Elementary numerosity and measures

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Finitely additive measures

Definition

A *finitely additive measure* is a triple \((\Omega, \mathcal{A}, \mu)\) where:

- The *space* \(\Omega\) is a non-empty set;
- \(\mathcal{A}\) is a *ring of sets* over \(\Omega\), i.e. a non-empty family of subsets of \(\Omega\) satisfying the conditions:
  \[ A, B \in \mathcal{A} \Rightarrow A \cup B, A \cap B, A \setminus B \in \mathcal{A}; \]
- \(\mu : \mathcal{A} \to [0, +\infty]_\mathbb{R}\) is an *additive function*, i.e.
  \[ \mu(A \cup B) = \mu(A) + \mu(B) \text{ whenever } A, B \in \mathcal{A} \text{ are disjoint.} \]
  We also assume that \(\mu(\emptyset) = 0\).

The measure \((\Omega, \mathcal{A}, \mu)\) is called *non-atomic* when all finite sets in \(\mathcal{A}\) have measure zero.
Elementary numerosities

Definition

An *elementary numerosity* on a set $\Omega$ is a function

$$n : \mathcal{P}(\Omega) \to [0, +\infty)_F$$

defined for all subsets of $\Omega$, taking values into the non-negative part of a non-archimedean field $F$, and satisfying the conditions:

- $n(\{x\}) = 1$ for every point $x \in \Omega$;
- $n(A \cup B) = n(A) + n(B)$ whenever $A$ and $B$ are disjoint.
Numerosity measures

Proposition

Let $n : \mathcal{P}(\Omega) \rightarrow [0, +\infty)_F$ be an elementary numerosity, and for every $\beta > 0$ in $F$ define the function $n_\beta : \mathcal{P}(\Omega) \rightarrow [0, +\infty)_{\mathbb{R}}$ by posing

$$n_\beta(A) = \text{sh} \left( \frac{n(A)}{\beta} \right).$$

Then $n_\beta$ is a finitely additive measure defined for all subsets of $\Omega$. Moreover, $n_\beta$ is non-atomic if and only if $\beta$ is an infinite number.
The main result (1)

**Theorem**

Let \((\Omega, \mathcal{A}, \mu)\) be a non-atomic finitely additive measure. Then there exist

- a non-archimedean field \(F \supseteq \mathbb{R}\);
- an elementary numerosity \(n : \mathcal{P}(\Omega) \to [0, +\infty)\_F\);

such that for every positive number of the form \(\beta = \frac{n(A^*)}{\mu(A^*)}\) one has

\[
\mu(A) = n_\beta(A) \quad \text{for all } A \in \mathcal{A}.
\]

Moreover, if \(\mathcal{B} \subseteq \mathcal{A}\) is a subring whose non-empty sets have all positive measure, then we can also assume that

\[
n(B) = n(B') \quad \text{for all } B, B' \in \mathcal{B} \text{ such that } \mu(B) = \mu(B').
\]
Idea of the proof

Let $\Lambda$ be the family of all finite subsets of $\Omega$. We need to find a suitable ultrafilter $\mathcal{U}$ over $\Lambda$ in a way that, if $\mathbb{F} = \mathbb{R}^\Lambda / \mathcal{U}$ is the ordered field obtained as the ultrapower of $\mathbb{R}$ modulo $\mathcal{U}$, the numerosity defined by by posing

$$n(X) = \langle |X \cap \lambda| : \lambda \in \Lambda \rangle_\mathcal{U}$$

satisfies the desired properties.
The main result (2)

**Theorem**

Let $\mathcal{A}$ be a ring of subsets of $\Omega$ and let $\mu : \mathcal{A} \to [0, +\infty]_\mathbb{R}$ be a non-atomic pre-measure. Then, along with the associated outer measure $\overline{\mu}$, there exists an “inner” finitely additive measure $\underline{\mu} : \mathcal{P}(\Omega) \to [0, +\infty]_\mathbb{R}$ such that:

1. There exists an elementary numerosity $n : \mathcal{P}(\Omega) \to \mathbb{F}$ such that $\underline{\mu} = n_\beta$ for every positive $\beta$ of the form $\beta = \frac{n(A^*)}{\mu(A^*)}$.
2. $\underline{\mu}(C) = \overline{\mu}(C)$ for all $C \in \mathcal{C}_\mu$, the Caratheodory $\sigma$-algebra associated to $\mu$.
3. $\underline{\mu}(X) \leq \overline{\mu}(X)$ for all $X \subseteq \Omega$. 
An application to Lebesgue measure

**Corollary**

Let \((\mathbb{R}, \mathcal{L}, \mu_L)\) be the Lebesgue measure over \(\mathbb{R}\). Then there exists an elementary numerosity \(n : \mathcal{P}(\mathbb{R}) \to \mathbb{F}\) such that:

- \(n([x, x + a)) = n([y, y + a))\) for all \(x, y \in \mathbb{R}\) and for all \(a > 0\).
- \(n([x, x + a)) = a \cdot n([0, 1))\) for all rational numbers \(a > 0\).
- \(sh \left( \frac{n(X)}{n([0,1]))} \right) = \mu_L(X)\) for all \(X \in \mathcal{L}\).
- \(sh \left( \frac{n(X)}{n([0,1]))} \right) \leq \overline{\mu}_L(X)\) for all \(X \subseteq \mathbb{R}\).
Some ideas for further research

- From Lebesgue measure to Lebesgue integral;
- representing more measures with the same numerosity (e.g. Hausdorff measures);
- applications to (non-archimedean) probability.