Elementary numerosity and measures

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Finitely additive measures

Definition

A *finitely additive measure* is a triple $(\Omega, \mathfrak{A}, \mu)$ where:

- The *space* Ω is a non-empty set;
- $\mathfrak{A}$ is a *ring of sets* over Ω, *i.e.* a non-empty family of subsets of Ω satisfying the conditions: $A, B \in \mathfrak{A} \Rightarrow A \cup B, A \cap B, A \setminus B \in \mathfrak{A}$;
- $\mu : \mathfrak{A} \to [0, +\infty]_{\mathbb{R}}$ is an *additive function*, *i.e.* $\mu(A \cup B) = \mu(A) + \mu(B)$ whenever $A, B \in \mathfrak{A}$ are disjoint. We also assume that $\mu(\emptyset) = 0$.

The measure $(\Omega, \mathfrak{A}, \mu)$ is called *non-atomic* when all finite sets in $\mathfrak{A}$ have measure zero.

Elementary numerosities

Definition

An *elementary numerosity* on a set Ω is a function

$$\mathfrak{n} : \mathcal{P}(\Omega) \to [0, +\infty)_{\mathbb{F}}$$

defined for all subsets of Ω, taking values into the non-negative part of a non-archimedean field $\mathbb{F}$, and satifying the conditions:

- $\mathfrak{n}(\{x\}) = 1$ for every point $x \in \Omega$;
- $\mathfrak{n}(A \cup B) = \mathfrak{n}(A) + \mathfrak{n}(B)$ whenever A and B are disjoint.

Numerosity measures

Proposition

Let $\mathfrak{n} : \mathcal{P}(\Omega) \to [0, +\infty)_{\mathbb{F}}$ *be an elementary numerosity, and for every* $\beta > 0$ *in* $\mathbb{F}$ *define the function* $\mathfrak{n}_\beta : \mathcal{P}(\Omega) \to [0, +\infty]_{\mathbb{R}}$ *by posing*

$$\mathfrak{n}_\beta(A) \; = \; sh\left(\frac{\mathfrak{n}(A)}{\beta}\right).$$

Then $\mathfrak{n}_\beta$ *is a finitely additive measure defined for all subsets of* Ω. *Moreover,* $\mathfrak{n}_\beta$ *is non-atomic if and only if* β *is an infinite number.*

The main result (1)

Theorem

Let $(\Omega, \mathfrak{A}, \mu)$ be a non-atomic finitely additive measure. Then there exist

- *a non-archimedean field $\mathbb{F} \supseteq \mathbb{R}$;*
- *an elementary numerosity $\mathfrak{n} : \mathcal{P}(\Omega) \to [0, +\infty)_{\mathbb{F}}$;*

such that for every positive number of the form $\beta = \frac{\mathfrak{n}(A^*)}{\mu(A^*)}$ one has

$$\mu(A) \ = \ \mathfrak{n}_\beta(A) \text{ for all } A \in \mathfrak{A}.$$

Moreover, if $\mathfrak{B} \subseteq \mathfrak{A}$ is a subring whose non-empty sets have all positive measure, then we can also assume that

$$\mathfrak{n}(B) \ = \ \mathfrak{n}(B') \ \text{ for all } B, B' \in \mathfrak{B} \text{ such that } \mu(B) = \mu(B').$$

Idea of the proof

Let Λ be the family of all finite subsets of Ω. We need to find a suitable ultrafilter $\mathcal{U}$ over Λ in a way that, if $\mathbb{F} = \mathbb{R}^{\Lambda}/\mathcal{U}$ is the ordered field obtained as the ultrapower of $\mathbb{R}$ modulo $\mathcal{U}$, the numerosity defined by by posing

$$\mathfrak{n}(X) \; = \; \langle |X \cap \lambda| : \lambda \in \Lambda \rangle_{\mathcal{U}}$$

satisfies the desired properties.

The main result (2)

Theorem

Let $\mathfrak{A}$ be a ring of subsets of Ω and let $\mu : \mathfrak{A} \to [0, +\infty]_{\mathbb{R}}$ be a non-atomic pre-measure. Then, along with the associated outer measure $\overline{\mu}$, there exists an "inner" finitely additive measure

$$\underline{\mu} : \mathcal{P}(\Omega) \to [0, +\infty]_{\mathbb{R}}$$

such that:

1. *There exists an elementary numerosity $\mathfrak{n} : \mathcal{P}(\Omega) \to \mathbb{F}$ such that $\underline{\mu} = \mathfrak{n}_\beta$ for every positive β of the form $\beta = \frac{\mathfrak{n}(A^*)}{\mu(A^*)}$.*
2. *$\underline{\mu}(C) = \overline{\mu}(C)$ for all $C \in \mathfrak{C}_\mu$, the Caratheodory σ-algebra associated to μ.*
3. *$\underline{\mu}(X) \leq \overline{\mu}(X)$ for all $X \subseteq \Omega$.*

An application to Lebesgue measure

Corollary

Let $(\mathbb{R}, \mathfrak{L}, \mu_L)$ *be the Lebesgue measure over* $\mathbb{R}$. *Then there exists an elementary numerosity* $\mathfrak{n} : \mathcal{P}(\mathbb{R}) \to \mathbb{F}$ *such that:*

- $\mathfrak{n}([x, x+a)) = \mathfrak{n}([y, y+a))$ *for all* $x, y \in \mathbb{R}$ *and for all* $a > 0$.
- $\mathfrak{n}([x, x+a)) = a \cdot \mathfrak{n}([0,1))$ *for all rational numbers* $a > 0$.
- $sh\left(\frac{\mathfrak{n}(X)}{\mathfrak{n}([0,1))}\right) = \mu_L(X)$ *for all* $X \in \mathfrak{L}$.
- $sh\left(\frac{\mathfrak{n}(X)}{\mathfrak{n}([0,1))}\right) \leq \overline{\mu}_L(X)$ *for all* $X \subseteq \mathbb{R}$.

Some ideas for further research

- From Lebesgue measure to Lebesgue integral;
- representing more measures with the same numerosity (e.g. Hausdorff measures);
- applications to (non-archimedean) probability.