I can’t remember my title

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Outline

1. General Nonsense.
2. A short, inadequate introduction to/review of Nonstandard Analysis.
3. Nonstandard measure spaces; Loeb construction
4. Discrete arguments in Analysis via NSA
5. Applications of NSA to uniform and asymptotic properties in discrete math, additive number theory, etc

A remarkable result, not about combinatorics?

Theorem 0.1. *(G. Keller, 1972) Let $\mathcal{V}$ be a variety of groups. Then $\mathcal{V}$ is uniformly amenable iff every group in $\mathcal{V}$ is amenable.*

- What does *uniformly amenable* mean? (What does *amenable* mean?)
- What properties of *variety* are used here? Does it hold for other classes of groups?
- Are there similar results for mathematical properties other than amenability, or objects other than groups? What is really going on here?

Relevance for this audience:

- The proof is a recipe for proving results of the form:
  If *every* element of a class $\mathfrak{C}$ has property P, then $\mathfrak{C}$ is uniformly P.
- The proof shares elements with recent nonstandard proofs of results in combinatorics and additive number theory (cf Jin, Elek, Szegedy, Tao, *et al*)

A group G is *amenable* if there is a left-invariant, finitely-additive probability measure μ on $(G, \mathcal{P}(G))$ with $P(G) = 1$.

Finite groups are amenable; abelian groups are amenable; homomorphic images and subgroups of amenable groups are amenable;...

A *variety* of groups is a class determined by a (finite) set of words

Theorem 0.2. *(Følner): G is amenable if and only if:*

$$\forall A \subseteq G \text{ finite } \forall \epsilon > 0 \exists E \subseteq G \text{ finite } \forall a \in A \ \frac{||E \triangle aE||}{||E||} < \epsilon$$

A group G is *uniformly Følner*, or *uniformly amenable* if $||E||$ can be chosen to depend only on $||A||$ and ϵ, that is, if there is a function $F : \mathbb{N} \times (0,1) \longrightarrow \mathbb{N}$ such that

$$\forall n \in \mathbb{N} \ \forall A \subseteq G \text{ s.t. } ||A|| < n \ \forall \epsilon > 0$$

$$\exists E \subseteq G \text{ s.t. } ||E|| < F(n, \epsilon) \ \& \ \forall a \in A \ \frac{||E \triangle aE||}{||E||} < \epsilon$$

A class $\mathcal{D}$ of groups is *uniformly amenable* if there is a single function $F : \mathbb{N} \times (0,1) \longrightarrow \mathbb{N}$ that witnesses amenability for all the groups in $\mathcal{D}$

Remark. This is not the best definition of *uniformly amenable*. Better is:

G is uniformly amenable if $^*G$ is amenable.

$\mathcal{D}$ is uniformly amenable if *every* G in $^*\mathcal{D}$ is amenable.

TWO FOLK EQUATIONS

NONSTANDARD ANALYSIS::ULTRAFILTERS

=

REAL NUMBERS::DEDEKIND CUTS

DISCRETE OR FINITARY ARGUMENTS + NONSTANDARD METHODS

=

INFINITARY OR CONTINUOUS RESULTS IN STANDARD MATHEMATICS

Examples

(Anderson et al) Continuous time stochastic processes "are" random walks

(R) The Riesz Representation Theorem "is" the Farkas Lemma

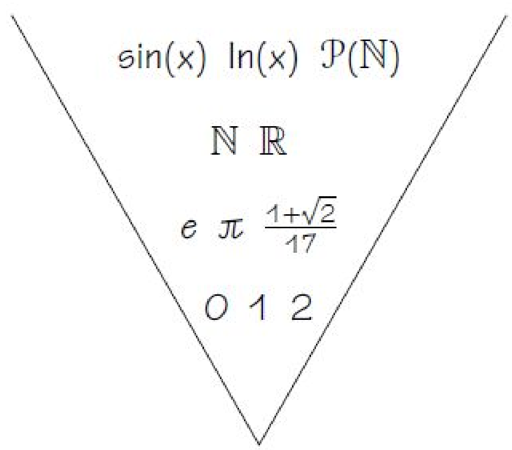
(Artstein; R) Selecting from a random closed set "is" a Marriage Lemma

(R) Marriage Lemma can be used to build invariant measures on topological spaces

Review: Components of the nonstandard model

(I) Start with a (standard) mathematical universe (or **superstructure**) V:

- A large set containing *every* other mathematical object we might want to talk about, such as all natural numbers $0, 1, 2, \ldots$; real numbers $\sqrt{2}, \pi, e, \ldots$; the *set* $\mathbb{N}$ of natural numbers *as an object*; the set $\mathbb{R}$ of real numbers; *every* function from $\mathbb{R}$ to $\mathbb{R}$, and the set of all such functions; *etc.*
- We call the elements of this mathematical universe **standard**.



(II) An infinite cardinal κ bigger than $\aleph_0$. It is often convenient to take $\kappa >$card(V).

(III) A first-order language $\mathcal{L}_V$ with constant, function, and relation symbols for *every* constant, function, and relation in V.

The nonstandard model consists of a new (bigger) superstructure $^*V$, and an injection $* : V \longrightarrow {}^*V$, satisfying transfer and ($\kappa-$)saturation:

Transfer:

If S is a statement about objects in V, then S is true in V if and only if it true in ${}^*V$.

(Technically: *every bounded first-order $\mathcal{L}_V$-sentence holds in V if and only if it holds in ${}^*V$.*)

Examples:

"$\mathbb{N}$ is Well-ordered":

$$[\forall A \in \mathcal{P}(\mathbb{N})]\ [(A \neq \emptyset) \Rightarrow (\exists n \in A \forall x \in A n \leq x)]$$

"$^*\mathbb{N}$ is *-Well-ordered":

$$[\forall A \in {}^*\mathcal{P}(\mathbb{N})]\ [(A \neq \emptyset) \Rightarrow (\exists n \in A \forall x \in A n {}^*\!\leq x)]$$

"$^*\mathbb{N}$ is Well-ordered" (false):

$$[\forall A \subseteq {}^*\mathbb{N}]\ [(A \neq \emptyset) \Rightarrow (\exists n \in A \forall x \in A n {}^*\!\leq x)]$$

More Examples:

If G is a group, then $^*G$ is a group. Any *group is a group.

If $\mathcal{A}$ is an algebra of sets, then so is $^*\mathcal{A}$

Combinatorial results such as the *Marriage Lemma holds of appropriate objects in the nonstandard model.

Remark: We might imagine a standard mathematician living in universe V, and a nonstandard mathematician living in $^*V$. The transfer principle says that both these mathematicians experience exactly the same true statements. The reason this is possible is that they both speak the same language - the language of V. In particular, the 'nonstandard' mathematician will not be able to refer to any particular element of $^*V$ that is not the $*-$image of an element of V.

κ-Saturation:

Suppose that $\mathcal{S}$ is a collection of fewer than κ statements about an *object* X, and that for *every* finite subcollection of $\mathcal{S}$ there is an *object* in $^*V$ for which they hold; then there is an *object* in $^*V$ for which **all** the statements in $\mathcal{S}$ hold simultaneously.

Example: Consider the statements: "x is a natural number," "$x \geq 1$", "$x \geq 2$", "$x \geq 3$",$\cdots$

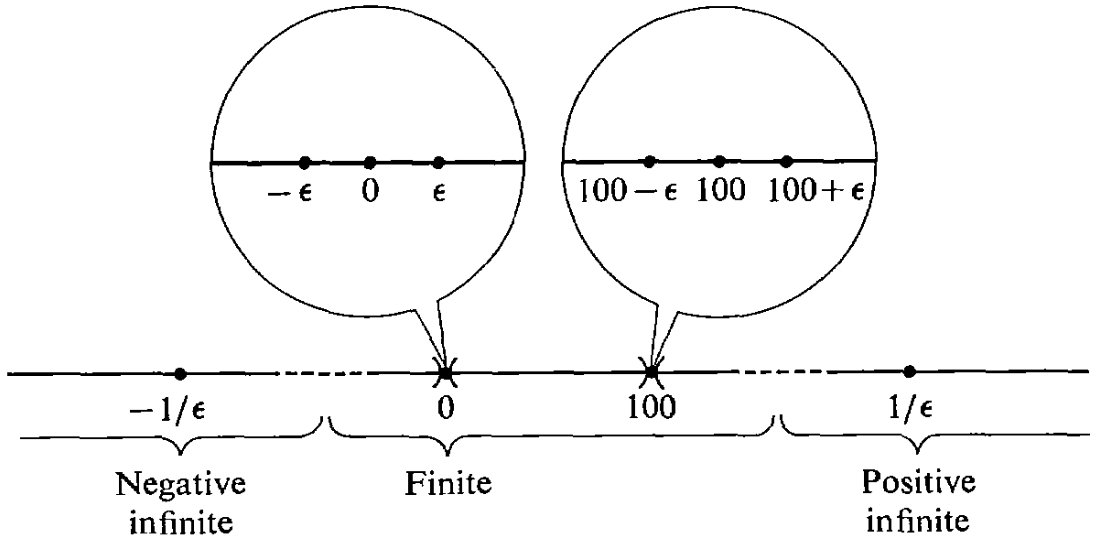
Any finite subset of these statements refers to a largest number N which satisfies this finite set of statements.

It follows that there is a an *element* H *of* ${}^*\mathbb{N}$ satisfying all the statements, that is, such that for *every* (standard) natural number n, $H > n$

Such an H is an infinite *hyperfinite number.*

Since $1/H$ is less than *every* standard real number, it is a positive infinitesimal in ${}^*\mathbb{R}$.

Since ${}^*\mathbb{R}$ (sometimes called the set of *"hyperreal numbers"*) is, like the usual set of real numbers, closed under the basic arithmetic operations, it is a non-Archimedean field, and looks roughly like this :

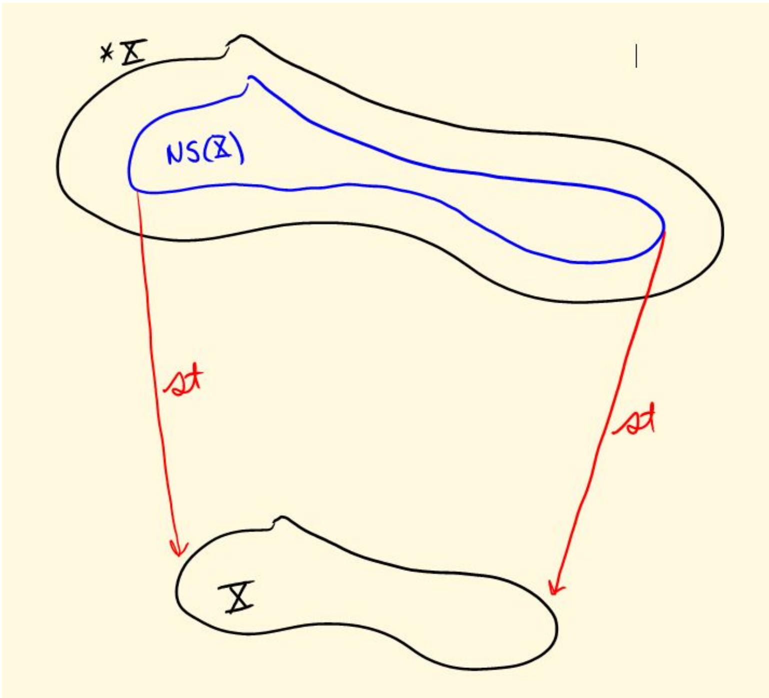


(figure taken from Keisler's Calculus text)

It turns out that *every* finite hyperreal s differs infinitesimally from some unique standard real r; call r the *standard part* of s, $r = st(s)$.

In other words, st() takes any finite hyperreal to the closest standard real number.

There is a similar situation for any reasonable nice topological space:



κ-**Saturation (alternate):**

There is an equivalent formulation in terms of *internal sets*. The **internal sets** are a distinguished subcollection of sets in $^*V$ that includes (i) $^*A$ for *every* $A \in V$; (ii) Every $\mathcal{L}_V$-definable set in $^*V$; (iii) Every element of another internal set.

In terms of internal sets, κ-saturation becomes: Suppose $\{A_i\}_{i<\lambda}$ is a collection of sets such that

(a) $\lambda < \kappa$;

(b) The sets A_i are internal; and

(c) The collection $\{A_i\}_{i<\lambda}$ satisfies the *finite intersection property*:

$$\text{for any } i_1 < i_2 < \cdots i_N < \lambda, \qquad (A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_N}) \neq \emptyset$$

Then $\bigcap_{i<\lambda} A_i \neq \emptyset$.

More easy consequences of saturation:

$^*\mathbb{N}$ is not well-ordered. (There is no least infinite hyperinteger.)

if $A_1 \supseteq A_2 \subseteq \cdots A_n \supseteq$ are internal and $A = \bigcap_{n\in\mathbb{N}} A_n$ is internal then for some n $A_n = A$.

If A is a standard infinite set then $^*A$ is strictly bigger than A

If A is any standard set then there is a hyperfinite $\hat{A}$ such that $A \subseteq \hat{A} \subseteq {}^*A$

Hyperfinite sets

Definition: A set E in $^*V$ is *hyperfinite* if there is a *one-to-one correspondence between E and $\{0, 1, 2, \ldots, H\}$ for some H in $^*\mathbb{N}$. Equivalently, if the mathematical statement "E is finite" holds in $^*V$.

Examples: 1. Every finite set is hyperfinite.

2. If H is an infinite integer, $\{0, 1, 2, \cdots, H\} = \{n \in {}^*\mathbb{N} : n \leq H\}$ is a hyperfinite subset of $^*\mathbb{N}$

3. If H is an infinite integer, $\{0, \frac{1}{H}, \frac{2}{H}, \cdots, \frac{H-1}{H}, 1\}$is a hyperfinite subset of $^*[0.1]$

Theorem: If A is an infinite set in V then there is a hyperfinite set $\hat{A} \subseteq {}^*A$ in ${}^*V$ such that *every* element of A is in $\hat{A}$

Proof: Consider the statements: (i) X is finite; (ii) $a \in X$ (*one such statement for every element a of A*); (iii) $X \subseteq A$.

Given any finite number of these statements, a corresponding finite number $\{a_1, \ldots, a_n\}$ of elements of A are mentioned, so $X = \{a_1, \ldots, a_n\}$ satisfies those statements. By the saturation principle there is therefore a set X in ${}^*V$ satisfying all the statements simultaneously; let $\hat{A}$ be this X.

Corollary: There is a hyperfinite set containing $\mathbb{R}$.

This *gives another way of proving that* ${}^*\mathbb{R}$ has infinitesimals. If $\hat{\mathbb{R}}$ is a hyperfinite set *extending* $\mathbb{R}$, *then the least element of* $\hat{\mathbb{R}} \cap {}^*(0, \infty)$ is a *positive* infinitesimal.

"Nonstandard analysis is the art of making infinite sets finite by extending them." —M. Richter

Nonstandard measure spaces

Recall that a finite measure space is a triple $(X, \mathcal{A}, \mu)$ such that

1. $\mathcal{A}$ is a σ-algebra on X
2. $\mu : \mathcal{A} \longrightarrow [0, \infty)$ is an additive set function with $\mu(\emptyset) = 0$
3. μ is σ-additive (= countably additive):

$$\mu(\bigcup_{i\in\mathbb{N}} A_i) = \sum_{i\in\mathbb{N}} \mu(A_i)$$

provided the sets A_i are all disjoint

It follows that a *measure space would be a triple $(X, \mathcal{A}, \mu)$ such that

1. $\mathcal{A}$ is a $^*\sigma$-algebra on $^*X$ (whatever that means; note that "algebra"="*algebra")
2. $\mu : \mathcal{A} \longrightarrow {}^*[0, \infty)$ is an *additive set function with $\mu(\emptyset) = 0$
3. μ is $^*\sigma$-additive:

$$\mu(\bigcup_{i \in {}^*\mathbb{N}} A_i) = {}^* \sum_{i \in {}^*\mathbb{N}} \mu(A_i)$$

   provided the sets A_i are all disjoint

What *do* the last two mean? (2) says that $P\mu$ is additive for any standardly finite sequence of sets, *and* for a hyperfinite sequence of sets, so is stronger than standard additivity. (3) says that μ is additive *even* for a sequence of sets indexed by the nonstandard natural numbers; however, it generally will *not* be additive for a sequence indexed by the standard natural numbers (such a sum will generally not even be defined).

Nonstandard measure spaces can arise in two natural ways:

I. As the star of a standard measure space. If $(X, \mathcal{A}, \mu)$ is a standard measure space, then $({}^*X, {}^*\mathcal{A}, {}^*\mu)$ is a nonstandard measure space.

II. As something constructed internally. For example, if N is an infinite integer, we could define $(\underline{\Omega}, \mathcal{P}(\underline{\Omega}), \mu)$ by

- $\underline{\Omega} = \{H, T\}^N =$ all possible sequences of coin flips of length N
- $\mu(A) = \frac{||A||}{2^N}$

Conversion to standard spaces

Observation If $(X, \mathcal{A}, \mu)$ is a finite-valued nonstandard measure space, $\mathrm{st}(\mu)$ is a real-valued function on $(X, \mathcal{A})$ satisfying all the properties of a standard measure space except the countable ones.

In fact, if $A_0 \supseteq A_1 \supseteq A_2 \supseteq A_3 \cdots$ are elements of $\mathcal{A}$, then $\cap_{n\in\mathbb{N}} A_n$ can only be in $\mathcal{A}$ if the sequence is eventually constant; this is an immediate consequence of κ—saturation. It follows that $\mathcal{A}$ is not a σ-algebra except in trivial cases.

Loeb (1972) noticed that this means that $\mathrm{st}(\mu)$ is trivially σ-additive when restricted to $\mathcal{A}$, and by the Carathéodory Extension Theorem from measure theory $\mathrm{st}(\mu)$ has a natural, well-behaved extension μ_L to a σ-algebra $\mathcal{A}_L$ extending $\mathcal{A}$. The standard probability space $(X, \mathcal{A}_L, \mu_L)$ (now called the *Loeb Space*) retains many of the properties of the nonstandard space from which it is obtained.

Example Let $(\Omega, \mathcal{P}(\Omega), P)$ be the infinite coin-flipping example from before (Ω=all sequences of length N from $\{H, T\}$, where N is infinite, and $\mathcal{P}(\Omega)$ is the *internal* power set of Ω, namely the internal algebra of all internal subsets of Ω). Then for any given sequence $\omega \in \Omega$ of coin flips, $P(\{\omega\}) = 2^{-N}$, but $P_L(\{\omega\}) = 0$.

Some observations about Loeb measures

- It requires AC in an *essential* way
- In practice, the nonstandard measure constructed is often taken to be an internal weighted counting *measure on a hyperfinite set, and is often based on an intuitive finitary approximation of the problem. (EG, random walk to approximate Brownian Motion.)
- When using these measures one moves frequently back and forth between the hyperreal-valued measure μ and the real-valued counterpart μ_L, depending on need at the moment.

Typical applications to analysis

See any modern (post 1975) graduate-level text in nonstandard analysis.

Examples of nonstandard arguments

Theorem 0.3. *(Ramsey)*

1. If $\mathbb{N}$ is k-colored there is an infinite monochromatic subset
2. For all $m \in \mathbb{N}$ there exists an $n \in \mathbb{N}$ such that if $\{0, \ldots, n-1\}$ is k-colored then there is a monochromatic subset of size m

Proof. (Joram Hirshfeld, 1988) Mauro proved (1) yesterday, let's prove that (1) $\Rightarrow$ (2).

Suppose m witnesses failure of (2).

For all n there is a k-coloring of n with no monochrome subset of size m.

By saturation there is an infinite $n \in {}^*\mathbb{N}$ and a k-coloring $\mathcal{C}$ of n with no (internal) monochrome subset of size m.

The restriction of $\mathcal{C}$ to $\mathbb{N}$ is a k-coloring of $\mathbb{N}$, so by (1) $\mathbb{N}$ has an infinite monochrome subset.

Any finite subset of size $> m$ from this monochrome subset is an internal subset of ${}^*\mathbb{N}$ which is monochrome with respect to $\mathcal{C}$, witnessing a contradiction.

$\dashv$

Theorem 0.4. *(van der Waerden)*

1. *If $\mathbb{N}$ is k-colored there are arbitrarily long monochromatic arithmetic progressions*
2. *For all $m \in \mathbb{N}$ there exists an $n \in \mathbb{N}$ such that if $\{0, ..., n-1\}$ is k-colored then there is a monochromatic arithmetic progression of size m*

Proof. Note (2) $\Rightarrow$ (1). Let's prove that (1) $\Rightarrow$ (2).

Suppose m witnesses failure of (2).

For all n there is a k-coloring of n with no monochrome arithmetic progression of size m.

By saturation there is an infinite $n \in {}^*\mathbb{N}$ and a k-coloring $\mathcal{C}$ of n with no monochrome *-arithmetic progression of size m.

The restriction of $\mathcal{C}$ to $\mathbb{N}$ is a k-coloring of $\mathbb{N}$, so by (1) $\mathbb{N}$ has arbitrarily large monochrome arithmetic progressions.

Any such monochromatic arithmetic progression of size $> m$ is (since a finite vector of standard numbers) a monochrome *-arithmetic progression, witnessing a contradiction.

$\dashv$

Results like (2) can of course be proved in more general partition regularity situations.

Theorem 0.5. *(Szemeredi) Let $E \subseteq \mathbb{N}$ have positive upper density, that is, $\limsup_n \frac{||E \cap [0,n]||}{n} = \delta > 0$. Then E contains arbitrarily long arithmetic sequences.*

Proof. Let ℓ be arbitrary.

For some infinite N, $\frac{||{}^*E \cap [0,N-1]||}{N} \approx \delta$

Let $\Omega = \{1, 2, \ldots, N\}, \mathcal{A} = {}^*\mathcal{P}(\Omega), \mu(A) = \frac{||A||}{N}$ for $A \in \mathcal{A}$.

$(\Omega, \mathcal{A}, \mu)$ is a finite *measure space, can convert to standard measure space $(\Omega, \mathcal{A}_L, \mu_L)$ using the Loeb construction.

Note $\mu_L(E_N) = \delta$, where $E_N = {}^*E \cap [0, N-1]$

Define $T : \Omega \longrightarrow \Omega$ by $T(x) = x + 1 \mod N$. Then $(\Omega, \mathcal{A}_L, \mu_L, T)$ is a standard dynamical system.

Recall Furstenberg's *Multiple Recurrence Theorem*: If $(X, \mathcal{B}, \nu)$ is a finite measure space and $T_1, \ldots, T_\ell$ are commuting measure-preserving transformations of $(X, \mathcal{B}, \nu)$ then for any set A

with $\nu(A) > 0$ there is an integer $n \geq 1$ with

$$\nu(A \cap T_1^{-n} A \cap T_2^{-n} A \cap \cdots \cap T_\ell^{-n} A) > 0$$

Like yesterday, let $T_1 = T, T_2 = T^2, \ldots, T_\ell = T^\ell$, these obviously commute.

Apply Furstenburg recurrence to thee Ts and $A = E_N$ to get a (standard, finite) n

Let $e \in E_N \cap T_1^{-n} E_N \cap T_2^{-n} E_N \cap \cdots \cap T_\ell^{-n} E_N$. (Assume that e is not finite.)

Exercise: show that $e, e + n, e + 2n, \ldots, e + \ell n \in E_N \subseteq {}^*E$

So: "$(\exists e \in {}^*\mathbb{N})(e \in {}^*E \bigwedge e + n \in {}^*E \bigwedge e + 2n \in {}^*E \bigwedge \cdots \bigwedge e + \ell n \in {}^*E)$" is true.

By transfer, "$(\exists e \in \mathbb{N})(e \in E \bigwedge e + n \in E \bigwedge e + 2n \in E \bigwedge \cdots \bigwedge e + \ell n \in E)$" is true.

$\dashv$

Remark: substituting "Banach upper density" for "upper density" in the above does not change the proof, except that $\Omega = [M, M + N - 1]$ for some infinite M. R. Jin has exploited in several results the fact that the nonstandard version of these proofs makes it clear when hypotheses like this can be weakened in this way.