $G$-chain isomorphism $\varphi^\bullet : B^\bullet(G) \to B(G^{\bullet+1})$ defined by
\[
\varphi^n(f)(g_0, \ldots, g_n) = f(g_n^{-1} g_n^{-1} \cdots g_n^{-1} g_0^{-1})
\]
(its inverse is given by $(\varphi^n)^{-1}(f)(g_0, \ldots, g_n) = f(g_n^{-1} g_{n-1}^{-1} g_{n-2}^{-1} \cdots g_1^{-1} g_0^{-1})$). We also observe that the contracting homotopy described in Proposition 8.4 is conjugated by $\varphi^\bullet$ into Ivanov’s contracting homotopy for the complex $(B(G^\bullet), d_\bullet)$ (which is defined in [Iva87]).

Our notation is much closer to Monod’s one. In fact, in [Mon01] the more general case of a topological group $G$ is addressed, and the $n$-th module of the standard $G$-resolution of $\mathbb{R}$ is inductively defined by setting
\[
C^0_b(G, \mathbb{R}) = C_b(G, \mathbb{R}), \quad C^n_b(G, \mathbb{R}) = C_b(G, C^{n-1}_b(G, \mathbb{R}))
\]
where $C_b(G, E)$ denotes the space of continuous bounded maps from $G$ to the Banach space $E$. However, as observed in [Mon01, Remarks 6.1.2 and 6.1.3], the case when $G$ is an abstract group may be recovered from the general case just by equipping $G$ with the discrete topology. In that case, our notion of standard resolution coincides with Monod’s one (see also [Mon01, Remark 7.4.9]).

The following result can be proved by means of standard homological algebra arguments (see [Iva87], [Mon01, Lemmas 7.2.4 and 7.2.6] for full details):

**Theorem 8.6.** Let $\alpha : E \to F$ be a $G$-map between normed $R[G]$-modules, let $(E, E^\bullet, \delta^\bullet_E)$ be a strong resolution of $E$, and suppose $(F, F^\bullet, \delta^\bullet_F)$ is an augmented complex such that $F_i$ is relatively injective for every $i \geq 0$. Then $\alpha$ extends to a chain map $\alpha^\bullet$, and any two extensions of $\alpha$ to chain maps are $G$-homotopic.

**Corollary 8.7.** Let $V$ be a normed $R[G]$-module, and let $(V, V^\bullet, \delta^\bullet_V)$ be a relatively injective strong resolution of $V$. Then for every $n \in \mathbb{N}$ there is a canonical isomorphism
\[
H^n_b(G, V) \cong H^n_b(V^\bullet).
\]
Moreover, this isomorphism is bi-Lipschitz with respect to the seminorms of $H^n_b(G, V)$ and $H^n(V^\bullet)$.

**Proof.** By Proposition 8.4, the standard resolution of $V$ is also a relatively injective strong resolution of $V$ over $G$. Therefore, Theorem 7.4 provides chain maps between $C^\bullet_b(G, V)$ and $V^\bullet$, which are one the $G$-homotopy inverse of the other. Therefore, these chain maps induce isomorphisms in
cohomology. The conclusion follows from the fact that bounded chain maps induce bounded maps in cohomology.

By Corollary 8.7, every relatively injective strong resolution of $V$ induces a seminorm on $H^b_\bullet(G,V)$. Moreover, the seminorms defined in this way are pairwise equivalent. However, in many applications, it is important to be able to compute the exact canonical seminorm of elements in $H^b_\bullet(G,V)$. Unfortunately, it is not possible to capture the isometry type of $H^b_\bullet(G,V)$ via arbitrary relatively injective strong resolutions. Therefore, a special role is played by those resolutions which compute the canonical seminorm. The following fundamental result is due to Ivanov, and implies that these distinguished resolutions are in some sense extremal:

**Theorem 8.8.** Let $V$ be a normed $\mathbb{R}[G]$-module, and let $(V,V^\bullet,\delta^\bullet)$ be any strong resolution of $V$. Then the identity of $V$ can be extended to a chain map $\alpha^\bullet$ between $V^\bullet$ and the standard resolution of $V$, in such a way that $\|\alpha^n\| \leq 1$ for every $n \geq 0$. In particular, the canonical seminorm of $H^b_\bullet(G,V)$ is not bigger than the seminorm induced on $H^b_\bullet(G,V)$ by any relatively injective strong resolution.

**Proof.** One can define $\alpha^n$ by induction setting, for every $v \in E^n$ and $g_j \in G$: $\alpha^n(v)(g_0, \ldots, g_n) = \alpha^{n-1}(g_0(k^n(g_0^{-1}(v))))(g_1, \ldots, g_n),$ where $k^\bullet$ is a contracting homotopy for the given resolution $(V,V^\bullet,\delta^\bullet)$. It is not difficult to prove by induction that $\alpha^\bullet$ is indeed a norm-decreasing chain $G$-map (see [Iva87], [Mon01, Theorem 7.3.1] for the details).

**Corollary 8.9.** Let $V$ be a normed $\mathbb{R}[G]$-module, let $(V,V^\bullet,\delta^\bullet)$ be a relatively injective strong resolution of $V$, and suppose that the identity of $V$ may be extended to a chain map $\alpha^\bullet : C^b_\bullet(G,V) \to V^\bullet$ such that $\|\alpha^n\| \leq 1$ for every $n \in \mathbb{N}$. Then $\alpha^\bullet$ induces an isometric isomorphism between $H^b_\bullet(G,V)$ and $H^\bullet(V^\bullet)$. In particular, the seminorm induced by the resolution $(V,V^\bullet,\delta^\bullet)$ coincides with the canonical seminorm on $H^b_\bullet(G,V)$.

9. **More on amenability**

The following result establishes an interesting relationship between the amenability of $G$ and the relative injectivity of normed $\mathbb{R}[G]$-modules.

**Proposition 9.1.** The following facts are equivalent:

1. The group $G$ is amenable.
2. Every dual normed $\mathbb{R}[G]$-module is relatively injective.
3. The trivial $\mathbb{R}[G]$-module $\mathbb{R}$ is relatively injective.

**Proof.** (1) $\Rightarrow$ (2): Let $W$ be a normed $\mathbb{R}[G]$-module, and let $V = W'$ be the dual normed $\mathbb{R}[G]$-module. We first construct a left inverse (over $\mathbb{R}[G]$) of
the augmentation map \( \varepsilon : V \to C^0_b(G, V) \). We fix an invariant mean \( m \) on \( G \). For \( f \in C^0_b(G, V) \) and \( w \in W \) we consider the function

\[
f_w : G \to \mathbb{R}, \quad f_w(g) = f(g)(w).
\]

It follows from the definitions that \( f_w \) is an element of \( \ell^\infty(G) \), so we may define a map \( \mathcal{R} : C^0_b(G, V) \to V \) by setting \( \mathcal{R}(f)(w) = m(f_w) \). It is immediate to check that \( \mathcal{R} \) is indeed a bounded functional on \( W \), whose norm is bounded by \( \|f\|_\infty \). In other words, the map \( r \) is well-defined and norm non-increasing. The \( G \)-invariance of the mean \( m \) implies that \( \mathcal{R} \) is \( G \)-equivariant, and an easy computation shows that \( \mathcal{R} \circ \varepsilon = \text{Id}_V \).

Let us now consider the diagram

\[
\begin{array}{ccc}
0 & \longrightarrow & A \\
\varepsilon \downarrow & & \downarrow \alpha \\
V & \mathcal{R} \downarrow & \mathcal{R} \\
\varepsilon \downarrow & & \downarrow \beta \\
C^0_b(G, V) & \mathcal{I} \downarrow & \mathcal{I} \\
\end{array}
\]

By Lemma 8.3, \( C^0_b(G, V) \) is relatively injective, so there exists a bounded \( \mathbb{R}[G] \)-map \( \beta' \) such that \( \|\beta'\| \leq \|\varepsilon \circ \alpha\| \leq \|\alpha\| \) and \( \beta' \circ \ell = \varepsilon \circ \alpha \). The \( \mathbb{R}[G] \)-map \( \beta := \mathcal{R} \circ \beta' \) satisfies \( \|\beta\| \leq \|\alpha\| \) and \( \beta \circ \ell = \alpha \). This shows that \( V \) is relatively injective.

(2) \( \Rightarrow \) (3) is obvious, so we are left to show that (3) implies (1). If \( \mathbb{R} \) is relatively injective and \( \sigma : \ell^\infty(G) \to \mathbb{R} \) is the map defined by \( \sigma(f) = f(1) \), then there exists an \( \mathbb{R}[G] \)-map \( \beta : \ell^\infty(G) \to \mathbb{R} \) such that \( \|\beta\| \leq 1 \) and the following diagram commutes:

\[
\begin{array}{ccc}
\mathbb{R} & \mathcal{I} \downarrow & \mathcal{I} \\
\sigma \downarrow & & \downarrow \beta \\
\ell^\infty(\mathbb{R}) & \ell^\infty(\mathbb{R}) \\
\end{array}
\]

Using that \( \beta(1_G) = 1 \) and \( \|\beta\| \leq 1 \) it is easy to show that \( \beta \) is a mean. Being an \( \mathbb{R}[G] \)-map, \( \beta \) is \( G \)-invariant, whence the conclusion.

The previous proposition allows us to provide an alternative proof of Theorem 6.6, which we recall here for the convenience of the reader:

**Theorem 9.2.** Let \( G \) be an amenable group, and let \( V \) be a dual normed \( \mathbb{R}[G] \)-module. Then \( H^n_b(G, V) = 0 \) for every \( n \geq 1 \).

**Proof.** The complex

\[
0 \longrightarrow V \xrightarrow{\text{Id}} V \longrightarrow 0
\]

provides a relatively injective strong resolution of \( V \), so the conclusion follows from Corollary 8.7.
9.1. **Amenable spaces.** The notion of amenable space was introduced by Zimmer [Zim78] in the context of actions of topological groups on standard measure spaces (see e.g. [Mon01, Section 5.3] for several equivalent definitions). In our case of interest, i.e. when \( G \) is a discrete group acting on a set \( S \) (which may be thought as endowed with the discrete topology), the amenability of \( S \) as a \( G \)-space amounts to the amenability of the stabilizers in \( G \) of elements of \( S \) [AEG94, Theorem 5.1]. Therefore, we may take this characterization as a definition:

**Definition 9.3.** A left action \( G \times S \to S \) of a group \( G \) on a set \( S \) is **amenable** if the stabilizer of every \( s \in S \) is an amenable subgroup of \( G \). In this case, we equivalently say that \( S \) is an amenable \( G \)-set.

The importance of amenable \( G \)-sets is due to the fact that they may be exploited to isometrically compute the bounded cohomology of \( G \). If \( S \) is any \( G \)-set and \( V \) is any normed \( \mathbb{R}[G] \)-module, then we denote by \( \ell^\infty(S^{n+1},V) \) the space of bounded functions from \( S^{n+1} \) to \( V \). This space may be endowed with the a structure of a normed \( \mathbb{R}[G] \)-module via the action

\[
(g \cdot f)(s_0, \ldots, s_n) = g \cdot (f(g^{-1}s_0, \ldots, g^{-1}s_n)).
\]

The differential \( \delta^n : \ell^\infty(S^{n+1}) \to \ell^\infty(S^n) \) defined by

\[
\delta^n(f)(s_0, \ldots, s_{n+1}) = \sum_{i=0}^n (-1)^i f(s_0, \ldots, \hat{s}_i, \ldots, s_n)
\]

endows the pair \((\ell^\infty(S^{n+1},V), \delta^n)\) with the structure of a normed \( \mathbb{R}[G] \)-complex. Together with the augmentation \( \varepsilon : V \to \ell^\infty(S,V) \) given by \( \varepsilon(v)(s) = v \) for every \( s \in S \), such a complex provides a strong resolution of \( V \):

**Lemma 9.4.** *The augmented complex*

\[
0 \to V \to \ell^\infty(S,V) \xrightarrow{\delta^0} \ell^\infty(S^2,V) \xrightarrow{\delta^1} \ell^\infty(S^3,V) \xrightarrow{\delta^2} \ldots
\]

*provides a strong resolution of \( V \).*

*Proof.* Let \( \overline{s} \) be a fixed element of \( S \). Then the maps

\[
k^n : \ell^\infty(S^{n+1},V) \to \ell^\infty(S^n,V), \quad k^n(f)(s_0, \ldots, s_{n-1}) = f(\overline{s}, s_0, \ldots, s_{n-1})
\]

provide the required contracting homotopy. \( \square \)

**Lemma 9.5.** *Suppose that \( S \) is an amenable \( G \)-space, and that \( V \) is a dual normed \( \mathbb{R}[G] \)-module. Then \( \ell^\infty(S^{n+1},V) \) is relatively injective for every \( n \geq 0 \).*

*Proof.* Of course, if \( S \) is amenable, then the same is true for \( S^n \), \( n \geq 1 \), so we may assume \( n = 0 \). Moreover, let \( W \) be the normed \( \mathbb{R}[G] \)-module such that \( V = W' \).
Let us consider the extension problem described in Definition 8.1, with $\ell^\infty(S,V)$:

\[
\begin{array}{c}
0 \longrightarrow A \xrightarrow{\sigma} B \\
\alpha \quad \beta
\end{array}
\]

\[\ell^\infty(S,V)\]

We denote by $R \subseteq S$ a set of representatives for the action of $G$ on $S$, and for every $r \in R$ we denote by $G_r$ the stabilizer of $r$, endowed with the invariant mean $\mu_r$. Moreover, for every $s \in S$ we choose an element $g_s \in G$ such that $g_s^{-1}(s) = r_s \in R$. Then $g_s$ is uniquely determined up to right multiplication by elements in $G_r$.

Let us fix an element $b \in B$. In order to define $\beta(b)$, for every $s \in S$ we need to know the value taken by $\beta(b)(s)$ on every $w \in W$. Therefore, we fix $s \in S$, $w \in W$, and we set

\[\beta(b)(s))(w) = \mu_{r_s}(f_b),\]

where $f_b \in \ell^\infty(G_{r_s}, \mathbb{R})$ is defined by

\[f_b(g) = ((g_s g) \cdot \alpha(\sigma(g^{-1}g_s^{-1}b))) (s)(w)\]

Since $\|\sigma\| \leq 1$ we have that $\|\beta\| \leq \|\alpha\|$, and the behaviour of means on constant functions implies that $\beta \circ \iota = \alpha$.

Observe that the element $\beta(b)(s)$ does not depend on the choice of $g_s \in G$. In fact, if we replace $g_s$ by $g_s g'$ for some $g' \in G_{r_s}$, then the function $f$ defined above is replaced by the function

\[f'_b(g) = ((g_s g') \cdot \alpha(\sigma(g^{-1}g_s^{-1}b))) (s)(w) = f_b(g'g),\]

and $\mu_{r_s}(f'_b) = \mu_{r_s}(f_b)$ by the invariance of the mean $\mu_{r_s}$. This fact allows us to prove that $\beta$ is a $G$-map. In fact, let us fix $\overline{g} \in G$ and let $\overline{\sigma} = \overline{g}^{-1}(s)$. Then we may assume that $g_r = \overline{g}^{-1}g_s$, so

\[(\overline{g})(\beta(b))(s))(w) = \beta(b)(\overline{g})(\overline{g}^{-1}w) = \mu_{r_s}(\overline{f}),\]

where $\overline{f} \in \ell^\infty(G_{r_s}, \mathbb{R})$ is given by

\[\overline{f}_b(g) = ((\overline{g}^{-1}g_s g) \cdot \alpha(\sigma(g^{-1}g_s^{-1}\overline{g}b))) (\overline{g})(\overline{g}^{-1}w) = ((g_s g) \cdot \alpha(\sigma(g^{-1}g_s^{-1}\overline{g}b))) (s)(w) = \overline{f}_b(g).\]

This concludes the proof. \[\square\]

As anticipated above, we are now able to show that amenable spaces may be exploited to compute bounded cohomology:

**Theorem 9.6.** Let $S$ be an amenable $G$-set and let $V$ be a dual normed $\mathbb{R}[G]$-module. Then the homology of the complex

\[0 \longrightarrow \ell^\infty(S,V)^G \xrightarrow{\delta^0} \ell^\infty(S^2,V)^G \xrightarrow{\delta^1} \ell^\infty(S^3,V)^G \xrightarrow{\delta^2} \ldots\]
is canonically isometrically isomorphic to $H^\bullet(G,V)$.

**Proof.** The previous lemmas imply that the augmented complex $(V,\ell^\infty(S^{*+1}),\delta^*)$ provides a relatively injective strong resolution of $V$, so Corollary 8.7 implies that the homology of the $G$-invariants of $(V,\ell^\infty(S^{*+1}),\delta^*)$ is isomorphic to the bounded cohomology of $G$ with coefficients in $V$. By Corollary 8.9, in order to prove that such an isomorphism is isometric we are left to construct a norm non-increasing chain map

$$\alpha^\bullet : C^\bullet_b(G,V) \to \ell^\infty(S^{*+1},V).$$

We keep notation from the proof of the previous lemma, i.e. we fix a set of representatives $R$ for the action of $G$ on $S$, and for every $s \in S$ we choose an element $g_s \in G$ such that $g_s^{-1}s \in R$. For every $r \in R$ we also fix an invariant mean $\mu_r$ on the stabilizer $G_r$.

Let us fix an element $f \in C^0_b(G,V)$, and take $(s_0,\ldots,s_n) \in S^{n+1}$. If $V = W'$, we also fix an element $w \in W$. For every $i$ we denote by $r_i \in R$ the representative of the orbit of $s_i$, and we consider the invariant mean $\mu_r \times \cdots \times \mu_r$ on $G_{r_1} \times \cdots \times G_{r_n}$ (see Remark 6.5). Then we consider the function $f_{s_0,\ldots,s_n} \in \ell^\infty(G_{r_1} \times \cdots \times G_{r_n},\mathbb{R})$ defined by

$$f_{s_0,\ldots,s_n}(g_0,\ldots,g_n) = f(g_{s_0}g_0,\ldots,g_{s_n}g_n)(w).$$

By construction we have $\|f_{s_0,\ldots,s_n}\|_\infty \leq \|f\|_\infty \cdot \|w\|_W$, so we may set

$$(\alpha^n(f)(s_0,\ldots,s_n))(w) = (\mu_{r_1} \times \cdots \times \mu_{r_n})(f_{s_0,\ldots,s_n}),$$

thus defining an element $\alpha^n(f)(s_0,\ldots,s_n) \in W' = V$ such that $\|\alpha^n(f)\|_V \leq \|f\|_\infty$. We have thus shown that $\alpha^n : C^0_b(G,V) \to \ell^\infty(S^{*+1},V)$ is a well-defined norm non-increasing linear map. The fact that $\alpha^n$ commutes with the action of $G$ follows from the invariance of the means $\mu_r$, $r \in R$, and the fact that $\alpha^\bullet$ is a chain map is obvious.

Let us prove some direct corollaries of the previous results. Observe that, if $W$ is a dual normed $\mathbb{R}[H]$-module and $\psi : G \to H$ is a homomorphism, then the induced module $\psi^{-1}(W)$ is a dual normed $\mathbb{R}[G]$-module.

**Theorem 9.7.** Let $\psi : G \to H$ be a surjective homomorphism with amenable kernel, and let $W$ be a dual normed $\mathbb{R}[H]$-module. Then the induced map

$$\psi^\bullet : H^\bullet_b(H,W) \to H^\bullet_b(G,\psi^{-1}W)$$

is an isometric isomorphism.

**Proof.** The action $G \times H \to H$ defined by $(g,h) \mapsto \psi(g)h$ endows $H$ with the structure of an amenable $G$-set. Therefore, Theorem 9.6 implies that the bounded cohomology of $G$ with coefficients in $\psi^{-1}W$ is isometrically isomorphic to the cohomology of the complex $\ell^\infty(H^{*+1},\psi^{-1}W)^G$. However, since $\psi$ is surjective, we have a (tautological) isometric identification between $\ell^\infty(H^{*+1},\psi^{-1}W)^G$ and $C^\bullet_b(G,W)^G$, whence the conclusion.  \(\square\)
Corollary 9.8. Let $\psi: G \to H$ be a surjective homomorphism with amenable kernel. Then $H^n_b(G, \mathbb{R})$ is isometrically isomorphic to $H^n_b(H, \mathbb{R})$ for every $n \in \mathbb{N}$.

9.2. Alternating cochains.

10. Bounded cohomology of topological spaces

Let $X$ be a topological space, and let $R = \mathbb{Z}, \mathbb{R}$. Recall that $C^\bullet(X, R)$ (resp. $C^\bullet(X, R)$) denotes the usual complex of singular chains (resp. cochains) on $X$ with coefficients in $R$. For $i \in \mathbb{N}$, we let $S_i(X)$ be the set of singular $i$–simplices in $X$. We also regard $S_i(X)$ as a subset of $C_i(X, R)$, so that for any cochain $\varphi \in C_i(X, R)$ it makes sense to consider its restriction $\varphi|_{S_i(X)}$. For every $\varphi \in C^i(X, R)$, we set

$$\|\varphi\| = \|\varphi\|_\infty = \sup \{|\varphi(s)| : s \in S_i(X)\} \in [0, \infty].$$

We denote by $C^i_b(X, R)$ the submodule of bounded cochains, i.e. we set

$$C^i_b(X, R) = \{\varphi \in C^i(X, R) : \|\varphi\| < \infty\}.$$

Since the differential takes bounded cochains into bounded cochains, $C^i_b(X, R)$ is a subcomplex of $C^\bullet(X, R)$. We denote by $H^\bullet(X, R)$ and $H^\bullet_b(X, R)$ respectively the homology of the complexes $C^\bullet(X, R)$ and $C^\bullet_b(X, R)$. Of course, $H^\bullet(X, R)$ is the usual singular cohomology module of $X$ with coefficients in $R$, while $H^\bullet_b(X, R)$ is the bounded cohomology module of $X$ with coefficients in $R$. Just as in the case of groups, the norm on $C^i(X, R)$ descends (after the suitable restrictions) to a seminorm on each of the modules $H^\bullet(X, R)$, $H^\bullet_b(X, R)$. More precisely, if $\varphi \in H$ is a class in one of these modules, which is obtained as a quotient of the corresponding module of cocycles $Z$, then we set

$$\|\varphi\| = \inf \{|\psi| : \psi \in Z, [\psi] = \varphi \text{ in } H\}.$$ 

This seminorm may take infinite values on elements in $H^\bullet(X, R)$ and may be null on non-zero elements in $H^\bullet_b(X, R)$ (but not on non-zero elements in $H^\bullet(X, R)$): this is clear in the case with integer coefficients, and it is a consequence of the Universal Coefficient Theorem in the case with real coefficients, since a real cohomology class with vanishing seminorm has to be null on any cycle, whence null in $H^\bullet(X, \mathbb{R})$.

The inclusion of bounded cochains into possibly unbounded cochains induce a map in cohomology

$$c^\bullet : H_b^\bullet(X, R) \to H^\bullet(X, R),$$

which is called \textit{comparison map}.

10.1. Basic properties of bounded cohomology of spaces. Bounded cohomology enjoys some of the fundamental properties of classical singular cohomology: for example, $H_i^b(\{\text{pt.}\}, R) = 0$ if $i > 0$, and $H_0^b(\{\text{pt.}\}, R) = R$ (more in general, $H_b^0(X, R)$ is canonically isomorphic to $\ell^\infty(S)$, where $S$ is the set of the path connected components of $X$). The usual proof of
the homotopy invariance of singular homology is based on the construction of an algebraic homotopy which takes every \( n \)-simplex onto the sum of at most \( n + 1 \) \((n + 1)\)-dimensional simplices. As a consequence, the homotopy operator induced in cohomology preserves bounded cochains. This implies that bounded cohomology is a homotopy invariant of topological spaces. Moreover, if \((X, Y)\) is a topological pair, then there is an obvious definition of \( H^*_{\text{b}}(X, Y)\), and it is immediate to check that the analogous of the long exact sequence of the pair in classical singular cohomology also holds in the bounded case.

Perhaps the most important peculiarity of bounded cohomology with respect to classical singular cohomology is the lacking of any Mayer-Vietoris sequence (or, equivalently, of any excision theorem). This is also at the basis of the phenomenon for which spaces with finite-dimensional bounded cohomology may be tamely glued to each other to get spaces with infinite-dimensional bounded cohomology (see Remark 10.5 below).

Recall from Subsection 1.2 that \( H^n(X, R) \cong H^n(\pi_1(X), R)\) for every aspherical CW-complex \( X \). We are now going to prove an analogous result in the context of bounded cohomology. As anticipated in Subsection 2.3, a fundamental result by Gromov provides an isometric isomorphism \( H^n_{\text{b}}(X, \mathbb{R}) \cong H^n_{\text{b}}(\pi_1(X), \mathbb{R})\) even without any assumption on the asphericity of \( X \). This section is devoted to a proof of Gromov’s result. We will closely follow Ivanov’s argument [Iva87], which deals with the case when \( X \) is (homotopically equivalent to) a countable CW-complex. Before going into the general case, we will concentrate our attention on the easier case of aspherical spaces.

### 10.2. Bounded singular cochains as relatively injective modules.

Henceforth, we assume that \( X \) is a path connected topological space admitting the universal covering \( \tilde{X} \), we denote by \( G \) the fundamental group of \( X \), and we fix an identification of \( G \) with the group of covering automorphisms of \( \tilde{X} \). Just as we did in Subsection 7.4, we endow \( C^*_{\text{b}}(\tilde{X}, R) \) with the structure of a normed \( R[G] \)-module. Our arguments are based on the obvious but fundamental isometric identification

\[
C^*_{\text{b}}(X, R) \cong C^*_{\text{b}}(\tilde{X}, R)^G,
\]

which induces a canonical isometric identification

\[
H^*_b(X, R) \cong H^*(C^*_{\text{b}}(\tilde{X}, R)^G).
\]

As a consequence, in order to prove the isomorphism \( H^*_b(X, R) \cong H^*_b(G, R)\) it is sufficient to show that the complex \( C^*_{\text{b}}(\tilde{X}, R) \) provides a relatively injective strong resolution of \( R \) (some more care is needed to prove that this isomorphism is also isometric). The relative injectivity of the modules \( C^*_{\text{b}}(\tilde{X}, R) \) can be easily deduced from the argument described in the proof of Lemma 7.6, which applies verbatim in the context of bounded singular cochains:
Lemma 10.1. For every $n \in \mathbb{N}$, the bounded cochain module $C^n_b(\tilde{X}, R)$ is relatively injective.

Therefore, in order to show that the bounded cohomology of $X$ is isomorphic to the bounded cohomology of $G$ we need to show that the (augmented) complex $C^\dot{b}_\cdot(\tilde{X}, R)$ provides a strong resolution of $R$. We will show that this is the case if $\tilde{X}$ is contractible. In the general case, it is not even true that $C^\dot{b}_\cdot(\tilde{X}, R)$ is acyclic (see Remark 10.10). However, in the case when $R = \mathbb{R}$ a deep result by Ivanov shows that the complex $C^\dot{b}_\cdot(\tilde{X}, \mathbb{R})$ indeed provides a strong resolution of $\mathbb{R}$. A sketch of Ivanov’s proof will be given in Subsection 10.4.

Before going on, we point out that we always have a norm non-increasing map from the bounded cohomology of $G$ to the bounded cohomology of $X$:

Lemma 10.2. Let us endow the complexes $C^\dot{b}_\cdot(G, R)$ and $C^\dot{b}_\cdot(\tilde{X}, R)$ with the obvious augmentations. Then, there exists a norm non-increasing chain map

$$r^\bullet : C^\dot{b}_\cdot(G, R) \to C^\dot{b}_\cdot(\tilde{X}, R)$$

extending the identity of $R$.

Proof. Let us choose a fundamental region $R$ for the action of $G$ on $\tilde{X}$. We consider the map $r_0 : S_0(\tilde{X}) = \tilde{X} \to G$ taking a point $x$ to the unique $g \in G$ such that $x \in g(R)$. For $n \geq 1$, we define $r_n : S_n(\tilde{X}) \to C^{n+1}$ by setting $r_n(s) = (r_0(s(e_0)), \ldots, r_n(s(e_n)))$, where $s \in S_n(\tilde{X})$ and $s(e_i)$ is the $i$-th vertex of $s$. Finally, we extend $r_n$ to $C_n(\tilde{X}, R)$ by $R$-linearity and define $r^n$ as the dual map of $r_n$. Since $r_n$ takes any single simplex to a single $(n+1)$-tuple, it is readily seen that $r^n$ is norm non-increasing (in particular, it takes bounded cochains into bounded cochains). The fact that $r^n$ is a $G$-equivariant chain map is obvious. □

The following result will prove useful later:

Lemma 10.3. Let $Y$ be a path connected topological space. If $Y$ is aspherical, then the augmented complex $C^\dot{b}_\cdot(Y, R)$ admits a contracting homotopy (so it is a strong resolution of $R$). If $\pi_i(Y) = 0$ for every $i \leq n$, then then there exists a partial contracting homotopy

$$\mathbb{R} \overset{k_0}{\longrightarrow} C^0_b(Y, R) \overset{k_1}{\longrightarrow} C^1_b(Y, R) \overset{k_2}{\longrightarrow} \cdots \overset{k^n}{\longrightarrow} C^n_b(Y, \mathbb{R}) \overset{k^{n+1}}{\longrightarrow} C^{n+1}_b(Y, R)$$

(where we require that the equality $\delta^{m-1}k^m + k^{m+1}\delta^m = \text{Id}_{C^n_b(Y, R)}$ holds for every $m \leq n$, and $\delta^{-1} = \varepsilon$ is the usual augmentation).

Proof. For every $-1 \leq m \leq n$ we construct a map $T_m : C_m(Y, R) \to C_{m+1}(Y, R)$ such that $d_{m+1}T_m + T_{m-1}d_m = \text{Id}_{C_m(Y, R)}$, where we understand that $C_{-1}(Y, R) = R$ and $d_0 : C_0(Y, R) \to R$ is the augmentation map $d_0(\sum r_1y_i) = \sum r_i$. Let us fix a point $y_0 \in Y$, and define $T_{-1} : R \to C_0(Y, R)$ by setting $T_{-1}(r) = ry_0$. For $m \geq 0$ we define $T_m$ as the $R$-linear extension
of a map $T_m : S_m(Y) \to S_{m+1}(Y)$ having the following property: for every $s \in S_m(Y)$, the 0-th vertex of $T_m(s)$ is equal to $y_0$, and has $s$ as opposite face. We proceed by induction, and suppose that $T_i$ has been defined for every $-1 \leq i \leq m$. Take $s \in S_m(Y)$. Then, using the fact that $\pi_0(Y) = 0$ and the properties of $T_{m-1}$, it is not difficult to show that a simplex $s' \in S_{m+1}(Y)$ exists which satisfies both the equality $d_{m+1} s' = s - T_{m-1}(d_ms)$ and the additional requirement described above. We set $T_{m+1}(s) = s'$, and we are done.

Since $T_{m-1}$ sends any single simplex to a single simplex, its dual map $k^m$ sends bounded cochains into bounded cochains, and has operator norm equal to one. Therefore, the maps $k^m : C^m_b(Y, R) \to C^{m-1}_b(Y, R)$, $m \leq n+1$, provide the desired (partial) contracting homotopy.

10.3. The aspherical case. We are now ready to show that, under the assumption that $X$ is aspherical, the bounded cohomology of $X$ is isometrically isomorphic to the bounded cohomology of $G$ for any ring of coefficients:

**Theorem 10.4.** Let $X$ be an aspherical space, i.e. a path connected topological space such that $\pi_n(X) = 0$ for every $n \geq 2$. Then $H^b_n(X, R)$ is isometrically isomorphic to $H^b_n(G, R)$ for every $n \in \mathbb{N}$.

**Proof.** Lemmas 10.1 and 10.3 imply that the complex

$$
0 \to R \xrightarrow{\varepsilon} C^0_b(X, R) \xrightarrow{\delta^0} C^1_b(X, R) \xrightarrow{\delta^1} \ldots
$$

provides a relatively injective strong resolution of $R$ as a trivial $R[G]$-module, so $H^b_n(X, R)$ is canonically isomorphic to $H^b_n(G, R)$ for every $n \in \mathbb{N}$. The fact that the isomorphism $H^b_n(X, R) \cong H^b_n(G, R)$ is isometric is a consequence of Corollary 8.9 and Lemma 10.2.

**Remark 10.5.** Let $S^1 \vee S^1$ be the wedge of two copies of the circle. Then Theorem 10.4 implies that $H^b_2(S^1 \vee S^1, \mathbb{R}) \cong H^b_2(F_2, \mathbb{R})$ is infinite-dimensional, while $H^b_2(S^2) \cong H^b_2(\mathbb{Z}) = 0$. This shows that bounded cohomology of spaces cannot satisfy any Mayer-Vietoris principle.

10.4. Ivanov’s contracting homotopy. We now come back to the general case, i.e. we do not assume that $X$ is contractible. In order to show that $H^b_n(X, R)$ is isometrically isomorphic to $H^b_n(G, R)$ we need to prove that the complex of singular bounded cochains on $X$ provides a strong resolution of $R$. In the case when $R = \mathbb{Z}$, this is false in general, since the complex $C^\bullet_b(X, \mathbb{Z})$ may be even non-exact (see Remark 10.10). On the other hand, a deep result by Ivanov ensures that $C^\bullet_b(\tilde{X}, \mathbb{R})$ indeed provides a strong resolution of $\mathbb{R}$.

Ivanov’s argument makes use of sophisticated techniques from algebraic topology, which work under the assumption that $X$, whence $\tilde{X}$, is a countable CW-complex (but see Remark 10.9). We begin with the following:

**Lemma 10.6** ([Iva87], Theorem 2.2). Let $p : Z \to Y$ be a principal $H$-bundle, where $H$ is a topological group. Then there exists a chain map
Therefore, we get a sequence of maps $S_n$ are $\Delta^n$-invariant, this map does not depend on the chosen identification.

More precisely, let $K_n = F(\Delta^n, H)$ be the space of continuous functions from the standard $n$-simplex to $H$, and define on $K_n$ the operation given by pointwise multiplication of functions. With this structure, $K_n$ is an abelian group, so it admits an invariant mean $\mu_n$. Observe that the permutation group $\mathfrak{S}_{n+1}$ acts on $\Delta^n$ via affine transformations. This action induces an action on $K_n$, whence on $\ell^\infty(K_n)$, and on the space of means on $K_n$, so there is an obvious notion of $\mathfrak{S}_{n+1}$-invariant mean on $K_n$. By averaging over the action of $\mathfrak{S}_{n+1}$, the space of $K_n$-invariant means may be retracted onto the space $\mathcal{M}_n$ of $\mathfrak{S}_{n+1}$-invariant $K_n$-invariant means on $K_n$, which, in particular, is non-empty. Finally, observe that any affine identification of $\Delta^n$ with a face of $\Delta^n$ induces a map $\mathcal{M}_n \to \mathcal{M}_{n-1}$. Since elements of $\mathcal{M}_n$ are $\mathfrak{S}_{n+1}$-invariant, this map does not depend on the chosen identification. Therefore, we get a sequence of maps

$$
\mathcal{M}_0 \leftarrow \mathcal{M}_1 \leftarrow \mathcal{M}_2 \leftarrow \mathcal{M}_3 \leftarrow \cdots
$$

Recall now that the Banach-Alaouglu Theorem implies that every $\mathcal{M}_n$ is compact. This easily implies that there exists a sequence $\{\mu_n\}$ of means such that $\mu_n \in \mathcal{M}_n$ and $\mu_n \to \mu_{n-1}$ under the map $\mathcal{M}_n \to \mathcal{M}_{n-1}$. We say that such a sequence in compatible.

Let us now fix $s \in S_n(Y)$, and observe that there is a bijection between $P_s$ and $K_n$. This bijection is uniquely determined up to the choice of an element in $P_s$, up to left multiplication by an element of $K_n$. In particular, if $f \in \ell^\infty(P_s)$, and $\mu$ is a left invariant mean on $K_n$, then there is a well-defined value $\mu(f)$.

Let us now choose a compatible sequence of means $\mu_0, \ldots, \mu_n, \ldots$. We define the operator $A^n$ by setting

$$
A^n(\varphi)(s) = \mu_n(\varphi|_{P_s}) \quad \text{for every } \varphi \in C^*_b(Z), \ s \in S_n(Y).
$$

Since the sequence $\{\mu_n\}$ is compatible, the sequence of maps $A^n$ is a chain map. The inequality $\|A^n\| \leq 1$ is obvious, and the fact that $A^n \circ p^*$ is the identity of $C^*_b(Y, \mathbb{R})$ may be deduced from the behaviour of means on constant functions.

Theorem 10.7 (Iv87). Let $X$ be a path connected CW-complex with universal covering $\tilde{X}$. Then the (augmented) complex $C^*_b(\tilde{X}, \mathbb{R})$ provides a relatively injective strong resolution of $\mathbb{R}$.

Proof. We only sketch Ivanov’s argument, referring the reader to [Iva87] for full details. Building on results by Dold and Thom [DT58], Ivanov constructs
an infinite tower of bundles

\[ X_1 \xleftarrow{p_1} X_2 \xleftarrow{p_2} X_3 \xleftarrow{\ldots} X_n \xleftarrow{\ldots} \]

where \( X_1 = \tilde{X} \), \( \pi_i(X_m) = 0 \) for every \( i \leq m \), \( \pi_i(X) = \pi_i(X) \) for every \( i > m \) and each map \( p_m : X_{m+1} \to X_m \) is a principal \( H_m \)-bundle for some topological connected abelian group \( H_m \), which has the homotopy type of a \( K(\pi_{n+1}(X), n) \).

By Lemma 10.3, for every \( n \) we may construct a partial contracting homotopy

\[ \mathbb{R} \xleftarrow{k^0} C_b^0(X_n, \mathbb{R}) \xleftarrow{k_1} C_b^1(X_n, \mathbb{R}) \xleftarrow{k_2} \ldots \xleftarrow{k^{n+1}} C_b^{n+1}(X_n, \mathbb{R}). \]

Moreover, Lemma 10.6 implies that for every \( m \in \mathbb{N} \) the chain map \( p^*_m : C_b^*(X_m, \mathbb{R}) \to C_b^*(X_{m+1}, \mathbb{R}) \) admits a left inverse chain map \( A^*_m : C_b^*(X_{m+1}, \mathbb{R}) \to C_b^*(X_m, \mathbb{R}) \) which is norm non-increasing. This allows us to define a partial contracting homotopy

\[ \mathbb{R} \xleftarrow{k^0} C_b^0(X, \mathbb{R}) \xleftarrow{k_1} C_b^1(X, \mathbb{R}) \xleftarrow{k_2} \ldots \xleftarrow{k^{n+1}} C_b^{n+1}(X, \mathbb{R}) \xleftarrow{\ldots} \]

via the formula

\[ k^i = A_i^{-1} \circ \ldots \circ A_{i-1} \circ k_n \circ p_{n-1} \circ \ldots \circ p_1 \circ p_1 \quad \text{for every } i \leq n + 1. \]

The existence of such a partial homotopy is sufficient for all our applications. In order to construct a complete contracting homotopy a further argument is needed, for which we refer the reader to [Iva87].

10.5. **Gromov’s Theorem.** The discussion in the preceding subsection implies the following:

**Theorem 10.8** ([Gro82, Iva87]). Let \( X \) be a countable CW-complex. Then \( H_b^*(X, \mathbb{R}) \) is canonically isomorphic to \( H_b^*(G, \mathbb{R}) \).

**Proof.** Observe that, if \( X \) is a countable CW-complex, then \( G = \pi_1(X) \) is countable, so \( \tilde{X} \) is also a countable CW-complex. Therefore, Lemma 10.1 Theorem 10.7 imply that the complex

\[ 0 \to R \xrightarrow{e} C_b^0(\tilde{X}, \mathbb{R}) \xrightarrow{\delta^0} C_b^1(\tilde{X}, \mathbb{R}) \xrightarrow{\delta^1} \ldots \]

provides a relatively injective strong resolution of \( \mathbb{R} \) as a trivial \( \mathbb{R}[G] \)-module, so \( H_b^*(X, \mathbb{R}) \) is canonically isomorphic to \( H_b^*(G, \mathbb{R}) \) for every \( n \in \mathbb{N} \). The fact that the isomorphism \( H_b^*(X, \mathbb{R}) \cong H_b^*(G, \mathbb{R}) \) is isometric is a consequence of Corollary 8.9 and Lemma 10.2. \( \square \)

**Remark 10.9.** Theo Bülher has communicated to the author that Ivanov’s argument may be generalized to show that \( C_b^*(Y, \mathbb{R}) \) is a strong resolution of \( \mathbb{R} \) whenever \( X \) is a simply connected topological space. As a consequence, Theorem 10.8 holds even when the assumption that \( X \) is a countable CW-complex is replaced by the weaker condition that \( X \) is path connected and admits a universal covering.
Remark 10.10. Theorem 10.8 does not hold for bounded cohomology with integer coefficients. In fact, if $X$ is any topological space, just as in the proof of Proposition 4.9 the short exact sequence $0 \to \mathbb{Z} \to \mathbb{R} \to \mathbb{R}/\mathbb{Z} \to 0$ induces an exact sequence

$$H^n_b(X, \mathbb{R}) \to H^n(X, \mathbb{R}/\mathbb{Z}) \to H^{n+1}_b(X, \mathbb{Z}) \to H^{n+1}_b(X, \mathbb{R}).$$

If $X$ is a simply connected CW-complex, then $H^n_b(X, \mathbb{R}) = 0$ for every $n \geq 1$, so we have

$$H^n_b(X, \mathbb{Z}) \cong H^{n-1}_b(X, \mathbb{R}/\mathbb{Z}) \quad \text{for every } n \geq 2.$$

For example, in the case of the 2-dimensional sphere we have $H^3_b(S^2, \mathbb{Z}) \cong H^2(S^2, \mathbb{R}/\mathbb{Z}) \cong \mathbb{R}/\mathbb{Z}$.

11. $\ell^1$-homology and duality

Suppose that $V$ is any $\mathbb{Z}[G]$-module, and let $C_n(G, \mathbb{Z})$ be the free $\mathbb{Z}$-module admitting the set $G^{n+1}$ as a basis. The diagonal action of $G$ onto $G^{n+1}$ defined by $g \cdot (g_0, \ldots, g_n) = (gg_0, \ldots, gg_n)$ extends to an action of $G$ on $C_n(G, \mathbb{Z})$, which, therefore, is endowed with the structure of a $\mathbb{Z}[G]$-module. Moreover, the elements of the form $(1, g_1, g_1g_2, \ldots, g_1 \cdots g_n)$ freely generate $C_n(G, \mathbb{Z})$ as a $\mathbb{Z}[G]$-module, so $C_n(G, \mathbb{Z})$ is a projective $\mathbb{Z}[G]$-module. The map

$$C^{n+1} \to C_{n-1}(G, \mathbb{Z}) \quad (g_0, \ldots, g_n) \to \sum_{i=0}^{n} (g_0, \ldots, \hat{g}_i, \ldots, g_n)$$

extends to a $\mathbb{Z}[G]$-map $d_n : C_n(G, \mathbb{Z}) \to C_{n-1}(G, \mathbb{Z})$, and it is immediate to check that $\ker d_n = \text{Im} d_{n+1} = 0$ for every $n \geq 1$. Moreover, together with the augmentation map

$$\varepsilon : C_0(G, V) \to \mathbb{Z}, \quad \varepsilon \left( \sum_{g \in G} a_g g \right) = \sum_{g \in G} a_g,$$

the complex $(C_\bullet(G, \mathbb{Z}), d_\bullet)$ provides a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}[G]$:

$$0 \longleftarrow \mathbb{Z} \longleftarrow C_0(G, \mathbb{Z}) \xrightarrow{d_1} C_1(G, \mathbb{Z}) \xrightarrow{d_2} \cdots \xrightarrow{d_n} C_n(G, \mathbb{Z}) \xrightarrow{d_{n+1}} \cdots$$

Let now $V$ be a $\mathbb{Z}[G]$-module. Then there is an obvious $\mathbb{Z}[G]$-isomorphism between the module $\text{Hom}_{\mathbb{Z}}(C_n(G, \mathbb{Z}), V)$ and the cochain module

$$C^n(G, V) = \{ f : G^{n+1} \to V \}$$

(which was introduced above in the case when $V$ is a vector space). This shows that the definition of group cohomology with coefficients in an $\mathbb{R}[G]$-module given in the previous section just specializes the general definition of group cohomology with coefficients in a $\mathbb{Z}[G]$-module.

Theorem 11.1. AAA
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