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DUAL PRESENTATIONS OF THE GROUP OF A KNOT

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In 1934 it was proved by H. Seifert [4] that the Alexander polynomial of a single knot is always a symmetric polynomial of even degree. In 1951 G. Torres [5] extended Seifert's method to multiple knots, and showed that the Alexander polynomial of a multiple knot has an analogous "symmetry" property. Although the proof of Seifert's theorem actually allows one to conclude the stronger statement that the equivalence class of Alexander matrices [2] is itself symmetric in a certain sense, the generalization of Seifert's method does not seem to lead to any information about the matrix class of a multiple knot.

In this paper we attack the "symmetry problem" by an entirely different method, based on the discovery of a basic "duality" in the group of any knot, single or multiple. By this new method the previous results of Seifert and Torres are obtained (Corollaries 2 and 3) more easily and in better perspective.

For a multiple knot the "symmetry property" of the matrix class turns out to be somewhat complicated, and this is doubtless the reason why generalization of Seifert's method failed to uncover it.

The notations of [2] are used throughout.

1. Statement of results

A *presentation* [2] of a group G is a set of letters $x_1, \cdots, x_n$ and a set of words $\eta_1(x_1, \cdots, x_n), \cdots, \eta_m(x_1, \cdots, x_n)$ in these letters such that G is isomorphic to $X/(\eta)$, where X is the free group generated by $x_1, \cdots, x_n$ and (η) is the consequence of $\eta_1, \cdots, \eta_m$ in X; G is the image of X under the associated homomorphism ϕ, whose kernel is (η). Along with G we consider also its commutator quotient group H and the associated homomorphism ψ of G upon H. The homomorphisms $\phi: X \rightarrow G$ and $\psi: G \rightarrow H$ extend linearly to homomorphisms, denoted also by ϕ and ψ, of the (integral) group rings: $\phi: JX \rightarrow JG$, $\psi: JG \rightarrow JH$.

The linear extension to the group ring JF of the anti-isomorphism $f \rightarrow f^{-1}$ of any group F will be called *conjugation* and denoted by a bar. Thus $\overline{\sum a_i f_i} = a_i f_i^{-1}$, $a_i \in J$, $f_i \in F$.

Two presentations $(x_1, \cdots, x_n : \eta_1, \cdots, \eta_n)$ and $(y_1, \cdots, y_n : \xi_1, \cdots, \xi_n)$ of G will be called *dual* if

$$(1) \qquad x_i \equiv y_i^{-1} \pmod{\psi\phi}, \qquad i = 1, \cdots, n;$$

$$(2) \qquad \frac{\partial \eta_j}{\partial x_i}(x_i - 1) \equiv \overline{\frac{\partial \xi_i}{\partial y_j}(y_j - 1)} \pmod{\psi\phi}, \qquad i, j = 1, \cdots, n.$$

Theorem. *The group of any tame knot in 3-space has a dual pair of presentations.*

COROLLARY 1. *The group of a tame single knot in 3-space has an Alexander matrix that is equivalent to the conjugate of its transpose. Consequently the elementary ideals* $\mathfrak{E}_1, \mathfrak{E}_2, \cdots$ *are invariant under conjugation.*

COROLLARY 2. *The Alexander polynomial* Δ *of a tame single knot in 3-space has the property*

$$(*_1) \qquad \Delta(t) = t^m \Delta(t^{-1}),$$

where $m \equiv 0 \pmod 2$.

COROLLARY 3. *The Alexander polynomial* Δ *of a tame knot of multiplicity* $\mu > 1$ *in* 3-*space has the property*

$$(*_\mu) \qquad \Delta(t_1, \cdots, t_\mu) = (-1) t_1^{m_1 - 1} \cdots t_\mu^{m_\mu - 1} \Delta(t_1^{-1}, \cdots, t_\mu^{-1}),$$

where $m_i \equiv$ *the linking number of the* i^{th} *component with the sum of the other components* (mod 2).

2. The over- and under-presentations

In this section we shall indicate how to define presentations of a knot group that are adapted to our purpose. A tame knot in 3-space R^3 is represented by the union K of $\mu \geqq 1$ disjoint simple closed polygons $K_1, \cdots, K_\mu$. We may assume that each component is oriented and that the projection of K vertically on a given horizontal plane R^2 is regular [3]. A double point of the projection K' corresponds to two points of K, of which the higher is called the *overcrossing point* and the lower the *undercrossing point*. On each component of K we select a positive even number of ordinary (i.e. not over- or undercrossing points), points, thus dividing the component into two classes of subarcs that alternate around it. The points of subdivision are to be chosen in such a way that one class of subarcs, called *overpasses*, contain no undercrossing point, and the other class of subarcs, called *underpasses*, contain no overcrossing point. This can, of course, be done in many different ways. We denote the overpasses of K by $A_1, \cdots, A_n$ and indicate them in the figures by heavy lines; the underpasses are indicated by light lines and denoted by $B_1, \cdots, B_n$. The ordering is arbitrary except that, for each i, A_i and B_i are to belong to the same component of K. By a semi-linear isotopic deformation of K that displaces points vertically we can arrange it that the overpasses lie above R^2 and the underpasses below R^2 (except, of course, their end-points, the original points of subdivision $p_1, p_2, \cdots, p_{2n}$, which lie in R^2).

The projection of A_i is denoted by A_i' and that of B_j by B_j'. We write $A = A_1 \cup \cdots \cup A_n$, $A' = A_1' \cup \cdots \cup A_n'$, $B = B_1 \cup \cdots \cup B_n$, $B' = B_1' \cup \cdots \cup B_n'$, $p = p_1 \cup p_2 \cup \cdots \cup p_{2n}$. The set of points of R^3 lying over (under) A, B, p etc. will be denoted $A^\#$, $B^\#$, $p^\#$ etc. ($A^\flat$, $B^\flat$, $p^\flat$ etc.) and oriented coherently with respect to A, B, p etc.

The fundamental groups of $R^3 - K \cup B^\flat$ and $R^3 - K \cup A^\#$ are denoted by X and Y respectively; the base point for X is infinitely high and the base point for Y is infinitely low. To any path w in $R^2 - p$ whose initial point w_0 and end-point w_1 lie in $R^2 - K'$ we associate the element $\#(w)$ of X represented by the path $w_0^\# \cdot w \cdot (w_1^\#)^{-1}$ and the element $\flat(w) = w_0^\flat \cdot w \cdot (w_1^\flat)^{-1}$. In particular if w is a path disjoint to K' except that it crosses A'_i once from left to right the associated element $\#(w)$ of X is denoted by x_i; similarly y_j denotes the element $\flat(w)$ of Y where w is a path disjoint to K' except that it crosses B'_j once from left to right.

The complement of $K \cup B^\flat$ is of the same homotopy type as the complement of $A \cup p^\flat$; hence X is seen [6] to be the free group generated by $x_1, \cdots, x_n$. Similarly Y is the free group generated by $y_1, \cdots, y_n$.

The elements $\#(w)$ of X and $\flat(w)$ of Y may be expressed as words in $x_1, \cdots, x_n$ and $y_1, \cdots, y_n$ respectively. Assuming that w is in general position with respect to K' it is easy to write these words down. In fact $\#(w) = x_{i_1}^{\varepsilon_1} x_{i_2}^{\varepsilon_2} \cdots x_{i_l}^{\varepsilon_l}$ if w crosses the projected overpasses $A'_{i_1}, A'_{i_2}, \cdots, A'_{i_l}$ in the order named, and $\varepsilon_k = +1$ or -1 according as w is crossing A'_{i_k} from left to right or from right to left; similarly $\flat(w) = y_{j_1}^{\varepsilon_1} y_{j_2}^{\varepsilon_2} \cdots y_{j_l}^{\varepsilon_l}$ if w crosses the projected underpasses $B'_{j_1}, B'_{j_2}, \cdots, B'_{j_l}$ in the order named, and $\varepsilon_k = +1$ or -1 according as w is crossing B'_{j_k} from left to right or from right to left.

Let $U_1, \cdots, U_n$ be disjoint Jordan regions in R^2 such that $A'_i \subset U_i$, and let $V_1, \cdots, V_n$ be disjoint Jordan regions in R^2 such that $B'_j \subset V_j$. The boundary curves u_i of U_i and v_j of V_j are to be in general position with respect to K' and so oriented that from above u_i appears clockwise and v_j counterclockwise. Let r_i be a path in $R^2 - B'$ to u_i from a fixed exterior point e, and s_j a path in $R^2 - A'$ from e to v_j. These paths $r_1, \cdots, r_n, s_1, \cdots, s_n$ should be in general position with respect to K', and may, of course, be chosen in several ways. Let $\eta_j = \#(s_j \cdot v_j \cdot s_j^{-1})$, $\xi_i = \flat(r_i \cdot u_i \cdot r_i^{-1})$, $\rho_i = \flat(r_i)$, $\sigma_j = \#(s_j)$.

The group X is mapped onto $G = \pi(R^3 - K)$ by cutting $K \cup B^\flat$ with a plane R_-^2 parallel to R^2 and below K. More precisely $X = \pi(R^3 - K \cup B^\flat) \to \pi(R^3 - (K \cup B^\flat - R_-^2)) \approx \pi(R^3 - K) = G$. The kernel of this homomorphism ϕ is easily seen [6] to be just the consequence of the elements $\eta_1, \cdots, \eta_n$. Similarly the group Y is mapped onto G by cutting $K \cup A^\#$ with a plane R_+^2 parallel to R^2 and above K; the kernel of this homomorphism, which we also denote by ϕ, is the consequence of the elements $\xi_1, \cdots, \xi_n$. Thus we have derived, after a number of arbitrary choices, two presentations $(x_1, \cdots, x_n : \eta_1, \cdots, \eta_n)$ and $(y_1, \cdots, y_n : \xi_1, \cdots, \xi_n)$ of G. We call these the *over presentation* and the *under presentation* respectively. It will be observed that the description of either one is obtainable from the description of the other by reversing the orientation of K and interchanging the meanings of "up" and "down."

It may also be observed that one can obtain the classical Wirtinger presentation [3] as an over presentation by choosing as the underpasses d short subarcs of K, one containing each of the d undercrossing points. Unless the projection is alternating the Wirtinger presentation is not the most economical over presentation.

3. An example

Diagram 1 shows a (non-alternating) projection of a knot of multiplicity $\mu = 2$; it is divided, as economically as possible, into overpasses and underpasses. Only the eight overpasses are labelled. For each $i = 1, \cdots, 8$ the underpass following A_i is to be B_i. By way of illustration of the preceding constructions the path $s_2v_2s_2^{-1}$ is indicated by dotted lines. An over-presentation has the eight generators $x_1, \cdots, x_8$ and eight relators $\eta_1, \cdots, \eta_8$ where

$$\begin{aligned}
\eta_1 &= x_1x_3x_2^{-1}x_3^{-1}\\
\eta_2 &= x_1(x_6x_4x_2x_4^{-1}x_6^{-1}x_3^{-1})x_1^{-1}\\
\eta_3 &= x_3x_1x_4^{-1}x_1^{-1}\\
\eta_4 &= x_3(x_7x_2x_4x_2^{-1}x_7^{-1}x_1^{-1})x_3^{-1}\\
\eta_5 &= x_1(x_5x_8x_6^{-1}x_8^{-1})x_1^{-1}\\
\eta_6 &= x_3(x_2x_6x_2^{-1}x_7^{-1})x_3^{-1}\\
\eta_7 &= x_1(x_4x_7x_4^{-1}x_5x_8^{-1}x_5^{-1})x_1^{-1}\\
\eta_8 &= x_1(x_8x_6x_5^{-1}x_6^{-1})x_1^{-1}.
\end{aligned}$$

The overpasses $A_1, \cdots, A_4$ belong to K_1 and the overpasses $A_5, \cdots, A_8$ belong to K_2; accordingly $x_i^{\psi\phi} = t_1$ if $i \leqq 4$ and $x_i^{\psi\phi} = t_2$ if $i \geqq 5$. The Alexander matrix is

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
η_1	1	$-t_1$	$t_1 - 1$	0	0	0	0	0
η_2	0	$t_1^2t_2$	$-t_1$	$t_1t_2(1 - t_1)$	0	$t_1(1 - t_1)$	0	0
η_3	$t_1 - 1$	0	1	$-t_1$	0	0	0	0
η_4	$-t_1$	$t_1t_2(1 - t_1)$	0	$t_1^2t_2$	0	0	$t_1(1 - t_1)$	0
η_5	0	0	0	0	t_1	$-t_1t_2$	0	$t_1(t_2 - 1)$
η_6	0	$t_1(1 - t_2)$	0	0	0	t_1^2	$-t_1$	0
η_7	0	0	0	$t_1(1 - t_2)$	$t_1(t_2 - 1)$	0	t_1^2	$-t_1t_2$
η_8	0	0	0	0	$-t_1t_2$	$t_1(t_2 - 1)$	0	t_1

which is equivalent to

$$\begin{pmatrix} 1 - t_1 + t_1^2 & 0 & 0 \\ 0 & (1 - t_1)(1 - 2t_1 - 2t_1^2t_2 + t_1^3t_2) & (1 - t_2)(1 - 2t_1 - 2t_1^2t_2 + t_1^3t_2) \end{pmatrix}.$$

Thus [2] the Alexander polynomial is

$$\Delta(t_1, t_2) = (1 - t_1 + t_1^2)((1 - 2t_1) - t_1^2t_2(2 - t_1)).$$

The elementary ideal of deficiency 2 is

$$\mathfrak{E}_2 = (1 - t_1 + t_1^2, (1 + t_1)(1 + t_2)).$$

A corresponding under presentation may be constructed by the reader.

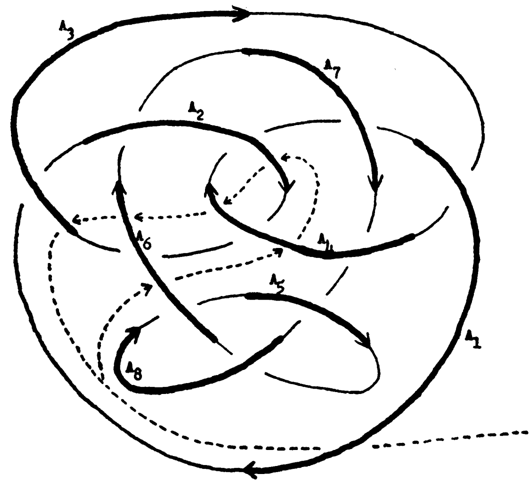

DIAGRAM 1

4. Proof of the theorem

The commutator quotient group H of G is free abelian of rank μ and is generated by elements $t_1, \cdots, t_\mu$ where t_i is represented by a path whose linking number with K_j is δ_{ij}. From the description of x_i and y_j it is clear that $x_i^{\psi\phi} = y_i^{-\psi\phi} = t_{k(i)}$ where $K_{k(i)}$ is the component of K to which A_i and B_i belong. Thus (1) is satisfied.

To prove (2) we shall consider one particular crossing of K, say one of the crossings of A_i over B_j, and calculate the contributions of that crossing to the two members of (2). We may assume that in the neighborhood of this crossing the Jordan regions U_i and V_j intersect in a Jordan region W. Denote by $u_i^{\cdot}$ and u_i^* the two arcs of u_i outside W, and by $v_j^{\cdot}$ and v_j^* the two arcs of v_j outside W. We may assume that r_i and s_i end at the point of $u_i \cap v_j$ indicated on diagram 2. (If necessary r_i may be prolonged around u_i and s_j around v_j.) Then we have

$$\xi_i = \rho_i y_j^{\varepsilon} \alpha_i y_j^{-\varepsilon} \alpha_i^* \rho_i^{-1},$$

$$\eta_j = \sigma_j x_i^{\delta} \beta_j x_i^{-\delta} \beta_j^* \sigma_j^{-1},$$

where $\varepsilon = \pm 1$, $\delta = \pm 1$, $\alpha_i = \flat(u_i^{\cdot})$, $\alpha_i^* = \flat(u_i^*)$, $\beta_j = \sharp(v_j^{\cdot})$, $\beta_j^* = \sharp(v_j^*)$, and the common end-point of r_i and s_j falls on the common end-point of α_i^* and β_j^*.

Suppose $\rho_i^{\psi\phi} = \prod_{k=1}^{\mu} t_k^{\lambda_k}$ and $\sigma_j^{\psi\phi} = \prod_{k=1}^{\mu} t_k^{\theta_k}$. Since the intersection number of two oriented closed curves in R^2, such as $r_i s_j^{-1}$ and K_k', is equal to zero, and since r_i crosses only projected overpasses while s_j crosses only projected underpasses, it follows from (1) that $\lambda_k + \theta_k = 0$. Thus $\rho_i \equiv \sigma_j^{-1}$ (mod $\psi\phi$). Applying the same reasoning to the oriented closed curves that make up the boundaries of the Jordan region $U_i - W$ and $V_j - W$, we get $\alpha_i \equiv x_i^{-\delta}$, $\alpha_i^* \equiv x_i^{\delta}$, $\beta_j \equiv y_j^{-\varepsilon}$,

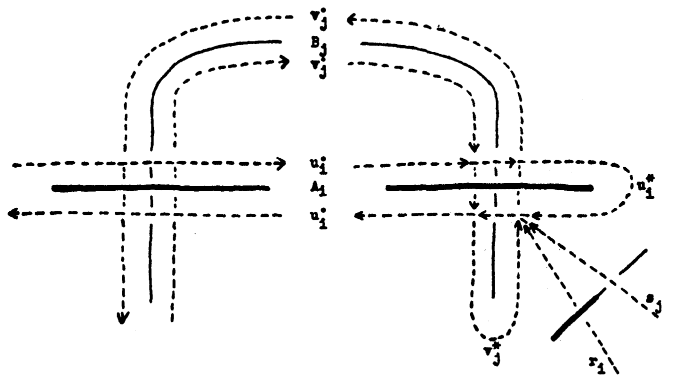

DIAGRAM 2

$\beta_j^* \equiv y_j^\varepsilon \pmod{\psi\phi}$. Thus

$$\frac{\partial \xi_i}{\partial y_j} \equiv \rho_i(1 - \alpha_i) \frac{y_j^\varepsilon - 1}{y_j - 1} + \cdots \pmod{\psi\phi},$$

$$\frac{\partial \eta_j}{\partial x_i} \equiv \sigma_j(1 - \beta_j) \frac{x_i^\delta - 1}{x_i - 1} + \cdots \pmod{\psi\phi},$$

where the remainder terms on the right are contributions from other crossings of A_i' and B_j'. It follows that

$$\begin{aligned} \overline{\frac{\partial \xi_i}{\partial y_j}(y_j - 1)} &\equiv \rho_i^{-1}(1 - \alpha_i^{-1})(y_j^{-\varepsilon} - 1) + \cdots \\ &\equiv \sigma_j(1 - x_i^\delta)(\beta_j - 1) + \cdots \\ &\equiv \frac{\partial \eta_j}{\partial x_i}(x_i - 1) \qquad \pmod{\psi\phi}. \end{aligned}$$

5. Proof of corollary 1

If K is a single knot, $\mu = 1$ and H is the infinite cyclic group generated by t, where $x_i^{\psi\phi} = t$ and $y_j^{\psi\phi} = t^{-1}$. Hence (2) yields $\overline{\partial \xi_i / \partial y_j} \equiv \partial \eta_j / \partial x_i \pmod{\psi\phi}$. Consequently the Alexander matrix $\| \partial \xi_i / \partial y_j \|^{\psi\phi}$ of the presentation $(y\colon \xi)$ is the conjugate transpose of the Alexander matrix $\| \partial \eta_j / \partial x_i \|^{\psi\phi}$ of the presentation $(x\colon \eta)$. Furthermore the conjugate transpose of any minor of order $n - d$ of one of the matrices is a minor of order $n - d$ of the other matrix. Hence $\mathfrak{E}_d = \bar{\mathfrak{E}}_d$ for each $d = 0, 1, 2, \cdots$.

6. Proof of corollary 2

The ideal $\mathfrak{E}_1$ is a principal ideal if the tame knot is single [2] and the Alexander polynomial Δ is, by definition, a generator of this ideal. (Δ is only determined up to a factor $\pm t^r$.) Since $\mathfrak{E}_1 = \bar{\mathfrak{E}}_1$, we have $\Delta(t) = \varepsilon t^m \Delta(t^{-1})$ for some integer m, where $\varepsilon = \pm 1$. It is known [1] that $\Delta(1) = \pm 1$; it follows immediately that $\varepsilon = +1$. Writing $\Delta(t) = c_0 + c_1 t + \cdots + c_m t^m$, we have $c_i = c_{m+i}$ for $i = 1, 2, \cdots$. Therefore m must be even, for if m were odd we could have $1 \equiv \Delta(1) \equiv 2(c_0 + c_1 + \cdots + c_{\frac{1}{2}(m-1)}) \equiv 0 \pmod 2$.

7. Proof of corollary 3

If we delete from each of the two matrices the last row and the last column, say, the determinants of the two minors obtained [2] will be $\Delta(t_1, \cdots, t_\mu)(t_k - 1)$ and $\Delta(t_1, \cdots, t_\mu)(t_k^{-1} - 1)$, where A_n and B_n belong to K_k. Since Δ is only determined up to a factor $\pm t_1^{r_1} \cdots t_\mu^{r_\mu}$ it follows from the theorem that

$$(3_\mu) \qquad \Delta(t_1, \cdots, t_\mu) = \varepsilon t_1^{m_1 - 1} \cdots t_\mu^{m_\mu - 1} \Delta(t_1^{-1}, \cdots, t_\mu^{-1}).$$

It remains to prove that $\varepsilon = (-1)^\mu$ and $m_i \equiv \sum_{j=1}^{\mu} l_{ij} \pmod 2$, where l_{ij} is the linking number of K_i and K_j if $i \neq j$ and $l_{ii} = 0$. The proof of these facts depends on the following property of the Alexander polynomial of a multiple knot [5]:

$$(4_\mu) \qquad \begin{aligned} \Delta(t_1, 1) &= \frac{t_1^{l_{12}} - 1}{t_1 - 1} \Delta(t_1) && \text{if } \mu = 2, \\ \Delta(t_1, \cdots, t_{\mu-1}, 1) &= (t_1^{l_{1\mu}} \cdots t_{\mu-1}^{l_{\mu-1\,\mu}} - 1)\Delta(t_1, \cdots, t_{\mu-1}) && \text{if } \mu > 2, \end{aligned}$$

where $\Delta(t_1, \cdots, t_\mu)$ is an Alexander polynomial of $K_1 \cup \cdots \cup K_\mu$ and $\Delta(t_1, \cdots, t_{\mu-1})$ is a properly chosen Alexander polynomial of $K_1 \cup \cdots \cup K_{\mu-1}$. From (4_μ) it follows that

$$\Delta(t_1, 1, \cdots, 1) = (t_1^{l_{1\mu}} - 1)(t_1^{l_{1\mu-1}} - 1) \cdots (t_1^{l_{13}} - 1) \frac{t_1^{l_{12}} - 1}{t_1 - 1} \Delta(t_1),$$

so that $\Delta(t_1, \cdots, t_\mu)$ is certainly different from zero if $l_{1i} \neq 0$ for $i = 2, 3, \cdots, \mu$.

(I) Suppose first that $l_{1i} \neq 0$ for $i = 2, 3, \cdots, \mu$. If $\mu = 2$ we obtain from (3_2) and (4_2) that

$$\frac{t_1^{l_{12}} - 1}{t_1 - 1} \Delta(t_1) = \varepsilon t_1^{m_1 - 1} \frac{t_1^{-l_{12}} - 1}{t_1^{-1} - 1} \Delta(t_1^{-1}).$$

Using Corollary 2, it follows that $\varepsilon = +1$ and $m_1 \equiv l_{12} \pmod 2$. Proceeding by induction on μ we assume $\mu > 2$ and obtain similarly from (3_μ) and (4_μ) that

$$\left(\prod_{i=1}^{\mu-1} t_i^{l_{i\mu}} - 1\right)\Delta(t_1, \cdots, t_{\mu-1}) = -\varepsilon \sum_{i=1}^{\mu-1} t_i^{m_i - l_{i\mu} - 1} \left(\prod_{i=1}^{\mu-1} t_i^{l_{i\mu}} - 1\right) \Delta(t_1^{-1}, \cdots, t_{\mu-1}^{-1}).$$

Since $l_{1\mu} \neq 0$ the factor $\left(\prod t_i^{l_{i\mu}} - 1\right)$ may be cancelled from both sides. Since $l_{1i} \neq 0$ for $i = 2, 3, \cdots, \mu - 1$ it follows from the inductive hypothesis that

$$\Delta(t_1, \cdots, t_{\mu-1}) = (-1)^{\mu-1} t_1^{n_1 - 1} \cdots t_{\mu-1}^{n_{\mu-1} - 1} \Delta(t_1^{-1}, \cdots, t_{\mu-1}^{-1}),$$

where $n_i \equiv \sum_{j=1}^{\mu-1} l_{ij} \pmod 2$. Since we are assured that $\Delta(t_1, \cdots, t_{\mu-1})$ is not zero in this case, we conclude easily that $\varepsilon = (-1)^\mu$ and $m_i \equiv \sum_{j=1}^{\mu} l_{ij}$ (mod 2), completing the induction.

(II) Next we suppose that not all of $l_{12}, \cdots, l_{1\mu}$ are different from zero. We adjoin a simple closed polygon K_0, disjoint to K, such that the linking number l_{0i} of K_0 and K_i is different from zero for each $i = 1, \cdots, \mu$. Then, by case (I),

$$\Delta(t_0, \cdots, t_\mu) = (-1)^{\mu+1} t_0^{n_0-1} \cdots t_\mu^{n_\mu - 1} \Delta(t_0^{-1}, \cdots, t_\mu^{-1}),$$

where $n_i \equiv \sum_{j=0}^{\mu} l_{ij} \pmod 2$. Applying $(4_{\mu+1})$, we get

$$\prod_{i=1}^{\mu} (t_i^{l_{0i}} - 1)\Delta(t_1, \cdots, t_\mu) = -(-1)^{\mu+1} \prod_{i=1}^{\mu} t_i^{n_i - l_{0i} - 1} \left(\prod_{i=1}^{\mu} t_i^{l_{0i}} - 1 \right) \Delta(t_1^{-1}, \cdots, t_\mu^{-1}).$$

Since $\prod_{i=1}^{\mu} (t_i^{l_{0i}} - 1) \neq 0$, there follows (3_μ), where $\varepsilon = (-1)^\mu$ and $m_i \equiv \sum_{j=1}^{\mu} l_{ij}$ (mod 2).

7. Concluding remarks

Seifert has shown that a polynomial $\Delta(t)$ that satisfies $\Delta(t) = t^m \Delta(t^{-1})$ and $\Delta(1) = \pm 1$ is an Alexander polynomial of some single knot. Thus any ideal $\mathfrak{E}$ in the ring JH of the infinite cyclic group H that is invariant under conjugation and is mapped by o onto all of J is the elementary ideal of deficiency 1 of some (single) knot. This suggests the more general problem: Given a chain of ideals $\mathfrak{E}_1 \subset \mathfrak{E}_2 \subset \mathfrak{E}_3 \subset \cdots$ in the ring of the infinite cyclic group such that each $\mathfrak{E}_d$ is invariant under conjugation and is mapped by o onto all of J, under what conditions is this the chain of elementary ideals of some single tame knot?

For multiple knots the situation is more complicated. Corollary 3 describes a property of $\mathfrak{E}_1$, a weakened form of which is the statement that $\mathfrak{E}_1$ is invariant under conjugation. Does some analogous property hold for $\mathfrak{E}_d$ when $d > 1$? In particular, are the ideals $\mathfrak{E}_2, \mathfrak{E}_3, \cdots$ invariant under conjugation?

Finally, if a polynomial $\Delta(t_1, \cdots, t_\mu)$ satisfies $(*_\mu)$, under what additional conditions is it the Alexander polynomial of some knot?

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