A PURELY ALGEBRAIC CHARACTERIZATION OF THE HYPERREAL NUMBERS

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Abstract. The hyper-real numbers of nonstandard analysis are characterized in purely algebraic terms as homomorphic images of a suitable class of rings of functions.

1. Introduction

Since the seminal classical work by E. Hewitt [7] appeared over sixty years ago, the algebraic/topological study of rings of functions has been constantly alive in the literature (see e.g. [6], [1], [10], [4] and [9]). Recently, in their book [5], G. Dales and H. Woodin gave new insights to the subject by deeply investigating a class of totally ordered real fields, namely the *superreal fields*. Among them, the so-called *hyperreal fields* and the *ultrapowers*. Now, all *ultrapowers* are hyperreals ${}^*\mathbb{R}$ of nonstandard analysis (nonstandard reals), but the two notions of hyperreal fields are different.

The very definition of nonstandard reals ${}^*\mathbb{R}$ as usually given in the literature, requires notions from mathematical logic. Precisely, such definition is formulated by means of the *Leibniz transfer principle*, an elementary embedding property for bounded quantifier formulas in the language of set theory.

The goal of this paper is to provide an alternative equivalent definition of ${}^*\mathbb{R}$ in purely algebraic (and elementary) terms. Precisely, we shall characterize the hyperreal fields of nonstandard analysis as homomorphic images of *composable rings* $\mathcal{F}$ of real-valued functions ("composable" means closed under compositions with any function $f : \mathbb{R} \to \mathbb{R}$.)

From a philosophical point of view, our proposed definition of ${}^*\mathbb{R}$ could be justified by the following facts.

- The operations on $\mathcal{F}$ are defined point-wise, hence the operations on ${}^*\mathbb{R}$ are directly inherited from the usual field operations on $\mathbb{R}$.
- A crucial feature of the nonstandard real numbers ${}^*\mathbb{R}$ is that every function $f : \mathbb{R} \to \mathbb{R}$ has a *nonstandard extension* ${}^*f : {}^*\mathbb{R} \to {}^*\mathbb{R}$ that satifies the same "elementary" properties. Thanks to composability, the nonstandard extension ${}^*f$ can be defined in a natural way, by means of its natural "lifting" $\widehat{f} : \mathcal{F} \to \mathcal{F}$ given by $\widehat{f}(\varphi) = f \circ \varphi$.
- There is no need to postulate the *Leibniz transfer principle*, because that logical principle follows from our definition.

As a side result, this definition makes it possible to naturally accomodate the nonstandard reals in the Dales-Woodin's algebraic hierarchy of *superreal fields* [5].

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2. The Hyperreal Numbers of Nonstandard Analysis

For a detailed presentation of the *superstructure approach* to nonstandard analysis, and for the unexplained notions and notation, we refer to [3] §4.4. For completeness, we briefly recall here the crucial definitions.

Definition 2.1. For any set X of atoms, the *superstructure over* X is the set $V(X) = \bigcup_{n\in\mathbb{N}} V_n(X)$, where $V_0(X) = X$ and $V_{n+1}(X) = \mathcal{P}(V_n(X))$ is the powerset of $V_n(X)$. A *nonstandard embedding* is a mapping $* : V(\mathbb{R}) \to V({}^*\mathbb{R})$ that satisfies the *Leibniz transfer principle*, i.e. for every bounded quantifier formula $\sigma(x_1, \ldots, x_n)$ and elements $a_1, \ldots, a_n \in V(\mathbb{R})$, $\sigma(a_1, \ldots, a_n) \Leftrightarrow \sigma({}^*a_1, \ldots, {}^*a_n)$. It is assumed that ${}^*r = r$ for every $r \in \mathbb{R}$ and that ${}^*\mathbb{R}$ is a set of atoms.

By *bounded quantifier formula* we mean a first-order formula in the language $\mathcal{L} = \{\in\}$ of set theory, where all quantifiers occur in the bounded forms $\forall x \in y \ldots$ (i.e. $\forall x\ x \in y \to \ldots$) or $\exists x \in y \ldots$ (i.e. $\exists x\ x \in y \wedge \ldots$). In the literature, nonstandard embeddings also satisfy the condition $\mathbb{N} \neq {}^*\mathbb{N}$, but for simplicity we do not assume it here. In particular, also the identity map on $V(\mathbb{R})$ is allowed as (the trivial) nonstandard embedding.

Definition 2.2. A field $\mathbb{F}$ is a set of *nonstandard reals* (or *hyperreal numbers of nonstandard analysis*) if there exists a nonstandard embedding $* : V(\mathbb{R}) \to V({}^*\mathbb{R})$ where ${}^*\mathbb{R} = \mathbb{F}$.

In [8], H.J. Keisler showed that, up to isomorphisms, the nonstandard reals are precisely the *limit ultrapowers* of $\mathbb{R}$.[1] The characterization theorem we present in the next section could be proved by taking that result as a starting point. However, we prefer to give a direct proof, in order to make this paper self-contained and to keep our treatment as close to the basic language of algebra as possible.

3. The Characterization Theorem

Definition 3.1. Let R be a given ring. For any set I, denote by R^I the ring of all functions $\varphi : I \to R$ where operations are defined pointwise. A *ring of (R-valued) functions* $\mathcal{F}$ is a subring of some R^I. It is assumed that a ring of functions contains all constant functions.

The crucial notion we shall use in the sequel is the following.

Definition 3.2. A ring of functions $\mathcal{F} \subseteq R^I$ is *composable* if for every $\varphi \in \mathcal{F}$ and for every $f : R \to R$, the composition $f \circ \varphi : I \to R$ is in $\mathcal{F}$.

Rings of the form R^I are trivially composable. Other examples are

$$\mathcal{F} = \{\varphi : I \to R \mid |\operatorname{ran}\varphi| \leq \aleph_0\},$$

the rings of those functions taking at most countably many values. We remark that composability can be seen as a *lifting* property, because it allows extending each function $f : R \to R$ to a function $\widehat{f} : \mathcal{F} \to \mathcal{F}$ by putting $\widehat{f}(\varphi) = f \circ \varphi$.

We are now ready to prove the characterization theorem that we propose as an alternative definition of nonstandard reals.

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[1] The limit ultrapowers are a generalization of the ultrapowers. Definitions and basic results can be found in [3] §6.4.

Theorem 3.3. *A field $\mathbb{F}$ is a set of nonstandard reals if and only if it is a homomorphic image of some composable ring $\mathcal{F}$ of real-valued functions.*

Proof. Assume first that there is a surjective ring-homomorphism $J : \mathcal{F} \twoheadrightarrow \mathbb{F}$ where $\mathcal{F} \subseteq \mathbb{R}^I$ is a composable ring of real-valued functions. Without loss of generality we can assume that $J(c_r) = r$ for all $r \in \mathbb{R}$, where c_r denotes the constant function with value r. We have to show that there is a nonstandard embedding $* : V(\mathbb{R}) \to V(\mathbb{F})$.

For every $\varphi \in \mathcal{F}$, denote by $Z(\varphi) = \{i \in I \mid \varphi(i) = 0\}$ its *zero set*. Then the family $\{Z(\varphi) \mid J(\varphi) = 0\}$ is a filter base that can be extended to an ultrafilter $\mathcal{U}$ on I. On the set of functions:

$$\mathcal{G} = \{\varphi : I \to A \mid A \in V(\mathbb{R}) \text{ and } \exists \varphi' \in \mathcal{F}\ \exists h \text{ with } \varphi = h \circ \varphi'\}$$

consider the equivalence relation: $\varphi \sim \psi \ \Leftrightarrow \{i \in I \mid \varphi(i) = \psi(i)\} \in \mathcal{U}$ and the *pseudo-membership relation*: $\psi \lhd \varphi \ \Leftrightarrow \{i \in I \mid \psi(i) \in \varphi(i)\} \in \mathcal{U}$. Then define the mapping $\Psi : \mathcal{G}/\!\sim \to V(\mathbb{F})$ by putting

$$\Psi([\varphi]) = J(\vartheta) \text{ if } \varphi \sim \vartheta \in \mathcal{F}, \text{ and } \Psi([\varphi]) = \{\Psi([\psi]) \mid \psi \lhd \varphi\} \text{ otherwise.}$$

Without loss of generality, we are assuming that $\mathbb{F}$ is a set of atoms. It can be directly verified that the above definition is well-posed. The mapping Ψ satisfies the following version of *Łos theorem*. For every $\varphi_1, \ldots, \varphi_n \in \mathcal{G}$ and for every bounded quantifier formula $\sigma(x_1, \ldots, x_n)$:

$$\sigma(\Psi([\varphi_1]), \ldots, \Psi([\varphi_n])) \ \Leftrightarrow \ \{i \in I \mid \sigma(\varphi_1(i), \ldots, \varphi_n(i))\} \in \mathcal{U}.$$

The proof is by induction on the complexity of formulas. Everything is straightforward, except one implication at the quantifier step, where the *composability* property of $\mathcal{F}$ is used in an essential way. Precisely, let $\varphi'_s \in \mathcal{F}$ and let $\varphi_s = h_s \circ \varphi'_s \in \mathcal{G}$ for $s = 0, \ldots, n$. Assume that

$$\Lambda = \{i \in I \mid \exists x \in \varphi_0(i)\ \sigma(x, \varphi_1(i), \ldots, \varphi_n(i))\} \in \mathcal{U}.$$

Let $\mathcal{B}$ be a base of $\mathbb{R}$ as a vector space on $\mathbb{Q}$. Since $\mathcal{B}$ has the power of the continuum, we can find 1-1 maps $f_s : \mathbb{R} \to \mathcal{B}$ with pairwise disjoint ranges. By the composability of $\mathcal{F}$, the function $\psi = (\sum_{i=0}^{s} f_s \circ \varphi'_s) \in \mathcal{F}$. Notice that, by linear independency, $\psi(i) = \psi(j) \Rightarrow (f_s \circ \varphi'_s)(i) = (f_s \circ \varphi'_s)(j)$ for all $s \Rightarrow \varphi'_s(i) = \varphi'_s(j)$ for all s, hence $\varphi_s(i) = \varphi_s(j)$ for all s. In particular, there exists a function ζ such that:

- For every $i \in \Lambda$, $\zeta(i) \in \varphi_0(i)$ witnesses $\sigma(\zeta(i), \varphi_1(i), \ldots, \varphi_n(i))$,
- $\zeta(i) = \zeta(j)$ whenever $\psi(i) = \psi(j)$.

As a straight consequence of the latter property, there is a function h with $\zeta = h \circ \psi$, hence $\zeta \in \mathcal{G}$. We can now apply the inductive hypothesis and obtain

$$\Psi([\zeta]) \in \Psi([\varphi_0]) \wedge \sigma(\Psi([\zeta]), \Psi([\varphi_1]), \ldots, \Psi([\varphi_n])).$$

Now define $* : V(\mathbb{R}) \to V(\mathbb{F})$ as the mapping where ${}^*r = r$ if $r \in \mathbb{R}$, and ${}^*A = \{\Psi([\varphi]) \mid \varphi : I \to A\}$ otherwise. The definition is well-posed and ${}^*\mathbb{R} = \mathbb{F}$. Since Ψ satisfies Łos theorem, it is easily seen that the *Leibniz transfer principle* holds and so $*$ is the desired nonstandard embedding.[2]

[2] The particular case $\mathcal{F} = \mathbb{R}^I$ of this implication was treated in [2].

Vice versa, assume that $*: V(\mathbb{R}) \to V({}^*\mathbb{R})$ is a nonstandard embedding. Let I be the set of all finite collections of hyperreal numbers and real functions $f : \mathbb{R}^k \to \mathbb{R}$ (of several variables).

Our next goal is to find a function $\Phi : {}^*\mathbb{R} \to \mathbb{R}^I$ and a maximal ideal M in such a way that the composition $K = \pi \circ \Phi : {}^*\mathbb{R} \to \mathbb{R}^I/M$ is a 1-1 ring-homomorphism ($\pi : \mathbb{R}^I \to \mathbb{R}^I/M$ is the canonical projection).

We claim that for every $i \in I$, one can find a mapping $\chi_i : {}^*\mathbb{R} \to \mathbb{R}$ such that:

(1) $\chi_i(x) = x$ for all real numbers $x \in i$;
(2) $\chi_i({}^*f(a_1, \ldots, a_k)) = f(\chi_i(a_1), \ldots, \chi_i(a_k))$ for all k-variable functions $f \in i$, and for all hyperreals $a_1, \ldots, a_k \in i$.

Enumerate all the equalities ${}^*f_j(a_{j1}, \ldots, a_{jk_j}) = {}^*g_j(b_{j1}, \ldots, b_{jh_j})$ for $j = 1, \ldots, n$, where the functions $f_j : \mathbb{R}^{k_j} \to \mathbb{R}$ and $g_j : \mathbb{R}^{h_j} \to \mathbb{R}$ are in i, and the hyperreal numbers $a_{jl}, b_{jl} \in i$. Then the following bounded formula is true:

$$\exists x_{11}, \ldots, x_{nk_n}, y_{11}, \ldots, y_{nh_n} \in {}^*\mathbb{R} \left(\bigwedge_{j=1}^{n} {}^*f_j(x_{j1}, \ldots, x_{jk_j}) = {}^*g_j(y_{j1}, \ldots, y_{jh_j}) \right).$$

By the *Leibniz transfer principle*, there are $r_{jl}, s_{jl} \in \mathbb{R}$ that satisfy all the corresponding standard equalities $f_j(r_{j1}, \ldots, r_{jk_j}) = g_j(s_{j1}, \ldots, s_{jh_j})$ for $j = 1, \ldots, n$. Notice that, by simple modifications of the above formula (if needed), we can assume the following:

(a) $r_{jl} = a_{jl}$ (and $s_{jl} = b_{jl}$) whenever $a_{jl} \in \mathbb{R}$ (or $b_{jl} \in \mathbb{R}$, respectively).
(b) $r_{jl} = r_{j'l'}$ (and $s_{jl} = s_{j'l'}$) whenever $a_{jl} = a_{j'l'}$ (or $b_{jl} = b_{j'l'}$, respectively).
(c) $r_{jl} = s_{j'l'}$ whenever $a_{jl} = b_{j'l'}$.

Namely, property (a) can be obtained by omitting those existential quantifiers that correspond to *real* numbers (and by directly considering them as parameters). As for (b) and (c), one adds to the formula the corresponding equalities $x_{jl} = x_{j'l'}$, $y_{jl} = y_{j'l'}$, and $x_{jl} = y_{j'l'}$.

As a consequence of (a), (b) and (c), a mapping $\chi : {}^*\mathbb{R} \to \mathbb{R}$ can be defined in such a way that $\chi_i(a_{jl}) = r_{jl}$ and $\chi_i(b_{jl}) = s_{jl}$. In particular, the required properties (1) and (2) are fulfilled.

Now define $\Phi : {}^*\mathbb{R} \to \mathbb{R}^I$ by putting $\Phi(a) = \Phi_a$, where $\Phi_a(i) = \chi_i(a)$ for all $i \in I$. By condition (1), $\Phi(r) = c_r$ for every $r \in \mathbb{R}$.

For any given $a, b \in {}^*\mathbb{R}$, let $j(a, b) \in I$ be the finite collection which consists of a, b and of the sum and product functions $+, \cdot : \mathbb{R}^2 \to \mathbb{R}$.

By condition (2), for all $i \supseteq j(a, b)$:

- $\Phi_a(i) + \Phi_b(i) = \chi_i(a) + \chi_i(b) = \chi_i(a + b) = \Phi_{a+b}(i)$;
- $\Phi_a(i) \cdot \Phi_b(i) = \chi_i(a) \cdot \chi_i(b) = \chi_i(a \cdot b) = \Phi_{a \cdot b}(i)$.

In particular, for all $a, b \in {}^*\mathbb{R}$, both $\Phi_a + \Phi_b - \Phi_{a+b}$ and $\Phi_a \cdot \Phi_b - \Phi_{a \cdot b}$ belongs to the following ideal:

$$P = \left\{ \varphi \in \mathbb{R}^I \mid \exists j \in I \text{ such that } \varphi(i) = 0 \text{ for all } i \supseteq j \right\}.$$

Now pick M any maximal ideal extending P, and let $\pi : \mathbb{R}^I \twoheadrightarrow \mathbb{R}^I/M$ be the canonical projection onto the corresponding quotient field. Then the composition $K = \pi \circ \Phi : {}^*\mathbb{R} \to \mathbb{R}^I/M$ is a ring-homomorphism because $K(a) + K(b) - K(a+b) = 0$ and $K(a) \cdot K(b) - K(a \cdot b) = 0$ for all $a, b \in {}^*\mathbb{R}$. Notice that K is necessarily 1-1. In order to get an isomorphism out of K, consider the following family of functions:

$$\mathcal{F} = \begin{array}{l}\big\{\varphi \in \mathbb{R}^I \mid \exists\, a_1, \ldots, a_n \in {}^*\mathbb{R} \text{ such that} \\ \text{if } \chi_i(a_s) = \chi_j(a_s) \text{ for all } s = 1, \ldots, n \ \text{ then } \varphi(i) = \varphi(j)\big\}\end{array}$$

A straightforward verification proves that $\mathcal{F}$ is a composable subring of $\mathbb{R}^I$. Since trivially $\operatorname{ran}\Phi \subseteq \mathcal{F}$, it makes sense to consider the composition

$$K' = \pi' \circ \Phi : {}^*\mathbb{R} \to \mathbb{F},$$

where $\pi' : \mathcal{F} \twoheadrightarrow \mathbb{F} = \mathcal{F}/M'$ is the restriction of π that projects $\mathcal{F}$ onto its quotient field modulo $M' = M \cap \mathcal{F}$. K' is a 1-1 ring-homomorphism because K is.

In order to prove that K is an isomorphism, we are left to show that for every $\varphi \in \mathcal{F}$, there exists $b \in {}^*\mathbb{R}$ with $\pi(\varphi) = \pi(\Phi_b)$. By the definition of $\mathcal{F}$, there are finitely many hyperreals $a_1, \ldots, a_n$ such that $\varphi(i) = \varphi(j)$ whenever $\chi_i(a_s) = \chi_j(a_s)$ for all $s = 1, \ldots, n$. But then we can pick a function $f : \mathbb{R}^n \to \mathbb{R}$ such that $f(\chi_i(a_1), \ldots, \chi_i(a_n)) = \varphi(i)$ for all $i \in I$. If $b = {}^*f(a_1, \ldots, a_n)$, by the condition (2) above, $\Phi_b(i) = \varphi(i)$ for all $i \supseteq \{a_1, \ldots, a_n, f\} \in I$, hence $\pi(\Phi_b) = \pi(\varphi)$ as desired. The composition $J = {K'}^{-1} \circ \pi' : \mathcal{F} \twoheadrightarrow {}^*\mathbb{R}$ is the surjective ring-homomorphism we were looking for. □

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References

[1] M.Y. Antonovskij, D.V. Chudnovsky, G.V. Chudnovsky and E. Hewitt, *Rings of real-valued continuous functions. II*, Math. Z. **176** (1981), 151–186.

[2] V. Benci and M. Di Nasso, *A ring homomorphism is enough to get nonstandard analysis*, Bull. Belg. Math. Soc. – S. Stevin, to appear.

[3] C.C. Chang and H.J. Keisler, *Model Theory* (3rd edition), North-Holland, 1990.

[4] G.L. Cherlin and M.A. Dickmann, *Real closed rings I. Residue rings of rings of continuous functions*, Fund. Math. **126** (1986), 147–183.

[5] H.G. Dales and W.H. Woodin, *Super-real Fields*, Oxford University Press, 1996.

[6] L. Gillman and M. Jerison, *Rings of Continuous Functions*, Van Nostrandt, 1960.

[7] E. Hewitt, *Rings of real-valued continuous functions*, Trans. Amer. Math. Soc. **64** (1948), 45–99.

[8] H.J. Keisler, *Foundations of Infinitesimal Calculus*, Prindle, Weber & Schmidt, 1976.

[9] J.J. Moloney, *Residue class domains of the ring of convergent sequences and of* $C^\infty([0, 1], \mathbb{R})$, Pacific J. Math. **143** (1990), 79–153.

[10] J. Roitman, *Non-isomorphic hyper-real fields from non-isomorphic ultrapowers*, Math. Z. **181** (1982), 93–96.

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