THE EIGHTFOLD PATH TO NONSTANDARD ANALYSIS

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Abstract. This paper consists of a quick introduction to the "hyper-methods" of nonstandard analysis, and of a review of eight different approaches to the subject, which have been recently elaborated by the authors.

Those who follow the noble Eightfold Path are freed from the suffering and are led ultimately to Enlightenment.

(Gautama Buddha)

Contents

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Introduction. Since the original works [42, 43] by Abraham Robinson, many diffent presentations to the methods of nonstandard analysis have been proposed over the last forty years. The task of combining in a satisfactory manner rigorous theoretical foundations with an easily accessible exposition soon revealed very difficult to be accomplished. The first pioneering work in this direction was W.A.J. Luxemburg's lecture notes [38]. Based on a direct use of the ultrapower construction, those notes were very popular in the "nonstandard" community in the sixties. Also Robinson himself gave a contribution to the sake of simplification, by reformulating his initial type-theoretic approach in a more familiar set-theoretic framework. Precisely, in his joint work with E. Zakon [45], he introduced the *superstructure approach*, by now the most used foundational framework.

To the authors' knowledge, the first relevant contribution aimed to make the "hyper-methods" available even at a freshman level, is Keisler's book [35], which is a college textbook for a first course of elementary calculus. There, the principles of nonstandard analysis are presented axiomatically in a nice and elementary form (see the accompanying book [34] for the foundational aspects). Among the more recent works, there are the "gentle" introduction by W. Henson [28], R. Goldblatt's lectures on the hyperreals [27], and K. Stroyan's textbook [49].

Recently the authors investigated several different frameworks in algebra, topology, and set theory, that turn out to incorporate explicitly or implicitly the "hyper-methods". These approaches show that nonstandard extensions naturally arise in several quite different contexts of mathematics. An interesting phenomenon is that some of those approaches lead in a straightforward manner to ultrafilter properties that are independent of the axioms of Zermelo-Fraenkel set theory ZFC.

Contents. This article is divided into two parts. The first part consists of an introduction to the hyper-methods of nonstandard analysis, while the second one is an overview of eight different approaches to the subject recently elaborated by the authors. Most proofs are omitted, but precise references are given where the interested reader can find all details.

Part I contains two sections. The longest Section 1 is a soft introduction to the basics of nonstandard analysis, and will be used as a reference for the remaing sections of this article. The three fundamental "hypertools" are presented, namely the *star-map*, the *transfer principle*, and the *saturation property*, and several examples are given to illustrate their use in the practice. The material is intentionally presented in an elementary (and sometimes semi-formal) manner, so that it may also serve as a quick presentation of nonstandard analysis for newcomers. Section 2 is focused on the connections between the hyper-extensions of nonstandard analysis and ultrapowers. In particular, a useful characterization of the models of hyper-methods is presented in purely algebraic terms, by means of limit ultrapowers.

Each of the eight Sections 3–10 in Part II presents a different possible "path" to nonstandard analysis. The resulting eight approaches, although not strictly equivalent to each other, are all suitable for the practice, in that each of them explicitly or implicitly incorporates the fundamental "hypertools" introduced in Section 1.

Section 3 is about a modified version of the so-called *superstructure approach*, where a single superstructure is considered both as the standard and the nonstandard universe (see [3].) In Section 4, we present a purely algebraic approach presented in [7, 8], which is based on the existence of a "special" ring homomorphism. Starting from such a homomorphism, we define in a direct manner a superstructure model of the hyper-methods, as defined in Section 3.

In Section 5, the axiomatic theory *ZFC of [18] is presented, that can be seen as an extension of the superstructure approach to the full generality of set theory. Section 6 is dedicated to the so-called *Alpha Theory*, an axiomatic presentation that postulates five elementary properties for an "ideal" (infinite) natural number α (see [5].) These axioms suffice for defining a star-map on the universal class of all mathematical objects.

Section 7 deals with *topological extensions*, a sort of "topological completions" of a given set X, introduced and studied in [10, 20]. These structures are spaces ${}^\star X$ where any function $f : X \to X$ has a continuous $\star$-extension, and where the $\star$-extension ${}^\star A$ of a subset $A \subseteq X$ is simply its closure in ${}^\star X$. Hyper-extensions of nonstandard analysis, endowed with a natural topology, are characterized as those topological extensions that satisfy two simple additional properties. Moreover, several important features of nonstandard extensions, such as the enlarging and saturation properties, can be naturally described in this topological framework. Section 8, following [26], further simplifies the topological approach of the preceding section. By assuming that the $\star$-extensions of unary functions satisfy three simple "preservation properties" having a purely functional nature, one obtains all possible hyper-extensions of nonstandard analysis.

Section 9 deals with natural ring structures that can be given to suitable subspaces of $\beta\mathbb{Z}$, the Stone-Čech compactification of the integers $\mathbb{Z}$ (see [22].) Such rings turn out to be sets of hyperintegers with special properties that are independent of ZFC. In the final Section 10, we consider a new way of counting that has been proposed in [6] and which maintains the ancient principle that "the whole is larger than its parts". This counting procedure is suitable for all those countable sets whose elements are "labelled" by natural numbers. We postulate that this procedure satisfies three natural "axioms of compatibility" with respect to inclusion, disjoint union, and Cartesian product. As a consequence, sums and products of numerosities can be defined, and the resulting semi-ring of numerosities becomes a special set of hypernatural numbers, whose existence is independent of ZFC.

Disclaimer. A disclaimer is in order. By no means the approaches presented here have been choosen because they are better than others, or because they provide an exhaustive picture of this field of research. Simply, this article surveys the authors' contributions to the subject over the last decade. In particular, throughout the paper we stick to the so-called *external* viewpoint of nonstandard methods, based on the existence of a star-map $*$ providing an hyper-extension $^*A$ for each standard object A. This is to be confronted with the *internal* approach of Nelson's IST [39], and other related nonstandard set theories where the *standard predicate* `st` is used in place of the star-map (cf. e.g. the recent book [32]; see also Hrbàček's article in this volume). Extensive treatments of nonstandard analysis based on the internal approach are given e.g. in the books [23, 24, 41].

Part I – The "Hyper-methods"

§1. What are the "hyper-methods"? Roughly, nonstandard analysis essentially consists of two fundamental tools: the *star-map* $*$ and the *transfer principle*. In most applications, a third fundamental tool is also considered, namely the *saturation property*.

There are several different frameworks where the methods of nonstandard analysis (the "hyper-methods") can be presented. The goal of this section is to introduce the basic notions in such a way that their formulations do not depend on the specific approach that one is adopting. Of course, there is a price we have to pay to reach this generality. Sometimes, the definitions as given here are not entirely formalized (at least from the point of view of a logician). However we are confident that they are still sufficiently clear and unumbiguous to the point that some "practitioners" may find them suitable already. To reassure the suspicious reader, we anticipate that each of the eight Sections 3–10 consists of a specific approach where all notions presented here are given rigorous foundations.

Besides the fundamental tools, this section also contains the definition of internal element, sketchy proofs of the first consequences of the definitions, as well as a bunch of relevant examples. It is not a complete introduction (e.g. overspill and hyperfinite sets are not treated), but it may be used as a first reading for beginners interested in nonstandard analysis.

1.1. The basic definitions. In order to correctly formulate the fundamental tools of hyper-methods, we need the following

Definition 1.1. A *universe* $\mathbb{U}$ is a nonempty collection of "mathematical objects" that is closed under subsets (i.e. $a \subseteq A \in \mathbb{U} \Rightarrow a \in \mathbb{U}$) and closed under the basic mathematical operations. Precisely, whenever $A, B \in \mathbb{U}$, we require that also the *union* $A \cup B$, the *intersection* $A \cap B$, the *set-difference* $A \setminus B$, the *ordered pair* (A, B), the *Cartesian product* $A \times B$, the *powerset* $\mathcal{P}(A) = \{a \mid a \subseteq A\}$, the *function-set* $B^A = \{f \mid f : A \to B\}$, all belong to $\mathbb{U}$.[1] A universe $\mathbb{U}$ is also assumed to contain (copies of) all sets of numbers $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C} \in \mathbb{U}$, and to be *transitive*, i.e. members of members of $\mathbb{U}$ belong to $\mathbb{U}$ (in formulæ: $a \in A \in \mathbb{U} \Rightarrow a \in \mathbb{U}$).

[1] Clearly, here we implicitly assume that A and B are sets, otherwise these operation don't make sense. The only exception is the ordered pair, that makes sense for *all* mathematical objects A and B.

The notion of "mathematical object" includes all objects used in the ordinary practice of mathematics, namely: numbers, sets, functions, relations, ordered tuples, Cartesian products, *etc.* It is well-known that all these notions can be defined as sets and formalized in the foundational framework of Zermelo-Fraenkel axiomatic set theory ZFC.[2] For sake of simplicity, here we consider them as primitive concepts not necessarily reduced to sets.

Hyper-Tool # 1: STAR-MAP.
The star-map is a function $* : \mathbb{U} \to \mathbb{V}$ *between two universes that associates to each object* $A \in \mathbb{U}$ *its hyper-extension (or nonstandard extension)* ${}^*A \in \mathbb{V}$*. It is also assumed that* ${}^*n = n$ *for all natural numbers* $n \in \mathbb{N}$*, and that the* properness *condition* ${}^*\mathbb{N} \neq \mathbb{N}$ *holds.*

It is customary to call *standard* any object $A \in \mathbb{U}$ in the domain of the star-map, and *nonstandard* any object $B \in \mathbb{V}$ in the codomain. The adjective standard is also often used in the literature for hyper-extensions ${}^*A \in \mathbb{V}$.

We remark rightaway that one could directly consider a single universe $\mathbb{U} = \mathbb{V}$. Doing so, the traditional distinction between standard and nonstandard objects is overcome.[3] We point out that in all approaches appeared in the literature, the standard universe is taken to be large enough so as to include all mathematical objects under consideration.

We are now ready to introduce the second powerful tool of nonstandard methods. It states that the star-map preserves a large class of properties.

Hyper-Tool # 2: TRANSFER PRINCIPLE.
Let $P(a_1, \ldots, a_n)$ *be a property of the standard objects* $a_1, \ldots, a_n$ *expressed as an "elementary sentence". Then* $P(a_1, \ldots, a_n)$ *is true if and only if the same sentence is true about the corresponding hyper-extensions* ${}^*a_1, \ldots, {}^*a_n$*. That is:*

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[2] E.g. in ZFC, an ordered pair (a, b) is defined as the *Kuratowski pair* $\{\{a\}, \{a, b\}\}$; an n-tuple is inductively defined by $(a_1, \ldots, a_n, a_{n+1}) = ((a_1, \ldots, a_n), a_{n+1})$; an n-place relation R on A is identified with the set $R \subseteq A^n$ of n-tuples that satisfy it; a function $f : A \to B$ is identified with its graph $\{(a, b) \in A \times B \mid b = f(a)\}$; and so forth. As for numbers, complex numbers $\mathbb{C} = \mathbb{R} \times \mathbb{R}/\approx$ are defined as equivalence classes of ordered pairs of real numbers, and the real numbers $\mathbb{R}$ are defined as equivalence classes of suitable sets of rational numbers (namely, Dedekind cuts or Cauchy sequences). The rational numbers $\mathbb{Q}$ are a suitable quotient $\mathbb{Z} \times \mathbb{Z}/\approx$, and the integers $\mathbb{Z}$ are in turn a suitable quotient $\mathbb{N} \times \mathbb{N}/\approx$. The natural numbers of ZFC are defined as the set ω of von Neumann naturals: $0 = \emptyset$ and $n + 1 = n \cup \{n\}$ (so that each natural number $n = \{0, 1, \ldots, n - 1\}$ is identified with the set of its predecessors.) We remark that these definitions are almost compulsory in order to obtain a set theoretic reductionist foundation, but certainly they are not needed in the ordinary development of analysis.

[3] This matter will be discussed in Section 3 (see Definition 3.3) and Section 5.

$$P(a_1, \ldots, a_n) \Longleftrightarrow P({}^*a_1, \ldots, {}^*a_n)$$

The *transfer principle* (also known as *Leibniz principle*) is given a rigorous formulation by using the formalism of mathematical logic and, in particular, by appealing to the notion of *bounded quantifier formula* in the first-order language of set theory. Here we only give a semi-formal definition, and refer the reader to §4.4 of [13] for a fully rigorous treatment.

DEFINITION 1.2. We say that a property $P(x_1, \ldots, x_n)$ of the objects $x_1, \ldots, x_n$ is expressed as an *elementary sentence* if the following two conditions are fulfilled:

(1) Besides the usual logic connectives ("not", "and", "or", "if ... then", "if and only if") and quantifiers ("there exists", "for all"), only the basic notions of function, value of a function at a given point, relation, domain, codomain, ordered n-tuple, i-th component of an ordered tuple, and membership $\in$, are involved.
(2) The scopes of all universal quantifiers $\forall$ ("for all") and existential quantifiers $\exists$ ("there exists") are "bounded" by some set.

A quantifier is *bounded* when it occurs in the form "for every $x \in X$" or "there exists $y \in Y$", for some specified sets X, Y. Thus, in order to correctly apply the *transfer principle*, one has to stick to the following rule.

> *Rule of the thumb.* Whenever considering quantifiers: "$\forall x \ldots$" or "$\exists y \ldots$", we must always specify the range of the variables, i.e. we must specify sets X and Y and reformulate: "$\forall x \in X \ldots$" and "$\exists y \in Y \ldots$". In particular, all quantifications on subsets: "$\forall x \subseteq X \ldots$" or "$\exists x \subseteq X \ldots$", must be reformulated in the form "$\forall x \in \mathcal{P}(X) \ldots$" and "$\exists x \in \mathcal{P}(X) \ldots$" respectively, where $\mathcal{P}(X)$ is the powerset of X. Similarly, all quantifications on functions $f : A \to B$, must be bounded by B^A, the set of all functions from A to B.

We are now ready to give the

FUNDAMENTAL DEFINITION:

A model of hyper-methods (or a model of nonstandard analysis) is a triple $\langle * ; \mathbb{U} ; \mathbb{V} \rangle$ *where* $* : \mathbb{U} \to \mathbb{V}$ *is a star-map satisfying the transfer principle.*

1.2. Some applications of transfer. We now show a few simple applications of the *transfer principle*, aimed to clarify the crucial notion of elementary sentence.

EXAMPLE 1.3. By condition (1) of Definition 1.2, the following are all elementary sentences: "f is a function with domain A and codomain B";

"b is the value taken by f at the point a"; "R in an n-place relation on A"; "C is the Cartesian product of A and B". Thus by *transfer*, we get that "${}^*f : {}^*A \to {}^*B$ is a function with domain ${}^*A$ and codomain ${}^*B$"; "${}^*b = {}^*f({}^*a)$ is the value taken by ${}^*f$ at the point ${}^*a$", i.e. ${}^*(f(a)) = {}^*f({}^*a)$; "${}^*R$ is an n-place relation on ${}^*A$"; and "${}^*C = {}^*A \times {}^*B$ is the Cartesian product of ${}^*A$ and ${}^*B$".

Example 1.4. The inclusion and all basic operations on sets are preserved under the star-map, with the only relevant exceptions of the powerset and the function-set (see Example 1.9 below). In fact the properties: "$A \subseteq B$"; "$C = A \cup B$"; "$C = A \cap B$"; and "$C = A \setminus B$" can all be formulated as elementary sentences. For instance, "$A \subseteq B$" means that "$\forall x \in A. x \in B$", *etc.* By *transfer* we obtain that "${}^*A \subseteq {}^*B$"; "${}^*C = {}^*A \cup {}^*B$"; "${}^*C = {}^*A \cap {}^*B$"; and "${}^*C = {}^*A \setminus {}^*B$".

Example 1.5. Let $f : A \to B$ be any given standard function. Then the *images* $f(A') = \{f(a) \mid a \in A'\}$ of subsets $A' \subseteq A$, and the *preimages* $f^{-1}(B') = \{a \in A \mid f(a) \in B'\}$ of subsets $B' \subseteq B$, are both preserved under the star-map, i.e. ${}^*(f(A')) = {}^*f({}^*A')$ and ${}^*(f^{-1}(B')) = {}^*f^{-1}({}^*B')$. In particular, ${}^*\mathrm{Range}(f) = \mathrm{Range}({}^*f)$, and so f is onto if and only if ${}^*f$ is. It is also easily shown that f is 1-1 if and only if ${}^*f$ is.

Two more relevant properties are the following: ${}^*\{a \in A \mid f(a) = g(a)\} = \{\alpha \in {}^*A \mid {}^*f(\alpha) = {}^*g(\alpha)\}$, and ${}^*\mathrm{Graph}(f) = \mathrm{Graph}({}^*f)$. All these properties are proved by direct applications of the *transfer principle*. E.g. the last equality is proved by transferring the elementary sentence:

"$x \in \mathrm{Graph}(f)$ if and only if there exist $a \in A$ and $b \in B$ such that $b = f(a)$ and $x = (a, b)$".

Example 1.6. Let A be a nonempty standard set, and consider the property: "$<$ is a linear ordering on A". Notice first that $<$ is a binary relation, hence ${}^*\!<$ is a binary relation on ${}^*A$. By definition, $<$ is a linera ordering if and only if it satisfies the following three properties, that are expressed by means of bounded quantifiers.

$$\forall x \in A\ (x \not< x)$$
$$\forall x, y, z \in A\ (x < y \text{ and } y < z) \Rightarrow x < z$$
$$\forall x, y \in A\ (x < y \text{ or } y < x \text{ or } x = y)$$

Then we can apply the *transfer principle* and get that "${}^*\!<$ is a linear ordering on ${}^*A$".

Example 1.7. It directly follows from condition (1) of Definition 1.2 that the hyper-extension of an n-tuple of standard objects $A = (a_1, \dots, a_n)$ is ${}^*A = ({}^*a_1, \dots, {}^*a_n)$. Similarly, if $A = \{a_1, \dots, a_n\}$ is a finite set of standard objects, then its star-extension is ${}^*A = \{{}^*a_1, \dots, {}^*a_n\}$. This fact is proved by applying *transfer* to the following elementary sentence:

"$a_1 \in A$ and ... and $a_n \in A$ and for all $x \in A$, $x = a_1$ or ... or $x = a_n$"

Notice that for every standard set A, $\{{}^*a \mid a \in A\} \subseteq {}^*A$ (apply *transfer* to all sentences "$a \in A$"). In the last example we have seen that the inclusion is actually an equality when A is finite. But this is never the case when A is infinite, as a consequence of the *properness condition* ${}^*\mathbb{N} \neq \mathbb{N}$.

PROPOSITION 1.8. *Let A be an infinite standard set A. Then the inclusion $\{{}^*a \mid a \in A\} \subset {}^*A$ is proper.*

PROOF. Fix a standard map $f : A \to \mathbb{N}$ which is onto. Then ${}^*f : {}^*A \to {}^*\mathbb{N}$ is onto as well. Now assume by contradiction that all elements in ${}^*A$ are of the form ${}^*a$ for some $a \in A$. Then:

$$ {}^*\mathbb{N} = \{{}^*f({}^*a) \mid a \in A\} = \{{}^*(f(a)) \mid a \in A\} = \{{}^*n \mid n \in \mathbb{N}\} = \mathbb{N}, $$

against the properness condition ${}^*\mathbb{N} \neq \mathbb{N}$. ⊣

EXAMPLE 1.9. Let A and B be any standard sets. By transferring the sentences: "$\forall x \in \mathcal{P}(A),\ \forall y \in x,\ y \in A$" and "$\forall f \in B^A,\ f$ is a function with domain A and codomain B", it is proved that ${}^*\mathcal{P}(A) \subseteq \mathcal{P}({}^*A)$, and ${}^*(B^A) \subseteq {}^*B^{{}^*A}$, respectively. Arguing similarly as in Example 1.7, one easily shows that these inclusions are equalities whenever both A and B are finite. In the infinite case, the inclusions are proper (cf. Proposition 1.25).

1.3. The basic sets of hypernumbers. Let us now concentrate on the hyper-extensions of sets of numbers.

DEFINITION 1.10. The elements of ${}^*\mathbb{N}$, ${}^*\mathbb{Z}$, ${}^*\mathbb{Q}$, ${}^*\mathbb{R}$ and ${}^*\mathbb{C}$ are called *hypernatural*, *hyperinteger*, *hyperrational*, *hyperreal*, and *hypercomplex* numbers, respectively.

Besides natural numbers, for convenience it is also customary to assume that ${}^*z = z$ for *all* numbers z. In this case, we have the inclusions $\mathbb{N} \subset {}^*\mathbb{N}$, $\mathbb{Z} \subset {}^*\mathbb{Z}$, $\mathbb{Q} \subset {}^*\mathbb{Q}$, $\mathbb{R} \subset {}^*\mathbb{R}$, and $\mathbb{C} \subset {}^*\mathbb{C}$ (the inclusions are proper by Proposition 1.8). Whenever confusion is unlikely, some asterisks will be omitted. For instance, we shall use the same symbols $+$ and $\cdot$ to denote both the sum and product operations on $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ and the corresponding operations defined on the hyper-extensions ${}^*\mathbb{N}, {}^*\mathbb{Z}, {}^*\mathbb{Q}, {}^*\mathbb{R}, {}^*\mathbb{C}$. Similarly for the ordering $\leq$.

In the next proposition we itemize the first properties of hypernumbers, all obtained as straighforward applications of the *transfer principle*.

PROPOSITION 1.11.

1. *${}^*\mathbb{Z}$ is a commutative ring, ${}^*\mathbb{Q}$ is an ordered field, ${}^*\mathbb{R}$ is a real-closed field, and ${}^*\mathbb{C}$ is an algebraically closed field;*[4]

[4] Recall that an ordered field is *real-closed* if every positive element is a square, and every polynomial of odd degree has a root. A field is *algebraically closed* if all non-constant polynomials have a root.

2. *Every non-zero hypernatural number* $\nu \in {}^*\mathbb{N}$ *has a successor* $\nu+1$ *and a predecessor* $\nu - 1$;[5]
3. $(\mathbb{N}, \leq)$ *is an initial segment of* $({}^*\mathbb{N}, \leq)$, *i.e. if* $\nu \in {}^*\mathbb{N} \setminus \mathbb{N}$, *then* $\nu > n$ *for all* $n \in \mathbb{N}$;
4. *For every positive* $\xi \in {}^*\mathbb{R}$ *there exists a unique* $\nu \in {}^*\mathbb{N}$ *such that* $\nu \leq \xi < \nu + 1$. *In particular,* ${}^*\mathbb{N}$ *is unbounded in* ${}^*\mathbb{R}$;
5. *The hyperrational numbers* ${}^*\mathbb{Q}$, *as well as the hyperirrational numbers* ${}^*(\mathbb{R} \setminus \mathbb{Q}) = {}^*\mathbb{R} \setminus {}^*\mathbb{Q}$, *are dense in* ${}^*\mathbb{R}$;[6]
6. *Let* Z *be any of the sets* $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ *or* $\mathbb{R}$, *and consider the open interval* $(a, b) = \{x \in X \mid a < x < b\}$ *determined by numbers* $a < b$ *in* Z. *Then the hyper-extension* ${}^*(a, b) = \{\xi \in {}^*Z \mid a < \xi < b\}$. *Similar equalities also hold for intervals of the form* $[a, b), (a, b], (a, b), (-\infty, b]$ *and* $[a, +\infty)$.

As a consequence of property (3) above, the elements of ${}^*\mathbb{N} \setminus \mathbb{N}$ are called *infinite*. More generally:

DEFINITION 1.12. A hyperreal number $\xi \in {}^*\mathbb{R}$ is *infinite* if either $\xi > \nu$ or $\xi < -\nu$ for some $\nu \in {}^*\mathbb{N} \setminus \mathbb{N}$. Otherwise we say that ξ is *finite*. We call *infinitesimal* those hyperreal numbers $\varepsilon \in {}^*\mathbb{R}$ such that $-r < \varepsilon < r$ for all positive reals $r \in \mathbb{R}$. In this case we write $\varepsilon \sim 0$.

The following properties are easily seen:[7] $\varepsilon \neq 0$ is infinitesimal if and only if its reciprocal $1/\varepsilon$ is infinite; if ξ and ζ are finite, then also $\xi + \zeta$ and $\xi \cdot \zeta$ are finite; if $\varepsilon, \eta \sim 0$, then also $\varepsilon + \eta \sim 0$; if $\varepsilon \sim 0$ and ξ is finite, then $\varepsilon \cdot \xi \sim 0$; if ω is infinite and ξ is *not* infinitesimal, then $\omega \cdot \xi$ is infinite; if $\varepsilon \neq 0$ is infinitesimal but ξ is *not* infinitesimal, then ξ/ε is infinite; if ω is infinite and ξ is finite, then $\xi/\omega \sim 0$; *etc.*
Infinitesimal and infinite numbers can be seen as formalizations of the intuitive ideas of "small" number and "large" number, respectively. Also the idea of "closeness" can be formalized as follows.

DEFINITION 1.13. The hyperreal numbers ξ and ζ are *infinitely close* if $\xi - \zeta$ is infinitesimal. In this case, we write $\xi \sim \zeta$.

Clearly, $\sim$ is an equivalence relation. The completeness of the real numbers $\mathbb{R}$ yields the following result.

THEOREM 1.14 (Standard part). *For every finite* $\xi \in {}^*\mathbb{R}$, *there exists a unique real number* $r \in \mathbb{R}$ *(called the "standard part" of* ξ*) such that* $\xi \sim r$.

[5] We say that ξ' is the *successor* of ξ (or ξ is the *predecessor* of ξ') if $\xi < \xi'$ and there exist no elements ζ such that $\xi < \zeta < \xi'$.
[6] I.e., for all $\xi < \zeta$ in ${}^*\mathbb{R}$, there exist $x \in {}^*\mathbb{Q}$ and $y \in {}^*\mathbb{R} \setminus {}^*\mathbb{Q}$ such that $\xi < x, y < \zeta$.
[7] In fact, they hold in any *non-archimedean* field (the archimedean property is defined in Example 1.18).

PROOF. The least upper bound $r = \sup\{a \in \mathbb{R} \mid a \leq \xi\}$ has the desired property. $\dashv$

The next interesting result shows that in a way the hyperrationals already "incorporate" the real numbers (see e.g. [48] Thm. 4.4.4 and [15] Ch.II Thm. 2).

THEOREM 1.15. *Let* ${}^*\mathbb{Q}_b$ *be the ring of finite hyperrationals, and let* $\mathfrak{I}$ *be the maximal ideal of its infinitesimals. Then* $\mathbb{R}$ *and* ${}^*\mathbb{Q}_b/\mathfrak{I}$ *are isomorphic as ordered fields.*

1.4. Correctly applying the transfer principle. From the examples presented so far, one might (wrongly) guess that applying the *transfer principle* merely consists in putting asterisks $*$ all over the place. It is not so, because – as we already pointed out – only elementary sentences can be transferred. We now give three relevant examples aimed to clarify this matter.

EXAMPLE 1.16. Recall the *well-ordering* property of $\mathbb{N}$:

"Every nonempty subset of $\mathbb{N}$ *has a least element".*

By applying the *transfer principle* to this formulation, we would get that "Every nonempty subset of ${}^*\mathbb{N}$ has a least element". But this is clearly false (e.g. the collection ${}^*\mathbb{N} \setminus \mathbb{N}$ of infinite hypernaturals has no least element, because if ν is infinite, then $\nu - 1$ is infinite as well). We reached a wrong conclusion because we transferred a sentence which is *not* elementary (the universal quantifier is not bounded). However, we can easily overcome this problem by reformulating the well-ordering property as the following elementary sentence: "*Every nonempty element of* $\mathcal{P}(\mathbb{N})$ *has a least element*", where $\mathcal{P}(\mathbb{N})$ is the powerset of $\mathbb{N}$. (Notice that the property "X has a least element" is elementary, because it means: "there exists $x \in X$ such that for all $y \in X$, $x \leq y$".) We can now correctly apply the *transfer* and get: "*Every nonempty element of* ${}^*\mathcal{P}(\mathbb{N})$ *has a least element*", where it is intended that the ordering on ${}^*\mathbb{N}$ is the hyper-extension of the ordering on $\mathbb{N}$. The crucial point here is that ${}^*\mathcal{P}(\mathbb{N})$ is *properly included* in $\mathcal{P}({}^*\mathbb{N})$ (see Proposition 1.25 below).

EXAMPLE 1.17. Recall the *completeness property* of real numbers:

"Every nonempty subset of $\mathbb{R}$ *which is bounded above, has a l.u.b."*

As in the previous example, if we directly apply *transfer* to this formulation, we reach a false conclusion, namely: "Every nonempty subset of ${}^*\mathbb{R}$ which is bounded above, has a l.u.b." (e.g. the set of infinitesimals is bounded above but has no least upper bound). Again, the problem is that the sentence above is *not* elementary because it contains a quantification over subsets. To fix the problem, we simply have to consider the powerset $\mathcal{P}(\mathbb{R})$

and reformulate: "*Every nonempty element of* $\mathcal{P}(\mathbb{R})$ *which is bounded above has a l.u.b.*". Thus, by the *transfer principle*, we have a least upper bound for each upper-bounded element of ${}^*\mathcal{P}(\mathbb{R})$ (which is a *proper* subset of $\mathcal{P}({}^*\mathbb{R})$, see Proposition 1.25 below).

As suggested by the last examples, restricting to elementary sentences is not a limitation, because virtually all mathematical
properties can be equivalently rephrased in elementary terms.

Another delicate aspect that needs some caution, is the possibility of misreading a transferred sentence, once all asterisks $*$ have been put in the right place. A relevant example is given by the archimedean property.

EXAMPLE 1.18. The *archimedean property* of real numbers can be expressed in this elementary form:

"*For all positive* $x \in \mathbb{R}$*, there exists* $n \in \mathbb{N}$ *such that* $n \cdot x > 1$".

By *transfer*, we obtain: "*For all positive* $\xi \in {}^*\mathbb{R}$*, there exists* $\nu \in {}^*\mathbb{N}$ *such that* $\nu \cdot \xi > 1$". Notice that this sentence does *not* express the archimedean property of ${}^*\mathbb{R}$, because the element ν could be an *infinite* hypernatural.

Clearly, the hyperreal field ${}^*\mathbb{R}$ is not archimedean (in fact, an ordered field is non-archimedean if and only if it contains non-zero infinitesimals). In particular $\mathbb{R}$ and ${}^*\mathbb{R}$ are not isomorphic. We remark that this phenomenon of non-isomorphic mathematical structures that cannot be distinguished by any elementary sentence, is indeed the very essence of nonstandard analysis (and more generally, of model theory, a branch of mathematical logic).

1.5. Internal elements. We now introduce a fundamental class of objects in nonstandard analysis.

DEFINITION 1.19. An *internal* object is any element $\xi \in {}^*X$ belonging to some hyper-extension ${}^*X$. An element $\xi \in \mathbb{V}$ of the nonstandard universe is *external* if it is not internal.

Notice that all hyper-extensions ${}^*X$ are internal, because e.g. ${}^*X \in {}^*Y$, where $Y = \{X\}$ is the singleton of X. We remark that in most foundational approaches proposed in the literature, the collection of internal objects is assumed to be *transitive*, i.e. if $b \in B$ and B is internal, then b is internal as well.[8]

The following useful theorem is a straightforward consequence of the *transfer principle* and of the definition of internal object (see e.g. [13] Prop. 4.4.14).

[8] The matter of transitivity of the class of internal sets gives rise to interesting considerations in the foundations of nonstandard set theories (cf. Hrbàček's remarks in Subsection 3.3 of [31].)

Theorem 1.20 (Internal Definition Principle). *If $P(x, x_1, \ldots, x_n)$ is an elementary sentence and $B, B_1, \ldots, B_n$ are internal objects, then also the set $\{x \in B \mid P(x, B_1, \ldots, B_n)\}$ is internal.*

By direct applications of this principle, the following is proved.

Proposition 1.21.

1. *The collection $\mathcal{I}$ of internal sets is closed under union, intersection, set-difference, finite sets and tuples, Cartesian products, and under images and preimages of internal functions;*
2. *For every standard set A, ${}^*\mathcal{P}(A) = \mathcal{P}({}^*A) \cap \mathcal{I}$ is the set of all internal subsets of ${}^*A$;*
3. *For all standard sets A and B, ${}^*(B^A) = ({}^*B^{{}^*A}) \cap \mathcal{I}$ is the set of all internal functions from ${}^*A$ to ${}^*B$;*
4. *If $C, D \in \mathcal{I}$ are internal, then $\mathcal{P}(C) \cap \mathcal{I}$ (the set of all internal subsets of C) and $(D^C) \cap \mathcal{I}$ (the set of all internal functions from C to D) are internal.*

The notion of internal set is useful to correctly apply the *transfer principle*. In fact, any quantification on subsets or functions, can be transferred to a quantification on *internal* subsets or *internal* functions, respectively. For instance, let us go back to Examples 1.16 and 1.17. The *well-ordering* of $\mathbb{N}$ is transferred to: "Every nonempty *internal* subset of ${}^*\mathbb{N}$ has a least element". The *completeness* of $\mathbb{R}$ transfers to: "Every nonempty *internal* subset of ${}^*\mathbb{R}$ that is bounded above has a l.u.b.".

Another example is the following.

Example 1.22. The well-ordering property of $\mathbb{N}$ implies that: "There is no decreasing function $f : \mathbb{N} \to \mathbb{N}$". Then, by *transfer*, "There is no *internal* decreasing function $g : {}^*\mathbb{N} \to {}^*\mathbb{N}$".[9]

In general, we can state the following

> *Rule of the thumb.* Properties about *subsets* or about *functions* of standard objects, transfer to the corresponding properties about *internal subsets* or *internal functions*, respectively.

We can use the above considerations to prove that certain objects are external.

Example 1.23. The set ${}^*\mathbb{N} \setminus \mathbb{N}$ of the *infinite* hypernatural numbers is *external*, because it has no least element. Also $\mathbb{N}$ is external, otherwise the set-difference ${}^*\mathbb{N} \setminus \mathbb{N}$ would be internal.[10] The set of infinitesimal hyperreal

[9] We remark that there are models of hyper-methods where (external) decreasing functions $g : {}^*\mathbb{N} \to {}^*\mathbb{N}$ exist.

[10] Here $\mathbb{N} \subset {}^*\mathbb{N}$ is seen as an element of the nonstandard universe.

numbers is another external collection, because it is bounded above but with no least upper bound.

An easy example of external function is the following.

EXAMPLE 1.24. Let $g : {}^*\mathbb{N} \to {}^*\mathbb{N}$ be the function such that $g(n) = n$ if $n \in \mathbb{N}$, and $g(\nu) = 0$ if $\nu \in {}^*\mathbb{N} \setminus \mathbb{N}$. Then g is external, otherwise its range $\mathbb{N}$ would be internal.

As a consequence of Proposition 1.21, the above Examples 1.23 and 1.24 show that ${}^*\mathcal{P}(\mathbb{N}) \neq \mathcal{P}({}^*\mathbb{N})$ and ${}^*(\mathbb{N}^{\mathbb{N}}) \neq {}^*\mathbb{N}^{{}^*\mathbb{N}}$. More generally, we have

PROPOSITION 1.25.

1. *Every infinite internal set has at least the size of the continuum, hence it cannot be countable. In particular, for every infinite standard set A, the inclusion ${}^*\mathcal{P}(A) \subset \mathcal{P}({}^*A)$ is proper;*
2. *If the standard set A is infinite and B contains at least two elements, then the inclusion ${}^*(B^A) \subset {}^*B^{{}^*A}$ is proper.*

We warn the reader that getting familiar with the distinction between internal and external objects is probably the hardest step in learning nonstandard analysis.

1.6. The saturation principle. The star-map and the *transfer principle* suffice to develop the basics of nonstandard analysis, but for more advanced applications a third tool is also necessary, namely:

> **Countable Saturation Principle.** *Suppose $\{B_n\}_{n\in\mathbb{N}} \subseteq {}^*A$ is a countable family of internal sets with the "finite intersection property". Then the intersection $\bigcap_{n\in\mathbb{N}} B_n \neq \emptyset$ is nonempty.*

Recall that a family of sets $\mathcal{B}$ has the *finite intersection property* if $\bigcap_{i=1}^n B_i \neq \emptyset$ for all choices of finitely many $B_1, \ldots, B_n \in \mathcal{B}$. In several contexts, stronger saturation principles are considered where also families of larger size are allowed. Precisely, let κ be a given uncountable cardinal.

> **Fundamental Tool # 3: κ-SATURATION PROPERTY.**
> *Suppose $\mathcal{B} \subseteq {}^*A$ is a family of internal subsets of some hyperextension ${}^*A$, and suppose $|\mathcal{B}| < \kappa$. If $\mathcal{B}$ has the "finite intersection property", then $\bigcap \mathcal{B} \neq \emptyset$.*

In this terminology, countable saturation is $\aleph_1$-saturation. The next example illustrates a relevant use of saturation.

EXAMPLE 1.26. Let (X, τ) be a Hausdorff topological space with *character* κ, hence each point $x \in X$ has a base of neighborhoods $\mathcal{N}_x$ of size at most κ. Clearly, the family of internal sets $\mathcal{B}_x = \{{}^*I \mid I \in \mathcal{N}_x\}$ has the finite intersection property. If we assume κ^+-saturation,[11] the intersection $\mu(x) = \bigcap_{I \in \mathcal{N}_x} {}^*I \neq \emptyset$. In the literature, $\mu(x)$ is called the *monad* of x. Notice that $\mu(x) \cap \mu(y) = \emptyset$ whenever $x \neq y$, since X is Hausdorff. Monads are the basic ingredient in applying the hyper-methods to topology, starting with the following characterizations (see e.g. [37] Ch.III):

- $A \subseteq X$ is *open* if and only if for every $x \in A$, $\mu(a) \subseteq {}^*A$;
- $C \subseteq X$ is *closed* if and only if for every $x \notin C$, $\mu(x) \cap {}^*C = \emptyset$;
- $K \subseteq X$ is *compact* if and only if ${}^*K \subseteq \bigcup_{x \in K} \mu(x)$.

Sometimes in the literature, the following weakened version of saturation is considered, where only families of hyper-extensions are allowed.

DEFINITION 1.27. *κ-enlarging property*: Suppose $\mathcal{F} \subseteq A$ is a family of subsets of some standard set A, and suppose that $|\mathcal{F}| < \kappa$. If $\mathcal{F}$ has the "finite intersection property", then $\bigcap_{F \in \mathcal{F}} {}^*F \neq \emptyset$.[12]

Notice that the κ^+-enlarging property suffices to prove that the monads $\mu(x)$ of the above Example 1.26 are nonempty.

§2. Ultrapowers and hyper-extensions. In this section we deal with the connections between ultrapowers and the hyper-extensions of nonstandard analysis. In particular, we will see that, up to isomorphisms, hyper-extensions are precisely suitable subsets of ultrapowers, namely the proper limit ultrapowers. This characterization theorem will be used in Part II of this article to show that the given definitions actually yield models of the hyper-methods.

2.1. Ultrafilters and ultrapowers. Recall that a *filter* $\mathcal{F}$ on a set I is a nonempty family of subsets of I that is closed under intersections and supersets, i.e.

- If $A, B \in \mathcal{F}$ then $A \cap B \in \mathcal{F}$;
- If $A \in \mathcal{F}$ and $B \supseteq A$, then also $B \in \mathcal{F}$.

A typical example of filter on a set I is the *Frechet filter* $\mathcal{F}r$ of cofinite subsets.

$$\mathcal{F}r = \{A \subseteq I \mid I \setminus A \text{isfinite}\}.$$

DEFINITION 2.1. An *ultrafilter* $\mathcal{U}$ on I is a filter that satisfies the additional property: $A \notin \mathcal{U} \Leftrightarrow I \setminus A \in \mathcal{U}$.

It is easily shown that ultrafilters on I are those non-trivial filters with are maximal with respect to inclusion.[13] As a consequence of the definition,

[11] κ^+ denotes the successor cardinal of κ. Thus $|\mathcal{B}| < \kappa^+$ is the same as $|\mathcal{B}| \leq \kappa$.

[12] We remark that the enlarging property is strictly weaker than saturation, in the sense that there are models of the hyper-methods where the κ-enlarging property holds but κ-saturation fails.

[13] By the *trivial filter* on I we mean the collection $\mathcal{P}(I)$ of all subsets of I.

if a finite union $A_1 \cup \cdots \cup A_n \in \mathcal{U}$ belongs to an ultrafilter, then at least one of the $A_i \in \mathcal{U}$.

First examples are the *principal* ultrafilters $\mathcal{U}_i = \{A \subseteq I \mid i \in A\}$, where i is a fixed element of I. Notice that an ultrafilter is non-principal if and only if it contains no finite sets (hence, if and only if it includes the Frechet filter). The existence of non-principal ultrafilters is proved by a straight application of Zorn's lemma.

Given an ultrafilter $\mathcal{U}$ on the set I, consider the following equivalence relation $\equiv_{\mathcal{U}}$ on functions with domain I:

$$f \equiv_{\mathcal{U}} g \Longleftrightarrow \{i \in I \mid f(i) = g(i)\} \in \mathcal{U}.$$

The *ultrapower* of a set X modulo $\mathcal{U}$ is the quotient set:

$$X^I_{\mathcal{U}} = \{[f]_{\mathcal{U}} \mid f : I \to X\}$$

where we denoted by $[f]_{\mathcal{U}} = \{g \in X^I \mid f \equiv_{\mathcal{U}} g\}$ the equivalence class of f. When the ultrafilter $\mathcal{U}$ is clear from the context, we simply write $[f]$. X is canonically embedded into its ultrapower $X^I_{\mathcal{U}}$ by means of the *diagonal map* $d : x \mapsto [c_x]$, where $c_x : I \to X$ is the constant function with value x.

The ultrapower construction is commonly used to obtain models of hyper-methods. Indeed, models of hyper-methods are fully characterized by means the generalized notion of limit ultrapower (see Theorem 2.10 below.)

Ultrafilters naturally arise in hyper-extensions.

Definition 2.2. Let X be any standard set, and let $\alpha \in {}^*X$. The *ultrafilter generated* by $\alpha \in {}^*X$, is the following family of subsets of X:

$$\mathcal{U}_\alpha = \{A \subseteq X \mid \alpha \in {}^*A\}.$$

It is readily verified that $\mathcal{U}_\alpha$ is actually an ultrafilter on X. Moreover, $\mathcal{U}_\alpha$ is non-principal if and only if $\alpha \neq {}^*x$ for all $x \in X$.

2.2. Complete structures. In order to formulate the next results, we need the

Definition 2.3. A *X-complete structure* is a system

$$\mathfrak{A}(X) = \langle X_{\mathfrak{A}};\ \{F_{\mathfrak{A}} \mid F : X^n \to X\};\ \{R_{\mathfrak{A}} \mid R \subseteq X^n\}\rangle$$

that consists of a superset $X_{\mathfrak{A}}$ of X, of an n-place function $F_{\mathfrak{A}} : (X_{\mathfrak{A}})^n \to X_{\mathfrak{A}}$ for each $F : X^n \to X$, and of an n-place relation $R_{\mathfrak{A}} \subseteq (X_{\mathfrak{A}})^n$ for each $R \subseteq X^n$.

Ultrapowers and hyper-extensions of X provide natural examples of X-complete structures.

Example 2.4. A crucial feature of ultrapowers of a given set X, is that all functions $F : X^n \to X$ and all relations $R \subseteq X^n$ can be naturally extended to functions $\widetilde{F} : (X^I_{\mathcal{U}})^n \to X^I_{\mathcal{U}}$ and relations $\widetilde{R} \subseteq (X^I_{\mathcal{U}})^n$, respectively. Precisely, we set:

$$\widetilde{F}([f_1], \ldots , [f_n]) = [g] \Leftrightarrow \{i \in I \mid F(f_1(i), \ldots , f_n(i)) = g(i)\} \in \mathcal{U}$$
$$\widetilde{R}([f_1], \ldots , [f_n]) \Leftrightarrow \{i \in I \mid R(f_1(i), \ldots , f_n(i))\} \in \mathcal{U}.$$

(The above definitions are well-posed as a consequence of the properties of filter.) If we identify every $x \in X$ with its diagonal image $d(x) \in X^I_{\mathcal{U}}$, then the ultrapower $X^I_{\mathcal{U}}$ becomes a X-complete structure:

$$\mathfrak{X}^I_{\mathcal{U}} = \langle X^I_{\mathcal{U}};\ \{\widetilde{F} \mid F : X^n \to X\};\ \{\widetilde{R} \mid R \subseteq X^n\}\,\rangle.$$

Example 2.5. Let $\langle\, * \,; \mathbb{U} \,; \mathbb{V}\,\rangle$ be a model of hyper-methods, and take any $X \in \mathbb{U}$. If every $x \in X$ is identified with ${}^*x \in {}^*X$, then

$${}^*\mathfrak{X} = \langle\, {}^*X; \{{}^*F \mid F : X^n \to X\}; \{{}^*R \mid R \subseteq X^n\}\,\rangle$$

is a X-complete structure, called the *hyper-structure induced by* $\langle\, * \,; \mathbb{U} \,; \mathbb{V}\,\rangle$.

Another important example is the following

Example 2.6. Let $\langle\, * \,; \mathbb{U} \,; \mathbb{V}\,\rangle$ be a model of hyper-methods, take $X \in \mathbb{U}$ and $\alpha \in {}^*X$. Define the *subspace generated by* α in ${}^*X$ as

$$X_\alpha = \{{}^*f(\alpha) \mid f : X \to X\,\}.$$

Notice that, if $F : X^n \to X$ is any n-place function, and ${}^*f_i(\alpha) \in X_\alpha$ for $i = 1, \ldots , n$, then also the image ${}^*F({}^*f_1(\alpha), \ldots , {}^*f_n(\alpha)) = {}^*g(\alpha) \in X_\alpha$, where g is the function defined by $x \mapsto F(f_1(x), \ldots , f_n(x))$. Thus, by restricting the structure ${}^*\mathfrak{X}$ of the example 2.5 above we obtain a X-complete structure

$$\mathfrak{X}_\alpha = \langle\, X_\alpha; \{{}^*F \restriction {X_\alpha}^n \mid F : X^n \to X\}; \{{}^*R \cap {X_\alpha}^n \mid R \subseteq X^n\}\,\rangle.$$

The natural notion of isomorphism for X-complete structures is the following:

Definition 2.7. Let $\mathfrak{A}(X)$ and $\mathfrak{B}(X)$ be X-complete structures. A bijection $\Theta : X_{\mathfrak{A}} \to X_{\mathfrak{B}}$ is an *isomorphism* of X-complete structures if for every $F : X^n \to X$, for every $R \subseteq X^n$, and for every $x_1, \ldots , x_n \in X_{\mathfrak{A}}$, the following hold:

$$\Theta(F_{\mathfrak{A}}(x_1, \ldots , x_n)) = F_{\mathfrak{B}}(\Theta(x_1), \ldots , \Theta(x_n)) \ \text{ and}$$
$$(x_1, \ldots , x_n) \in R_{\mathfrak{A}} \Leftrightarrow (\Theta(x_1), \ldots , \Theta(x_n)) \in R_{\mathfrak{B}}.$$

In this case, we say that $\mathfrak{A}(X)$ and $\mathfrak{B}(X)$ are *completely isomorphic.*

A relevant example of isomorphic complete structures is given by the next proposition, whose proof is straightforward from the examples above.

Proposition 2.8. *Let* $\langle\, * \,; \mathbb{U} \,; \mathbb{V}\,\rangle$ *be a model of hyper-methods, let* $X \in \mathbb{U}$*, and pick* $\alpha \in {}^*X$*. Let* X_α *and* $\mathcal{U}_\alpha$ *be the subspace and the ultrafilter generated by* α*. Then the map* $\Theta : X_\alpha \to X^X_{\mathcal{U}_\alpha}$ *defined by* $\Theta({}^*f(\alpha)) = [f]_{\mathcal{U}_\alpha}$ *is an isomorphism between the* X*-complete structures* $\mathfrak{X}_\alpha$ *and* $\mathfrak{X}^X_{\mathcal{U}_\alpha}$.

2.3. The characterization theorem. Hyper-extensions have an algebraic characterization as suitable subsets of ultrapowers. To this end, we recall the following generalization of ultrapowers.

DEFINITION 2.9. Let I be a set, $\mathcal{U}$ an ultrafilter on I and $\mathcal{F}$ a filter on the product $I \times I$. For every set X, the *limit ultrapower* $X^I_{\mathcal{U}}|\mathcal{F}$ is the subset of the ultrapower $X^I_{\mathcal{U}}$ that consists of all equivalence classes $[f]$ of functions $f : I \to X$ that are "piecewise constant" with respect to $\mathcal{F}$, i.e. such that $\{(i, i') \in I \times I \mid f(i) = f(i')\} \in \mathcal{F}$.[14] We say that the triple $(I, \mathcal{U}, \mathcal{F})$ is *proper* when the diagonal embedding $d : \mathbb{N} \to \mathbb{N}^I_{\mathcal{U}}|\mathcal{F}$ is not onto.[15]

Notice that, when $\mathcal{F} = \mathcal{P}(I \times I)$ is the trivial filter, then $X^I_{\mathcal{U}}|\mathcal{F} = X^I_{\mathcal{U}}$. Thus limit ultrapowers generalize ultrapowers. Similarly as ultrapowers, also limit ultrapowers provide complete structures, according to the Example 2.4above. The following characterization holds (cf. Theorem 3.4).

THEOREM 2.10 (Keisler's characterization). *Let*

$$^\star\mathfrak{X} = \langle\, ^\star X; \{^\star F \mid F : X^n \to X\}; \{^\star R \mid R \subseteq X^n\}\,\rangle.$$

be a X-complete structure. Then the following are equivalent:

1. $^\star\mathfrak{X} = {}^*\mathfrak{X}$ *is induced by a model of hyper-methods* $\langle\, * \,; \mathbb{U}\,; \mathbb{V}\,\rangle$*;*
2. $^\star\mathfrak{X}$ *is isomorphic to some limit ultrapower* $\mathfrak{X}^I_{\mathcal{U}}|\mathcal{F}$ *where* $(I, \mathcal{U}, \mathcal{F})$ *is proper;*
3. *A is properly included in $^\star A$ for all infinite $A \subseteq X$, and the transfer principle holds: If σ is an elementary formula involving functions $F_1, \ldots, F_m$ and relations $R_1, \ldots, R_k$, then, for all $x_1, \ldots, x_n \in X$,*

$$\sigma(x_1, \ldots, x_n, F_1, \ldots, F_m, R_1, \ldots, R_k) \Leftrightarrow$$
$$\sigma(x_1, \ldots, x_n, {}^\star F_1, \ldots, {}^\star F_m, {}^\star R_1, \ldots, {}^\star R_k).$$

The above result was proved by H.J. Keisler in the context of superstructures, as an application of his characterization theorem of *complete extensions* as (isomorphic copies of) limit ultrapowers (see [13], Thms. 6.4.10 and 6.4.17). An alternative proof of this result, based on the subspaces X_α and the ultrafilters $\mathcal{U}_\alpha$ generated by α, can be reconstructed from arguments in [20, 26], and will appear in full details in [9].

Part II – The Eightfold Path

§3. The superstructure approach. The approach that is most commonly adopted by practitioners of nonstandard methods is the so-called *superstructure approach.* It was first elaborated by A. Robinson jointly

[14] Limit ultrapowers have been introduced in the early sixties by H.J. Keisler [33].
[15] Equivalently, when the diagonal embedding $d : A \to A^I_{\mathcal{U}}|\mathcal{F}$ is not onto for *any* infinite A.

with E. Zakon in [45]. For a detailed exposition of this approach, we refer to Section 4.4 of [13], where all the proofs omitted here can be found.

By the axioms of Zermelo-Fraenkel set theory ZFC, all existing "objects" are sets. As already pointed out in the Footnote 2, numbers, ordered tuples, sets, Cartesian products, relations, functions, as well as virtually all mathematical objects, can in fact be coded as sets. Following the common practice with superstructures, here we adopt as a foundational framework the (slightly) modified version of ZFC that allows also the existence of "atoms". (By *atoms* we mean objects that can be elements of sets but are not sets themselves, and are "empty" with respect to $\in$.) This is consistent with everyday practice, where one never considers, say, π or Napier's constant e as sets.

3.1. The definitions. The basic notion is the following.

Definition 3.1. Let X be a set of atoms. The *superstructure* over X is the increasing union

$$V(X) = \bigcup_{n\in\mathbb{N}} V_n(X)$$

where $V_0(X) = X$ and, by induction, the $(n+1)$-th stage $V_{n+1}(X) = V_n(X) \cup \mathcal{P}(V_n(X))$ adds all subsets of the n-th stage. It is assumed that (a copy of) the natural numbers $\mathbb{N} \subseteq X$.[16]

Notice that superstructures are suitable to formalize the notion of *universe* of Definition 1.1. Suppose we want to investigate some mathematical object Z. Then, all what is needed in the study of Z belongs to any superstructure $V(X)$, provided X includes (a copy of) Z. E.g. in real analysis, the real functions, the usual spaces of functions and functionals, the norms and so forth, as well as the involved topologies, are *all* elements of $V(\mathbb{R})$. The point is that superstructures are closed under *all* the usual mathematical constructions. Namely, if $A, B \in V(X)$ are sets, then the union $A \cup B$, the intersection $A \cap B$, the set-difference $A \setminus B$, the ordered pair (A, B), the Cartesian product $A \times B$, the set B^A of all functions from A to B, any n-place relation R on A, the powerset $\mathcal{P}(A) \in V(X)$, *etc.*, all belong to $V(X)$.

The following definition is the one most commonly adopted by practitioners of nonstandard analysis.

Definition 3.2 ([13] §4.4). A *superstructure model of (nonstandard or) hyper-methods* is a triple $\langle * \,;\, V(X) \,;\, V(Y)\rangle$ where:

1. $V(X)$ and $V(Y)$ are superstructures;

---

[16] We remark that superstructures $V(X)$ over sets of atoms can be also implemented in the "pure" set theory ZFC. This can be done by taking X as a set of nonempty sets x that "behave" as atoms with respect to $V(X)$, i.e. such that $x \cap V(X) = \emptyset$ (see [13] §4.4, where such X are called *base sets*).

2. ${}^*X = Y$;
3. ${}^*n = n$ for all $n \in \mathbb{N}$, and $\mathbb{N}$ is properly included in ${}^*\mathbb{N}$;
4. $* : V(X) \to V(Y)$ satisfies the *transfer principle*.

We propose here a modified version of the above definition, where a single superstructure $V(X)$ is considered instead of two.[17]

Definition 3.3. We say that a triple $\langle * \,;\, V(X) \,;\, V(X) \rangle$ is a *single superstructure model of (nonstandard or) hyper-methods* if:

1. $V(X)$ is a superstructure;
2. ${}^*X = X$;
3. ${}^*n = n$ for all $n \in \mathbb{N}$, and $\mathbb{N}$ is properly included in ${}^*\mathbb{N}$;
4. $* : V(X) \to V(X)$ satisfies the *transfer principle*.

One advantage of this definition is that the traditional distinction between standard and nonstandard objects is overcome. Each object under consideration is in fact standard, and one can consider its hyper-extension. For instance, in this context, one could take the set of hyper-hypernatural numbers ${}^{**}\mathbb{N}$, the set of hyper-infinitesimals, and so forth. Moreover, all possible hyper-extensions are obtained in some single superstructure model, as shown by the following theorem. (The proof is obtained by suitably modifying the construction in [3]. See also [9].)

Theorem 3.4. *Let X be a set, and let $\langle \star \,;\, \mathbb{U} \,;\, \mathbb{V} \rangle$ be any model of hyper-methods with $X \in \mathbb{U}$. Then there exists a* single superstructure *model of hyper-methods $\langle * \,;\, V(X') \,;\, V(X') \rangle$ with $X \subseteq X'$ and such that the two X-complete structures ${}^\star\mathfrak{X}$ and ${}^*\mathfrak{X}$ are isomorphic.*

3.2. A characterization theorem.

The following result shows that the *transfer principle* can be equivalently reformulated as a closure property of the star-map under basic set operations. It can be used as an alternative definition that may be more appealing to those mathematicians who are not familiar with the notion of elementary sentence. A proof can be obtained by adapting the arguments used to prove Theorem 3.2. of [45].

Theorem 3.5. *A map $* : V(X) \to V(Y)$ between superstructures satisfies the transfer principle if and only if the following finite list of properties is satisfied for all $A, B \in V(X)$:*[18]

1. ${}^*\{A, B\} = \{{}^*A, {}^*B\}$;
2. ${}^*(A \cup B) = {}^*A \cup {}^*B$;

[17] This idea was first persued by V. Benci in [3].
[18] This list could easily be reduced (some of the itemized properties can be derived from the others). However, for the sake of completeness and clarity, we decided to include all basic operations. Clearly, with the exception of item 1, A and B are assumed to be sets.

3. ${}^*(A \cap B) = {}^*A \cap {}^*B$;
4. ${}^*(A \setminus B) = {}^*A \setminus {}^*B$;
5. ${}^*(A \times B) = {}^*A \times {}^*B$;
6. ${}^*(\bigcup A) = \bigcup {}^*A$, *i.e.* ${}^*\{x \mid \exists y \in A.\ x \in y\} = \{\xi \mid \exists \eta \in {}^*A.\ \xi \in \eta\}$;
7. ${}^*\{(x,x) \mid x \in A\} = \{(\xi,\xi) \mid \xi \in {}^*A\}$;
8. ${}^*\{(x,y) \mid x \in y \in A\} = \{(\xi,\eta) \mid \xi \in \eta \in {}^*A\}$;
9. ${}^*\{x \mid \exists y.\ (x,y) \in A\} = \{\xi \mid \exists \eta.(\xi,\eta) \in {}^*A\}$;
10. ${}^*\{y \mid \exists x.\ (x,y) \in A\} = \{\eta \mid \exists \xi.\ (\xi,\eta) \in {}^*A\}$;
11. ${}^*\{(x,y) \mid (y,x) \in A\} = \{(\xi,\eta) \mid (\eta,\xi) \in {}^*A\}$;
12. ${}^*\{(x,y,z) \mid (x,z,y) \in A\} = \{(\xi,\eta,\zeta) \mid (\xi,\zeta,\eta) \in {}^*A\}$.

§4. The algebraic approach. We think there is a very simple "path" to nonstandard analysis, which is suitable to students who know the basics of elementary algebra. It is an algebraic approach based on the existence of a "special" homomorphism of algebras. Precisely:

DEFINITION 4.1. The map $J : \mathcal{R}^{\mathbb{N}} \to \mathcal{R}$ is a *hyper-homomorphism*[19] if the following conditions are satisfied:

1. $\mathcal{R}$ is a *superfield* of the real numbers $\mathbb{R}$.
2. $J : \mathcal{R}^{\mathbb{N}} \to \mathcal{R}$ is a *surjective homomorphism* of $\mathbb{R}$-algebras, where $\mathcal{R}^{\mathbb{N}}$ is the ring of sequences $\varphi : \mathbb{N} \to \mathcal{R}$, with operations defined pointwise.
3. The kernel of J is *non-principal*.

4.1. The star-map. We now sketch how to obtain a model of hyper-methods out of an hyper-homomorphism $J : \mathcal{R}^{\mathbb{N}} \to \mathcal{R}$.

For convenience, without loss of generality we assume that $\mathcal{R}$ is a set of atoms. Let $V(\mathcal{R}) = \bigcup_{k\in\mathbb{N}} V_k(\mathcal{R})$ be the superstructure over $\mathcal{R}$, and consider the family of sequences $\mathcal{F} = \bigcup_{k\in\mathbb{N}}(V_{k+1}(\mathcal{R}) \setminus V_k(\mathcal{R}))^{\mathbb{N}}$. Inductively extend the map J to a map $\widetilde{J} : \mathcal{F} \cup \mathcal{R}^{\mathbb{N}} \to V(\mathcal{R})$ as follows.

$$\widetilde{J}(\varphi) \ = \ \begin{cases} J(\varphi) & \text{if } \varphi : \mathbb{N} \to \mathcal{R} \\ \{\widetilde{J}(\psi) \mid \forall n\,.\,\psi(n) \in \varphi(n)\} & \text{if } \varphi : \mathbb{N} \to (V_{k+1}(\mathcal{R}) \setminus V_k(\mathcal{R})) \end{cases}$$

Let $c_A \in \mathcal{F} \cup \mathcal{R}^{\mathbb{N}}$ denote the constant sequence with value $A \in V(\mathcal{R})$. Define the map $* : V(\mathcal{R}) \to V(\mathcal{R})$ by setting ${}^*A = \widetilde{J}(c_A)$ for every $A \in V(\mathcal{R})$.

Notice that, for any $\xi \in \mathcal{R}$, we have ${}^*\xi = \widetilde{J}(c_\xi) = J(c_\xi) \in \mathcal{R}$. In particular, if $x \in \mathbb{R}$, then ${}^*x = J(c_x) = J(x \cdot c_1) = x \cdot J(c_1) = x \cdot 1 = x$. Moreover, for every set $A \in V(\mathcal{R})$, we have that ${}^*A = \{\widetilde{J}(\varphi) \mid \varphi \in A^{\mathbb{N}}\}$. A suitable modification of arguments in [7] proves that the map $*$ satisfies the *transfer principle*, as well as the other properties of Definition 3.3:

[19] This notion of hyper-homomorphism is different from that given in [7].

Theorem 4.2. *The triple* $\langle * ; V(\mathcal{R}) ; V(\mathcal{R})\rangle$ *is a single superstructure model of hyper-methods that satisfies the countable saturation property.*

More details can be found in [9].

4.2. Construction of a hyper-homomorphism. We define by transfinite induction an increasing κ-sequence of fields $\langle \mathcal{R}_\beta \mid \beta < \kappa\rangle$ and an increasing κ-sequence of maps $\langle J_\beta \mid \beta < \kappa\rangle$ such that, for all $\beta < \kappa$, $J_\beta : (\mathcal{R}_\beta)^{\mathbb{N}} \to \mathcal{R}_{\beta+1}$ is a surjective homomorphism of $\mathbb{R}$-algebras. If the length κ of the chains has uncountable cofinality, e.g. if $\kappa = \omega_1$, then

$$\bigcup_{\beta<\kappa} (\mathcal{R}_\beta)^{\mathbb{N}} = (\bigcup_{\beta<\kappa} \mathcal{R}_\beta)^{\mathbb{N}}.$$

Thus, by taking $\mathcal{R} = \bigcup_{\beta<\kappa} \mathcal{R}_\beta$ and $J = \bigcup_{\beta<\kappa} J_\beta$, the conditions of Definition 4.1 are fulfilled.

The construction is the following. Fix a non-principal maximal ideal $\mathfrak{m}$ in $\mathbb{R}^{\mathbb{N}}$. Notice that, for every superfield K of $\mathbb{R}$, the ideal generated by $\mathfrak{m}$ in $K^{\mathbb{N}}$ is a non-principal maximal ideal $\mathfrak{m}_K$. Moreover if $K \subseteq K'$ then $\mathfrak{m}_K = K^{\mathbb{N}} \cap \mathfrak{m}_{K'}$.[20] Now put

- $\mathcal{R}_0 = \mathbb{R}$ and, for limit γ, $\mathcal{R}_\gamma = \bigcup_{\beta<\gamma} \mathcal{R}_\beta$.
- For all $\beta < \kappa$, let $\mathfrak{m}_\beta$ be the maximal ideal generated by $\mathfrak{m}$ in $(\mathcal{R}_\beta)^{\mathbb{N}}$. Put $\mathcal{R}_{\beta+1} = (\mathcal{R}_\beta)^{\mathbb{N}}/\mathfrak{m}_\beta$, and for $\gamma < \beta$ and $\varphi \in (\mathcal{R}_\gamma)^{\mathbb{N}}$ identify the classes of φ modulo $\mathfrak{m}_\gamma$ and modulo $\mathfrak{m}_\beta$, so as to get $\mathcal{R}_{\gamma+1} \subseteq \mathcal{R}_{\beta+1}$.
- For every $\beta < \kappa$, let $J_\beta : (\mathcal{R}_\beta)^{\mathbb{N}} \to \mathcal{R}_{\beta+1}$ be the canonical homomorphism onto the quotient.

The κ-chains $\langle \mathcal{R}_\beta \mid \beta < \kappa\rangle$ and $\langle J_\beta \mid \beta < \kappa\rangle$ satisfy the desired properties.

4.3. A characterization of the hyperreal numbers. A modified notion of hyper-homomorphism is suitable to characterize all hyper-extensions ${}^*\mathbb{R}$ of the real numbers.

Definition 4.3. A *composable ring of real-valued functions* is a subring $\Psi \subseteq \mathbb{R}^I$ (where I is any set) that is closed under compositions, i.e. if $\psi \in \Psi$ and $f : \mathbb{R} \to \mathbb{R}$, then $f \circ \psi \in \Psi$.[21]

Similarly to hyper-homomorphisms, starting from any surjective homomorphism of $\mathbb{R}$-algebras $J : \Psi \to K$, where K is a field, one can construct a superstructure model of hyper-methods $\langle *;\, V(\mathbb{R});\, V(K)\rangle$, according to Definition 3.2. In particular, $K = {}^*\mathbb{R}$ is a set of hyperreal numbers. Moreover, *all* possible sets of hyperreals are obtained in that way. namely:

Theorem 4.4 ([8], Thm. 3.3). *A field* $K = {}^*\mathbb{R}$ *is a set of hyperreal numbers if and only if it is the homomorphic image of some composable ring of real-valued functions.*

[20] In fact $\mathfrak{m}_K = \{\psi \in K^{\mathbb{N}} \mid \exists \varphi \in \mathfrak{m}\, (\psi(n) = 0 \iff \varphi(n) = 0)\,\}$.

[21] As usual, the ring operations on $\mathbb{R}^I$ are defined pointwise.

§5. The nonstandard set theory $^*\mathsf{ZFC}$. In this section we present the axiomatic system $^*\mathsf{ZFC}$ that incorporates the hyper-methods in the full generality of set theory.[22] Precisely, $^*\mathsf{ZFC}$ generalizes the superstructure approach of Section 3, by taking as universe the universal class V of *all* sets. This general approach is aimed to include the methods of nonstandard analysis jointly with the usual principles of mathematics, within a unified axiomatic system. In the resulting nonstandard set theory, there is no need to consider different universes to treat different problems (as it is customary with superstructures). In the universal class V all mathematical entities coexist, and the distinction between standard and nonstandard objects as members of different universes is overcome.

5.1. The first three groups of axioms. The theory $^*\mathsf{ZFC}$ presented here consists of five groups of axioms, formulated in the usual first-order language of set theory, with an additional function symbol $* : V \to V$ for the star-map. It is a modified version of the theory presented in [18], to which we refer for details and proofs. The axioms of $^*\mathsf{ZFC}$ are the following.

Axiom 1. ZFC^-, i.e. all axioms of Zermelo-Fraenkel set theory with choice, with the exception of the axiom of *regularity*. The *separation* and *collection* schemata are assumed also for those formulas where the symbol $*$ occurs.[23]

Axiom 2. The class $\mathcal{I} = \{x \mid \exists y.\, x \in {}^*y\}$ of internal objects is *transitive*, i.e. elements of internal sets are internal.

Axiom 3. The star-map $* : V \to V$ preserves all Gödel's operations as itemized in Theorem 3.5.

By Axiom 1, we can say that all arguments of ordinary mathematics can be formalized within $^*\mathsf{ZFC}$.[24] Axiom 2 postulates a convenient (and natural) property of internal sets (cf. Footnote 8). Axiom 3 is a convenient formulation of the *transfer principle*. In fact, by assuming Axioms 1-2, Axiom 3 holds if and only if the star-map $* : V \to V$ satisfies the *transfer*

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[22] The theory presented here is just one of several *nonstandard set theories* that have been proposed over the last thirty years. For an overview of this interesting subject, we refer the reader to the survey by K. Hrbàček in this volume. See also [16].

[23] Recall the *axiom schema of collection*: For every formula $\sigma(x,y)$,

$$(\forall x \in A\, \exists y\, \sigma(x,y)) \to (\exists B\, \forall x \in A\, \exists y \in B\, \sigma(x,y)).$$

In ZF, collection and replacement are equivalent. In a non-wellfounded context, the latter axiom is weaker than the former. Notice that collection rather than replacement is needed in several mathematical applications.

[24] Actually, the axiom of *regularity* (also known as *foundation*) is rarely used (if ever) in ordinary mathematics beyond set-theory itself. On the other hand, regularity cannot be assumed in nonstandard set theory, since, e.g., the hyper-extension $^*\omega$ of von Neumann natural numbers is necessarily non-wellfounded. (All set theoretic results depending on regularity can be reformulated as *properties of wellfounded sets*.)

principle.(See the similar Theorem 3.5 in the framework of superstructures.) As a consequence, the triple $\langle * \,; V \,; V \rangle$ matches the Fundamental Definition of model of hyper-methods given in Section 1.

5.2. Saturation. A core problem every nonstandard set theory has to face is the so-called *Hrbàček paradox* (first presented in [30]), namely the inconsistency of the hyper-methods in full set theory, in the presence of unlimited levels of saturation (or even of enlarging property). E.g., $|{}^*\mathbb{N}|^+$-saturation fails necessarily, because $\{{}^*\mathbb{N}\setminus\{\nu\} \mid \nu \in {}^*\mathbb{N}\}$ is a family of internal sets, with the finite intersection property, that has empty intersection. Most nonstandard set theories avoid this inconsistency by weakening or giving up with some set theoretic axioms, such as the power-set axiom, or the replacement schema, or the axiom of choice. The theory $^*\mathsf{ZFC}$ postulates *all* usual principles of set theory, and overcomes Hrbàček paradox by restricting to "definable saturation".

Axiom 4. The κ-saturation property holds for all "$\in$-definable" cardinals κ. (See [18] for a precise formulation.)

We remark that all cardinals that are used in practice, i.e. those cardinals that are "explicitly mentioned" (e.g. 17, ω, $\aleph_{13}$, the first inaccessible cardinal, *etc.*) are "definable" by some elementary formula in the $\in$-language of set theory. Thus, say, the $\aleph_{\omega_1}$-saturation property holds in $^*\mathsf{ZFC}$. As a result, roughly speaking we can say that $^*\mathsf{ZFC}$ retains the flavour of unlimited saturation. More precisely, let $P(x)$ be any property that is expressed in elementary form without using the symbol $*$. Suppose that, for every cardinal κ, κ-saturation implies $P(a)$ for every set a of cardinality less than κ. Then we can conclude rightaway that $P(a)$ is proved for *all* a. In fact, by assuming the contrary, the least size κ of a counter-example b would be an "$\in$-definable" cardinal. Then κ^+-saturation would hold, contradicting the failure of $P(b)$.

The following is a typical example.

Example 5.1. The characterization of compactness for a topological space X mentioned in Example 1.26, is proved as an application of the κ^+-enlarging property, with κ the size of a base of neighborhoods of x, for every $x \in X$. In particular, $|X|^+$-saturation suffices to prove the following.

> "Let (X, τ) be a Hausdorff topological space. Then X is *compact* if and only if ${}^*X = \bigcup_{x\in X} \mu(x)$, where $\mu(x)$ is the *monad* of x".

Making use of this characterization, one can produce a nice and short "nonstandard" proof of *Tychonoff theorem* (see e.g. [37] III.2.7). Precisely, the following is proved:

> "For all families $\{X_i : i \in I\}$ of compact topological Hausdorff spaces with cardinalities $|I|, |X_i| < \kappa$, the product space $\prod_i X_i$ is compact".

By the above considerations, we can thereby conclude that Tychonoff theorem is proved for *all* topological Hausdorff spaces (without restrictions on cardinalities).

5.3. Foundational remarks. The last axiom of $^*\mathsf{ZFC}$ is a weak form of regularity that can be retained in this context:

Axiom 5. The universal class is the increasing union of an ordinal-indexed sequence of sets $V = \bigcup_\alpha W_\alpha$, inductively defined by $W_0 = \emptyset$, $W_{\alpha+1} = \mathcal{P}(W_\alpha \cup {}^*W_\alpha)$, and $W_\alpha = \bigcup_{\gamma<\alpha} W_\gamma$ for limit α.

So V is arranged in a Von Neumann-like cumulative hierarchy, where every set is obtained from the empty set by iterating powersets and hyper-extensions. This "minimality" axiom has purely set theoretic interest, but it has no effects on the practice of hyper-methods.[25]

It is an interesting fact (proved in [18]) that, notwithstanding the apparent strength of $^*\mathsf{ZFC}$, an $\in$-sentence σ is a theorem of ZFC if and only if its relativization σ^{WF} to the class of wellfounded sets is a theorem of $^*\mathsf{ZFC}$. It follows that the two theories ZFC and $^*\mathsf{ZFC}$ are equiconsistent.

§6. The Alpha Theory. This approach is grounded on the introduction of a new mathematical object, called α. We can think of α as an "ideal" (infinitely large) natural number added to $\mathbb{N}$, in a similar way as the imaginary unit i can be seen as a new ideal number added to the real numbers $\mathbb{R}$. We proceed axiomatically. First, we postulate that all $\mathbb{N}$-sequences can be extended so as to take an "ideal" ultimate value at α. Such ideal values are then ruled by four properties, all expressed in elementary terms.

In the following, *all* "usual" principles of mathematics are implicitly assumed, in the form of Zermelo-Fraenkel set theory without regularity.[26] Moreover, following a common practice, we allow a set of atoms $\mathcal{A}$ that includes all natural numbers. The proofs of all results stated in this section can be found in [5].

6.1. The axioms. The Alpha-Theory consists of five axioms as given below.[27] By *sequence* we mean any function whose domain is the set $\mathbb{N}$ of natural numbers.

[25] This idea of a "minimality" axiom was first introduced in [19].

[26] As we are treating the hyper-methods in the generality of full set theory, the regularity axiom *cannot* be assumed (cf. Footnote 24).

[27] The axioms $\alpha 1$-$\alpha 5$ are formulated "informally". A rigorous presentation of the Alpha-Theory can be given as a nonstandard set theory in the first-order language that consists of the membership relation symbol $\in$, of a unary relation symbol $\mathcal{A}$ for atoms, and of a function symbol J. In fact, by denoting $J(\varphi) = \varphi[\alpha]$ for sequences φ, axioms $\alpha 1$-$\alpha 5$ are easily rephrased in this formal language (cf. §6 of [5]).

AXIOM α1. (Extensions). *Every sequence φ is uniquely extended to $\mathbb{N} \cup \{\alpha\}$. The corresponding value at α is denoted by $\varphi[\alpha]$ and called the value of φ at "infinity".*

Notice that we do not assume $\varphi[\alpha] \in A$ when $\varphi : \mathbb{N} \to A$. In fact, in general, this will not be the case. The next axiom gives a natural property of "coherence":

AXIOM α2. (Compositions). *If φ and ψ are sequences and f is any function such that both compositions $f \circ \varphi$ and $f \circ \psi$ are defined, then*

$$\varphi[\alpha] = \psi[\alpha] \Rightarrow (f \circ \varphi)[\alpha] = (f \circ \psi)[\alpha].$$

The remaining three axioms rule the (possible) values at infinity.

AXIOM α3. (Atoms).

- *If $\varphi(n) \in \mathcal{A}$ is an atom for all n, then $\varphi[\alpha] \in \mathcal{A}$ is an atom as well.*
- *If c_k is the constant sequence with value $k \in \mathbb{N}$, then $c_k[\alpha] = k$.*
- *If $\imath_{\mathbb{N}} : \mathbb{N} \to \mathbb{N}$ is the identity sequence on $\mathbb{N}$, then $\imath_{\mathbb{N}}[\alpha] = \alpha \notin \mathbb{N}$.*

AXIOM α4. (Sets).

- *If $c_\emptyset$ is the constant sequence with value $\emptyset$, then $c_\emptyset[\alpha] = \emptyset$.*
- *If φ is a sequence of nonempty sets, then*

$$\varphi[\alpha] = \{ \psi[\alpha] \mid \forall n \,.\, \psi(n) \in \varphi(n) \}.$$

AXIOM α5. (Pairs).

- *If $\varphi(n) = \{\psi(n), \vartheta(n)\}$ for all $n \in \mathbb{N}$, then $\varphi[\alpha] = \{\psi[\alpha], \vartheta[\alpha]\}$.*

Thus, for natural numbers, the notions of constant sequence and identity sequence are preserved at infinity. The condition $\alpha \notin \mathbb{N}$ simply says that the ideal number α is actually a *new* number. Notice that $\imath_{\mathbb{N}}$ provides a first example of a sequence φ such that $\varphi[\alpha] \notin \mathrm{Range}(\varphi)$.

The Axiom α4 postulates that the membership relation is preserved at infinity. Besides, all elements of $\varphi[\alpha]$ are obtained as values at infinity of sequences which are pointwise members of φ. In particular, we have the "transitivity" property that elements of values at infinity are values at infinity. The last Axiom α5 gives the "expected" values at infinity to sequences of pairs. For instance, suppose that $\chi : \mathbb{N} \to \{0, 1\}$ is a characteristic function. Then this axiom guarantees that either $\chi[\alpha] = 0$ or $\chi[\alpha] = 1$.

As a straight consequence of the axioms, two sequences that are "eventually" equal (i.e. equal for all but finitely many n) take equal values at infinity. Similarly, if two sequences are "eventually" different, then they take different values at infinity (see [5], Prop. 1.3). Moreover all basic set-operations (with the exception of power-set) are preserved at infinity ([5], Prop. 1.1). E.g., if $\varphi(n) = \psi(n) \cup \vartheta(n)$ for all n, then $\varphi[\alpha] = \psi[\alpha] \cup \vartheta[\alpha]$, and similarly for unions, set-differences, ordered pairs, Cartesian products, *etc.*

6.2. The star-map. We can now define hyper-extensions of all objects, namely:

DEFINITION 6.1. The *hyper-extension* ${}^*A$ of an object A is the value at infinity of the constant sequence $c_A : n \mapsto A$, i.e. ${}^*A = c_A[\alpha]$.

A few remarks about the above definition.

- If A is any nonempty set, ${}^*A = \{\, \varphi[\alpha] \mid \varphi \in A^{\mathbb{N}}\,\}$. In particular, *internal* objects are precisely the values at infinity of sequences.
- $\mathbb{N}$ is properly included in ${}^*\mathbb{N}$. In fact, by Axiom $\alpha 3$, ${}^*n = n$ for all $n \in \mathbb{N}$, and $\alpha \in {}^*\mathbb{N} \setminus \mathbb{N}$.
- Recall that in a set-theoretic framework, any function $f : A \to B$ is identified with its graph $\{(a, f(a)) \mid a \in A\} \subseteq A \times B$. Consistently, the hyper-extension ${}^*f$ defined as above, is actually a function from ${}^*A$ to ${}^*B$. More precisely, for every sequence $\varphi \in A^{\mathbb{N}}$, ${}^*f(\varphi[\alpha]) = (f \circ \varphi)[\alpha]$ (see [5], Prop. 2.3).
- The value at infinity $\varphi[\alpha]$ of a sequence φ, is actually the value at α of the hyper-extension ${}^*\varphi$, because ${}^*\varphi(\alpha) = {}^*\varphi(\imath_{\mathbb{N}}[\alpha]) = (\varphi \circ \imath_{\mathbb{N}})[\alpha] = \varphi[\alpha]$.

The main results of the Alpha Theory can be summarized as follows:

THEOREM 6.2 ([5], Thms. 6.2, 6.7, and 4.4). *Let V be the universal class of all objects (sets and atoms). Then*

1. *The map $* : V \to V$ satisfies the transfer principle, hence the triple $\langle\, * \,; V \,; V\,\rangle$ matches the Fundamental Definition of model of hypermethods, as given in Section 1;*
2. *For every X, the X-complete structure $\langle\, {}^*X; \{{}^*F \mid F : X^n \to X\}; \{{}^*R \mid R \subseteq X^n\}\rangle$ is isomorphic to the ultrapower $X^{\mathbb{N}}_{\mathcal{U}_\alpha}$ modulo the ultrafilter $\mathcal{U}_\alpha = \{A \subseteq \mathbb{N} \mid \alpha \in {}^*A\}$ generated by α;*
3. *The countable saturation property holds.*

6.3. Cauchy's principles. Cauchy's conception of an infinitesimal as a "variable converging to zero" has been interpreted as "infinitesimal hyperreal number" (see e.g. Lakatos' article [36]). The Alpha Theory seems an appropriate framework for accomodating this idea. Let us consider the following principles:

Cauchy's Infinitesimal Principle (CIP)**:** *Every infinitesimal number $\varepsilon \in {}^*\mathbb{R}$ is the value at infinity of some infinitesimal real sequence φ (i.e. $\varepsilon = \varphi[\alpha]$ where $\lim_{n\to\infty} \varphi(n) = 0$).*

Strong Cauchy's Infinitesimal Principle (SCIP)**:** *Every hyperreal number is the value at infinity of some monotone sequence of reals.*

Within the Alpha Theory AT, the former principle CIP holds if and only if the ultrafilter $\mathcal{U}_\alpha = \{A \subseteq \mathbb{N} \mid \alpha \in {}^*A\}$ is a *P-point*, whereas the latter principle SCIP holds if and only if $\mathcal{U}_\alpha$ is *selective*.[28] As a consequence, the theory AT+SCIP is consistent (provided ZFC is), but not even CIP can be proved by AT. See the discussion in Section 6.4 of [5].

§7. The topological approach. A main feature shared by compactifications and completions in topology and by nonstandard models of analysis is the existence of a "canonical" extension ${}^\star f : {}^\star X \to {}^\star X$ for each function $f : X \to X$. Given an arbitrary set X, we consider here a topological extension of X as a sort of "completion" ${}^\star X$, where the "$\star$" operator provides a *distinguished continuous extension* of each function $f : X \to X$. We shall see that the "$\star$" operator can also be extended to subsets of X as the *closure* operator. A detailed exposition of this topic has been given in [20].

7.1. Topological extensions. We introduce our fundamental definition:

Definition 7.1. Let X be a *dense* subspace of the T_1 topological space[29] ${}^\star X$. Assume that a distinguished *continuous extension* ${}^\star f : {}^\star X \to {}^\star X$ is associated to every function $f : X \to X$. We say that ${}^\star X$ is a *topological extension* of X if:

(c) ${}^\star g \circ {}^\star f = {}^\star(g \circ f)$ for all $f, g : X \to X$, and
(i) if $f(x) = x$ for all $x \in A \subseteq X$, then ${}^\star f(\xi) = \xi$ for all $\xi \in \overline{A}$.

The topological extension ${}^\star X$ of X is a *topological hyperextension*[30] if

(a) for all $f, g : X \to X$
$f(x) \neq g(x)$ for all $x \in X \implies {}^\star f(\xi) \neq {}^\star g(\xi)$ for all $\xi \in {}^\star X$;
(p) for all $\xi, \eta \in {}^\star X$ there exist $\zeta \in {}^\star X$ and $p, q : X \to X$ such that $\xi = {}^\star p(\zeta)$ and $\eta = {}^\star q(\zeta)$.

Notice that, if a topological extension ${}^\star X$ of X is Hausdorff, then ${}^\star f$ is the *unique* continuous extension of f, for X is dense. Therefore properties (c) and (i) are automatically satisfied, and our definition would have required

[28] Many equivalent properties can be used in defining *P-points* and *selective* (or Ramsey) ultrafilters over $\mathbb{N}$ (see, e.g. [11] or [12]). Here the following are pertinent:
- $\mathcal{U}$ is a *P-point* if and only if every $f : \mathbb{N} \to \mathbb{N}$ is either equivalent mod $\mathcal{U}$ to a constant or to a finite-to-one function.
- $\mathcal{U}$ is *selective* if and only if every $f : \mathbb{N} \to \mathbb{N}$ is either equivalent mod $\mathcal{U}$ to a constant or to a 1-1 function.

Clearly selective ultrafilters are P-points, but, surprisingly enough, the converse implication is independent of ZFC. E.g., assuming the Continuum Hypothesis, there are plenty of selective and non-selective P-points. On the other hand, there are models of ZFC without P-points, models with many P-points but no selective ultrafilters, and even models with a unique (up to isomorphism) P-point that is selective (see [47]).

[29] Recall that a topological space is T_1 if its points are closed.

[30] Topological hyperextensions are indeed *hyper-extensions* in the sense of Section 2, by Theorem 7.3 below.

only (a) and (p) (see [10], where Hausdorff topological extensions have been introduced and studied). However considering only Hausdorff spaces would have turned out too restrictive. In fact, Hausdorff hyperextensions of X amount to a very restricted class of subspaces of the Stone-Čech compactification βX of the discrete space X, as characterized in Theorem 7.4 below. Moreover, according to Theorem 7.5, no Hausdorff topological hyperextension can be $(2^{\aleph_0})^+$-enlarging. Last but not least, the existence of such extensions, although consistent, cannot be proved in ZFC alone (see [20] and [2]). These are the reasons why we only require that topological extensions be T_1 spaces.

Topological extensions already satisfy several important cases of the *transfer principle*. E.g., if f is constant, or injective, or surjective, then so is ${}^\star f$. Moreover the extension of the characteristic function of any subset $A \subseteq X$ is the characteristic function of the closure $\overline{A}$ of A in ${}^\star X$, and so we can put ${}^\star A = \overline{A}$ ([20], Lemmata 1.2 and 1.3). However other basic cases of the *transfer principle* may fail, because topological extensions embrace at once all possible nonstandard models together with more general structures. In order to obtain the full *transfer principle*, we postulated the additional properties (a) and (p), called *analiticity* and *coherence* in [20]. The property (a) isolates a fundamental feature that marks the difference between nonstandard extensions and ordinary continuous extensions of functions: "disjoint functions have disjoint extensions".[31] The property (p) provides a sort of "internal coding of pairs", useful for extending multivariate functions "parametrically": this possibility is essential to get hyper-extensions in the sense of Section 1.[32] Compare with the well known fact that there are functions of two variables that do not have continuous extensions to the Stone-Čech compactification.

Since a finite set cannot have *nontrivial* topological extensions, we are interested only in infinite sets, and for convenience we stipulate that $\mathbb{N} \subseteq X$. It is always assumed by nonstandard analysts that *all infinite sets* are

[31] Clearly (a) follows from the principle "standard functions behave like germs"

(e) for all $f, g : X \to X$ and all $\xi \in {}^\star X$

$$ {}^\star f(\xi) = {}^\star g(\xi) \iff \exists A \subseteq X.\ \xi \in \overline{A}\ \ \&\ \forall x \in A\ \ f(x) = g(x), $$

which can be rephrased as a sort of "preservation of equalizers", namely

$$ \{\xi \in {}^\star X \mid {}^\star f(\xi) = {}^\star g(\xi)\} = {}^\star\{x \in X \mid f(x) = g(x)\}. $$

[32] Notice that the properties (i), (c), (a), (e) are obviously instances of the *transfer principle*. This could seem *prima facie* not to be the case of the condition (p). On the contrary, a *strong uniform version* of that property can be obtained by transfer. Simply compose any bijection $\delta : X \to X \times X$ with the ordinary projections $\pi_1, \pi_2 : X \times X \to X$, and obtain "projections" $p, q : X \to X$ satisfying:

for all $\xi, \eta \in {}^*X$ there exists a unique $\alpha \in {}^*X$ such that ${}^*p(\alpha) = \xi,\ {}^*q(\alpha) = \eta$.

indeed *extended*. Call *proper* those topological extensions where $A = {}^\star A$ if and only if A is finite. We can give three topological characterizations of proper extensions.

THEOREM 7.2 ([20], Thm. 6.4). *Let ${}^\star X$ be a topological extension of X. Then ${}^\star X$ is proper if and only if any of the following equivalent properties is fulfilled:*

(i) *${}^\star X$ is Weierstraß;*[33]
(ii) *$\mathbb{N}$ is not closed in ${}^\star X$;*
(iii) *there exists a sequence of clopen subsets of ${}^\star X$ whose intersection is not open.*

As remarked in [20], the property *(iii)* fails if and only if X carries a countably complete (σ-additive) ultrafilter. Hence *nontrivial improper extensions* require *uncountable measurable cardinals*.

7.2. Topological hyperextensions are hyper-extensions. The interest in topological hyperextensions lies in the fact that combining the "analytic" property (a) with the "pair-coding" condition (p), yields the strongest *transfer principle*, thus providing hyper-extensions in the sense of Section 2. In order to apply Theorem 2.10, one has to extend all n-place functions and relations. The *ratio* of considering only unary functions lies in the following facts that hold in every topological hyperextension ${}^\star X$ of X (see Section 5 of [20]).

1. *For all $\xi_1, \dots, \xi_n \in {}^\star X$ there exist $p_1, \dots, p_n : X \to X$ and $\zeta \in {}^\star X$ such that ${}^\star p_i(\zeta) = \xi_i$ for $i = 1, \dots, n$.*
2. *Let $p_1, \dots, p_n, q_1, \dots, q_n : X \to X$ and $\xi, \eta \in {}^\star X$ satisfy ${}^\star p_i(\xi) = {}^\star q_i(\eta)$ for $i = 1, \dots, n$. Then, for all $F : X^n \to X$*
$$ {}^\star(F \circ (p_1, \dots, p_n))(\xi) = {}^\star(F \circ (q_1, \dots, q_n))(\eta). $$

It follows that there is a unique way of assigning an extension ${}^\star F$ to every function $F : X^n \to X$ in such a way that all compositions are preserved:[34]
$$ {}^\star F(\xi_1, \dots, \xi_n) = {}^\star(F \circ (p_1, \dots, p_n))(\zeta), \text{ where } {}^\star p_i(\zeta) = \xi_i \text{ for } i = 1, \dots, n.\text{[35]} $$

By using the characteristic functions in n variables one can assign an extension ${}^\star R$ also to all n-place relations R on X. Thus one obtains a X-complete structure ${}^\star\mathfrak{X} = \langle {}^\star X; \{{}^\star F \mid F : X^n \to X\}; \{{}^\star R \mid R \subseteq X^n\} \rangle$.

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[33] A topological space is *Weierstraß* if all continuous real-valued functions are *bounded*. Hausdorff spaces that are Weierstraß are called *pseudocompact* (see [25]).

[34] I.e. for all $m, n \geq 1$, for all $F : X^n \to X$, and for all $G_1, \dots, G_n : X^m \to X$, ${}^\star F \circ ({}^\star G_1, \dots {}^\star G_n) = {}^\star(\varphi \circ (\psi_1, \dots, \psi_n))$.

[35] *Caveat*: For all $n > 1$ there are functions of n variables whose extensions *cannot be continuous w.r.t. the product topology.* This fact marks an important difference between the topological notion of *compactification* (where, e.g., $\beta\mathbb{N} \times \beta\mathbb{N}$ is quite different from $\beta(\mathbb{N}\times\mathbb{N})$) and the notion of *nonstandard model* (where ${}^*\mathbb{N}\times{}^*\mathbb{N}$ is identified with ${}^*(\mathbb{N}\times\mathbb{N})$).

Theorem 5.5 of [20] states that the *transfer principle* holds, hence points 1 and 3 of Theorem 2.10 yield that every topological hyperextension ${}^\star X$ is a (nonstandard) *hyper-extension*.

Every topological extension ${}^\star X$ being a T_1 space, we know that all sets of the form $E(f,\eta) = \{\xi \in {}^\star X \mid {}^\star f(\xi) = \eta\}$, for $f : X \to X,\ \eta \in {}^\star X$, are closed in ${}^\star X$. The (arbitrary) intersections of finite unions of such sets are the *closed* sets of a topology, which is the *coarsest* T_1 *topology on* ${}^\star X$ *that makes all functions* ${}^\star f$ *continuous*. We call it the *Star topology* of ${}^\star X$, and we say that ${}^\star X$ is a *star extension* if it has the Star topology. *Vice versa*, one can topologize any hyper-extension ${}^*X$ of X with the corresponding Star topology: then X is dense in ${}^*X$. Summing up we have:

Theorem 7.3 ([20], Thm. 3.2). *Any (nonstandard) hyper-extension* ${}^*X$ *of* X*, when equipped with the Star topology, becomes a topological star hyperextension of* X*. Conversely, any topological hyperextension* ${}^\star X$ *of* X *is a (nonstandard) hyper-extension, possibly endowed with a topology finer than the Star topology.*

Whenever the interest focuses on the "nonstandard behaviour" of the topological extension ${}^\star X$, one can therefore assume w.l.o.g. to deal with a star extension.[36]

7.3. Hausdorff topological extensions. Any topological extension of X is canonically mappable into the *Stone-Čech compactification* βX of the discrete space X.[37] Given a topological extension ${}^\star X$ of X, define the *canonical map* $\upsilon : {}^\star X \to \beta X$ by $\upsilon(\xi) = \mathcal{U}_\xi = \{A \subseteq X \mid \xi \in {}^\star A\}$, which is an ultrafilter over X. Then we have ([20], Thm. 2.1):

1. *The canonical map* $\upsilon : {}^\star X \to \beta X$ *is the unique continuous extension of the embedding* $e : X \to \beta X$*, and* $\upsilon \circ {}^\star f = \overline{f} \circ \upsilon$ *for all* $f : X \to X$.[38]
2. *The map* υ *is injective if and only if* ${}^\star X$ *is Hausdorff.*
3. *The map* υ *is surjective if and only if the* S*-topology of* ${}^\star X$ *is quasi-compact (equivalently if every clopen filter has nonempty intersection).*

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[36] In nonstandard analysis one considers the *S-topology* of hyper-extensions ${}^*X$, i.e. the topology generated by the (clopen) sets ${}^*A$ for $A \subseteq X$. Unfortunately, the S-topology is usually coarser than the star topology of ${}^*X$. In fact we have ([20], Thm. 1.4):
1. *The* S*-topology of* ${}^*X$ *is either* 0*-dimensional or not* T_0.
2. *The star topology of* ${}^*X$ *is Hausdorff if and only if the* S*-topology is* T_1.
3. *The star topology and the* S*-topology of* ${}^*X$ *agree if and only if any of them is Hausdorff (actually* 0*-dimensional).*

[37] For various definitions and properties of the Stone-Čech compactification see [25]. If X is a discrete space, we identify βX with the set of all ultrafilters over X, endowed with the topology having as basis $\{\mathcal{O}_A \mid A \in \mathcal{P}(X)\}$, where $\mathcal{O}_A$ is the set of all ultrafilters containing A. (The embedding $e : X \to \beta X$ is given by the principal ultrafilters.)

[38] Here $\overline{f}$ is the unique continuous extension to βX of $f : X \to X$. (In terms of ultrafilters, $\overline{f}$ can be defined by putting $A \in \overline{f}(\mathcal{U}) \Leftrightarrow f^{-1}(A) \in \mathcal{U}$.)

Whenever ${}^\star X$ is Hausdorff, the map υ can always be turned into a homeomorphism, either by endowing $\upsilon({}^\star X)$ with a suitably finer topology, or by taking on ${}^\star X$ the (coarser) S-topology. Actually, any Hausdorff extension makes use of the same "function-extending mechanism" as the Stone-Čech compactification. In particular we can characterize all *Hausdorff topological hyperextensions* by means of a reformulation in terms of ultrafilters of the condition (e) of Footnote 31.

Call an ultrafilter $\mathcal{U}$ on X *Hausdorff* if, for all $f, g : X \to X$,

$$(\mathsf{H}) \qquad \overline{f}(\mathcal{U}) = \overline{g}(\mathcal{U}) \iff \{\, x \in X \mid f(x) = g(x)\} \in \mathcal{U}.$$ [39]

Call *invariant* a subspace Y of βX such that $\overline{f}(\mathcal{U}) \in Y$ for all $\mathcal{U} \in Y$ and all $f : X \to X$. Call Y *accessible* if property (p) holds in Y, i.e. for all $\mathcal{U}, \mathcal{V} \in Y$ there exist $\mathcal{W} \in Y$ and $p, q : X \to X$ such that $\mathcal{U} = \overline{p}(\mathcal{W})$ and $\mathcal{V} = \overline{q}(\mathcal{W})$. Then any topological extension ${}^\star X$ is mapped by υ onto an invariant subspace of βX, and we have

Theorem 7.4 ([10], Thm. 1.5). *Every invariant subspace $Y \subseteq \beta X$ is a Hausdorff topological extension of X with the S-topology. Moreover Y is a hyperextension if and only if Y is accessible and all ultrafilters in Y are Hausdorff. Conversely, ${}^\star X$ is a Hausdorff topological extension of X if and only if the map υ is a continuous bijection of ${}^\star X$ onto an invariant subspace of βX. Moreover ${}^\star X$ is a hyperextension if and only if $\upsilon({}^\star X)$ is accessible and contains only Hausdorff ultrafilters.*

7.4. Bolzano extensions and saturation. In our topological context, the enlargement and saturation properties are related to weak compactness properties of the S- and Star topologies. In order to investigate the saturation properties we should isolate a topological counterpart of the notion of internal set. However in the following we only need the obvious assumption that the *basic closed sets* $E(f, \eta)$ of the Star topology are "internal".

Theorem 7.5 ([20], Thm. 6.5). *Let ${}^\star X$ be a topological extension of X.*
1. If ${}^\star X$ is $(2^{|X|})^+$-saturated, then the star topology of ${}^\star X$ is Bolzano.[40] *In particular every set X has Bolzano hyperextensions.*
2. ${}^\star X$ is a $(2^{|X|})^+$-enlargement if and only if the S-topology of ${}^\star X$ is quasi-compact.
3. A hyperextension cannot be simultaneously $(2^{\aleph_0})^+$-enlarging and Hausdorff. In particular there exist no countably compact hyperextensions.

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[39] The property (H) has been introduced in [14] under the name (C). Hausdorff ultrafilters are studied in [21] and [2].

[40] Call *Bolzano* a topological space where every infinite subset has *cluster points*, or equivalently every countable open cover has a finite subcover (so *countably compact* means Bolzano and Hausdorff).

Thus sufficiently saturated hyper-extensions are Bolzano. Every Bolzano extension is necessarily *proper*, hence Weierstraß, by Theorem 7.2. Therefore, in our context, Bolzano-Weierstraß together do not yield *countable compactness.* It seems to us a very interesting consequence of Theorem 7.5 that three important classes of topological spaces, namely *hyper, Hausdorff,* and *Bolzano* extensions have pairwise nonempty intersection, but no common element.

7.5. Simple and homogeneous extensions. We conclude this section by focusing on an interesting class of "minimal" topological extensions of X. We say that ${}^\star X$ is *simple* if it has *no nontrivial invariant subspaces.* We can give various characterizations of simple extensions:

THEOREM 7.6 ([20], Thm. 6.7). *A topological extension ${}^\star X$ of X is simple if and only if any of the following equivalent properties is fulfilled:*

(*i*) *${}^\star X$ is Hausdorff and homogeneous;*[41]
(*ii*) *${}^\star X$ is Hausdorff and all ultrafilters in $\upsilon({}^\star X \setminus X)$ are isomorphic;*[42]
(*iii*) *there exists $\alpha \in {}^\star X$ such that ${}^\star X = \{{}^\star f(\alpha) \mid f : X \to X\}$ and the ultrafilter $\mathcal{U}_\alpha = \upsilon(\alpha)$ is selective.*[43] (*In fact any $\alpha \in {}^\star X \setminus X$ has this property.*)

In particular all simple extensions are Hausdorff hyperextensions.

The *hypernatural* and *hyperreal numbers* obtained *via* simple topological extensions share the following remarkable properties, already underlined in [10], and emphasized in Sections 6 and 10 of this article:

- for any $\alpha \in {}^*\mathbb{N} \setminus \mathbb{N}$, ${}^*\mathbb{N} = \{{}^*g(\alpha) \mid g : \mathbb{N} \to \mathbb{N} \text{ strictly increasing}\}$;
- ${}^*\mathbb{N}$ is a set of *numerosities* in the sense of the Section 10 below;
- ${}^*\mathbb{R}$ satisfies the "Strong Cauchy Infinitesimal Principle" of Section 6 above.

The existence of *simple topological extensions*, corresponding to that of *selective* ultrafilters, is problematic. Many possibilities are consistent with ZFC: that any infinite set has $2^{2^{\aleph_0}}$ *nonisomorphic proper simple extensions*, or that there are *no simple extensions*, or even that any infinite set has *a unique proper simple extension* (see e.g. [12, 13]). The third possibility might be intriguing, yielding as it does a *unique minimal "prime" hyperextension* ${}^*X$ for any infinite set X.

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[41] A convenient notion of *homogeneous* topological extension is obtained by requiring that *any two points* of ${}^\star X \setminus X$ are connected by a homeomorphism of ${}^\star X$ onto itself. (Any such homeomorphism induces a bijection of X, so no topological extension can be topologically homogeneous *stricto sensu.*)

[42] Recall that the ultrafilter $\mathcal{U}$ over I is isomorphic to the ultrafilter $\mathcal{V}$ over J if there is a bijection $\tau : I \to J$ such that $A \in \mathcal{U} \iff \tau(A) \in \mathcal{V}$.

[43] See Footnote 28.

§8. The functional approach. A reflexion on the topological approach to nonstandard models sketched in Section 7 shoud make it apparent that only in Hausdorff extensions, where every function $f : X \to X$ has a *unique* continuous extension ${}^{\star}f : {}^{\star}X \to {}^{\star}X$, the topology is really responsible of the nonstandard structure. In the general case, when uniqueness of continuous extensions gets lost, it is rather the choice of a *distinguished continuous extension* made by the "$\star$" operator that "induces" a topology on ${}^{\star}X$. These considerations suggest that "purely functional" conditions could characterize the hyper-extensions of an arbitrary set, without any mention of topologies. We follow here [26], to which we refer for more details and complete proofs. In that paper, simple supersets ${}^{\star}X$ of X are considered, together with an operator $\star : X^X \to {}^{\star}X^{{}^{\star}X}$, providing a *distinguished extension* of each function $f : X \to X$. Three simple and natural algebraic conditions on the "$\star$" operator are then isolated, namely:

- *preservation of compositions*

(comp) ${}^{\star}g \circ {}^{\star}f = {}^{\star}(g \circ f)$ for all $f, g : X \to X$;

- *preservation of the diagonal*[44]

(diag) if $\chi : X \times X \to \{0, 1\}$ is the characteristic function of the diagonal (i.e. $\chi(x, y) = 1 \iff x = y$), then for all $f, g : X \to X$ and all $\xi \in {}^{\star}X$

$$ {}^{\star}(\chi \circ (f, g))(\xi) = \begin{cases} 1 & \text{if } {}^{\star}f(\xi) = {}^{\star}g(\xi), \\ 0 & \text{otherwise;} \end{cases} $$

- *accessibility of pairs*

(acc) for all $\xi, \eta \in {}^{\star}X$ there exist $\alpha \in {}^{\star}X$ and $p, q : X \to X$ s. t.

$$ {}^{\star}p(\alpha) = \xi, \ {}^{\star}q(\alpha) = \eta. $$

The main theorem of [26] then states that any map $\star : X^X \to {}^{\star}X^{{}^{\star}X}$ satisfying the above conditions can be *uniquely expanded* to all n-ary functions and relations so as to provide a X-complete structure ${}^{\star}\mathfrak{X}$, satisfying the full *transfer principle.*

8.1. The functional extensions. The condition (acc) above seems *prima facie* not to have the same flavour of a "preservation property" shared by the preceding ones.[45] So we adopt the following definition:

[44] We assume that $0, 1 \in X$, in order to have at disposal the usual *characteristic functions.*

[45] However, notwithstanding its apparent *second-order* character, also (acc) shares this feature. As we already remarked for the corresponding property (p) of topological extensions, a *strong uniform version* of accessibility can be obtained as an instance of transfer, appealing to suitable "projections" (see Footnote 32).

Definition 8.1. A superset ${}^{\star}X$ of the set X is a *functional extension* of X if to every function $f : X \to X$ is associated a distinguished $\star$-*extension* ${}^{\star}f : {}^{\star}X \to {}^{\star}X$ in such a way that (comp) and (diag) hold. The functional extension ${}^{\star}X$ is a *hyperextension* if also condition (acc) holds.

It turns out that preserving compositions and diagonal suffices to "well-preserve" characteristic functions, so as to allow for "well-extending" all subsets of X. Namely, if ${}^{\star}X$ is a functional extension of X, then ([26], Thm. 1.2):

1. *If $\chi_A : X \to X$ is the characteristic function of $A \subseteq X$, then the extension ${}^{\star}\chi_A$ is the characteristic function in ${}^{\star}X$ of a set ${}^{\star}A \supseteq A$.*
2. *The map $\star : A \mapsto {}^{\star}A$ commutes with binary union, intersection and complement. Moreover ${}^{\star}A \cap X = A$, hence $\star$ is a boolean isomorphism of $\mathcal{P}(X)$ onto a subfield of $\mathcal{P}({}^{\star}X)$.*

Having so defined $\star$-extensions of sets, the property (diag) gives immediately a sort of "preservation of equalizers", which corresponds to a basic idea of nonstandard analysis that we have already met before, namely that "standard functions behave like germs" (see Footnote 31):

3. ([26], Cor. 1.3) *For all $f, g : X \to X$*
$$\{\xi \in {}^{\star}X \mid {}^{\star}f(\xi) = {}^{\star}g(\xi)\} = {}^{\star}\{x \in X \mid f(x) = g(x)\}$$
or equivalently, for all $\xi \in {}^{\star}X$,
$${}^{\star}f(\xi) = {}^{\star}g(\xi) \iff \exists A \subseteq X. (\xi \in {}^{\star}A \ \ \& \ \forall x \in A \,. f(x) = g(x)\,).$$

Notice that the identity map *may not be preserved* by functional extensions. If $\imath : X \to X$ is the identity of X, in the general case one only obtains

4. ${}^{\star}f(\xi) = {}^{\star}f({}^{\star}\imath(\xi)) = {}^{\star}\imath({}^{\star}f(\xi))$ *for all $f : X \to X$ and all $\xi \in {}^{\star}X$.*

Thus ${}^{\star}\imath$ is the identity exactly on those points of ${}^{\star}X$ which are reached by some function ${}^{\star}f$, and any function ${}^{\star}f$ maps the points ξ and ${}^{\star}\imath(\xi)$ to the same point. When ${}^{\star}\imath$ is not the identity, the extension ${}^{\star}X$ can be considered "redundant", in the sense that the $\star$-extensions of all functions are completely determined by their restrictions to ${}^{\star}\imath({}^{\star}X)$, and the remaining elements of ${}^{\star}X$ are not attained by any function ${}^{\star}f$. Moreover ${}^{\star}\imath({}^{\star}X)$, equipped with the restrictions of all ${}^{\star}f$s, becomes a functional extension where the identity is preserved. Call *irredundant* a functional extension ${}^{\star}X$ of X if every point of ${}^{\star}X$ is in the range of some ${}^{\star}f$, i.e.

(acc_0) for all $\xi \in {}^{\star}X$ there exist $f : X \to X$ and $\eta \in {}^{\star}X$ such that ${}^{\star}f(\eta) = \xi$.

Important preservation properties are derivable only in irredundant extensions, e.g. ([26], Cor. 1.4), for all $f : X \to X$ and all $A \subseteq X$,

5. ${}^{\star}f({}^{\star}A) = {}^{\star}(f(A))$ (*in particular ${}^{\star}f$ is surjective if f is surjective*).
6. *If $f : X \to X$ is injective on A, then ${}^{\star}f$ is injective on ${}^{\star}A$.*
7. *Extensions of finite sets are trivial, i.e. $A = {}^{\star}A$ whenever A is finite.*

In order to make an effective use of functional extensions, all properties above are relevant, and all of them hold in any nonstandard model. So we have in mind essentially only irredundant extensions. We have not singled out irredundancy in Definition 8.1 because this condition is still too weak to obtain the full *transfer principle*. We have chosen instead the slightly stronger property (acc) that every *pair* of points is accessible from a single point. In fact (acc) does the job, as we shall see below.

8.2. The functional hyperextensions. As remarked above, also the property (acc) can be obtained by transfer, and so it has to hold in every nonstandard extension. On the other hand, in combination with (diag) and (comp), it provides *unique unambiguous extensions of all n-ary functions and relations* so as to obtain the full *transfer principle for all first order properties*. For any function $F : X^n \to X$ the $\star$-extension ${}^\star F$ is obtained in the same simple, natural, "parametric" way used in Subsection 7.2 for topological hyperextensions, namely (see [26], Thm. 2.5):

$${}^\star F(\xi_1, \dots, \xi_n) = {}^\star(F \circ (f_1, \dots, f_n))(\alpha),$$

where $f_i : X \to X$ and $\alpha \in {}^\star X$ are such that ${}^* f_i(\alpha) = \xi_i$ for $i = 1, \dots, n$.

For extending n-ary relations one simply appeals to the corresponding characteristic functions in n variables. In this way a *functional hyperextension* ${}^\star X$ of X gives rise to a X-complete structure ${}^\star\mathfrak{X}$ in the sense of Section 2. We could then prove the *transfer principle* for X, inductively on the complexity of the formula σ, and Theorem 2.10 would yield

Theorem 8.2. *A functional extension is a (nonstandard) hyper-extension if and only if it is a functional hyperextension.*

Having at our disposal the topological extensions of Section 7, we prefer to outline a proof of the above theorem obtained by suitably topologizing every functional extension.

8.3. The Star-topology of functional extensions. It is apparent that the properties ($\mathsf{c}, \mathsf{i}, \mathsf{a}, \mathsf{p}$) characterizing topological hyperextensions hold in any functional hyperextension. In fact it was that definition that suggested the choice of the defining properties ($\mathsf{comp}, \mathsf{diag}, \mathsf{acc}$), according to [26]. So it is easily found a topology that turns any functional hyperextension into a topological hyperextension, namely the corresponding *Star topology*, as defined in Subsection 7.2. Again, X is dense in ${}^\star X$ with respect to the Star topology, and we have

Theorem 8.3. *Every functional hyperextension ${}^\star X$ of X, when endowed with the Star topology, becomes a topological hyperextension of X.*

We stated in Theorem 7.3 that *all topological hyperextensions are (nonstandard) hyper-extensions.* Thus Theorem 8.2 can be viewed as an easy corollary of the above theorem. But of course the aim of this "functional approach" is rather that of showing that *a few clear, natural, purely functional* conditions are all what is needed for the strongest requirements of nonstandard models. To this aim either the "algebraic" proof given in [26], or the inductive proof suggested in the preceding subsection seem to be more appropriate.

§9. Hyperintegers as ultrafilters. In the early days of nonstandard analysis, the question was raised as to whether the Stone-Čech compactification $\beta\mathbb{N}$ of $\mathbb{N}$ might be turned into a *nonstandard model* of the natural numbers. Unfortunately, the answer was in the negative (see the discussion in A. Robinson's paper [44]). More important, no extensions of sum and product from $\mathbb{N}$ to $\beta\mathbb{N}$ can be *continuous* (see e.g. [29]). However it has been shown in [22] that extending the entire commutative semiring structure of $\mathbb{N}$ is in fact possible, provided one reduces to *convenient subsets* of $\beta\mathbb{N}$. Here we consider the Stone-Čech compactification $\beta\mathbb{Z}$ of the discrete space $\mathbb{Z}$ of the integers, construed as a space of ultrafilters.

- $\beta\mathbb{Z}$ is the collection of all ultrafilters $\mathcal{U}$ on $\mathbb{Z}$, where each integer is identified with the corresponding *principal* ultrafilter;
- the family of sets of the form ${}^*X = \{\mathcal{U} \in \beta\mathbb{Z} \mid X \in \mathcal{U}\}$ with $X \subseteq \mathbb{Z}$, is a *basis* of *(cl)open* subsets;
- for every $f : \mathbb{Z} \to \mathbb{Z}$, the corresponding (unique) *continuous* extension ${}^*f : \beta\mathbb{Z} \to \beta\mathbb{Z}$ is given by ${}^*f(\mathcal{U}) = \big\{\, X \subseteq \mathbb{Z} \mid f^{-1}(X) \in \mathcal{U} \big\}$.
- we write $f \equiv_{\mathcal{U}} g$ to mean that the functions f and g are *equal* $\mathcal{U}$*-almost everywhere*, i.e. $\{\, n \in \mathbb{Z} \mid f(n) = g(n)\} \in \mathcal{U}$;
- we say that a subset $A \subseteq \beta\mathbb{Z}$ is *invariant* if ${}^*f(\mathcal{U}) \in A$ for all $f : \mathbb{Z} \to \mathbb{Z}$ and all $\mathcal{U} \in A$.[46]

9.1. Ultrafilter rings. We introduce the notion of *ultrafilter ring* as a suitable subset of $\beta\mathbb{Z}$, where the sum and product operations of $\mathbb{Z}$ can be extended in a natural way that preserves the property of being an ordered ring. The resulting structures are then shown to satisfy the *transfer principle*, and so they are sets of *hyperintegers*. A corresponding treatment of *ultrafilter semirings* in $\beta\mathbb{N}$ is given in [22].

Definition 9.1. An ordered ring $(A, \oplus, \odot, <)$ is an *ultrafilter ring* if A is an invariant subspace of $\beta\mathbb{Z}$ such that, for all $f, g : \mathbb{Z} \to \mathbb{Z}$ and all $\mathcal{U} \in A$,

$$(*) \quad {}^*f(\mathcal{U}) \oplus {}^*g(\mathcal{U}) = {}^*(f+g)(\mathcal{U}) \text{ and } {}^*f(\mathcal{U}) \odot {}^*g(\mathcal{U}) = {}^*(f \cdot g)(\mathcal{U}).$$

[46] Recall the *Rudin-Keisler-preordering* $\leq_{RK}$ on ultrafilters: $\mathcal{U} \leq_{RK} \mathcal{V}$ if there exists $f : \mathbb{Z} \to \mathbb{Z}$ s.t. $\mathcal{U} = {}^*f(\mathcal{V})$. Then $A \subseteq \beta(\mathbb{Z})$ is invariant if and only if it is *RK-downward closed*, i.e. $\mathcal{U} \leq_{RK} \mathcal{V} \in A \Longrightarrow \mathcal{U} \in A$.

It is worth noticing that the sum and product operations are completely determined by the conditions $(*)$ above. Hence any invariant subset of $\beta\mathbb{Z}$ admits *at most one* structure of ultrafilter ring. As we identify integers with the corresponding principal ultrafilters, every ultrafilter ring is a superring of $(\mathbb{Z}, +, \cdot)$. It is in fact an *end-extension*:

LEMMA 9.2. *Every ultrafilter ring A is discretely ordered. Hence every $\mathcal{U} \in A$ has an immediate predecessor $\mathcal{U} \ominus 1$, and an immediate successor $\mathcal{U} \oplus 1$; in particular all nonprincipal ultrafilters in A are infinite elements of A.*

More precisely, we have the following characterization, which is the ring counterpart of Theorem 1.6 of [22]:

THEOREM 9.3. *An invariant subset A of $\beta\mathbb{Z}$ admits a (unique) structure of ultrafilter ring if and only if the following conditions are fulfilled:*[47]

(a) *A is accessible, i.e. for all $\mathcal{U}, \mathcal{V} \in A$ there exist $f, g : \mathbb{Z} \to \mathbb{Z}$ and $\mathcal{W} \in A$ such that ${}^*f(\mathcal{W}) = \mathcal{U}$ and ${}^*g(\mathcal{W}) = \mathcal{V}$;*

(b) *every ultrafilter $\mathcal{U} \in A$ satisfies the Hausdorff property*

(H) *for all $f, g : \mathbb{Z} \to \mathbb{Z}$, ${}^*f(\mathcal{U}) = {}^*g(\mathcal{U}) \iff f \equiv_{\mathcal{U}} g$.*

For sake of brevity, call AIH a subset of $\beta\mathbb{Z}$ satisfying the conditions of Theorem 9.3, i.e. an *accessible invariant collection of Hausdorff ultrafilters.* The fact that every ultrafilter ring is AIH can be derived directly from the properties $(*)$ of Definition 9.1, as done in [22] for the corresponding Lemmata 1.3 and 1.5. The converse implication follows from the stronger fact that every AIH subset of $\beta\mathbb{Z}$ comes naturally as a set of *hyperintegers*, satisfying the full *transfer principle*, and so in particular it is a ring satisfying both conditions $(*)$. Namely

THEOREM 9.4. *Every ultrafilter ring is a (nonstandard) hyper-extension of the integers.*

In order to obtain Theorem 9.4 we could simply refer to Theorem 7.4. In fact, the latter theorem states in full generality that if a set X is a discrete dense subspace of the Hausdorff space ${}^\star X$ and every function $f : X \to X$ has a continuous extension ${}^\star f : {}^\star X \to {}^\star X$, then ${}^\star X$ becomes a (nonstandard) hyper-extension in the sense of Section 2 if and only if it is homeomorphic to an AIH subset of βX. Alternatively, we can give a direct "logic" proof by appealing to Theorem 2.10. In fact any AIH subset A of $\beta\mathbb{Z}$ gives a $\mathbb{Z}$-complete structure $\mathfrak{A}(\mathbb{Z}) = \langle A; \{{}^*F \mid F : X^n \to X\}; \{{}^*R \mid R \subseteq X^n\}\rangle$ where:

[47] In terms of the Rudin-Keisler preordering, the condition (a) says that A is *upward directed*. It is the exact counterpart of properties (p) and (acc) of Sections 7 and 8. The Hausdorff condition (H) under (b) is exactly the same of Subsection 7.3.

- $k \in \mathbb{Z}$ is identified with the corresponding principal ultrafilter in A;
- for all $F : \mathbb{Z}^k \to \mathbb{Z}$, for all $\mathcal{U} \in A$ and for all $f_1, \dots, f_k : \mathbb{Z} \to \mathbb{Z}$,
$$^*F\,(^*f_1(\mathcal{U}), \dots, {}^*f_k(\mathcal{U})) = {}^*(F \circ (f_1, \dots, f_k))\,(\mathcal{U});$$[48]
- for all relation $R \subseteq \mathbb{Z}^k$, for all $\mathcal{U} \in A$ and for all $f_1, \dots, f_k : \mathbb{Z} \to \mathbb{Z}$
$$(^*f_1(\mathcal{U}), \dots, {}^*f_k(\mathcal{U})) \in {}^*R \iff \{\, n \in \mathbb{Z} \mid (f_1(n), \dots, f_k(n)) \in R\} \in \mathcal{U}.$$

The defining properties of AIH sets are all what is needed for the above definitions of $^*F$ and $^*R$ to be well posed (see Lemma 2.1 of [22], or Subsection 7.2 above). Now one can prove by induction on the formula σ:

THEOREM 9.5 (see [22], Thm. 2.2). *Let A be an AIH subset of $\beta\mathbb{Z}$ and let $\sigma(x_1, \dots, x_k, F_1, \dots, F_m, R_1, \dots, R_n)$ be an elementary formula. Then, for all $\mathcal{U} \in A$ and all $f_1, \dots, f_k : \mathbb{Z} \to \mathbb{Z}$,*
$$\sigma\,(^*f_1(\mathcal{U}), \dots, {}^*f_k(\mathcal{U}), {}^*F_1, \dots, {}^*F_m, {}^*R_1, \dots, {}^*R_n) \iff$$
$$\{\, n \in \mathbb{Z} \mid \sigma\,(f_1(n), \dots, f_k(n), F_1, \dots, F_m, R_1, \dots, R_n)\} \in \mathcal{U}.$$

It follows that the *transfer principle* for $\mathbb{Z}$ holds, and Theorem 2.10 applies.

9.2. The question of existence. By Theorem 9.3, the existence of nontrivial ultrafilter rings yields the existence of nonprincipal Hausdorff ultrafilters over $\mathbb{Z}$. The converse implication also holds. Given a nonprincipal Hausdorff ultrafilter $\mathcal{U}$ over $\mathbb{Z}$, the subspace $\mathbb{Z}_{\mathcal{U}} = \{^*f(\mathcal{U}) \mid f : \mathbb{Z} \to \mathbb{Z}\}$ of $\beta\mathbb{Z}$ is AIH, hence an ultrafilter ring. (Actually, it is isomorphic to the ultrapower $\mathbb{Z}^{\mathbb{Z}}_{\mathcal{U}}$.) The question of the exact set theoretic strength of this hypothesis has been posed in [20], and it is not yet completely settled. According to results of [21], assuming Martin's Axiom, one obtains a lot of ultrafilter rings isomorphic to iterated ultrapowers, as well as ultrafilter rings which are not ultrapowers. On the other hand, as a consequence of [2], the existence of ultrafilter rings is unprovable in ZFC alone.

§10. Hypernatural numbers as numerosities of countable sets.

A general process of counting needs a set of "numbers" $\mathcal{N}$ and a "counting function" ν that associates to any suitable set A the "number" $\nu(A) \in \mathcal{N}$ of its elements. Let us call *counting system* a triplet $\langle \mathcal{S}, \mathcal{N}, \nu \rangle$ where $\mathcal{S}$ is the family of sets whose "numerosity" is to be counted, $\langle \mathcal{N}, \leq \rangle$ is a *linearly ordered* set of numbers, and ν is a function from $\mathcal{S}$ onto $\mathcal{N}$. Of course we look for extending the finite counting system $\langle Fin, \mathbb{N}, |\cdot| \rangle$, where Fin is the class of all finite sets and $|\cdot|$ is any of the usual ways of counting finite collections.

In principle one would like that a counting system satisfy the following two basic principles:

C1 *if there is a bijection between A and B, then $\nu(A) = \nu(B)$;*

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[48] $F \circ (f_1, \dots, f_k) : \mathbb{Z} \to \mathbb{Z}$ is the function $n \mapsto F(f_1(n), \dots, f_k(n))$.

C2 *if A is a proper subset of B then $\nu(A) < \nu(B)$.*

Moreover one should introduce also the operations of addition and multiplication on $\mathcal{N}$, following the naive intuition that *sums* and *products* of "numbers" directly correspond to the "numerosities" of *disjoint unions* and *Cartesian products*, respectively. So a third principle should be considered, namely

C3 *if $\nu(A) = \nu(A')$ and $\nu(B) = \nu(B')$, then $\nu(A \uplus B) = \nu(A' \uplus B')$ (where $\uplus$ denotes disjoint union) and $\nu(A \times B) = \nu(A' \times B')$.*

Unfortunately, if we want $\mathcal{S}$ to contain infinite sets, it is well known that properties C1 and C2 cannot go together.

By weakening C2 to $\nu(A) \leq \nu(B)$, Cantor developed his theory of *cardinal numbers*, namely a counting system $\langle V, Card, |\cdot|\rangle$, where V is the class of *all sets*, $Card$ is the class of *cardinal numbers* (now commonly taken as initial Von Neumann ordinals), and $|A|$ is the *cardinality* of A (often identified with the least Von Neumann ordinal equipotent to A). Cantor's beautiful theory of cardinals made it possible to deal with infinitely large numbers, but, apart its violation of Aristotle's principle "The whole is larger than its parts", it is not suitable to define infinitely small numbers and develop infinitesimal analysis.[49] This latter negative fact can be viewed as a consequence of the somehow awkward behaviour of sums and products of cardinal numbers.

The question naturally arises as to whether there are alternative ways of counting elements of infinite sets so that property C2 of counting systems can be retained (together with a suitable weakening of C1). More important, can the sum and product operations (defined by means of disjoint unions and Cartesian products) satisfy the usual algebraic properties of natural numbers?[50] And still more demanding, can this extension of the natural numbers be taken as a basis for producing *hyperrational and hyperreal numbers* suitable for the practice of nonstandard analysis?

All these questions have been given a first possible positive answer in [6] where a suitably structurated class of (countable) sets is considered. Notice that putting a structure on the sets to be counted is a natural way of overcoming the contrast between the principles C1 and C2. In fact, in this case, only bijections and subsets "which preserve the structure" are considered. E.g. Cantor's theory of ordinals can be viewed as a counting system $\langle WO, Ord, \overline{\cdot}\,\rangle$, where WO is the class of *well-ordered sets*, Ord is the class of *ordinals*, and $\overline{A}$ is the *order-type* of A. Then both C1 and C2 hold, restricted to *order-isomorphisms* and to *initial segments*, respectively. Notice

[49] By "infinitesimal" analysis we mean analysis where actual "infinitesimal" numbers are available.

[50] *Videlicet* correspond to the *non-negative part of a discretely ordered ring.*

however that ordinal arithmetic is quite unusual: e.g., commutativity fails even for addition, and $1+\alpha=\alpha<\alpha+1$ for all infinite α.

The aim of this section is to shortly present the contents of the paper [6]. An alternative proof of the main results proved there are outlined below by using the language of the topological extensions of Section 7.

10.1. Counting labelled sets. We start from the observation that very often, in counting the "numerosity" of a given set, one previously splits it into parts to be counted separately, and then takes the "ultimate value" of the sequence of partial sums. (Obviously such a sequence is eventually constant if the given set is finite.) If we want to apply this procedure to an infinite set, we have to partition it into a sequence of finite parts.[51] Equivalently, we have to give each element a "label" in $\mathbb{N}$, say. We are thus led to the notion of *labelled set* of [6]:

Definition 10.1. A *labelled set* is a pair $\mathbf{A}=\langle A,\ell_A\rangle$ where A is a set (the *domain* of $\mathbf{A}$) and $\ell_A:A\to\mathbb{N}$ (the *labelling function* of $\mathbf{A}$) is finite-to-one.[52]

Thus the domain A is the union of the non-decreasing sequence of finite sets $A_n=\{a\in A\mid \ell_A(a)\le n\}$, whose finite cardinality $|A_n|$ is a sort of n^{th} approximation to the numerosity of $\mathbf{A}$. Following [6], we label disjoint unions and Cartesian products so as to be consistent with the corresponding finite approximations:[53]

Definition 10.2. The *disjoint union* and the *Cartesian product* of the labelled sets $\mathbf{A}=\langle A,\ell_A\rangle$ and $\mathbf{B}=\langle B,\ell_B\rangle$ are

$$\mathbf{A}\uplus\mathbf{B}=\langle A\uplus B,\ell_A\uplus\ell_B\rangle,\text{ with }(\ell_A\uplus\ell_B)(x)=\begin{cases}\ell_A(x) & \text{if } x\in A\\ \ell_B(x) & \text{if } x\in B\end{cases}$$
$$\mathbf{A}\times\mathbf{B}=\langle A\times B,\ell_A\vee\ell_B\rangle,\text{ with }(\ell_A\vee\ell_B)(x,y)=\max\{\ell_A(x),\ell_B(y)\}$$

We say that $\mathbf{A}=\langle A,\ell_A\rangle$ is a *labelled subset* of $\mathbf{B}=\langle B,\ell_B\rangle$, and write $\mathbf{A}\sqsubseteq\mathbf{B}$, if $A\subseteq B$ and $\ell_A(a)=\ell_B(a)$ for all $a\in A$. Similarly for the strict inclusion $\mathbf{A}\sqsubset\mathbf{B}$. If $\mathbf{A}\sqsubseteq\mathbf{B}$, then we denote by $\mathbf{B}\setminus\mathbf{A}$ the labelled subset of $\mathbf{B}$ whose domain is $B\setminus A$. An *isomorphism* (resp. *equivalence*) between the labelled sets $\mathbf{A}$ and $\mathbf{B}$ is a (almost) *label-preserving bijection* $\varphi:A\to B$ such that $\ell_B\circ\varphi=\ell_A$ (resp. $\ell_B(\varphi(a))=\ell_A(a)$ for all but finitely many $a\in A$). We write $\mathbf{A}\cong\mathbf{B}$ if $\mathbf{A}$ and $\mathbf{B}$ are isomorphic, and $\mathbf{A}\sim\mathbf{B}$ if they are equivalent. Call *equalizer* of $\mathbf{A}$ and $\mathbf{B}$ the set $E(\mathbf{A},\mathbf{B})=\{n\in\mathbb{N}\mid |A_n|=|B_n|\}$. Clearly $\mathbf{A}\cong\mathbf{B}$ (resp. $\mathbf{A}\sim\mathbf{B}$) holds if and only if $E(\mathbf{A},\mathbf{B})=\mathbb{N}$ (resp. $E(\mathbf{A},\mathbf{B})$ is cofinite in $\mathbb{N}$).

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[51] So we can deal only with countable sets.

[52] I.e., for any given n, there are only finitely many $a\in A$ such that $\ell_A(a)=n$.

[53] I.e. $|\{x:(\ell_{A\uplus B})(x)\le n\}|=|A_n|+|B_n|$ and $|\{x:(\ell_{A\times B})(x)\le n\}|=|A_n|\cdot|B_n|$ for all n.

The crucial definition of [6] is obtained by postulating, for labelled sets, ("slightly" strengthened) formulations of the basic principles C1-C3.[54]

Definition 10.3. A map $\nu : \mathcal{L} \to \mathcal{N}$ from the class $\mathcal{L}$ of labelled sets onto a linearly ordered set $\langle \mathcal{N}, \leq \rangle$ is a *numerosity* function if the following conditions are fulfilled:

(N1) if $|A_n| \leq |B_n|$ for all n, then $\nu(\mathbf{A}) \leq \nu(\mathbf{B})$;
(N2) $\nu(\mathbf{A}) < \nu(\mathbf{B})$ if and only if $\nu(\mathbf{A}) = \nu(\mathbf{B}')$ for some $\mathbf{B}' \sqsubset \mathbf{B}$;
(N3) if $\nu(\mathbf{A}) = \nu(\mathbf{A}')$ and $\nu(\mathbf{B}) = \nu(\mathbf{B}')$, then $\nu(\mathbf{A} \uplus \mathbf{B}) = \nu(\mathbf{A}' \uplus \mathbf{B}')$ and $\nu(\mathbf{A} \times \mathbf{B}) = \nu(\mathbf{A}' \times \mathbf{B}')$

All the defining properties above are suggested by the naive idea that the numerosity $\nu(\mathbf{A})$ of the labelled set $\mathbf{A}$ is the "ultimate value" of the sequence $\nu_{\mathbf{A}}$ of the cardinalities $\nu_{\mathbf{A}}(n) = |A_n|$ of the finite approximations of $\mathbf{A}$.[55] In particular *equivalent* labelled sets have *equal* numerosities and *proper* labelled subsets have *smaller* numerosities. The "only if" part of property (N2) postulates that the numerosities of the labelled subsets of any labelled set are an initial segment of $\mathcal{N}$. This supplementary assumption has surprisingly far reaching consequences, and it is actually responsible for the positive answer to the last two questions posed above. First of all we have that the set $\mathcal{N}$ of the numerosities inherites a "good" algebraic structure:

Theorem 10.4 ([6], Props. 1.4 and 2.3). *Put $\nu(\mathbf{A}) + \nu(\mathbf{B}) = \nu(\mathbf{A} \uplus \mathbf{B})$ and $\nu(\mathbf{A}) \cdot \nu(\mathbf{B}) = \nu(\mathbf{A} \times \mathbf{B})$: then $\mathcal{N}$ becomes the set of non-negative elements of a discretely ordered (commutative) ring. In particular $\mathbb{N}$ is (isomorphic to) an initial segment of $\mathcal{N}$.*

In fact a much stronger property holds, namely that $\mathcal{N}$ is a very special set of hypernatural numbers, as we shall see below.

10.2. From numerosities to hyper-extensions. In [6] it is proved that $\mathcal{N}$ is a set of numerosities if and only if $\mathcal{N} \cong \mathbb{N}^{\mathbb{N}}_{\mathcal{U}}$ where $\mathcal{U}$ is a nonprincipal *selective* ultrafilter.[56] The key lemma of the proof is the following:

Lemma 10.5 ([6], Props. 3.3-4). *Put $\mathcal{U} = \{E(\mathbf{A}, \mathbf{B}) \mid \nu(\mathbf{A}) = \nu(\mathbf{B})\}$. Then*

(*i*) *$\nu(\mathbf{A}) = \nu(\mathbf{B})$ if and only if $E(\mathbf{A}, \mathbf{B}) \in \mathcal{U}$;*
(*ii*) *$\mathcal{U}$ is a (nonprincipal) selective ultrafilter over $\mathbb{N}$.*

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[54] Direct reformulations of C1 and C2 should have equality instead of $\leq$ in (N1), and the sole "if" part in (N2).

[55] Notice that every *non-decreasing* sequence $\sigma \in \mathbb{N}^{\mathbb{N}}$ can be so obtained: put $\mathbf{A} = \langle \mathbb{N}, \ell_\sigma \rangle$, where $\ell_\sigma(n) = k$ for $\sigma(k-1) \leq n < \sigma(k)$.

[56] See Footnote 28.

Here, given a numerosity function $\nu : \mathcal{L} \to \mathcal{N}$, we intend to endow $\mathcal{N}$ with a topology that turns $\mathcal{N}$ into a *simple topological hyperextension* of $\mathbb{N}$, in the sense of Subsection 7.5 above.

Given $f : \mathbb{N} \to \mathbb{N}$ we can define the extension ${}^*f : \mathcal{N} \to \mathcal{N}$ by ${}^*f(\nu(\mathbf{A})) = \nu(\mathbf{B})$, where $\mathbf{B}$ is any labelled set such that $\nu_{\mathbf{B}} \equiv_{\mathcal{U}} f \circ \nu_{\mathbf{A}}$. Such a $\mathbf{B}$ exists by (ii), since any sequence is $\mathcal{U}$-equivalent to a non-decreasing one, and the definition is well-posed by (i). The extension so obtained satisfies property (iii) of Theorem 7.6, because every infinite numerosity $\nu_{\mathbf{A}}$ can be obtained as ${}^*f(\alpha)$, where α is the numerosity of $\mathbb{N}^+$ with the identity labelling, and $f \in \mathbb{N}^{\mathbb{N}}$ is any bijective function agreeing with $\nu_{\mathbf{A}}$ on some $E \in \mathcal{U}$.[57] (By definition the ultrafilter $\mathcal{U}_\alpha$ is $\mathcal{U}$, and so selective.)

In particular, recalling Theorems 7.4 and 7.6, we obtain that $\mathcal{N}$ is isomorphic to the invariant subspace generated by $\mathcal{U}$ in the Stone-Čech compactification $\beta\mathbb{N}$ of $\mathbb{N}$. This subspace is in turn isomorphic to the ultrapower $\mathbb{N}^{\mathbb{N}}_{\mathcal{U}}$. Within this framework, what the proof of Theorem 4.3 of [6] actually proves is the following characterization:

THEOREM 10.6. *Every homogeneous subspace Y of $\beta\mathbb{N}$ is a set of numerosities, and for any $\alpha \in Y \setminus \mathbb{N}$ there is exactly one numerosity function $\nu : \mathcal{L} \to Y$ such that $\nu(\mathbb{N}^+) = \alpha$ (and α corresponds to the ultrafilter of equalizers for ν). Conversely, every numerosity function has a set of values $\mathcal{N}$ canonically isomorphic to the homogeneous subspace of $\beta\mathbb{N}$ generated by its ultrafilter of equalizers.*

So in effect any set of numerosities provides a very special set of hypernatural numbers, namely a *simple topological hyperextension* of $\mathbb{N}$. We have already remarked, at the end of Section 7, that the existence of simple topological extensions is independent of Zermelo-Fraenkel set theory. So we conclude that, although numerosity functions are defined by means of elementary properties that are naturally satisfied by the intuitive process of counting, their existence cannot be proved in the usual axiomatic framework of mathematics. Given the well known strong incompleteness of ZFC, this fact might be used to evaluate possible candidates for new axioms.

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