NUMEROSITIES OF LABELLED SETS: A NEW WAY OF COUNTING

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Abstract. The notions of "labelled set" and "numerosity" are introduced to generalize the counting process of finite sets. The resulting numbers, called numerosities, are then used to develop nonstandard analysis. The existence of a numerosity function is equivalent to the existence of a selective ultrafilter, hence it is independent of the axioms of ZFC.

Introduction.

Similarly as cardinals and ordinals, the "numerosities" we present in this paper originate as an attempt to extending the notion of finite cardinality. By considering suitable "labellings", we show that a notion of numerosity for (countable) infinite sets can be defined in such a way that the usual properties of finite cardinalities are preserved. Most notably, the numerosity of a proper subset is strictly smaller than the numerosity of the whole set; the numerosity of a disjoint union is the sum of the numerosities; and the numerosity of a Cartesian product is the product of the numerosities. We remark that these properties together are neither satisfied by cardinal nor by ordinal algebras.

We show that the set $\mathcal{N}$ of numerosities can be identified with the set $\mathbb{N}^*$ of the hypernatural numbers of nonstandard analysis. In fact, starting from a numerosity function, a nonstandard embedding $* : V_\infty(\mathbb{N}) \to V_\infty(\mathcal{N})$ can be defined in such a way that the Leibniz transfer principle and the countable saturation property are satisfied.

Although defined by means of elementary properties that are naturally satisfied by the intuitive process of counting, quite surprisingly the existence of numerosities is independent of the axioms of ZFC. Precisely, we prove that a numerosity function exists if and only if there exists a selective ultrafilter.

For unexplained set theoretic notions and notation, we refer to [12].

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1. Labelled Sets and their Numerosity.

In order to count elements one needs a set of numbers $\mathcal{N}$ and a "counting function" ν that associates to a given set A the "numerosity" $\nu(A) \in \mathcal{N}$ of its elements. Precisely, we call *counting system* a triplet $\langle \mathcal{S}, \mathcal{N}, \nu \rangle$ where $\mathcal{S}$ is the family of sets

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whose "numerosity" is to be counted, $\langle \mathcal{N}, \leq \rangle$ is a linearly ordered set of numbers, and ν is a function from $\mathcal{S}$ onto $\mathcal{N}$. The following are some basic features one would like a counting system to satisfy.[1]

(1) If there is a bijection between A and B, then $\nu(A) = \nu(B)$;
(2) If A is a proper subset of B then $\nu(A) < \nu(B)$;
(3) If $\nu(A) = \nu(A')$ and $\nu(B) = \nu(B')$, then the corresponding disjoint unions and Cartesian products satisfy:

$$\nu(A \uplus B) = \nu(A' \uplus B');\ \ \nu(A \times B) = \nu(A' \times B')$$

Naive intuition suggests that sums and products of "numerosities" directly correspond to the "numerosities" of disjoint unions and Cartesian products, respectively. This motivates the latter property itemized above.

If we take $\mathcal{S} = \mathcal{S}_{\text{fin}}$ the class of finite sets, $\mathcal{N} = \mathbb{N}$ the set of natural numbers, and $\nu = \sharp$ the usual finite cardinality, we get a counting system $\langle \mathcal{S}_{\text{fin}}, \mathbb{N}, \sharp \rangle$ that satisfies all three properties itemized above. If we want $\mathcal{S}$ to contain infinite sets, it is well known that properties 1 and 2 cannot go together. By dropping 2, Cantor developed the theory of *cardinal numbers*, namely the theory of the counting system $\langle \mathcal{V}, Card, \|\ \| \rangle$, where $\mathcal{V}$ is the universal class of all sets, $Card$ is the class of cardinal numbers (i.e. initial ordinals), and $\|A\|$ is the cardinal number of A (i.e. the unique cardinal equipotent to A). Starting from the informal idea of arranging elements of a given set on a line, and then counting them "one by one", Cantor developed another theory, namely the theory of *ordinals*. In this case, the counting system he considered was $\langle \mathcal{WO}, Ord, |\ \ | \rangle$, where $\mathcal{WO}$ is the class of well-ordered sets, Ord the class of ordinals, and $|A|$ the order-type of A.

Ordinals and cardinals provide two different ways for counting infinite sets, so different symbols are used. For instance, the cardinal number of $\mathbb{N}$ is denoted by $\aleph_0$ (the first infinite cardinal number), while the ordinal number of $\mathbb{N}$ is denoted by ω (the first infinite ordinal number). In this paper we shall denote by α the "numerosity" of $\mathbb{N}$.

Cardinals and ordinals made it possible to deal with infinitely large numbers, but they are not suitable to define infinitely small numbers and develop infinitesimal analysis.[2] This latter negative fact is a consequence of the awkward behavior of the sum and product operations. For instance, if a and b are two infinite cardinals, then:

$$a + b = a \cdot b = \max\{a, b\}$$

Notice that these equalities contradict property 2 of counting systems. When counting by cardinal numbers, one may add elements to an infinite set without making it bigger! If a and b are infinite ordinal numbers, things are even worse. In fact, the sum and product operations are not even commutative. Furthermore, if one adds an element at the bottom of an infinite well-ordered set, its ordinal number does not change, while its ordinal number becomes bigger if the element is added at the top. For instance:

$$1 + \omega = \omega < \omega + 1$$

Now a question naturally arises. Is there an alternative way of counting elements of infinite sets so that property 2 of counting systems can be retained? And also, can this be done is such a way that the sum and product operations (defined by means

[1] See [9] for a discussion of pedagogical aspects of the idea of "counting".
[2] By "infinitesimal" analysis we mean analysis where actual "infinitesimal" numbers exist.

of disjoint sums and Cartesian products) satisfy the usual algebraic properties of natural numbers? As shown in the sequel, the answer is positive for countable sets.

Roughly speaking, the idea is to count the "numerosity" of a given set by suitably splitting it into finite parts, and then considering the sequence of partial sums. For example, in order to count the inhabitants of the world, one could first divide the world into nations, then count the inhabitants of each nation, and finally consider the "approximating" sequence given by the number of inhabitants of the first n nations.

If we want to apply this procedure to any set, we need a criterion to distinguish between elements. In other words, we need to give each element a "label", similarly as in the above example each inhabitant of the world is given a nationality. To this end we formalize a notion of labelled set. It consists of a set plus a labelling function that partitions it into a countable family of finite (possibly empty) subsets.

Definition 1.1. A *labelled set* is a pair $\mathbf{A} = \langle A, \ell_A \rangle$ where the *domain* A is a set and the *labelling function* $\ell_A : A \to \mathbb{N}$ is finite-to-one.[3]

Thus the domain A is obtained as the union of the non-decreasing sequence of finite sets:

$$A_0 \subseteq A_1 \subseteq \ldots \subseteq A_n \subseteq A_{n+1} \subseteq \ldots$$

where $A_n = \{a : \ell_A(a) \leq n\}$. We shall refer to the finite cardinality $\sharp A_n$ as the n^{th} *pproximation* to the actual "numerosity" of $\mathbf{A}$, and refer to the sequence $n \mapsto \sharp A_n$ as the *approximating sequence* (to the numerosity) of $\mathbf{A}$.

If not mentioned otherwise, subsets of $\mathbb{N}$ will always be given the canonical labelling $\ell(n) = n$. Similarly, subsets of $\mathbb{Z}$ will always be equipped with the "almost" canonical labelling given by the absolute value. Denote by $\mathbf{0}$ the empty labelled set, and for each $n \in \mathbb{N}$, denote by $\mathbf{n} = \{0, \ldots, n-1\}$. These are the *canonical* finite labelled sets. We say that $\mathbf{A} = \langle A, \ell_A \rangle$ is a *labelled subset* of $\mathbf{B} = \langle B, \ell_B \rangle$, and write $\mathbf{A} \subseteq \mathbf{B}$, if $A \subseteq B$ and $\ell_A(a) = \ell_B(a)$ for all $a \in A$. Similarly for the strict inclusion $\mathbf{A} \subset \mathbf{B}$. If $\mathbf{A} \subseteq \mathbf{B}$, then we denote by $\mathbf{B} \setminus \mathbf{A}$ the labelled subset of $\mathbf{B}$ whose domain is $B \setminus A$.

If $n = \sharp A$ and $m = \sharp B$ are cardinalities of finite sets, then $n+m$ is the cardinality of the disjoint union $A \uplus B$ and $n \cdot m$ is the cardinality of the Cartesian product $A \times B$. Accordingly, we give the following definition for labelled sets:

Definition 1.2. The *sum* of two labelled sets is $\mathbf{A} \oplus \mathbf{B} = \langle A \uplus B, \ell_A \oplus \ell_B \rangle$ where

$$(\ell_A \oplus \ell_B)(x) = \begin{cases} \ell_A(x) & \text{if } x \in A \\ \ell_B(x) & \text{if } x \in B \end{cases}$$

The *product* is the labelled set $\mathbf{A} \odot \mathbf{B} = \langle A \times B, \ell_A \odot \ell_B \rangle$ where

$$(\ell_A \odot \ell_B)(x, y) = \max\{\ell_A(x); \ell_B(y)\}$$

Notice that the above definitions are consistent with sums and products of the finite approximations. That is, $\sharp\{x : (\ell_A \oplus \ell_B)(x) \leq n\} = \sharp A_n + \sharp B_n$ and $\sharp\{x : (\ell_A \odot \ell_B)(x) \leq n\} = \sharp A_n \cdot \sharp B_n$ for all n.

We are now ready to give the crucial definition.

Definition 1.3. A *numerosity function* for the class $\mathcal{L}$ of labelled sets is a map

$$num : \mathcal{L} \twoheadrightarrow \mathcal{N}$$

[3]That is, for any given n, it can be $\ell_A(a) = n$ only for finitely many a.

onto a linearly ordered set $\langle \mathcal{N}, \leq \rangle$ of *numerosities* such that the following properties are satisfied:

(**1**) If $\sharp A_n \leq \sharp B_n$ for all n, then $num(\mathbf{A}) \leq num(\mathbf{B})$;
(**2**) $\xi < num(\mathbf{A})$ if and only if $\xi = num(\mathbf{B})$ for some $\mathbf{B} \subset \mathbf{A}$;
(**3**) If $num(\mathbf{A}) = num(\mathbf{A}')$ and $num(\mathbf{B}) = num(\mathbf{B}')$ then $num(\mathbf{A} \oplus \mathbf{B}) = num(\mathbf{A}' \oplus \mathbf{B}')$ and $num(\mathbf{A} \odot \mathbf{B}) = num(\mathbf{A}' \odot \mathbf{B}')$.

The first property formalizes the idea that if all finite approximations indicate that the numerosity of $\mathbf{A}$ is not greater then the numerosity of $\mathbf{B}$, then this is actually the case. In particular, if $\mathbf{A} \subseteq \mathbf{B}$ then $num(\mathbf{A}) \leq num(\mathbf{B})$. The second property postulates on the one hand that proper subsets have strictly smaller numerosity; on the other hand that every labelled set contains subsets for each numerosity which is smaller than its own. Finally, the third property simply says that numerosities are consistent with the sum and product operations of labelled sets. Hence sums and products of numerosities can be defined in terms of disjoint unions and Cartesian products, respectively. We remark that these three properties together are neither satisfied by cardinal nor by ordinal numbers.

The following proposition itemizes immediate consequences of the definition.

Proposition 1.4.

(i) *$\mathcal{N}$ has a least element $0 = num(\mathbf{0})$.*
(ii) *All labelled singletons have the same numerosity 1.*
(iii) *Every numerosity $\xi = num(\mathbf{A})$ has successor $\xi + 1$, namely the numerosity of $\mathbf{A} \oplus \{\star\}$ where $\{\star\}$ is any labelled singleton. Moreover, if $\mathbf{A} \neq \mathbf{0}$, then $\xi = num(\mathbf{A})$ has also predecessor $\xi - 1$.*[4]
(iv) *If $\mathbf{A} = \langle A, \ell_A \rangle$ is finite, then $num(\mathbf{A}) = \sharp A$ is the cardinality of A.*

Since $\mathcal{N}$ contains a proper initial segment that is order-isomorphic to the set of natural numbers $\mathbb{N}$, for simplicity we directly assumed $\mathbb{N} \subset \mathcal{N}$ and denoted by n the n^{th} successor of $num(\mathbf{0})$.

Proof. (i) directly follows from property 2, because the empty labelled set contains no proper subsets.
(ii) By contradiction, let $\{*\}$ and $\{\star\}$ be two labelled singletons with different numerosities, say $num(\{*\}) < num(\{\star\})$. By property 2, there is $\mathbf{A} \subset \{\star\}$ with $num(\mathbf{A}) = num(\{*\})$. As a proper subset of a singleton, it must be $\mathbf{A} = \mathbf{0}$. Hence $\mathbf{A} \subset \{*\}$ and $num(\mathbf{A}) < num(\{*\})$, a contradiction.
(iii) Assume by contradiction that $num(\mathbf{A}) < \xi < num(\mathbf{A} \oplus \{\star\})$ for some $\xi \in \mathcal{N}$. Take $\mathbf{B} \subset \mathbf{A} \oplus \{\star\}$ with $num(\mathbf{B}) = \xi$ and take $\mathbf{C} \subset \mathbf{B}$ with $num(\mathbf{C}) = num(\mathbf{A})$. Now pick $b \in B \setminus C$ and denote by $\{\mathbf{b}\}$ the labelled subset of $\mathbf{B}$ with domain $\{b\}$. By (ii), $num(\{\mathbf{b}\}) = num(\{\star\})$. Thus, by property 3, $num(\mathbf{C}) = num(\mathbf{A})$ implies that $num(\mathbf{C} \oplus \{\mathbf{b}\}) = num(\mathbf{A} \oplus \{\mathbf{b}\})$. But the finite approximations $\sharp(\mathbf{C} \oplus \{\mathbf{b}\})_n \leq \sharp B_n$ for all n, and so $num(\mathbf{A} \oplus \{\mathbf{b}\}) = num(\mathbf{C} \oplus \{\mathbf{b}\}) \leq num(\mathbf{B}) < num(\mathbf{A} \oplus \{\star\})$, a contradiction. If $a \in A$, then it is easily verified that $num(\mathbf{A} \setminus \{\mathbf{a}\})$ is the predecessor of $num(\mathbf{A})$.
(iv) Straightforward (proceed by induction on $n = \sharp A$ and use the previous properties). $\square$

[4]Recall that $\xi + 1$ is the *successor* of ξ if $\xi < \xi + 1$ and $\zeta < \xi + 1$ implies $\zeta \leq \xi$. We say that $\xi - 1$ is the *predecessor* of ξ if ξ is the successor of $\xi - 1$.

Thanks to property 3, the following natural definitions of sum and product of numerosities are well-posed:

$$num(\mathbf{A}) + num(\mathbf{B}) = num(\mathbf{A} \oplus \mathbf{B})\,;\ \ num(\mathbf{A}) \cdot num(\mathbf{B}) = num(\mathbf{A} \odot \mathbf{B})$$

2. Integer and Rational Numerosities

In order to investigate the algebraic structure given by the sum and product operation on the set of numerosities, we need to look more carefully to the class of labelled sets. A notion of isomorphism is defined in a natural way.

Definition 2.1. Two labelled sets $\mathbf{A} \cong \mathbf{B}$ are *isomorphic* if there exists a bijection $f : A \to B$ that preserves the labellings, i.e. such that $\ell_B \circ f = \ell_A$.

$$\begin{tikzcd} & \mathbb{N} & \\ A \arrow[ur, "\ell_A"] \arrow[rr, "f"] & & B \arrow[ul, "\ell_B"'] \end{tikzcd}$$

Clearly, $\mathbf{n} \cong \mathbf{m} \Leftrightarrow n = m$. We remark that there is plenty of non-isomorphic finite labelled sets. For instance, if $\mathbf{A} = \langle \{a\}, \ell_A \rangle$ and $\mathbf{B} = \langle \{b\}, \ell_B \rangle$ are two singletons, then $\mathbf{A} \cong \mathbf{B}$ if and only if $\ell_A(a) = \ell_B(b)$. The notion of isomorphism is consistent with sums, products, and numerosities of labelled sets. In fact, the following is easily checked from the definitions.

Proposition 2.2.

(i) $\mathbf{A} \cong \mathbf{B} \Leftrightarrow \sharp A_n = \sharp B_n$ *for all* n*;*

(ii) *If* $\mathbf{A} \cong \mathbf{B}$ *then* $num(\mathbf{A}) = num(\mathbf{B})$*;*

(iii) *If* $\mathbf{A} \cong \mathbf{A}'$ *and* $\mathbf{B} \cong \mathbf{B}'$ *then* $\mathbf{A} \oplus \mathbf{B} \cong \mathbf{A}' \oplus \mathbf{B}'$ *and* $\mathbf{A} \odot \mathbf{B} \cong \mathbf{A}' \odot \mathbf{B}'$.

A *semi-ring* is a a triple $\langle R, +, \cdot \rangle$ where $+$ and $\cdot$ are associative operations, $+$ is commutative and distributivity holds. Thus a ring is a semi-ring where inverse elements with respect to $+$ exist. A *partially ordered semi-ring* is a semi-ring together with a partial order $\le$ such that $x \le y$ implies $x+z \le y+z$ and $x \cdot z \le y \cdot z$ for all x, y, z. A *positive* semi-ring is a commutative p.o. semi-ring where $x \le y$ if and only if there exists a *unique* z such that $y = x + z$. The natural numbers $\mathbb{N}$ are the prototype of positive semi-ring. Anothere relavant example of positive semi-ring is the collection of all functions from $\mathbb{N}$ to $\mathbb{N}$. The subfamilies of non-decreasing and of increasing functions are p.o. semi-rings (but not positive).[5] The same holds for the set of numerosities $\mathcal{N}$.

Proposition 2.3. $\langle \mathcal{N}, +, \cdot, \mathbf{0}, \mathbf{1} \le, \rangle$ *is a positive semi-ring with neutral elements.*

Proof. That $\mathcal{N}$ is a semi-ring is proved by using the fact that isomorphic labelled sets have the same numerosity. In fact, distributivity holds because $\mathbf{A} \odot (\mathbf{B} \oplus \mathbf{C}) \cong (\mathbf{A} \odot \mathbf{B}) \oplus (\mathbf{A} \odot \mathbf{C})$. Commutativity and associativity similarly follow from the following facts: $\mathbf{A} \oplus \mathbf{B} \cong \mathbf{B} \oplus \mathbf{A}$, $\mathbf{A} \odot \mathbf{B} \cong \mathbf{B} \odot \mathbf{A}$, $\mathbf{A} \oplus (\mathbf{B} \oplus \mathbf{C}) \cong (\mathbf{A} \oplus \mathbf{B}) \oplus \mathbf{C}$ and $\mathbf{A} \odot (\mathbf{B} \odot \mathbf{C}) \cong (\mathbf{A} \odot \mathbf{B}) \odot \mathbf{C}$. Moreover, $\mathbf{0}$ and $\mathbf{1}$ are the neutral elements because $\mathbf{A} \oplus \mathbf{0} \cong \mathbf{A}$, $\mathbf{A} \odot \mathbf{0} \cong \mathbf{0}$ and $\mathbf{A} \odot \mathbf{1} \cong \mathbf{A}$. Now let us turn to the ordering. Assume that $num(\mathbf{A}) \le num(\mathbf{B})$, and let $\mathbf{C}$ be given. Pick $\mathbf{A}' \subseteq \mathbf{B}$ with $num(\mathbf{A}') = num(\mathbf{A})$. Then $\mathbf{A}' \odot \mathbf{C} \subseteq \mathbf{B} \odot \mathbf{C} \Rightarrow num(\mathbf{A}) \odot num(\mathbf{C}) = num(\mathbf{A}' \odot \mathbf{C}) \le num(\mathbf{B} \odot \mathbf{C}) = num(\mathbf{B}) \odot num(\mathbf{C})$. Similarly, $\le$ is compatible with sums. Now, if $\mathbf{D} = \mathbf{B} \setminus \mathbf{A}'$, then clearly $\mathbf{A}' \oplus \mathbf{D} \cong \mathbf{B}$ and so $num(\mathbf{A}) + num(\mathbf{D}) = num(\mathbf{A}' \oplus \mathbf{D}) = num(\mathbf{B})$.

[5] Operations and ordering on rings of functions are defined pointwise, i.e. $h = f + g$ is the function such that $h(n) = f(n) + g(n)$ for all n, etc.

Finally, uniqueness is easily seen by showing that that $num(\mathbf{A}) + num(\mathbf{D}') = num(\mathbf{A}) + num(\mathbf{D})$ implies $num(\mathbf{D}') = num(\mathbf{D})$. $\square$

Let $\alpha = num(\mathbb{N})$ denote the numerosity of natural numbers. Then the numerosity of the Cartesian product $\mathbb{N} \times \mathbb{N}$ is $num(\mathbb{N} \odot \mathbb{N}) = \alpha^2$. Notice also that $\mathbb{Z} \cong \mathbb{N} \oplus (\mathbb{N} \setminus \mathbf{0})$, thus $num(\mathbb{Z}) = 2\alpha - 1$ is the predecessor of $\alpha + \alpha$.

We remark that in general the assignment of numerosities to labelled sets is not uniquely determined.[6]

As a consequence of the previous Proposition, subtraction for numerosities $\xi \geq \eta$ is well posed: Let $\xi - \eta$ be the *unique* ζ such that $\eta + \zeta = \xi$.

By a well-known procedure, the integers $\mathbb{Z}$ can be presented as ordered pairs of natural numbers identified modulo the following equivalence relation:

$$\langle a, b\rangle \sim \langle a', b'\rangle \Leftrightarrow a + b' = a' + b$$

The pair $\langle a, b\rangle$ is to be thought as $a - b$. Accordingly, each natural number n is identified with the equivalence class of the pair $\langle n, 0\rangle$. The operations and the ordering are defined as follows:

$$\langle a, b\rangle + \langle c, d\rangle = \langle a + c, b + d\rangle;\ \langle a, b\rangle \cdot \langle c, d\rangle = \langle ac + bd, bc + ad\rangle;$$
$$\langle a, b\rangle \leq \langle c, d\rangle \Leftrightarrow a + d \leq b + c.$$

Since $\mathbb{N}$ is a linearly ordered positive semi-ring with neutral elements, it is proved that the above construction yields a linearly ordered commutative ring with identity, where $x \leq y \Leftrightarrow y = x + z$ for some $z \in \mathbb{N}$. In the same way, starting from $\mathcal{N}$, we get the linearly ordered commutative ring $\mathcal{Z}$ of *integer numerosities*. Notice that $\mathcal{Z}$ has no zero divisors (because $\mathcal{N}$ has not), and so we can consider its fraction field:

$$\mathcal{Q} = \left\{ \pm \tfrac{num(\mathbf{A})}{num(\mathbf{B})} : \mathbf{A}, \mathbf{B} \text{ labelled sets, } \mathbf{B} \neq \mathbf{0} \right\}$$

We call $\mathcal{Q}$ the ordered field of *rational numerosities*. An element $\xi \in \mathcal{Q}$ is *bounded* if $-n < \xi < n$ for some natural number n. We say that ε is *infinitesimal* if $-1/n < \varepsilon < 1/n$ for all $n > 0$. Clearly α is unbounded (i.e. not bounded) and its reciprocal $1/\alpha$ is infinitesimal. So $\mathcal{Q}$ is non-archimedean.

Now let $\mathcal{Q}_b$ be the collection of bounded elements, and $\imath$ the collection of infinitesimals. Notice that $\mathcal{Q}_b$ is a subring of $\mathcal{Q}$ and $\imath$ is a maximal ideal of $\mathcal{Q}_b$. In fact, $\imath$ is closed under addition; if ξ is bounded and ε is infinitesimal then the product $\xi \cdot \varepsilon$ is infinitesimal; and maximality holds because every bounded (non-zero) numerosity whose inverse is unbounded is infinitesimal. As a consequence, the quotient of $\mathcal{Q}_b$ modulo $\imath$ is an ordered field. But we can say more:

Theorem 2.4. *The quotient $\mathcal{Q}_b/\imath$ and the real numbers $\mathbb{R}$ are isomorphic as ordered fields.*[7]

In particular, if we denote by $\approx$ the equivalence relation of being infinitely close (i.e. $\xi \approx \eta \Leftrightarrow \xi - \eta$ is infinitesimal) then every real number r is represented (up to infinitesimals) as a fraction:

$$r \approx \pm \tfrac{num(\mathbf{A})}{num(\mathbf{B})}$$

Thus, while a rational number is the ratio of the numerosities of two finite sets, a real number is the ratio of the numerosities of two labelled sets (up to infinitesimals).

[6] See the remarks at the foot of Theorem 4.3.

[7] The proof of this Theorem will be obtained in the next section as a consequence of Proposition 3.8.

For example, in our context, the real number $\sqrt{2}$ is represented as follows:

$$\sqrt{2} \approx \frac{num(\{1^2, 2^2, 3^2, \ldots\})}{num(\{2\cdot 1^2, 2\cdot 2^2, 2\cdot 3^2, \ldots\})}$$

We remark that by isolating a few properties of the numerosity $\alpha = num(\mathbb{N})$, an alternative presentation of nonstandard analysis can be given in truly elementary terms (see [3],[8] and [4]).

3. From numerosities to nonstandard analysis

In this section we show that the numerosity function allows getting the full strength of the superstructure approach to nonstandard analysis. In order to make the exposition as smooth as possible, most of the proofs in this section will be postponed to the appendix at the foot of the paper.

Let us first briefly recall the basics of the superstructure approach.[8] The *superstructure* $V_\infty(X)$ is the union $V_\infty(X) = \bigcup_k V_k(X)$ of the finite levels of the cumulative hierarchy over X, namely $V_0(X) = X$, and $V_{k+1}(X) = V_k(X) \cup \wp(V_k(X))$ is the union of the elements and subsets of the previous step. In the literature, it is always assumed that $X \supseteq \mathbb{N}$ is a set of atoms, at least with respect to its superstructure.[9] A mapping $* : V_\infty(X) \to V_\infty(Y)$ between the *standard model* $V_\infty(X)$ and the *nonstandard model* $V_\infty(Y)$ is a *nonstandard embedding* if the following are satisfied:

(1) ${}^*x = x$ for all $x \in X$ and ${}^*X = Y$.
(2) $\mathbb{N} \neq {}^*\mathbb{N}$.
(3) *Transfer Principle*: An "elementary property" σ is true about standard elements $a_1, \ldots, a_n$ if and only if it is true about the corresponding nonstandard elements ${}^*a_1, \ldots, {}^*a_n$.

The *transfer principle* is formalized as follows. For every bounded formula $\sigma(x_1, \ldots, x_n)$ in the language of set theory, and for every $a_1, \ldots, a_n \in V_\infty(X)$, $V_\infty(X) \models \sigma(a_1, \ldots, a_n) \Leftrightarrow V_\infty(Y) \models \sigma({}^*a_1, \ldots, {}^*a_n)$.

A set $B \in V_\infty(Y)$ is called *internal* if $B \in {}^*A$ for some $A \in V_\infty(X)$. In particular, all sets of the form ${}^*A$ are internal. A nonstandard embedding is *countably saturated* if every countable family $\mathcal{B}$ of internal sets with the finite intersection property (i.e. such that $\bigcap \mathcal{B}' \neq \emptyset$ for all finite subfamilies $\emptyset \neq \mathcal{B}' \subset \mathcal{B}$) has nonempty intersection $\bigcap \mathcal{B} \neq \emptyset$. It is assumed that $\mathcal{B} \subseteq {}^*A$ for some A.

We already remarked that the family of functions:

$$\mathcal{F} = \{\varphi : \mathbb{N} \to \mathbb{N} : \varphi \text{ is non-decreasing }\}$$

is a p.o. semi-ring. For every labelled set $\mathbf{A}$, the approximating sequence $\gamma_A : n \mapsto \sharp A_n$ is non-decreasing, hence in $\mathcal{F}$. Vice versa, every $\varphi \in \mathcal{F}$ is the approximating sequence γ_A of some labelled set $\mathbf{A}$. In fact, let

$$A_{\varphi,0} = \{0, 1, \ldots, \varphi(0) - 1\} \text{ and } A_{\varphi,n} = \{0, 1, \ldots, \varphi(n) - \varphi(n-1) - 1\} \text{ for } n \geq 1$$

(some of these sets may be empty). Let $A_\varphi = \biguplus_n A_{\varphi,n}$ be the disjoint union of the sets $A_{\varphi,n}$ and let ℓ_φ be the labelling such that $\ell_\varphi(a) = n \Leftrightarrow a \in A_{\varphi,n}$. Clearly $\mathbf{A}_\varphi = \langle A_\varphi, \ell_\varphi \rangle$ is a labelled set and $\sharp(A_\varphi)_n = \varphi(n)$ for all n. Recall that two

[8] The reader is referred to [6] §4.4 for unexplained notions and notation. Organic presentation of nonstandard real analysis by the superstructure approach are in [14], [7], [11], [10]. Excellent surveys of recent applications of nonstandard methods can be found in [1].

[9] That is, $x \cap (V_\infty(X) \setminus X) = \emptyset$ for each $x \in X$. E.g., in a non-wellfounded context, one can take *autosingletons* $x = \{x\}$ as atoms.

labelled sets having the same approximating sequence have the same numerosity (and they are isomorphic). Thus the following definition is well-posed.

Definition 3.1. Let $\rho : \mathcal{F} \twoheadrightarrow \mathcal{N}$ be the map such that $\rho(\varphi) = num(\mathbf{A}_\varphi)$.

ρ is an homomorphism of p.o. semi-rings. In fact, it is easily checked that $\mathbf{A}_{\varphi+\psi} \cong \mathbf{A}_\varphi \oplus \mathbf{A}_\psi$ and $\mathbf{A}_{\varphi\cdot\psi} \cong \mathbf{A}_\varphi \odot \mathbf{A}_\psi$, hence $\rho(\varphi+\psi) = \rho(\varphi)+\rho(\psi)$ and $\rho(\varphi\cdot\psi) = \rho(\varphi)\cdot\rho(\psi)$. We are left to show that ρ is order-preserving. Let $\varphi \leq \psi$. Then for every n, $\sharp(\mathbf{A}_\varphi)_n = \varphi(n) \leq \psi(n) = \sharp(\mathbf{A}_\psi)_n$, and so $\rho(\varphi) = num(\mathbf{A}_\varphi) \leq num(\mathbf{A}_\psi) = \rho(\psi)$.

Starting from ρ, we want to define a countably saturated nonstandard embedding $* : V_\infty(\mathbb{N}) \longrightarrow V_\infty(\mathcal{N})$.[10]

Definition 3.2. A set $D \subseteq \mathbb{N}$ is *qualified* if there exist $\varphi, \psi \in \mathcal{F}$ with $\rho(\varphi) = \rho(\psi)$ and $D = \{n : \varphi(n) = \psi(n)\}$.

In other words, D is qualified if and only if there exist two equinumerous labelled sets $\mathbf{A}$ and $\mathbf{B}$ such that D is the set of levels n at which the finite approximations $\sharp A_n = \sharp B_n$ agree. The next Proposition shows that the qualified sets are the "large sets" with respect to a measure on $\mathbb{N}$.

Proposition 3.3. *Let D, E be subsets of $\mathbb{N}$. Then*
- (i) $D = \{n : \varphi(n) = \psi(n)\}$ *is qualified* $\Leftrightarrow \rho(\varphi) = \rho(\psi)$;
- (ii) *If D is qualified and $E \supseteq D$, then E is qualified.*
- (iii) *If D and E are qualified, then $D \cap E$ is qualified;*
- (iv) *D is qualified $\Leftrightarrow$ its complement $D^c = \mathbb{N} \setminus D$ is not;*
- (v) *If D is finite, then D is not qualified.*

Let $\mu(D) = 1$ if D is qualified, and $\mu(D) = 0$ otherwise. Then the above Proposition says that μ is a $\{0,1\}$-valued finitely additive measure on $\mathbb{N}$ where finite sets have measure zero.

The next result states that every sequence of natural numbers is non-decreasing μ-almost everywhere, i.e. is the approximating sequence of some labelled set μ-almost everywhere.

Proposition 3.4. *For every function $\varphi : \mathbb{N} \to \mathbb{N}$ there is a non-decreasing ψ such that $\{n : \varphi(n) = \psi(n)\}$ is qualified.*

We now extend the map ρ to all functions in $\mathcal{F}_\infty = \bigcup_k \mathcal{F}_k$, where $\mathcal{F}_k = \{\varphi \mid \varphi : \mathbb{N} \to V_k(\mathbb{N})\}$. For $\varphi, \psi \in \mathcal{F}_\infty$, we write $\varphi =_\mu \psi$ if φ equals ψ μ-a.e., i.e. if $\{n : \varphi(n) = \psi(n)\}$ is qualified. Notice that $=_\mu$ is an equivalence relation. Similarly, we write $\varphi \in_\mu \psi$ if $\{n : \varphi(n) \in \psi(n)\}$ is qualified.

Proposition 3.5. *There exists a unique map $\boldsymbol{\rho} : \mathcal{F}_\infty \to V_\infty(\mathcal{N})$ such that:*
- (i) $\boldsymbol{\rho}(\varphi) = \rho(\varphi)$ *for all $\varphi \in \mathcal{F}$, i.e. $\boldsymbol{\rho}$ extends ρ;*
- (ii) *If $c_\emptyset$ is the sequence constantly equal to the empty set, then $\boldsymbol{\rho}(c_\emptyset) = \emptyset$;*
- (iii) $\boldsymbol{\rho}(\varphi) = \boldsymbol{\rho}(\psi) \Leftrightarrow \varphi =_\mu \psi$;
- (iv) $\boldsymbol{\rho}(\varphi) \in \boldsymbol{\rho}(\psi) \Leftrightarrow \varphi \in_\mu \psi$.

Definition 3.6. For every $x \in V_\infty(\mathbb{N})$, let ${}^*x = \boldsymbol{\rho}(c_x)$ where c_x is the constant sequence with value x.

[10] Without loss of generality, we are assuming that all elements of $\mathcal{N} \supset \mathbb{N}$ are atoms.

As a straight consequence of the proof of the previous Proposition (see the appendix), we get:

$^*n = n$ if $n \in \mathbb{N}$ is an atom ; $^*A = \{\boldsymbol{\rho}(\varphi) \mid \varphi : \mathbb{N} \to A\}$ if $A \in V_\infty(\mathbb{N}) \setminus \mathbb{N}$ is a set.

By induction, it is also easily seen that $\varphi \in \mathcal{F}_k$ implies $\boldsymbol{\rho}(\varphi) \in V_k(\mathcal{N})$, hence $*$ takes values in the superstructure over $\mathcal{N}$.

Theorem 3.7. *The map $* : V_\infty(\mathbb{N}) \to V_\infty(\mathcal{N})$ is a countably saturated nonstandard embedding, whose collection of internal elements is precisely the range of $\boldsymbol{\rho}$.*

In particular, the set of numerosities $\mathcal{N} = \mathbb{N}^*$ is a set of hypernaturals (i.e. nonstandard natural numbers). Moreover, the set of integer numerosities $\mathcal{Z}$ and the set of rational numerosities $\mathcal{Q}$ coincide with the set of hyperintegers $^*\mathbb{Z}$ and hyperrationals $^*\mathbb{Q}$, respectively.

For $\xi, \eta \in \mathbb{Q}$, denote $\xi \approx \eta$ whenever $\xi - \eta$ is infinitesimal, and let

$$^*\mathbb{Q}_b = \{\xi \in {}^*\mathbb{Q} : |\xi| < n \text{ for some } n \in \mathbb{N}\}$$

be the collection of *bounded hyperrationals.* The following is a well known fact in nonstandard analysis (see [7] §2.2).

Proposition 3.8. *The quotient $^*\mathbb{Q}_b/\!\approx$ and the real numbers $\mathbb{R}$ are isomorphic as ordered fields.*

Since $\mathcal{Q} = {}^*\mathbb{Q}$, this proves Theorem 2.4.

4. On the existence of a numerosity function.

In this section we shall show that, quite surprisingly, the existence of a numerosity function is independent of the axioms of ZFC. It is remarkable that the three elementary properties of Definition 1.3 – postulated to describe a naive intuition of "counting" – are already outside the scope of the usual foundational framework of mathematics.

A nonempty family $\mathcal{U}$ of subsets of I is called *ultrafilter* over I if it is closed under supersets and under finite intersections, and if for every $A \subseteq I$, either $A \in \mathcal{U}$ or its complement $A^c \in \mathcal{U}$. If no finite subsets belongs to $\mathcal{U}$, then $\mathcal{U}$ is called *non-principal.*[11] By Proposition 3.3, the family of qualified sets is a non-principal ultrafilter over $\mathbb{N}$.[12]

We now concentrate on ultrafilters over $\mathbb{N}$ that satisfy additional properties.

Proposition 4.1. *Let $\mathcal{U}$ be a non-principal ultrafilter over $\mathbb{N}$. Then the following properties are equivalent:*

(i) *For every partition $\{X_n : n \in \mathbb{N}\}$ of $\mathbb{N}$ where $X_n \notin \mathcal{U}$ for every n, there exists a "selective" set $X \in \mathcal{U}$ such that each intersection $X \cap X_n$ contains at most one element;*
(ii) *For every function $\varphi : \mathbb{N} \to \mathbb{N}$ there exists $D \in \mathcal{U}$ such that the restriction $\varphi \restriction D$ is either constant or 1-1;*
(iii) *For every function $\varphi : \mathbb{N} \to \mathbb{N}$ there exists $D \in \mathcal{U}$ such that $\varphi \restriction D$ is either constant or increasing;*
(iv) *For every function $\varphi : \mathbb{N} \to \mathbb{N}$ there exists $D \in \mathcal{U}$ such that $\varphi \restriction D$ is non-decreasing.*

[11] Sometimes the term *free* ultrafilter is also used in the literature.

[12] For more on ultrafilters, we refer to [6] Ch. 4.

In the literature, an ultrafilter that satisfies the above properties is called a *selective* ultrafilter (see e.g. [5]).

Proof. $(i) \leftrightarrow (ii)$ Given $\varphi : \mathbb{N} \to \mathbb{N}$, for each n define $X_n = \{m : \varphi(m) = n\}$. If some $X_n \in \mathcal{U}$ then $f \upharpoonright X_n$ is constant. Otherwise there exists a *selective* set $X \in \mathcal{U}$ such that, for each n, $X \cap X_n$ contains at most one element. Then the restriction $\varphi \upharpoonright X$ is 1-1. Vice versa, let $\{X_n : n \in \mathbb{N}\}$ be a partition of $\mathbb{N}$ where $X_n \notin \mathcal{U}$ for every n. Define $\varphi(k) = n \Leftrightarrow k \in X_n$, and take $D \in \mathcal{U}$ such that $\varphi \upharpoonright D$ is 1-1. Then $X = D$ is the desired selective set.
$(ii) \to (iii)$ Let $D \in \mathcal{U}$ be such that $\varphi \upharpoonright D$ is 1-1 (if $\varphi \upharpoonright D$ is constant for some $D \in \mathcal{U}$ there is nothing to prove). Define by induction the following sequence:

$$n_0 = \min D;\; n_{k+1} = \max\{n \in D : \varphi(n) \leq \varphi(m) \text{ for some } m \leq \xi_k\}$$

where $\xi_k = \min\{n \in D : n > n_k\}$. The above definitions of max are well-posed. In fact, each of the considered sets is finite since $\varphi \upharpoonright D$ is 1-1. Clearly $n_{k+1} \geq \xi_\kappa > n_k$. Now let us consider the following sequence of nonempty intervals:

$$X_0 = [0, n_0]\,;\; X_{k+1} = [n_k + 1, n_{k+1}].$$

By hypothesis, there is a selective set $X \in \mathcal{U}$ for the partition $\{X_n : n \in \mathbb{N}\}$. Now let $X' = \bigcup\{X_{2k} : k \in \mathbb{N}\}$ and $X'' = \bigcup\{X_{2k+1} : k \in \mathbb{N}\}$. By the ultrafilter property, either $X' \in \mathcal{U}$ or $X'' \in \mathcal{U}$. Let us assume $X' \in \mathcal{U}$, hence $X'' \notin \mathcal{U}$ (in the other case the proof is similar), and consider $E = (X \cap X' \cap D) \in \mathcal{U}$. We have to check that $\varphi \upharpoonright E$ is increasing. If $x, y \in E$ with $x < y$, then $x \in X_{2k}$ for some k and $y \in X_{2h}$ for some $h > k$. Notice that $y > n_{2k+1}$ and $y \in D$. Thus, by the definition of n_{2k+1}, we have that $\varphi(y) > \varphi(m)$ for all $m \leq \xi_{2k} \leq n_{2k+1}$. In particular $\varphi(y) > \varphi(x)$.
$(iii) \to (iv)$ Trivial.
$(iv) \to (i)$ Let $\{X_n : n \in \mathbb{N}\}$ be a partition of $\mathbb{N}$ where each $X_n \notin \mathcal{U}$. Define $\varphi(m) = n \Leftrightarrow m \in X_n$. By hypothesis there is $D \in \mathcal{U}$ such that $\varphi \upharpoonright D$ is non-decreasing. Notice that each set $D_n = D \cap X_n$ is finite, otherwise $\varphi(k) = n$ for all $k \geq \min D_n$ and $X_n \in \mathcal{U}$, a contradiction. Let us say that $D_n = \{m_1^{(n)} < \ldots < m_{k_n}^{(n)}\}$. Now define a function $\psi : \mathbb{N} \to \mathbb{N}$ as follows. For every n and for every $i = 1, \ldots, k_n$, let $\psi(m_i^{(n)}) = k_n - i$; and let $\psi(n) = 0$ if $n \notin D$. Then by hypothesis there is $D' \in \mathcal{U}$ such that $\psi \upharpoonright D'$ is non-decreasing. In particular, for every n, $D' \cap D_n$ contains at most one point. If we take $X = D \cap D'$, then $X \in \mathcal{U}$ is a selcetive set for $\{X_n : n \in \mathbb{N}\}$. □

In the sequel, we shall need one more characterization of selective ultrafilters in terms of a Ramsey property. In [2], the result to follow is credited to K. Kunen. A detailed proof can be found in [12] §38.

Proposition 4.2. *Let $\mathcal{U}$ be a non-principal ultrafilter over $\mathbb{N}$. Then $\mathcal{U}$ is selective if and only if every finite partition $\{Y_1, \ldots, Y_k\}$ of the set $[\mathbb{N}]^n$ of n-element subsets of $\mathbb{N}$ (n a finite number) has an homogeneous set $H \in \mathcal{U}$. That is, $[H]^n \subseteq Y_i$ for some i.*

We are now ready to prove the following

Theorem 4.3. *There exists a numerosity function if and only if there exists a selective ultrafilter.*

Proof. By Proposition 3.3, the family $\mathcal{U} = \{A \subseteq \mathbb{N} : A \text{ is qualified }\}$ is a non-principal ultrafilter on $\mathbb{N}$. Selectiveness of $\mathcal{U}$ directly follows from Proposition 3.4 (and Proposition 4.1).

Let us turn to the converse and assume that $\mathcal{U}$ is a selective ultrafilter. Consider the $\mathcal{U}$-ultrapower of $\mathbb{N}$:[13]

$$\mathcal{N} = \mathbb{N}^{\mathbb{N}}_{\mathcal{U}} = \{[\varphi]_{\mathcal{U}} \mid \varphi : \mathbb{N} \to \mathbb{N}\}$$

where $[\varphi]_{\mathcal{U}}$ is the equivalence class of φ modulo the equivalence relation $\varphi \sim_{\mathcal{U}} \psi \Leftrightarrow \{n : \varphi(n) = \psi(n)\} \in \mathcal{U}$. If we set

$$[\varphi]_{\mathcal{U}} \leq [\psi]_{\mathcal{U}} \Leftrightarrow \{n : \varphi(n) \leq \psi(n)\} \in \mathcal{U}$$

then by the properties of ultrafilter it is easily seen that the definition is well-posed (i.e. it does not depend on the representatives chosen in the equivalence classes) and that $\langle \mathcal{N}, \leq \rangle$ is a linearly ordered set. For every labelled set $\mathbf{A}$, define $num(\mathbf{A}) = [\gamma_A]_{\mathcal{U}}$ as the $\mathcal{U}$-equivalence class of its approximating sequence $\gamma_A : n \mapsto \sharp A_n$. Since $\mathcal{U}$ is selective, every $\varphi : \mathbb{N} \to \mathbb{N}$ is $\mathcal{U}$-equivalent to some non-increasing sequence, hence to the approximating sequence of some labelled set. This proves that num is onto. In order to show that num is a numerosity function, we have to prove properties (1), (2) and (3) itemized in Definition 1.3.

The first property is trivial from the definitions, because if $\sharp A_n \leq \sharp B_n$ for all n, then clearly $num(\mathbf{A}) = [\gamma_A]_{\mathcal{U}} \leq [\gamma_B]_{\mathcal{U}} = num(\mathbf{B})$. If $num(\mathbf{A}) = num(\mathbf{A}')$ and $num(\mathbf{B}) = num(\mathbf{B}')$, i.e. if $\gamma_A \sim_{\mathcal{U}} \gamma_{A'}$ and $\gamma_B \sim_{\mathcal{U}} \gamma_{B'}$, then $\gamma_{A \oplus B} = \gamma_A + \gamma_B \sim_{\mathcal{U}} \gamma_{A'} + \gamma_{B'} = \gamma_{A' \oplus B'}$. Hence $num(\mathbf{A} \oplus \mathbf{B}) = num(\mathbf{A}' \oplus \mathbf{B}')$. Similarly, $num(\mathbf{A} \odot \mathbf{B}) = num(\mathbf{A}' \odot \mathbf{B}')$ and (3) is proved. We are left to show (2).

Suppose $\mathbf{B} \subset \mathbf{A}$. Then the set $\{n : \gamma_B(n) < \gamma_A(n)\}$ is cofinite, hence in $\mathcal{U}$, and so $num(\mathbf{B}) = [\gamma_B]_{\mathcal{U}} < [\gamma_A]_{\mathcal{U}} = num(\mathbf{A})$. Vice versa, let $[\varphi]_{\mathcal{U}} < num(\mathbf{A})$. Since U is selective, we can assume without loss of generality that φ is non-decreasing. We have to find a labelled proper subset $\mathbf{B} \subset \mathbf{A}$ such that $num(B) = [\varphi]_{\mathcal{U}}$. We first prove the following

- **Claim.** There exists a set $H = \{k_0 < k_1 < k_2 < \ldots\} \in \mathcal{U}$ such that $\varphi(k_n) - \varphi(k_{n-1}) \leq \gamma_A(k_n) - \gamma_A(k_{n-1})$ for all $n > 0$.

This follows from the Ramsey property of Proposition 4.2. In fact, let us consider the following subset of $[\mathbb{N}]^2$:

$$Y = \{\{m, m'\} : m > m' \text{ and } \varphi(m) - \varphi(m') \leq \gamma_A(m) - \gamma_A(m')\}$$

and pick a homogeneous set $H = \{k_0 < k_1 < k_2 < \ldots\} \in \mathcal{U}$ for the partition $\{Y, Y'\}$ where $Y' = [\mathbb{N}]^2 \setminus Y$. Notice that $[H]^2 \subseteq Y'$ is impossible, otherwise $\varphi(k_n) - \varphi(k_{n-1}) > \gamma_A(k_n) - \gamma_A(k_{n-1})$ for all $n > 0$ would imply $\varphi(k_n) - \varphi(k_0) \geq \gamma_A(k_n) - \gamma_A(k_0) + n$ for all $n > 0$, which in turn would imply $\varphi(k_n) > \gamma_A(k_n)$ for all but finitely many n, contradicting the hypothesis $\{n : \varphi(n) < \gamma_A(n)\} \in \mathcal{U}$. Thus it must be $[H]^2 \subseteq Y$, and H satisfies the claim.

We now want to define $\mathbf{B} \subset \mathbf{A}$ with $num(\mathbf{B}) = [\varphi]_{\mathcal{U}}$. Without loss of generality, we can assume that $\varphi(h) < \gamma_A(h)$ for all $h \in H$ (otherwise take $H' = \{h \in H : \varphi(h) < \gamma_A(h)\} \in \mathcal{U}$). Pick $A'_0 \subseteq \{a \in A : \ell_A(a) \leq k_0\}$ with $\sharp A'_0 = \varphi(k_0)$, and for $n > 0$ pick $A'_n \subseteq \{a \in A : k_{n-1} < \ell_A(a) \leq k_n\}$ with $\sharp A'_n = \varphi(k_n) - \varphi(k_{n-1})$. (This is possible by the property of H). Let $\mathbf{B}$ be the labelled subset of $\mathbf{A}$ obtained as the union of all A'_n. By the definition, its counting function γ_B is such that

[13] A treatment of ultrapowers can be found in [6] Ch.4.

$\gamma_B(h) = \varphi(h)$ for all $h \in H$. We conclude that $num(\mathbf{B}) = [\gamma_B]_{\mathcal{U}} = [\varphi]_{\mathcal{U}}$, as desired. $\square$

We remark that the above construction shows that the assignment of numerosities to labelled sets is not uniquely determined. For instance, let $\mathbf{Even} = \{2n : n \in \mathbb{N}_+\}$ and $\mathbf{Odd} = \{2n-1 : n \in \mathbb{N}_+\}$ be the labelled sets of *even* and *odd* natural numbers, respectively.[14] When the underlying selective ultrafilter $\mathcal{U}$ of qualified sets contains the set of even numbers, then $num(\mathbf{Even}) = num(\mathbf{Odd})$, hence $\alpha = num(\mathbb{N}) = num(\mathbf{Even}) + num(\mathbf{Odd})$ is even. On the contrary, if $\mathcal{U}$ contains the set of odd numbers, then $num(\mathbf{Odd}) = num(\mathbf{Even}) + 1$ and α is odd.

We remark that this example, as well as other similar ones, are easily overcome by postulating additional conditions on the numerosity α. For instance, one could impose the following two natural properties:

(α1) For every natural number $k > 0$, α is a multiple of k (i.e. $\alpha = k \cdot \beta$ for some numerosity β);

($\alpha\mathbf{2}$) For every natural number $k > 0$, α is a k^{th}-power (i.e. $\alpha = \beta^k$ for some numerosity β).

If we denote by $k\mathbb{N} = \{kn : n \in \mathbb{N}_+\}$ and $\mathbb{N}^{(k)} = \{n^k : n \in \mathbb{N}_+\}$, then it is proved that the properties above are equivalent to the conditions: $(\alpha\mathbf{1})'$ $num(k\mathbb{N}) = \alpha/k$, and $(\alpha\mathbf{2})'$ $num(\mathbb{N}^{(k)}) = \sqrt[k]{\alpha}$, respectively.

It is known that the existence of selective ultrafilters is independent of ZFC. In fact, on the one hand, selective ultrafilters exist if we assume the *continuum hypothesis* or even *Martin's axiom*, a strictly weaker condition (see e.g. [2]). On the other hand, K. Kunen [13] showed that there are models of ZFC with no selective ultrafilters. Thus the following holds.

Corollary 4.4. *The existence of a numerosity function is independent of ZFC.*

We leave as an open problem the possibility of suitably restricting or enlarging the class of labelled sets in such a way that the existence of a numerosity function can be proved by ZFC.

5. Appendix

In this appendix we prove results stated in section 3. Let us first see the following

Lemma 5.1.

(i) *The empty set $\emptyset$ is not qualified. That is, if $\varphi(n) \neq \psi(n)$ for all n, then $\rho(\varphi) \neq \rho(\psi)$;*

(ii) *Let ϑ_D be the non-decreasing function such that $\vartheta_D(n) = n$ if $n \in D$ and $\vartheta_D(n) = n+1$ otherwise. Then D is qualified if and only if $\rho(\vartheta_D) = \rho(1_{\mathbb{N}}) = \alpha$.*

Proof. (*i*) For each n, $[\,\varphi(n) - \psi(n)\,]^2 > 0 \Rightarrow \varphi^2(n) + \psi^2(n) \geq 2\varphi(n)\psi(n) + 1 \Rightarrow$ (because ρ is an homomorphism of p.o. semi-rings) $\rho(\varphi)^2 + \rho(\psi)^2 \geq 2\rho(\varphi)\rho(\psi) + 1 \Rightarrow \rho(\varphi) \neq \rho(\psi)$.
(*ii*) Notice that the identity function $1_{\mathbb{N}} : n \mapsto n$ is the approximating sequence of $\mathbb{N}$ (with the canonical labelling). Thus $\rho(1_{\mathbb{N}}) = num(\mathbb{N}) = \alpha$. By definition, if $\rho(\vartheta_D) = \rho(1_{\mathbb{N}})$, then $D = \{n : \vartheta_D(n) = 1_{\mathbb{N}}(n)\}$ is qualified. Vice versa, suppose D

[14] By $\mathbb{N}_+ = \mathbb{N} \setminus \{0\}$ we denote the set of *nonzero* natural numbers.

is qualified. Without loss of generality we can assume that $D = \{n : \varphi(n) = \psi(n)\}$ where $\rho(\varphi) = \rho(\psi)$ and both φ and ψ are (strictly) increasing (otherwise, consider $\varphi' = \varphi + 1_{\mathbb{N}}$ and $\psi' = \psi + 1_{\mathbb{N}}$). Consider the non-decreasing function τ such that $\tau(n) = \varphi(n)$ if $n \in D$ and $\tau(n) = \varphi(n) + 1$ otherwise. For all n, $\tau(n) \neq \psi(n) + 1$, thus by (i), $\rho(\tau) \neq \rho(\psi)+1 = \rho(\varphi)+1$. Since $\varphi \leq \tau \leq \varphi+1$, it must be $\rho(\tau) = \rho(\varphi)$. But $\tau + 1_{\mathbb{N}} = \varphi + \vartheta_D$, thus $\rho(\tau) + \rho(1_{\mathbb{N}}) = \rho(\varphi) + \rho(\vartheta_D)$, and so $\rho(1_{\mathbb{N}}) = \rho(\vartheta_D)$. □

Proposition 3.3

Proof. (i) One direction is the definition of qualified set. Notice that $\varphi \cdot 1_{\mathbb{N}} + \varphi + \psi \cdot \vartheta_D = \psi \cdot 1_{\mathbb{N}} + \psi + \varphi \cdot \vartheta_D$, hence

$$\rho(\varphi) \cdot \alpha + \rho(\varphi) + \rho(\psi) \cdot \rho(\vartheta_D) = \rho(\psi) \cdot \alpha + \rho(\psi) + \rho(\varphi) \cdot \rho(\vartheta_D)$$

If $\rho(\vartheta_D) = \alpha$, then clearly $\rho(\varphi) = \rho(\psi)$.
(iii) By hypothesis $\rho(\vartheta_D) = \rho(\vartheta_E) = \alpha$, hence $\rho(\vartheta_D \cdot \vartheta_E) = \alpha^2 = \rho(1_{\mathbb{N}} \cdot 1_{\mathbb{N}})$. Since $A \cap B = \{n : \vartheta_D(n) \cdot \vartheta_E(n) = n^2\}$, clearly $A \cap B$ is qualified.
(iv) One implication is trivial because if D and D^c are both qualified, then also their intersection $D \cap D^c = \emptyset$ is qualified, a contradiction. Vice versa, notice that $1_{\mathbb{N}} \leq \vartheta_D \leq 1_{\mathbb{N}} + 1$. If $\rho(\vartheta_D) \neq \alpha$ then $\rho(\vartheta_D) = \alpha + 1$. But $\vartheta_D + \vartheta_{D^c} = 1_{\mathbb{N}} + 1_{\mathbb{N}} + 1$ implies that $\rho(\vartheta_D) + \rho(\vartheta_{D^c}) = \alpha + \alpha + 1$, hence $\rho(\vartheta_{D^c}) = \alpha$.
(ii) If E is not qualified, then by (iii) and (iv), $\emptyset = E^c \cap D$ is qualified, a contradiction.
(v) It is enough to show that singletons are not qualified. Let k be given, and consider the labelled sets $\mathbf{A} = \langle \{0,1\}, \ell_A \rangle$ and $\mathbf{B} = \langle \{1\}, \ell_B \rangle$ where $\ell_A(0) = 0$; $\ell_A(1) = k+1$ and $\ell_B(1) = k$. If $\gamma_A : n \mapsto \sharp A_n$ and $\gamma_B : n \mapsto \sharp B_n$ are the approximating sequences, then $\{n : \gamma_A(n) = \gamma_B(n)\} = \{k\}$. Since $\rho(\gamma_A) = num(\mathbf{A}) = 2 \neq 1 = num(\mathbf{B}) = \rho(\gamma_B)$, $\{k\}$ is not qualified. □

Proposition 3.4

Proof. Define $\xi(n+1) = \zeta(n) = \sum_{i \leq n} \varphi(i)$ and $\xi(0) = 0$. Clearly ξ and ζ are non-decreasing functions such that $\xi + \varphi = \zeta$. In particular $\xi \leq \zeta$, so there are labelled sets $\mathbf{A} \subseteq \mathbf{B}$ with $\rho(\xi) = num(\mathbf{A})$ and $\rho(\zeta) = num(\mathbf{B})$. Let $\mathbf{C} = \mathbf{B} \setminus \mathbf{A}$. As usual, let us denote by γ_A, γ_B and γ_C the approximating sequences. $\mathbf{A} \oplus \mathbf{C} \cong \mathbf{B} \Rightarrow \rho(\gamma_{A \oplus C}) = \rho(\gamma_B)$, i.e. $\rho(\gamma_A) + \rho(\gamma_C) = \rho(\gamma_B)$. Now $\rho(\gamma_A) = num(\mathbf{A}) = \rho(\xi)$ and $\rho(\gamma_B) = num(\mathbf{B}) = \rho(\zeta)$. Thus $\rho(\xi + \gamma_C) = \rho(\zeta)$ implies that $\{n : \xi(n) + \gamma_C(n) = \zeta(n)\} = \{n : \gamma_C(n) = \varphi(n)\}$ is qualified; and the function $\psi = \gamma_C$ satisfies the thesis. □

Proposition 3.5

Proof. For simplicity, in the following we abuse notation and directly write A instead of the corresponding constant sequence c_A (for instance, we shall write $\varphi \in_\mu A$ to mean that $\{n : \varphi(n) \in A\}$ is qualified). If $\varphi \in_\mu \mathbb{N}$, pick φ' non-decreasing with $\varphi =_\mu \varphi'$, and let $\boldsymbol{\rho}(\varphi) = \rho(\varphi')$. If $\varphi =_\mu \emptyset$, then put $\boldsymbol{\rho}(\varphi) = \emptyset$. Now proceed by induction, and assume that $\boldsymbol{\rho}$ has already been defined for all $\varphi \in \mathcal{F}_\infty$ with $\varphi \in_\mu V_k(\mathbb{N})$ in such a way that properties $(i) - (iv)$ are satisfied. Let $\varphi \notin_\mu \mathbb{N} \cup \{\emptyset\}$ and consider $k = \min\{h : \varphi \in_\mu V_h(\mathbb{N})\} \geq 1$. Notice that if $\psi \in_\mu \varphi$ then $\psi \in_\mu V_{k-1}(\mathbb{N})$. Thus by the inductive hypothesis we can set

$\boldsymbol{\rho}(\varphi) = \{\boldsymbol{\rho}(\psi) : \psi \in_\mu \varphi\}$. Properties $(i)-(iv)$, as well as the uniqueness of $\boldsymbol{\rho}$, are then verified in a straightforward manner. $\square$

Theorem 3.7

Proof. First, let us prove the following fact.

($\bullet$) For every bounded formula $\sigma(x_1, \ldots, x_n)$ in the language of set theory, and for every $\varphi_1, \ldots, \varphi_n \in \mathcal{F}_\infty$:
$$\sigma(\boldsymbol{\rho}(\varphi_1), \ldots, \boldsymbol{\rho}(\varphi_n)) \Leftrightarrow \{k : \sigma(\varphi_1(k), \ldots, \varphi_n(k))\} \text{ is qualified}$$

The arguments are essentially the same as in the proof of the fundamental Theorem of ultrapowers (see [6] §4.1). For completeness, we rephrase those arguments in our context. Proceed by induction on the complexity of formulas. For atomic formulas $\boldsymbol{\rho}(\varphi_1) = \boldsymbol{\rho}(\varphi_2)$ and $\boldsymbol{\rho}(\varphi_1) \in \boldsymbol{\rho}(\varphi_2)$, the thesis is given by (iii) and (iv) of Proposition 3.5. The conjunction $\sigma_1 \wedge \sigma_2$ and the negation $\neg\sigma$ steps directly follow from properties (iii) and (iv) of Proposition 3.3, respectively. Let us turn to the existential quantifier, and assume that

$$\exists x \in \boldsymbol{\rho}(\varphi)\, \sigma(x, \boldsymbol{\rho}(\varphi_1)), \ldots, \boldsymbol{\rho}(\varphi_n)).$$

Then there exists ψ such that $\psi \in_\mu \varphi$ and $\sigma(\boldsymbol{\rho}(\psi), \boldsymbol{\rho}(\varphi_1)), \ldots, \boldsymbol{\rho}(\varphi_n))$. By the inductive hypothesis, $D = \{k : \psi(k) \in \varphi(k)\}$ and $E = \{k : \sigma(\psi(k), \varphi_1(k), \ldots, \varphi_n(k))\}$ are both qualified. Hence

$$\{k : \exists x \in \varphi(k)\, \sigma(x, \varphi_1(k), \ldots, \varphi_n(k))\}$$

is qualified as well, because it is a superset of $D \cap E$. Vice versa, let us assume that

$$D = \{k : \exists x \in \boldsymbol{\rho}(\varphi)\, \sigma(x, \varphi_1(k), \ldots, \varphi_n(k))\}$$

is qualified. For every $k \in D$, pick $\xi_k \in \varphi(k)$ with $\sigma(\xi_k, \varphi_1(k), \ldots, \varphi_n(k))$. Take $\psi \in \mathcal{F}_\infty$ a sequence such that $\psi(k) = \xi_k$ for all $k \in D$. Then $\psi \in_\mu \varphi$, and by the inductive hypothesis $\sigma(\boldsymbol{\rho}(\psi), \boldsymbol{\rho}(\varphi_1), \ldots, \boldsymbol{\rho}(\psi_n))$. We conclude that

$$\exists x \in \boldsymbol{\rho}(\varphi)\, \sigma(x, \boldsymbol{\rho}(\varphi_1)), \ldots, \boldsymbol{\rho}(\varphi_n))$$

Now let us turn to the Leibniz transfer principle. Recall the following well known fact in set theory: If $T \subseteq T'$ are transitive classes, $\tau(x_1, \ldots, x_n)$ is a bounded formula, and $t_1, \ldots, t_n \in T$, then $T \models \tau(t_1, \ldots, t_n) \Leftrightarrow T' \models \tau(t_1, \ldots, t_n)$.[15] In particular this is true when $T = V_\infty(\mathbb{N})$ or $T = V_\infty(\mathcal{N})$, and T' is the universe of all sets. Now let a bounded formula $\sigma(x_1, \ldots, x_n)$ and elements $a_1, \ldots, a_n \in V_\infty(\mathbb{N})$ be given. By using the above property ($\bullet$), we get the following equivalences:

$$\begin{aligned} \sigma({}^*a_1, \ldots, {}^*a_n) &\Leftrightarrow V_\infty(\mathcal{N}) \models \sigma({}^*a_1, \ldots, {}^*a_n) \\ &\Leftrightarrow \sigma(\boldsymbol{\rho}(c_{a_1}), \ldots, \boldsymbol{\rho}(c_{a_n})) \\ &\Leftrightarrow V_\infty(\mathbb{N}) \models \sigma(a_1, \ldots, a_n) \\ &\Leftrightarrow \sigma(a_1, \ldots, a_n) \end{aligned}$$

This proves the transfer principle. Now let us turn to saturation, and consider a countable family $\mathcal{B} \subseteq {}^*A$ of internal sets. By definition of internal element, it is easily seen that $\mathcal{B} = \{\boldsymbol{\rho}(\varphi_n) : n \in \mathbb{N}\}$ for suitable sequences $\varphi_n : \mathbb{N} \to A$. For

[15] See for instance [6] §4.4. Recall that a set T is *transitive* if $x \in y \in T \Rightarrow x \in T$. We remark that the same definition of transitivity and the same absoluteness property also hold in the presence of atoms.

each $n \in \mathbb{N}$, pick $x_n \in \bigcap_{i=0}^{n} \boldsymbol{\rho}(\varphi_i)$. Without loss of generality we can assume that $x_n = \boldsymbol{\rho}(\psi_n)$ where $\psi_n(k) \in \varphi_i(k)$ for all $k \in \mathbb{N}$ and for every $i = 0, \ldots, n$. Define $\vartheta(n) = \psi_n(n)$. Notice that $A \in V_m(\mathbb{N}) \Rightarrow \vartheta \in \mathcal{F}_m \subset \mathcal{F}_\infty$. Now, for each n, $\{k : \vartheta(k) \in \varphi_i(k)\} = \{k : k \geq n\}$ is cofinite, hence qualified, and so $\boldsymbol{\rho}(\vartheta) \in \bigcap \mathcal{B}$ is the element we were looking for. $\square$

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