On the Foundations of Nonstandard Mathematics

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Abstract

In this paper we survey various set-theoretic approaches that have been proposed over the last thirty years as foundational frameworks for the use of nonstandard methods in mathematics.

Introduction.
Since the early developments of calculus, infinitely small and infinitely large numbers have been the object of constant interest and great controversy in the history of mathematics. In fact, while on the one hand fundamental results in the differential and integral calculus were first obtained by reasoning informally with infinitesimal quantities, it was easily seen that their use without restrictions led to contradictions. For instance, Leibnitz constantly used infinitesimals in his studies (the differential notation dx is due to him), and also formulated the so-called *transfer principle*, stating that those laws that hold about the real numbers also hold about the extended number system including infinitesimals. Unfortunately, neither he nor his followers were able to give a formal justification of the transfer principle. Eventually, in order to provide a rigorous logical framework for the treatment of the real line, infinitesimal numbers were banished from calculus and replaced by the $\varepsilon\delta$-method during the second half of the nineteenth century. [1]

A correct treatment of the infinitesimals had to wait for developments of a new field of mathematics, namely mathematical logic and, in particular, of its branch called model theory. A basic fact in model theory is that every infinite mathematical structure has *nonstandard models*, i.e. non-isomorphic structures which satisfy the same elementary properties. In other words, there are different but equivalent structures, in the sense that they cannot be distinguished by means of the elementary properties they satisfy. In a slogan, one could say that in mathematics *"words are not enough to describe reality"*.

[1] An interesting review of the history of calculus can be found in Robinson's book [R2], chapter X.

Although the existence of nonstandard models was first shown by Thoralf Skolem in the late twenties, a strong interest in their properties arose only in the fifties, when an intensive study of nonstandard models of arithmetic began. The *"invention"* of nonstandard analysis can be dated back to 1960, when Abraham Robinson had the idea of systematically applying that model-theoretic machinery to analysis [R1]. By considering nonstandard extensions of the real number system, he was able to provide the use of infinitesimal numbers with rigorous foundations, thus giving a solution to a century-old problem. According to some authors, Robinson's achievement may be one of the major mathematical advances of the century.

The existence of nonstandard extensions ${}^*R$ of the real number system, called *hyperreal numbers*, may appear contradictory, in that they seem to be in conflict with the well-known characterization theorem for R. For instance, if we want the hyperreal systems to be ordered fields, then, as proper extensions of R, they necessarily are neither archimedean nor Dedekind-complete. Thus, what sense the equivalence of ${}^*R$ and R is to be intended? A correct answer to this question is the core of nonstandard analysis. In the context of mathematical logic, the notion of elementary property can be given a precise definition. Namely, a property is *elementary* if it can be formulated as a first order formula in a specified language. Roughly speaking, a first order formula is a finite expression where quantifications are permitted only over variables ranging over elements but not over subsets. Thus, in the usual language that consists of symbols for addiction, multiplication, neutral elements and order relation, the properties of ordered field are first order, while Dedekind-completeness and the archimedean property are not. Notice in fact that completeness talks about subsets, and the archimedean property requires an infinitely long formula to be expressed: "$\forall x > 0\ (x > 1 \vee x + x > 1 \vee x + x + x > 1 \ldots)$". [2] Once the language has been specified, Leibnitz' *transfer principle* can be given a rigorous formulation.

> *Every property one can write down as a first order formula is true of the real numbers R if and only if it is true of any hyperreal number system ${}^*R$.*

The typical strategy in nonstandard analysis is as follows. Assume we want to prove (or disprove) some conjecture P about the real numbers, or more generally, about some mathematical structure M. Formalize P as a first order formula φ. It can happen that it is easier to decide P in some nonstandard model ${}^*M$ where additional tools may be available (for instance, infinitesimals), rather than in the *standard model* M. Once the property P, as formally expressed by the formula φ, has been proved (or disproved) in ${}^*M$, by *transfer* it is true (or false) in the standard structure M as well.

[2] The formalization "$\forall x\, \exists n \in N\ n \cdot x > 0$" is not in the given language. In fact, an extra symbol N for the naturals is needed.

The use of nonstandard models to prove "standard" theorems, could be seen in a similar way as, say, the use of complex numbers C to prove results about real numbers. If one is only interested in real numbers, then complex numbers can be seen as nothing but a mere tool to carry out proofs. [3] Of course, the above comparison should not be taken literally. The technicalities involved in nonstandard methods are somewhat of a different nature because they require notions from mathematical logic to be fully justified. But still, the basic idea is similar. Nonstandard methods do not give rise to *nonstandard mathematics* to be contrasted with *standard mathematics*. On the contrary, they provide a new powerful tool which is applicable across the whole mathematical spectrum, and whose strength and potentiality are probably still far from being fully exploited. Unfortunately, there is still some diffidence in the mathematical community about the use of nonstandard methods. An historical reason for this is the fact that infinitesimals were used incorrectly in the early developments of calculus. Nowadays, an obstacle to a wider diffusion of nonstandard methods is probably the fact that mathematicians are often not comfortable with mathematical logic. This is why many attempts have been made to find an elementary presentation to nonstandard analysis (i.e. a presentation not involving technical notions from mathematical logic). In this regard, see the approach given by H.J. Keisler [K2] aimed at the average beginning calculus student, and the one recently given by C.W. Henson [He].

In this paper we give a (partial) report about the research on the foundations on nonstandard methods developed over the last thirty years. We mainly concentrate on nonstandard set theories, that is on those axiomatic approaches that are given in the full generality of set theory. This is a survey paper, thus only a few sketched proofs are included, but plenty of references are given for further studies. The reader is assumed to have some basic knowledge of set theory and nonstandard analysis, but every mathematician can easily read the paper by skipping the more technical parts. [4]

We like to conclude this introduction with some optimistic consideration by H.J. Keisler. *"At the present time, the hyperreal number system is regarded as somewhat of a novelty. But because of its broad potential, it may eventually become a part of the basic toolkit of mathematicians ... The current high degree of specialization in mathematics serves to inhibit the process, since few established mathematicians are willing to take the time to learn both mathematical logic and an area of application. However, in the long term, applications of mathematical logic ... should result in future generations of mathematicians who are better trained in logic, and therefore more able to take advantage of the hyperreal line*

[3] On the other hand, C is a beautiful object of study in its own right. Similarly, in our opinion, nonstandard models themselves are interesting mathematical objects to be investigated.

[4] The default reference for set theoretic notions is [Ku]. There, basic definitions such as those of ordinal and cardinal numbers are given without assuming regularity.

when the opportunity arises.". [5]

§1. The Superstructure Approach.
Abraham Robinson's original presentation is developed within a type-theoretical version of higher-order logic (see [R1], [R2]). As the logical formalism needed appeared unnecessarily complicated to most mathematicians, another approach became more popular in practice, namely the more "concrete" one based on the construction of ultrapowers of R. The use of ultrapowers in nonstandard analysis was popularized by Wilhelmus Luxemburg. His lecture notes [L1] widely circulated during the sixties, and they were used as a reference in the field by an entire generation of mathematicians. Up to now, the most popular presentation of nonstandard analysis is the so-called *superstructure approach*. It was elaborated by Abraham Robinson himself jointly with Elias Zakon, and it first appeared in the Proceedings of the International Symposium on the Applications of Model Theory to Algebra, Analysis and Probability, held in 1967 [RZ]. The original presentation was then improved by Zakon [ZA]. Historically, the superstructure approach is the first one to be purely set-theoretic in nature.

There is no need to go very far in analysis to realize that only considering the real number system is not enough. One needs to talk about intervals, functions, function spaces, norms, topologies, and so on. In the usual framework of set theory, functions are subsets of cartesian products, cartesian products are sets of ordered pairs and ordered pairs $\langle a, b\rangle = \{\{a\}, \{a, b\}\}$ are Kuratowski pairs. Thus, for instance, a space $\mathcal{F}$ of real functions is identified with a subset of subsets of subsets of subsets of real numbers, and a topology on $\mathcal{F}$ (as the collection of its open sets) is a subset of subsets of $\mathcal{F}$ etc. As a consequence of this kind of considerations, Robinson and Zakon had the idea of taking superstructures as universes for the practice of mathematics.

A *superstructure* $V(X)$ over an infinite set of *individuals* X is defined inductively as follows. [6]

$$V(X) = \bigcup_{n\in N} V_n(X) \text{ where } V_0(X) = X;\ V_{n+1}(X) = V_n(X) \cup \mathcal{P}(V_n(X)).$$

For instance, by taking $X = R$ the set of real numbers, the resulting superstructure $\langle V(R), \in\rangle$ is a suitable universe where all objects of real analysis are available. For convenience, individuals are usually supposed to be *atoms*, i.e. objects with no elements and different from the emptyset $\emptyset$. In practice, this is a natural assumption. For instance, in analysis a real number is always handled as a primitive entity rather than as a set. [7] A *nonstandard embedding* is a map-

[5] [K3] p.235.

[6] For any set A, $\mathcal{P}(A) = \{a : a \subseteq A\}$ is the *power-set* of A.

[7] We remark that some caution is needed in order to formalize the above within the usual set theoretic framework. Although atoms are not allowed in ZFC, nevertheless, suitable sets X can be considered which behave as atoms relative to its superstructures. Precisely, sets X can be taken with the property $x \cap A = \emptyset$ for all $x \in X$ and for all $A \in V(X)$. See [CK] §4.4

ping $* : V(X) \to V(Y)$ from a superstructure $V(X)$, called the *standard model*, into another superstructure $V(Y)$, called *nonstandard model*, that satisfies the following properties: [8]

- ${}^*X = Y$
- *Transfer Principle*
  For every bounded formula $\sigma(x_1, \ldots, x_n)$ and elements $a_1, \ldots, a_n \in V(X)$,

  $$\langle V(X), \in \rangle \models \sigma(a_1, \ldots, a_n) \Leftrightarrow \langle V(Y), \in \rangle \models \sigma({}^*a_1, \ldots, {}^*a_n)$$

  i.e. the property expressed by σ is true of the elements $a_1, \ldots, a_n$ in the standard model if and only if it is true of the elements ${}^*a_1, \ldots, {}^*a_n$ in the nonstandard model.
- *Non-triviality*
  For every infinite set $A \in V(X)$, $\{{}^*a : a \in A\}$ is a proper subset of ${}^*A$.

Several excellent expositions of the superstructure approach to nonstandard analysis can be found in the literature. [9] This paper is not intended as an introduction to nonstandard methods, thus we refer to those expositions for unexplained notions and basic results. Here, we shall concentrate instead on some foundational aspects.

Notice that in the transfer principle only *bounded formulas* are considered, i.e. formulas where all quantifiers occur in the bounded form $\forall x (x \in y \to \ldots)$ or $\exists x (x \in y \wedge \ldots)$. This limitation is a consequence of the following fact.

Proposition 1.1. *There are no nonstandard embeddings $* : V(X) \to V(Y)$ where the transfer principle holds for all formulas.*
Proof. Since X is infinite, we can assume that there is an isomorphic copy of the natural numbers $N \subseteq X$. The following holds:

$$\langle V(X), \in \rangle \models \forall n \in N \, \exists A_n \, \exists f : \{0, 1, \ldots, n\} \to A_n \text{ such that } \forall x, y \in \{0, 1, \ldots, n\} \ (x < y \leftrightarrow f(x) \in f(y))$$

In fact, one can take the Von Neumann natural number $n+1$ as A_n. [10] If by contradiction $*$ satisfies the transfer principle for all formulas, in particular the above (unbounded) formula is satisfied in the nonstandard model $\langle V(Y), \in \rangle$ for all $\xi \in {}^*N$. It is a basic fact in nonstandard analysis that every unbounded

where such sets are called *base sets*.

[8] In the original formulation [RZ], a list of preservation properties for $*$ (e.g. ${}^*\{x\} = \{{}^*x\}$; ${}^*(X \setminus Y) = ({}^*X \setminus {}^*Y)$ etc.) was postulated in place of the *transfer principle*. The latter then followed as a theorem. Our definition follows the presentation given in [ZA].

[9] See for instance [HL] or [LI]. See also [CK] §4.4, where emphasis is given on model-theoretic aspects.

[10] Recall that in ZFC natural numbers are coded as Von Neumann numbers. Precisely $0 = \emptyset$ and, inductively, $n + 1 = n \cup \{n\}$.

hypernatural ξ originates an infinite descending chain $\xi > \xi - 1 > \xi - 2 > \dots$ Thus, if F_ξ is the function obtained by transfer, one gets an $\in$-descending chain $F_\xi(\xi) \ni F_\xi(\xi - 1) \ni F_\xi(\xi - 2) \ni \cdots$ in $V(Y)$, a contradiction. $\dashv$

Recall that a *standard set* is any element of the standard model; an *internal set* is any x with $x \in {}^*A$ for some standard A. Sometimes, also elements of the form ${}^*A$ are called standard. To avoid confusion, here we shall call them *internal-standard*. An *external set* is an element of the nonstandard model that is not internal. It is easily seen that internal-standard elements are internal, and that the collection I of all internal elements is transitive, that is elements of internal sets are internal. As a consequence of this fact, one can give the transfer principle the following modified formulation, where an explicit mention of superstructures is avoided. "Let P be an elementary property, i.e. a property formalized by a bounded formula. Then P is true of the standard sets $a_1, \dots, a_n$ if and only if P is true of ${}^*a_1, \dots, {}^*a_n$ relative to internal sets". For instance, "all subsets of N have a least element" is transferred to "all *internal* subsets of ${}^*N$ have a least element".

- *Transfer Principle (for the internal universe).*
  For every bounded formula $\sigma(x_1, \dots, x_n)$ and for all standard elements $a_1, \dots, a_n$, $\sigma(a_1, \dots, a_n) \Leftrightarrow \sigma^I({}^*a_1, \dots, {}^*a_n)$. [11]

In fact, notice that $V(X)$, $V(Y)$ and I are all transitive sets. Then recall that bounded formulas are preserved between transitive sets and apply transfer. $\dashv$

Besides transfer, the other fundamental principle of nonstandard analysis is the *saturation property*. Saturation is a typical model-theoretic notion usually defined in terms of realizability of sets of formulas. However, in a set-theoretic context, saturation can be given an elementary formulation as an intersection property. Precisely, the following formulation is now usually considered in nonstandard analysis (κ a given uncountable cardinal).

- *κ-Saturation Property*
  Let $\mathcal{F}$ be a family of internal sets with the finite intersection property (FIP), i.e. such that all its finite subfamilies have nonempty intersection. If $\mathcal{F}$ has cardinality less than κ, then $\mathcal{F}$ has nonempty intersection.

The above property where only countable families $\mathcal{F}$ are considered (i.e. where $\kappa = \aleph_1$) is usually called *countable saturation*.

Historically, a weaker form of saturation was already considered in Robinson's original presentation of nonstandard analysis, where he used the notion

[11] σ^I is the restriction of σ to I, i.e. σ^I is the formula obtained from σ by replacing each quantifier $\forall x$ and $\exists x$ with its I-restricted form $\forall x\,(x \in I \to \cdots)$ and $\exists x\,(x \in I \wedge \cdots)$, respectively.

of *concurrent relation.* The first one to investigate saturation within a purely set-theoretic framework was H.J. Keisler early in the sixties. The formulation of countable saturation in terms of intersections of sets is somewhat implicit in his paper [K1]. However, it was W.A.J. Luxemburg who isolated the κ-saturation property as a fundamental tool in nonstandard analysis [L3]. He gave evidence for the fact that κ-saturated nonstandard models, for suitable κ, are the right setting for a nonstandard study of topological spaces. For instance, one can characterize compact sets K of a given topological space X as those sets that satisfy: "$\forall\xi \in {}^*X\ \exists x \in X$ such that $\xi \in {}^*U$ for all neighborhoods U of x". We remark that this characterization holds only if κ-saturation is assumed for some κ larger than the cardinality of a topological basis for X. [12]

§2. Foundational Limitations of the Superstructure Approach.
The superstructure approach is entirely developed within set theory, thus it presents no foundational problems. However, it reveals serious limitations if proposed as a framework for the use of nonstandard methods in mathematics in their full generality. Here is a tentative list of such limitations.

- *Superstructures model only a fragment of ZFC.*

Since superstructures only consist of sets of finite rank in the cumulative hierarchy, they do not satisfy the Infinity axiom. Besides, the Replacement axiom is not satisfied either, because sets of individuals are assumed to be infinite. [13] As a consequence, superstructures do not allow the full scope of mathematical techniques. The restriction to sets of finite rank is a reaction to the following fact.

Proposition 2.1. *Let A and B be two transitive sets. If $\langle A, \in\rangle \models$ Infinity then there are no nontrivial embeddings $* : A \to B$ that satisfy the transfer principle.*
Proof. By the infinity axiom, the set of Von Neumann naturals $\omega \in A$. Recall that ω is the set of naturals as coded in ZFC, its ordering being given by the membership relation $\in$. Now, assume by contradiction that $*$ is nontrivial and satisfies transfer. By well-known nonstandard arguments, ${}^*\omega$ is linearly ordered by $\in$ and contains descending chains $\xi \ni \xi - 1 \ni \xi - 2 \cdots$ for each (unbounded) $\xi \in {}^*\omega \setminus \{{}^*n : n \in \omega\}$, against the axiom of regularity. $\dashv$

- *Different superstructures are needed for different problems.*

Suppose we want to study a mathematical structure M by using nonstandard methods. First, we have to take a superstructure $V(X)$ right for the purpose.

[12] A survey of topology done by nonstandard methods is given in [Lo].

[13] For instance, take $N \subseteq X$ a copy of the natural numbers and consider the function F with domain N such that $n \mapsto \{\cdots\{\emptyset\}\cdots\}$ (n brackets). F is definable in $V(X)$ but range$(F) \notin V(X)$.

For instance, we may want the set of individuals X to include both a copy of M and a copy of the real number system R. Then we have to consider a nonstandard embedding $* : V(X) \to V(Y)$ that satisfies κ-saturation for a suitable cardinal κ, large enough so that all desired nonstandard arguments can be carried out. It is clear that different nonstandard embeddings must be chosen to deal with different problems.

- *In principle, nonstandard methods are not concerned with superstructures.*

In fact, the formulation and use of nonstandard methods have no connections with the technical set-theoretic notion of cumulative hierarchy of sets.

- *It seems esthetically desirable to include all nonstandard techniques within a unified axiomatic system.*

The aim of giving a general foundational framework in which virtually all of mathematics including nonstandard arguments can be embedded, led to the formulations of various *nonstandard set theories.*

§3. Looking for a Nonstandard Set Theory.
To the author's knowledge, historically the first person who explicitly considerated the possibility of axiomatic systems for nonstandard analysis was G. Kreisel in his 1969 paper [Kr]. [14] He asked: *"Is there a simple formal system ... in which existing practice of nonstandard analysis can be codified? And if the answer is positive: is this formal system a conservative extension of the current systems of analysis ... ?"*. In that paper, he proposed himself a formal system **NS** for nonstandard analysis, where axioms were formulated making use of the $*$ symbol borrowed from the superstructure approach notation. Unfortunately, **NS** is presented in the language of type theory, by now unfamiliar to most mathematicians, and some of the notions considered in those early years of nonstandard analysis are now superseeded. Moreover, **NS** is a system for nonstandard analysis and it has not the generality of a nonstandard set theory. For these reasons we will not include a presentation of **NS** here, but we like to mention this theory as a forerunner of nonstandard set theories.

As emerged from the various axiomatic systems that have been proposed since the seventies, an *ideal* nonstandard set theory T should have as many as possible of the following features. [15]

1. T is an extension of "standard" mathematics as formalized by the classic Zermelo-Fraenkel set theory ZFC.

[14] By the way, published in the same volume of proceedings where the superstructure approach by Robinson and Zakon appeared.

[15] We remark that idealization and standardization are formulated here only in a naive language. These principles have different formalizations depending on the underlying nonstandard set theory.

2. T postulates a *transfer principle* between the standard and the internal universe for formulas of the language of ordinary mathematics.

3. T includes a strong *saturation principle*, sometimes called *idealization*.

4. T allows *standardization*. That is, for any given set, one can take the set of all its standard elements, and the result is again a standard set.

5. T is *conservative* over ZFC. That is, a standard fact is proved by T if and only if it is proved by ZFC.

Some comments on the above desiderata. Item # 1 says that working in T is the same as working in ZFC when dealing with ordinary mathematics. Items # 2 and # 3 states that T allows performing nonstandard constructions in its full generality. As for standardization, it allows considering the (nonstandard) notion of standard set when defining sets. This seems to be a very natural property to be assumed. In fact, suppose one wants to take all standard elements of a given (possibly external) set so as to form a standard set, and then apply the usual standard arguments on it. Without standardization this is not possible. We remark that standardization trivially holds in the superstructure approach, while it is usually set as an axiom in nonstandard set theories. Finally, item # 5 gives a strong foundational justification to T, which can be seen (at least) as a short-cut to standard theorems. [16]

Unfortunately, a theory T satisfying all the above properties cannot exist. First of all, the axiom of regularity cannot be demanded by # 1 in the external universe. Proposition 2.1 already showed that regularity for the nonstandard universe is incompatible with the infinity axiom for the standard universe. If we want the universal class to contain external sets, and to be large enough to go beyond the finite levels in the hierarchy, we necessarily have to give up regularity.

There are different positions about the role of the regularity axiom in mathematics, and this is not the right place for a discussion. [17] Here I will only give some brief personal consideration, not necessarily shared by others. Probably, the main effect of regularity is to give a nice picture of the universe $\mathbf{V}$. In fact, in presence of other axioms of ZFC, it can be equivalently formulated as the equality $\mathbf{V} = \bigcup\{V_\alpha : \alpha \in \mathrm{ON}\}$, stating that every set is obtained at some level of the cumulative hierarchy over the emptyset. However, this nice picture can also be seen as a limitation, in that it restricts the universe of sets. My position is the following. The axiom of regularity is an unnatural restriction

[16] We remark that conservativity is not the only notion available to test the strength of a nonstandard set theory T with respect to ZFC. For instance, on the semantic side, extensions properties for models are usually considered.

[17] The reader interested in nonwellfounded sets and their use can consult for instance [Ac], [Hi] and [BM].

to the existence of sets and should be replaced by a suitable anti-foundation principle yielding plenty of nonwellfounded sets. This can be done in such a way that the new system proves the same theorems about wellfounded sets that are proved by ZFC. In this framework, if one believes that only wellfounded sets exist or that only wellfounded sets are worth considering in mathematics, fine. He/she will never go beyond the wellfounded part of the universe. In those cases where assuming regularity is essential in order to formulate or prove results, then one can simply reformulate them as results about the wellfounded part of the universe. The universe of classic "wellfounded" mathematics is not changed. Simply, it is expanded by additional sets one can consider whenever it is useful. From now on, we denote by ZFC^- the theory ZFC without regularity.

If giving up regularity can be seen as a minor problem (or as a non-problem), essential restrictions that deeply undermine the dream of formulating the "perfect" nonstandard set theory were first pointed out by Karel Hrbáček [H1]. In a naive formulation, its result can be stated as follows: [18]

> **Hrbáček's paradoxes.** *Let T be any nonstandard set theory where items # 2, 3 and 4 as above are satisfied. Assume all axioms of ZFC^- are included in T for the standard universe, and all axioms of ZFC^- except power-set PS and choice AC are considered for the whole universe. Then: (i) $T + PS$ is inconsistent; (ii) $T + AC$ is inconsistent.*

Thus any proposed nonstandard set theory has to either give up some of the axioms of ZFC^- in the external universe, or assume some of the principles of nonstandard analysis only in a weakened form.

§4. Nelson's Internal Set Theory.

The *Internal Set Theory IST* was presented by Edward Nelson in his 1977 paper [N1]. So far, it is the only nonstandard axiomatic framework to have been actually adopted by working mathematicians. [19] Internal Set Theory is an elegant theory formulated as a result of a precise philosophical position. With respect to usual set theory, an additional predicate `st`, called *"standard"*, is part of the formal language. This way the notion of *standard set* is of the same nature as the membership relation, i.e. it is a primitive concept not to be defined. *"The reason for not defining "standard" is that it plays a syntactical, rather than semantic, role in the theory. It is similar to the use of "fixed" in informal mathematical discourse. One does not define this notion ... But the predicate "standard" – unlike "fixed" – will be part of the formal language of our theory ...".* [20] The main philosophical position of the *internal* approach is that

[18] A precise formulation will be given in §5.

[19] See for instance [DR] for an introduction to the practice of nonstandard analysis within IST.

[20] [N3] p. 3.

"... we do not enlarge the world of mathematical objects in any way, we merely construct a richer language to discuss the same objects as before.". [21] Thus, in contrast to the *superstructure approach*, there is no distinction between a "standard" set, say the real numbers R, and its "nonstandard" version, namely $^*R$. One only has a single set of real numbers, the usual one. The point is that IST is provided with the new predicate `st` for "standard", so that one can more deeply investigate properties of the real line and realize, for instance, that it contains infinitesimals. In this sense standard objects, that is objects x such that $\mathtt{st}(x)$ holds, are those sets one can already consider by using the ordinary language of mathematics. The novelty is that standard sets may have non-standard elements, i.e. elements that need the richer language of IST to be detected and investigated. *"...very, very large natural numbers, and very, very small real numbers were there all along, and now we have a suitable language for discussing them."* [22]

A description of the common ground shared by the two main view-points to nonstandard analysis, namely the *external* $*$-approach by means of superstructures and the *internal* approach by means of a standardness predicate, can be found in [DS]. In that paper, F. Diener and K. Stroyan also give an interpretation of IST in a superstructure, thus supplying a bridge to followers of the two schools.

Let us fix some notation. The quantifier $\forall^{st} x$ means "for all standard x" and similarly $\exists^{st} x$ means "there exists a standard x". That is, for every formula φ, we write $\forall^{st} x\, \varphi$ as a short-hand for $\forall x\, (\mathtt{st}(x) \to \varphi)$ and $\exists^{st} x\, \varphi$ as a short-hand for $\exists x\, (\mathtt{st}(x) \wedge \varphi)$.

- *ZFC.*
  All axioms of Zermelo-Fraenkel set theory with choice (including regularity) are assumed. [23]

- *Transfer Principle (T)*
  For every $\in$-formula φ whose free variables are $x_1, \ldots, x_n, y$, the following is an axiom:

$$\forall^{st} x_1 \cdots \forall^{st} x_n \; (\forall^{st} y \; \varphi \leftrightarrow \forall y \; \varphi)$$

By $\in$-formula, we mean a formula in the usual language of set theory, i.e. not involving the standardness predicate `st`. The *transfer principle* states that any formula of set theory with standard parameters $x_1, \ldots, x_n$ that holds for every standard set also holds for all sets. *"The intuition behind (T) is that if something is true for a fixed, but arbitrary, x then it is true for all x."* [24]

[21] Ibid. p 3.
[22] Ibid p. 11.
[23] We remark that regularity can be consistently assumed because IST does not admit external sets.
[24] Ibid. p. 5.

Before formulating the remaining principles of IST, we need some more notation. The quantifier $\forall^{stfin} x$ means "for all standard finite x" and similarly $\exists^{stfin} x$ means "there exists a standard finite x". The distinction between standard finite sets and finite sets is crucial in IST. [25] For instance, each initial segment $[0, x]$ of the natural numbers is finite, but it is also standard finite if and only if x is standard.

- *Idealization Principle (I)*
  For every $\in$-formula φ, the following is an axiom:

$$\forall^{stfin} x' \, \exists y \, \forall x \in x' \, \varphi \;\leftrightarrow\; \exists y \, \forall^{st} x \, \varphi$$

"The intuition behind (I) is that we can only fix a finite number of objects at a time. To say that there is a y such that for all fixed x we have φ is the same as saying that for any fixed finite set of x's there is a y such that φ holds for all of them." [26] We remark that (I) gives an extremely strong form of saturation. For instance, it implies the existence of finite sets containing the entire standard universe. [27]

- *Standardization Property (S)*
  For every formula φ the following is an axiom:

$$\forall^{st} x \, \exists^{st} y \, \forall^{st} z \, [z \in y \leftrightarrow (z \in x \, \wedge \, \varphi)]$$

"The intuition behind (S) is that if we have a fixed set, then we specify a fixed subset of it by giving a criterion for judging whether each fixed element is a member of it or not." [28] As we already pointed out in §3, standardization seems to be a natural property to be assumed, and very useful in making definitions. For instance, standardization is essential to prove the following external version of induction.

- *Let φ be any formula, and suppose $\varphi(0)$ and $\forall^{st} n \in N \, \varphi(n) \rightarrow \varphi(n+1)$. Then $\forall^{st} n \in N \, \varphi(n)$.*

IST is the theory ZFC+(I)+(S)+(T). A crucial fact when working within IST is that one has to be careful about *illegal* set formation. As only $\in$-formulas are allowed in the definition of subsets, many basic collections considered in the practice of nonstandard analysis are not sets of IST. For instance the collection of

[25] In the language of the superstructure approach, they correspond to finite and *finite sets, respectively.

[26] Ibid. p. 7.

[27] The usual intuitive justification for ZFC views its universe as never finished, but better and better approximated by the levels V_α of the cumulative hierarchy. This intuitive picture seems to contrast with existence of sets containing the entire standard universe.

[28] Ibid. p. 12.

all infinitesimal numbers is illegal, in that the notion of infinitesimal is defined by a formula involving `st` ("x is infinitesimal iff $|x| < \varepsilon$ for all *standard* $\varepsilon > 0$"). In general, none of the external collections available in the superstructure approach is an actual object of IST. Thus item # 1 of desired nonstandard set theories is implemented at the price of dramatically restricting the universe to internal sets. As pointed out by H.J. Keisler: *"...because external sets are missing, developments such as the Loeb measure construction and hyperfinite descriptive set theory cannot be carried out in their full generality in IST."* [29] However, we will see in the sequel that various improvements on the internal approach have been proposed which allow the direct use of external collections.

Concerning the foundational strength of the IST, the following holds. [30]

Theorem 4.1.
IST is a conservative extension of ZFC. That is, for every $\in$-formula σ:

$$\mathrm{ZFC} \vdash \sigma \;\Leftrightarrow\; \mathrm{IST} \vdash \sigma$$

Nelson also proved interesting syntactical properties of IST. Notably, he gave a reduction algorithm which takes external formulas and rewrites them as equivalent Σ_2^{st}-formulas (see [N1] §2).

Theorem 4.2.
Let φ be any formula of IST whose quantifiers are all bounded by standard sets. Then there is an $\in$-formula $\vartheta(x,y)$ with

$$\mathrm{IST} \vdash \varphi \;\leftrightarrow\; \exists^{st} x\, \forall^{st} y\, \vartheta(x,y)$$

On the same matter, in his 1988 paper [N2] Nelson showed that a reduction algorithm can also be given to re-write every nonstandard proof in IST as a standard proof in ZFC[V], a simple conservative extension of ZFC which directly encloses a form of the reflection principle. This gives an answer to the old problem raised by A. Robinson himself: *"to devise a purely syntactical transformation which correlates standard and nonstandard proofs..."*.

§5. Hrbáček's Nonstandard Set Theories.

Independently of Nelson, Karel Hrbáček developed his nonstandard set theories $\mathcal{NS}_1$,$\mathcal{NS}_2$ and $\mathcal{NS}_3$ already in the first half of the seventies [H1]. [31] In his systems he explicitly considered external sets in the universe. To this end, he added two symbols to the usual language of set theory, namely a predicate $\mathtt{st}(x)$

[29] [K3] p. 232.

[30] This theorem, due to William C. Powell, is proved in the appendix of Nelson's original paper [N1]. See also [CK] §4.4, where the proof is presented in a simpler form.

[31] Historically, he was the first one to actually develop comprehensive nonstandard set theories. In fact, his paper [H1] (published 1978) was accepted in May 1975, while Nelson presented his IST in the summer of 1976.

for "x is standard" and a predicate $\mathtt{int}(x)$ for "x is internal". We will use a similar notation for internal-bounded quantifiers $\forall^{int}$ and $\exists^{int}$ as we already did for standard-bounded quantifiers $\forall^{st}$ and $\exists^{st}$ in IST. [32]

The common part $\mathcal{NS}_0$ of the three Hrbáček's theories is the system given by the following seven groups of axioms.

- (H1) *ZFC for the standard universe.*
  For every axiom φ of ZFC, its standard relativization φ^{st} is assumed. [33]
- (H2) *A fragment of ZFC for the external universe.*
  The axioms of extensionality, emptyset, pairing, union, infinity and separation are assumed for the universal class (i.e. without restricting quantifiers). The separation schema is also considered for formulas containing predicates $\mathtt{st}$ and $\mathtt{int}$.
- (H3) *All standard sets are internal.*
$$\forall^{st} x\, \mathtt{int}(x)$$
- (H4) *The universe of internal sets is transitive.*
$$\forall x\, \forall^{int} y\ (x \in y \to \mathtt{int}(x))$$
- (H5) *Transfer Principle.*
  For every $\in$-formula φ whose free variables are $x_1, \ldots, x_n$, the following is an axiom:
$$\forall^{st} x_1 \cdots \forall^{st} x_n\ (\varphi^{st}(x_1, \ldots, x_n) \ \leftrightarrow\ \varphi^{int}(x_1, \ldots, x_n))$$
- (H6) *Standardization Property.*
$$\forall a\, \exists^{st} b\, \forall^{st} x\ (x \in a \ \leftrightarrow\ x \in b)$$

Notice that this version of standardization is stronger than Nelson's, because in the above a can be any (not necessarily standard) set. The unique standard set having the same standard elements as A is usually denoted by ${}^{o}A$. Notice that, by separation, one can also consider the set ${}^{\sigma}A = \{x \in A : \mathtt{st}(x)\}$.

- (H7) *Idealization Principle.*
  For every $\in$-formula φ whose free variables are $x_1, \ldots, x_n, t, a, b$, the following is an axiom: [34]

[32] That is, we will write $\forall^{int} x\, \varphi$ as a short-hand for $\forall x\, (\mathtt{int}(x) \to \varphi)$ and $\exists^{int} x\, \varphi$ as a short-hand for $\exists x\, (\mathtt{int}(x) \wedge \varphi)$.

[33] φ^{st} is the formula obtained from φ by setting each quantifier $\forall x$ and $\exists x$ in its standard-bounded form $\forall^{st}$ and $\exists^{st}$. We will also use a similar notation for internal relativization φ^{int}.

[34] $\forall^{stfin} x$ means $\forall^{st} x\, (x \text{ finite} \to \cdots)$.

$$\forall^{int} x_1 \cdots \forall^{int} x_n \, \forall a \text{ of standard-size}$$
$$[\, \forall^{stfin} a' \subseteq a \, \exists^{int} b \forall^{int} t \in a' \; \varphi^{int}(t, a, b, x_1, \ldots, x_n) \,] \rightarrow$$
$$[\, \exists^{int} b \forall^{int} t \in a \, \varphi^{int}(t, a, b, x_1, \ldots, x_n) \,]$$

A set A has *standard-size* if there is a standard X and a function f such that $A = \{f(x) : x \in X \,\&\mathtt{st}(x)\}$. The above principle gives an unlimited amount of saturation with respect to the standard universe.

Roughly speaking, the universe of standard sets in $\mathcal{NS}_0$ can be seen as the "usual" mathematical universe satisfying ZFC, and the universe of internal sets as an "**ON**-saturated" elementary extension of it. Then, the internal universe is further extended by considering collections of internal sets that are not themselves internal. The resulting universal class (external universe) models a fragment of ZFC, so that some of the mathematical constructions allowed for standard and internal sets, are also allowed for all sets without restrictions. Standardization permits unlimited use of external sets in constructions of standard sets. Since in Hrbáček's systems an external notion of cardinality is available, the size of internal sets can be compared with the size of external sets. As a consequence of saturation, the (external) size of internal sets is huge, and the restriction to sets of standard-size in the idealization principle is needed in order to avoid straight contradictions.

As already mentioned in §3, Hrbáček was the first one to point out that the "perfect" nonstandard set theory cannot exist. Notice that in order to make $\mathcal{NS}_0$ "perfect", also the remaining axioms of replacement, power-set and choice should be postulated for the external universe. Unfortunately this is not possible. Precisely, the following holds.

Theorem 5.1. *Hrbáček's Paradoxes.* [35]
(i) $\mathcal{NS}_0$ *+ Replacement + Power-set is inconsistent.*
(ii) $\mathcal{NS}_0$ *+ Replacement + Choice is inconsistent.*

The proofs formalize the following idea. As a consequence of idealization, the infinite standard set N of natural numbers has a larger size than ${}^{\sigma}A$, for every standard A. If enough axioms of ZFC are available for the external universe, then one can obtain a set containing arbitrarily large standard ordinals. Finally, by applying standardization, one gets that the collection **ON** of all ordinals is a set. A contradiction.

Three different axiomatic systems $\mathcal{NS}_1$, $\mathcal{NS}_2$ and $\mathcal{NS}_3$ were proposed by Hrbáček as the best possible compromises to avoid the above inconsistency results. Namely,

- $\mathcal{NS}_1$ is the theory $\mathcal{NS}_0$ + Replacement.

[35] [H1] Theorem 3. In the replacement schema, also formulas containing the symbols `st` and `int` are considered.

- $\mathcal{NS}_2$ is the theory $\mathcal{NS}_0$ + Power-set + Choice.
- $\mathcal{NS}_3$ is the theory $\mathcal{NS}_0$ + Standard-sized Replacement + Power-set + Choice.

The standard-sized replacement is the schema obtained by restricting all instances of replacement to external sets of standard-size. [36]

Theorem 5.2. [37]
(i) $\mathcal{NS}_1$ is a conservative extension of ZFC. That is, for any $\in$-sentence σ:

$$ZFC \vdash \sigma \Leftrightarrow \mathcal{NS}_1 \vdash \sigma^{st}$$

(ii) $\mathcal{NS}_2$ is a conservative extension of ZFC.
(iii) $\mathcal{NS}_3$ is consistent relative to ZFC + "there exists an inaccessible cardinal", but it is not conservative.

Thus, by assuming the ground theory $\mathcal{NS}_0$, the picture is clear. For the external universe, one can either postulate replacement or power-set and choice (possibly with a weakened form of replacement) but further strengthening are inconsistent. In author's opinion, Hrbáček's contribution [H1] was a cornerstone in the field under review. His paradoxes made it clear that the "perfect" nonstandard set theory cannot exist, and they were the starting point of the following research in the field.

§6. Kawai's axiomatic system NST.

Important contributions to the foundations of nonstandard methods were given by Toru Kawai in a series of papers culminating in [Kw]. He was the first one to present axiomatic systems where all axioms of ZFC^- are assumed for the external universe. Hrbáček's inconsistency results are avoided by weakening standardization and by considering both the standard universe and the internal universe as true sets. To this end, the language of Kawai's theory is different in nature from the previous ones. In place of the predicate symbols `st` and `int`, two new constant symbols S and I for the standard and internal universe are considered instead.

The axioms of Kawai's *Nonstandard Set Theory NST* are the following.

- (K1) *ZFC for the standard universe.*
  For every axiom φ of ZFC, its standard-relativization φ^S is assumed. [38]

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[36] Precisely, in the formulation of replacement as

$$\forall A\, \exists B\, \forall a \in A\, [\exists x\, \varphi(x, a, A) \to \exists x \in B\, \varphi(x, a, A)]$$

the set A has to be of standard-size.

[37] Theorems 1, 2 and 4 of [H1].

[38] φ^S is the formula obtained from φ by replacing all quantifiers $\forall x$ and $\exists x$ with its restricted form $\forall^S$ and $\exists^S$, that is $\forall x\, (x \in S \to \cdots)$ and $\exists x\, (x \in S \wedge \cdots)$), respectively. A similar notation will be also adopted for restricted quantifiers $\forall^I$ and $\exists^I$ and internal-restrictions φ^I.

- (K2) *ZFC⁻ for the external universe + Weak Regularity.*
  All axioms of ZFC are assumed for the universal class with the exception of regularity, which is only assumed in the following weak form.

$$\forall x \neq \emptyset\ [x \cap I = \emptyset\ \rightarrow\ (\exists y \in x\ y \cap x = \emptyset)\,]$$

- (K3) *All standard sets are internal.*

$$\forall^S x\ x \in I$$

- (K4) *I is transitive.*

$$\forall x\, \forall^I y\ (x \in y\ \rightarrow\ x \in I)$$

- (K5) *Tranfer Principle.*
  For every $\in$-formula φ whose free variables are $x_1, \ldots, x_n$, the following is an axiom:

$$\forall^S x_1 \cdots \forall^S x_n\ (\,\varphi^S(x_1, \ldots, x_n)\ \leftrightarrow\ \varphi^I(x_1, \ldots, x_n)\,)$$

- (K6) *Standardization Property.* [39]

$$\forall a\, [(\exists^S t\ {}^\sigma A \subseteq t)\ \rightarrow\ \exists^S b\, \forall^S x\, (x \in a \leftrightarrow x \in b)]$$

- (K7) *Idealization Principle.*
  For every $\in$-formula φ all of whose free variables are among $x_1, \ldots, x_n, t, b$, the following is an axiom: [40]

$$\begin{gathered} \forall^I x_1 \cdots \forall^I x_n\, \forall a\ S\text{-sized} \\ [\,\forall^{Ifin} a' \subseteq a\, \exists^I b\, \forall^I t \in a'\ \varphi^I(t, b, x_1, \ldots, x_n)\,] \rightarrow \\ [\,\exists^I b\, \forall^I t \in a\, \varphi^I(t, b, x_1, \ldots, x_n)\,] \end{gathered}$$

A set a is *S-sized* if there is a function $f : S \rightarrow a$ onto. Kawai's *weak regularity* axiom was subsequently considerated in other nonstandard set theories. Similarly as regularity in ZFC, weak regularity gives a picture of the external universe, as a cumulative hierarchy over the internal sets. With respect to usual set theory, NST is a bit confusing when considering the notion of size. For instance, the huge collections S and I of standard and internal elements, are themselves external sets of the universe. Since in Kawai's system external sets can be of unlimited size, full standardization cannot hold. Notice in fact that Kawai's (K6) is weaker than Hrbáček's (H6), in that only those sets whose

[39] ${}^\sigma A = \{x \in A : x \in S\}$ denotes the (external) collection of all standard elements of A.
[40] $\forall^{Ifin} x$ means $\forall^I x\, (x \text{ finite} \rightarrow \cdots)$.

standard elements are already contained in a standard set can be standardized. In the practice, this makes it often uncertain whether a given set can be standardized or not. NST is justified by the following theorem.

Theorem 6.1. [41]
NST is a conservative extension of ZFC. That is, for any ∈-sentence σ:

$$\mathrm{ZFC} \vdash \sigma \Leftrightarrow \mathrm{NST} \vdash \sigma^S$$

In [Kw], an equivalent reformulation of NST, namely UNST, is also given. It is called a nonstandard *"axiom system from the usual viewpoint"*, because it is presented by axiomatizing a nonstandard embedding $*$.

§7. Fletcher's Stratified Nonstandard Set Theory SNST.
Fletcher's work [Fl] explored the possibility of avoiding Hrbáček's paradoxes, by dynamically changing the "picture" of the universe. Instead of a static fixed universe, his system provides a whole hierarchy of internal and external universes. His point in formulating the *Stratified Nonstandard Set Theory SNST* can be summarized as follows. Although a single formal system is desiderable, it does not have to describe a single universe. SNST postulates an increasing sequence of internal and external universes indexed over the cardinals, where varying levels of saturation are satisfied. Full idealization does not hold, but any given amount of saturation is made available by working in a suitable level of the hierarchy.

The language of SNST is different from the usual language of set theory, in that all quantifiers have α or ext, α superscripts. The informal meaning is the following. For each cardinal α there is an internal universe I_α together with its external universe E_α. Quantifiers $\forall^\alpha x$, $\exists^\alpha x$ are intended as restricted to I_α, and $\forall^{ext,\alpha} x$, $\exists^{ext,\alpha} x$ as restricted to E_α. The standard universe S is identified with I_0. Thus $\forall^S$ and $\exists^S$ stand for $\forall^0$ and $\exists^0$ respectively. We will write $\forall\alpha$ as a short-hand for "$\forall^0 \alpha\, (\alpha$ is a cardinal $\to \cdots)$". The axioms of SNST are the following.

- (F1) *ZFC for the standard universe.*
  For every axiom φ of ZFC, its standard-relativization φ^S is assumed.

- (F2) *ZFC$^-$ + weak regularity for the external universe.*
  For every axiom φ of ZFC$^-$, its relativization $\varphi^{ext,\alpha}$ to the external universe E_α is assumed for all α. Besides, this form of weak regularity holds:

$$\forall\alpha\, \forall^{ext,\alpha} x \neq \emptyset\ [\forall^\alpha y\ y \notin x\ \to\ (\exists^{ext,\alpha} z \in x\ z \cap x = \emptyset)\,]$$

- (F3) *$I_\alpha \subseteq E_\alpha$, $I_\alpha \subseteq I_\beta$ and $E_\alpha \subseteq E_\beta$ for all $\alpha \le \beta$.*

[41] Conservation theorem for NST in [Kw].

$$\forall\alpha\,\forall\beta\,[\alpha \leq \beta \to (\forall^\alpha x\,\exists^{ext,\alpha} y\ y = x) \wedge$$
$$(\forall^\alpha x\,\exists^\beta y\ y = x) \wedge (\forall^{ext,\alpha} x\,\exists^{ext,\beta} y\ y = x)$$

- (F4) *Each strictly external part $E_\alpha \setminus I_\alpha$ is transitive. The whole internal universe $I = \bigcup_\alpha I_\alpha$ is transitive.* [42]

$$\forall\alpha\,\forall^{ext,\alpha} x\,(\exists^\alpha t\ x = t\ \vee\ \forall\beta\,\forall^{ext,\beta} y \in x\,\exists^{ext,\alpha} z\, y = z)$$

$$\forall\alpha\,\forall\beta\,\forall^\alpha x\,\forall^{ext,\beta} y \in x\ \exists^\beta z\ y = z$$

- (F5) *Transfer Principle.*
For every $\in$-formula φ whose free variables are $x_1, \ldots, x_n$, the following is an axiom:

$$\forall\alpha\,\forall\beta\,[\,\alpha \leq \beta\ \to\ \forall^\alpha x_1 \cdots \forall^\alpha x_n\ (\,\varphi^\alpha(x_1 \ldots, x_n)\ \leftrightarrow\ \varphi^\beta(x_1, \ldots, x_n)\,)\,]$$

- (F6) *Standardization Property.*

$$\forall\alpha\,\forall^{ext,\alpha} a\,\exists^S b\,\forall^S x\,(x \in a \leftrightarrow x \in b)$$

- (F7) *Idealization Principle.*

$$\forall\alpha\,\forall^\alpha r\ [\,r \text{ is a binary relation} \wedge \exists^S f\ f({}^\sigma\alpha) = {}^\sigma r\,]\ \to$$
$$[\,(\forall^{Sfin} a \subseteq \mathrm{dom}(r)\,\exists^\alpha b\,\forall^S a' \in a\, r(a', b))\ \to\ (\exists^\alpha b\,\forall^S a \in \mathrm{dom}(r)\, r(a, b)\,]$$

"The way to visualize SNST is to regard any infinite internal set A as an inexhaustible source of elements. How many elements you find in A depends on how hard you look; $\forall^\alpha x \in A$ means "for all elements x of A discoverable by looking with degree of throughness α"." [43] As presented, SNST is not formalized in the usual first-order predicate calculus. And here stands the main criticism to Fletcher's system. The use of ranked quantifiers Q^α and $Q^{ext,\alpha}$ resembles the old formalism of type theory, by now unfamiliar to most mathematicians. Probably, an equivalent reformulation of SNST in a first-order language could be of help to make it better accepted as a satisfactory foundational system for the practice of nonstandard methods. [44]

Models of SNST are constructed by an ultralimit process. Roughly speaking, one starts with the standard universe $I_0 = S$ and, by transfinite induction,

[42] We remark that internal universes I_α are *not* transitive.

[43] [Fl] p. 1007.

[44] Such a first-order presentation of SNST is possible by considering extra symbols. For instance, one could take the language $\mathcal{L}$ that consists of the membership symbol $\in$, of a predicate $\mathtt{st}(x)$ for "x is standard" and of two binary relation symbols $I(x, y)$ and $E(x, y)$ whose informal meaning is the following. For each (standard) cardinal α, $I_\alpha = \{y : I(\alpha, y)\}$ and $E_\alpha = \{y : E(\alpha, y)\}$. Then, (with some caution) SNST can be given an equivalent axiomatization by sentences of $\mathcal{L}$.

defines I_α as an α-saturated extension of the direct limit of $\{I_\beta : \beta < \alpha\}$. Then, external sets are built up on each I_α, by a transfinite iteration of a "simulated" power-set operator for extensional structures. SNST is justified by the following

Theorem 7.1. [45]
(i) Every model of ZFC is embedded as the standard universe in some model of SNST. In particular, SNST is a conservative extension of ZFC. That is, for every $\in$-formula σ:

$$ZFC \vdash \sigma \Leftrightarrow SNST \vdash \sigma^S$$

§8. Ballard's Enlargement Set Theory EST.
Fletcher's approach of considering many universes in the same formal system, was further developed by David Ballard [Ba]. By presenting his *Enlargement Set Theory EST* in the framework of a class theory, Ballard was able to avoid the notational problems of a ranked language. Starting from a suitable notion of *universe*, the following is postulated. For every universe U and every U-cardinal κ, there exists a nonstandard κ-saturated enlargement $U' \supseteq U$ (such an U' is also a universe). This way any given level of saturation is available. In Ballard's words, the *"intended interpretation of EST is of a cosmos of sets undergoing perpetual expansions. The expansion will be twofold: First, there will be external expansion wherein standard set theoretic operations ... are used to form new sets from old. Secondly, there will be an internal expansion wherein any set x, unless finite, will continually pick up new elements through the "non" standard process of saturation."* [46]

EST is formulated in the language of the Gödel-Bernays class theory GB. [47] As usual, capital letters $A, B, C, \ldots$ will denote class variables and small letters $x, y, z, \ldots$ will denote set variables. Recall that *sets* are those classes that are elements of classes, and a *proper class* is a class that is not a set. If φ is a formula of class theory, and U is a class, we will write "$U \models \varphi(x_1, \ldots, x_n, A_1, \ldots, A_m)$" to mean

$$x_1, \ldots, x_n \in U \,\wedge\, A_1, \ldots A_m \subseteq U \,\wedge\, \varphi^U(x_1, \ldots, x_n, A_1, \ldots, A_m)$$

where φ^U is the U-relativization of φ, that is the formula obtained from φ by rewriting set quantifiers $\forall x$,$\exists x$ as $\forall x \in U$, $\exists x \in U$, and class quantifiers $\forall A$, $\exists A$ as $\forall A \subseteq U$, $\exists A \subseteq U$, respectively. To better understand the sequel, the reader is warned in advance that EST is not to be intended as an extension of (a slight modification of) GB. For instance, in EST the existence of a universal class containing all sets will not be assumed. EST should rather be thought as a vessel containing suitable actual classes satisfying special properties. Similarly

[45] [Fl], Theorem p. 1006.
[46] [Ba] p. 102.
[47] A concise presentation of GB can be found in [Je] p. 76.

as the ZFC universe contains superstructures, EST contains *universes*, on which attention will be focused.

We say that a class is *static* if it is a subclass of some proper class. A static class which is a subclass of a set is called *local*. [48] The central notion in EST is that of *universe*. Roughly speaking, a universe is a class which essentially models set theory and reflects all bounded formulas, so that a *transfer* principle will be available. By definition, a *universe* is a class U such that:

- *U is static.*
- *U is not local.*
- *A modified version of Gödel-Bernays axioms holds for U.*
  Namely, U must satisfy the following form of extensionality:

  $$\begin{gathered}\forall x, y \in U\, \forall A, B \subseteq U\, \forall t\ \ [(t \in x \leftrightarrow t \in y) \to x = y\,] \wedge \\ [(t \in A \leftrightarrow t \in B) \to A = B\,]\end{gathered}$$

  Comprehension axioms are assumed in the form that asserts the existence of *static* subclasses of U. Regularity is not postulated and choice is assumed in the stronger form of a well-ordering of the universe. See [Ba] for details. [49]
- *U satisfies full separation with respect to arbitrary classes:*

  $$\forall A\, \exists B\, \forall x\, [\, x \in B \ \leftrightarrow\ (x \in A \wedge x \in U)\,]$$

- *U is able to detect when a static subclass is local or when it is a set.*

  $$\forall A \subseteq U\, (\, A \text{ local } \ \to\ \text{"}U \models A \text{ local"}\,)\ \wedge\ \forall x\, (x \subseteq U \to x \in U)$$

- *For each Gödel operation $\mathcal{G}$:*

  $$\forall x_1 \in U \cdots \forall x_n \in U\ (\, \vartheta_{\mathcal{G}}(x_1, \ldots, x_n)\ \leftrightarrow\ \text{"}U \models \vartheta_{\mathcal{G}}(x_1, \ldots, x_n)\text{"}\,)$$

  with $\vartheta_{\mathcal{G}}(x_1, \ldots, x_n)$ a (bounded) formula defining $\mathcal{G}$. [50]

---

[48] Note that at this point *nothing*, including GB, is assumed. Indeed, were a universal class $\mathbf{V}$ to exist, all classes would be static. Proper subclasses of sets are called *semisets* in Vopěnka-Hájek's *Alternative Set Theory* AST, where they play a central role. (see [VH]).

[49] We remark that some caution is needed when relativizing axioms of GB to U, because the latter is not assumed to be transitive. For instance, call U-classes those classes such that $A \subseteq U$, and U-sets those sets such that $x \in U$. Then it may be that a U-set is not a U-class. Also, notice that from U's point of view, there may be a class A and a set x having the same elements, but $A \neq x$.

[50] We remark that there is a finite number of Gödel's operations. A complete list is given in §13 of this paper, when formulating axiom 3 of *ZFC.

Notice that, as a consequence of the latter property, a universe U reflects all bounded formulas. [51] Now, let $U' \supseteq U$ be universes. We say that U' in an *enlargement* of U, if

$$U' \models \forall x \in U \,(\text{"}U \models x \text{ is finite "} \rightarrow x \subseteq U)$$

Thus enlargements $U' \supseteq U$ have the property that a set of the first universe is seen as finite by either universe only if it is seen as finite by both. We are now ready to give the two axioms of EST.

- (B1) There exists a universe U.
- (B2) For any universe U and for every U-cardinal κ, there exists a κ-saturated enlargement $U' \supseteq U$.

Recall that GB admits a finite axiomatization. As a consequence, (B1) is actually expressed by a formula in the class language of EST. Since the notion of cardinal is not absolute between U and U', caution is needed to correctly formalize the second axiom. First, it is seen that to each U-cardinal κ (i.e. $\kappa \in U$ and $U \models$ "κ is a cardinal") canonically corresponds a unique U'-cardinal κ'. Then the transitive closure $\overline{U}$ of U (as seen by U') is considered; and finally (B2) is expressed in U' as the usual intersection property for subsets of $\overline{U}$ of cardinality less than κ'. [52]

Though presented by only two axioms, the system EST may not be easy to grasp for the average working mathematician, not familiar with class theories. The notion of universe is quite different from the usual ones adopted in set theory. [53] Some of the required properties look a bit too technical and beyond the common practice of nonstandard methods. Also, it may be a bit confusing not to have an external absolute notion of cardinality.

However, the fundamental feature of EST is that it allows one to layer nonstandard arguments, that is external sets in one universe can be the standard ones in another. This fact successfully matches Ballard's philosophical position: *"In designing the vehicle EST, I have deliberately ignored the needs of practitioners and sought instead to decisively illustrate the full implications of this relativistic mathematical ontology. The EST cosmos is in continual explosion. Its universes A are merely frozen frames of reference. Within these frames, familiar mathematics can be observed, but with the slightest slip along the chute of time — i.e., a passage to any further enlargement $A \subseteq A'$ — the usual develops*

[51] In fact, recall that (with suitable hypotheses) preservation of bounded formulas is equivalent to closure under Gödel operations ([Je] Theorem 30). Here, all variables occurring in bounded formulas are assumed to be set variables.

[52] See [Ba] for details.

[53] In the literature, several definitions of universe can be found. However, by universe is usually meant a transitive class closed under some elementary operations. Typically, only basic notions such as pair, union, subset, power-set are considered in the definition.

unusualness. The universe A will have its clear version ω_A of the natural numbers, but when viewed in a later universe A', this ω_A will be seen to have picked new, unusual elements which are infinite. Of course A' has its version $\omega_{A'}$ of the natural numbers, but this only means that the cycle of slipping "reality" is poised to repeat itself again.". [54]

The foundational justification of EST is given by the next theorem.

Theorem 8.1.
For every $\in$-sentence σ, the following are equivalent:
(i) ZFC $\vdash \sigma$;
(ii) EST $\vdash$ "there exists a universe U that satisfies the regularity axiom and $U \models \sigma$";
(iii) EST $\vdash$ "for each U that satisfies the regularity axiom, $U \models \sigma$".

§9. The "Near-standard" Zermelo-Fraenkel-Boffa Set Theory ZFBC. In their paper [BH], David Ballard and Karel Hrbáček showed that a modification of the classic axiomatic system of set theory can be adopted as a setting for the practice of nonstandard methods in its full generality. Namely, they proposed the more or less standard ("near-standard") Zermelo-Fraenkel-Boffa set theory ZFBC, where an additional binary relation symbol C is included in the language to give a well-ordering of the universe, and regularity is replaced by a strong anti-foundation principle. Precisely, the axioms of ZFBC are the following.

- ZFC$^-$.
  All axioms of ZFC except regularity are assumed. The separation and replacement schemata are also considered for formulas where the symbol C occurs.

- $C(x,y)$ is a functional relation yielding a bijection $C : \mathbf{ON} \to \mathbf{V}$ between the class of ordinals and the universe. More formally,

$$\forall x\, \forall x'\, \forall y\, \forall y'\ [C(x,y) \wedge C(x',y')] \ \to$$
$$[(x = x' \leftrightarrow y = y') \wedge \text{“}x \text{ is an ordinal”}] \wedge (\forall y\, \exists x\ C(x,y))$$

- Boffa's *Superuniversality* axiom. [55]
  Assume $f : \langle A, E\rangle \to \langle A', E'\rangle$ is an end extension of extensional structures. Then for each transitive collapse $\pi : \langle A, E\rangle \to \langle T, \in\rangle$ there is a transitive collapse $\pi' : \langle A', E'\rangle \to \langle T', \in\rangle$ such that the following commutes: [56]

[54] [Ba] p. 128.
[55] Maurice Boffa's pioneering work on anti-foundation axioms dates back to the late sixties. A construction of models for the *superuniversality* axiom was first given in [Bo].
[56] $\imath$ is the inclusion map.

$$
\begin{tikzcd}
\langle A, E\rangle \arrow[r, "\pi"] \arrow[d, "f"'] & \langle T, \in\rangle \arrow[d, "\imath"] \\
\langle A', E'\rangle \arrow[r, "\pi'"] & \langle T', \in\rangle
\end{tikzcd}
$$

Recall that an *extensional structure* is a model of the language of set theory that satisfies the extensionality axiom. For instance, $\langle T, \in\rangle$ is an extensional structure for every transitive set T. An *end extension* $f : \langle A, E\rangle \to \langle A', E'\rangle$ is an embedding where images pick no new elements, that is *(i)* $xEy \leftrightarrow f(x)E'f(y)$; *(ii)* $\xi E' f(y) \to \xi = f(x)$ for some xEy.

Notice that, by taking $\langle A, E\rangle = \langle\emptyset, \emptyset\rangle$ the empty structure, superuniversality yields the existence of transitive collapses for all extensional structures $\langle A', E'\rangle$, wellfounded or not. Thus Boffa's axiom is a generalized form of the classic Mostowski transitive collapse theorem. The next result gives a foundational justification of ZFBC.

Theorem 9.1.
Every countable model $M \models$ ZFC is embedded as the collection of wellfounded sets into a model $N \models$ ZFBC. In particular, formulas of ZFC have a faithful interpretation in ZFBC by relativizing to the class WF of well-founded sets. That is, for every $\in$-sentence σ

$$\mathrm{ZFC} \vdash \sigma \;\Leftrightarrow\; \mathrm{ZFBC} \vdash \sigma^{WF}$$

Proof. Put together a classic result by U. Felgner on models of a global version of the axiom of choice [Fe], with Boffa's construction of models of Superuniversality [Bo]. As for the conservativity, assume by contradiction that ZFBC $\nvdash \sigma^{WF}$. Let $N \models$ ZFBC with $N \models \neg\sigma^{WF}$, and consider $M = (WF)^N$ the wellfounded part of N. Then $M \models$ ZFC and $M \models \neg\sigma$, thus ZFC $\nvdash \sigma$. Vice versa, assume ZFC $\nvdash \sigma$. By applying the downward Lowenheim-Skolem theorem, there is a countable model $M \models$ ZFC with $M \models \neg\sigma$. If $N \models$ ZFBC is an extension of its with $M = (WF)^N$, then $N \models \neg\sigma^{WF}$, hence ZFBC $\nvdash \sigma^{WF}$. $\dashv$

Although only sets are considered in Zermelo-Fraenkel system, to simplify notation we informally use classes as extensions of formulas. That is, a class $\mathbf{C}$ is a collection of the form $\mathbf{C} = \{x : \sigma(x)\}$ where σ is a formula. We will use boldface letters to denote them. φ^C denotes the formula obtained from φ by restricting all quantifiers to the class $\mathbf{C}$. That is, all quantifiers $\forall x$, $\exists x$ are replaced by $\forall x\,(\sigma(x) \to \cdots)$ and $\exists x\,(\sigma(x) \wedge \cdots)$, respectively. We say that a formula φ holds in $\mathbf{C}$, and write $\mathbf{C} \models \varphi$, when φ^C holds. Recall that a class $\mathbf{U}$ is *almost universal* if for every set x with $x \subseteq \mathbf{U}$, it is $x \subseteq y \in \mathbf{U}$ for some set y. A *universe* is a transitive almost universal class that is closed under the Gödel operations. As a consequence, it is proved that all axioms of ZFC without regularity hold in $\mathbf{U}$. [57] Let κ be a given cardinal. Like in the superstructure approach, we say that a universe $\mathbf{U}$ is *κ-saturated* if every family $\mathcal{F} \subseteq \mathbf{U}$ of cardinality less than κ with the finite intersection property, has nonempty intersection.

From the point of view of nonstandard analysis, the fundamental theorem of ZFBC is the following. [58]

Theorem 9.2 *Extension Principle.*
Let $\mathbf{U}$ be a universe and κ an infinite cardinal. Then there exists a κ-saturated universe $\mathbf{W}$ and an embedding $* : \mathbf{U} \to \mathbf{W}$ that satisfies the transfer principle. That is, for every $\in$-formula φ and $a_1, \ldots, a_n \in \mathbf{U}$

$$\mathbf{U} \models \varphi(a_1, \ldots, a_n) \;\Leftrightarrow\; \mathbf{W} \models \varphi(^*a_1, \ldots, {}^*a_n)$$

The practice of nonstandard methods in ZFBC is very close to the superstructure approach but way far more general. *"... fix a full universe* $\mathbf{S}$*, henceforth referred to as the standard universe ... the standard universe contains all the usual mathematical objects. The extension principle provides a universe* $^*\mathbf{S}$*, henceforth called internal universe, and an elementary embedding* $* : \mathbf{S} \to {}^*\mathbf{S}$ *... The main point is that all three universes (*$\mathbf{S}$*,* $^*\mathbf{S}$ *and* [the universal class of all sets] $\mathbf{V}$*) satisfy the familiar axioms of set theory and hence mathematics can be conducted without any restrictions in each of them. ... One can now proceed to conduct nonstandard analysis in the style of Robinson and Zakon ... but without the restrictions on the levels of the cumulative hierarchy imposed by superstructures.* [59] Thus ZFBC provides the framework for constructing κ-saturated nonstandard embeddings for *any* given universe in the same way as ZFC provides the framework for constructing embeddings of superstructures. The concepts used in nonstandard analysis are defined, not presented by means of additional symbols in the language of the theory. This is the reason why this foundational approach to nonstandard methods can be described as a "near-standard" one.

§10. Improving on the Internal Approach: Theories BST and HST. Recently, Vladimir Kanovei and Michael Reeken have published a series of papers as a result of a deep study of Nelson's IST and related set-theoretic foundational topics of nonstandard mathematics ([KR1], [KR2]). Their work led to substantial improvements of the *internal* approach. A fundamental achievement of theirs can be summarized as follows. By considering a minor modification of IST, namely the *Bounded Set Theory* BST, one can code external sets into the internal universe. This way a nonstandard set theory HST (*Hrbáček Set*

[57] See [JE] §11. A list containing all Gödel operations is given in §13 of this paper, when formulating axiom 3 of *ZFC.

[58] It is the Extension Principle proved in [BH].

[59] [BH] p. 744-745.

Theory) is obtained which overcomes the main inconveniences of IST. [60]

The starting point of Kanovei and Reeken's joint work, is the problem of improving Nelson's internal approach, in such a way that its defects can be overcome. The most striking one is the absence of external sets: *"IST fails to handle a very important type of nonstandard mathematical objects, therefore fails to serve as a system of foundations for nonstandard mathematics in all its totality."*. [61] In the first part of [KR1], they proposed to modify IST by postulating that every set belongs to a standard set.

- *Boundedness (B).*
  $\forall x\, \exists^{st} y\ x \in y$

Since (B) directly contradicts idealization (I), the latter has to be assumed in a weakened form. [62]

- *Bounded Idealization Principle (BI).*
  For every $\in$-formula φ, the following is an axiom:

$$\forall^{st} y\ [\forall^{stfin} x'\, \exists y' \in y\, (\forall x \in x'\, \varphi \leftrightarrow \exists y' \in y, \forall^{st} x\, \varphi)]$$

The resulting *Bounded Internal Theory BST* is closely related both to Nelson's and Hrbáček's theories.

Theorem 10.1. [63]
(i) Each of Hrbáček's systems $\mathcal{NS}$ proves all axioms of BST for the internal universe.
(ii) BST has an inner model in IST given by the class $B = \{x : \exists^{st} y\ x \in y\}$ of all bounded sets. In particular, BST is a conservative extension of ZFC. That is, for every $\in$-sentence σ:

$$\mathrm{ZFC} \vdash \sigma \ \Leftrightarrow\ \mathrm{BST} \vdash \sigma^{st}$$

BST retains all those features of IST used in applications and, in some sense, it can be considered as equivalent to the latter as far as the known practice of nonstandard analysis is concerned. Moreover, BST improves on IST in several aspects, as shown by the following

Theorem 10.2. [64]
(i) Reduction to Σ_2^{st}-form. For any formula $\varphi(x_1, \dots, x_n)$, there is an $\in$-formula $\psi(a, b, x_1, \dots, x_n)$ such that

[60] The reference to Hrbáček is due to the fact that HST is an extension of the system $\mathcal{NS}_1$.
[61] [KR1] part 1, p. 231.
[62] Recall that (I) implies the existence of a set S containing all standard sets. If $S \in x$ for some standard x, then $x \in S \in x$, against regularity.
[63] [KR1] part 1, §2.1.
[64] [KR1] part 1, Theorem 1.5 and Theorem 2.4.

$$BST \vdash \forall x_1 \cdots \forall x_n \ (\varphi(x_1, \ldots, x_n) \ \leftrightarrow \ \exists^{st} a \, \forall^{st} b \, \psi(a, b, x_1, \ldots, x_n))$$

In particular, in BST every sentence is equivalent to ψ^{st} *for some* $\in$*-sentence* ψ*.* [65]
(ii) Every countable model of ZFC is embedded as the standard universe into a model of BST.

The above results are to be contrasted with the following, due to Kanovei [Kv].

Theorem 10.3
(i) There is a sentence which is not equivalent in IST to ψ^{st} *for any* $\in$*-sentence* ψ*.*
(ii) There are countable transitive models of ZFC that cannot be embedded as the class of standard sets in any model of IST.

Kanovei and Reeken realized that BST allows coding external sets in the internal universe. Making use of the reduction property (not holding in IST), they first showed that it is possible in BST to define all $\in$-`st`-definable subclasses by a single formula with parameters. As a consequence, those external sets whose elements are all internal can be incorporated in a consistent way. Then, by applying a general construction of external sets grounded on the idea of a cumulative hierarchy along wellfounded trees, they showed that a full external universe can be obtained, which yields a model for a comprehensive external set theory. As a result, Kanovei and Reeken proposed an axiomatic nonstandard system, namely *Hrbáček's Set Theory HST*, formulated in the $\in$-`st`-language.

In the following, by internal universe we mean the definable class $I = \{x : \exists^{st} y \ x \in y\}$; $x \in I$ will be a short-hand for $\exists^{st} y \ x \in y$ and we will write $\forall^{int} x$ to mean $\forall x (x \in I \to \cdots)$ and similarly for $\exists^{int}$.

- (KR1) *ZFC for the standard universe.*
  For every axiom φ of ZFC, its standard-relativization φ^{st} is assumed.

- (KR2) *Axioms for the external universe.*
  The axioms of extensionality, pairing, union, infinity, separation and replacement are assumed for the universal class (i.e. without restricting quantifiers). The separation and replacement schemata are also considered for formulas containing the predicate `st`. Moreover, the following *weak regularity* axiom is assumed:

  $$\forall X \neq \emptyset \, \exists x \in X \ x \cap X \subseteq I$$

  and the following *standard-size choice* axiom is also assumed:

[65] We remark that the Σ_2^{st}-form reduction property holds in IST only for formulas where the standardness predicate occurs only in the form $\forall^{st} x \in y$, $\exists^{st} x \in y$, with y standard (see theorem 4.2).

$$\forall x \text{ of standard-size } \forall f \text{ function with } \mathrm{dom}(f) = x$$
$$(\forall t \in x\, f(t) \neq \emptyset) \rightarrow (\exists g \text{ function with } \forall t \in x\ g(t) \in f(t)$$

- (KR3) *All standard sets are internal.*

$$\forall^{st} x\, x \in I$$

- (KR4) *The universe I of internal sets is transitive.*

$$\forall^{int} x\, \forall y \in x\ y \in I$$

- (KR5) *Transfer Principle.*
For every $\in$-formula φ whose free variables are $x_1, \ldots, x_n$, the following is an axiom:

$$\forall^{st} x_1 \cdots \forall^{st} x_n\ (\,\varphi^{st}(x_1, \ldots, x_n) \leftrightarrow \varphi^{int}(x_1, \ldots, x_n)\,)$$

- (KR6) *Standardization Property.*

$$\forall a\, \exists^{st} b\, \forall^{st} x\ (x \in a \leftrightarrow x \in b)$$

- (KR7) *Saturation Principle.*
If a family of internal sets has standard-size and has the finite intersection property, then it has nonempty intersection.

HST turns out to be a strengthening of one of Hrbáček's systems; precisely it is equivalent to $\mathcal{NS}_1$ + weak regularity + standard-size choice. [66] It is a powerful theory, and several set-theoretic topics, such as constructibility and forcing, can be developed in it, thus suggesting HST as an interesting subject of foundational studies (see [KR2]). For instance, Kanovei and Reeken have recently proved in [KR3] that the *isomorphism property* (a useful tool in nonstandard analysis) can be consistently added to HST. *"... HST codifies the techniques used by present-day nonstandard analysis and, as a natural extension of a slight modification of IST, should be easily accessible to the practitioners familiar with the latter."* [67] A limitation of HST as a nonstandard set theory is that its universe is so large that it does support neither the power-set axiom nor the full axiom of choice, two fundamental set-theoretic tools. In the third part of

[66] In fact, notice that (with minor differences in the formalism) axioms (KR1), (KR3), (KR4), (KR5), (KR6) and (KR7) of HST correspond to axioms (H1), (H3), (H4), (H5), (H6) and (H7) of $\mathcal{NS}_0$, respectively. Besides, the axiom group $(KR2)$ is enclosed in $\mathcal{NS}_1$, with the only exception of weak regularity and standard-size choice. Although the additional axioms also holds in the model of $\mathcal{NS}_1$ given in [H1], the contruction [KR1] by Kanovei and Reeken is much simpler.

[67] [H3]

[KR1], a practical solution to this problem is proposed. By considering suitable subuniverses H_κ of HST, the power-set axiom can be saved at the price of reducing standard-sized properties to κ-standard-sized properties, with κ any given (standard) cardinal. Moreover, other subuniverses H'_κ can be considerd where choice for external sets is available, and in fact all sets are of standard size. Thus, in order to carry out a nonstandard argument where power-set is needed, one can fix a cardinal κ large enough for the purpose, and argue in one of those H_κ.

HST is justified by the following

Theorem 10.4.[68]
Every countable model of ZFC is embedded as the standard universe into a model of HST. In particular, HST is a conservative extension of ZFC. That is, for every $\in$-formula σ:

$$\mathrm{ZFC} \vdash \sigma \;\Leftrightarrow\; \mathrm{HST} \vdash \sigma^{st}$$

§11. Péraire's Relative Set Theory.

The Relative Set Theory RST was introduced by Yves Pèraire in [P1], and further developed in [P2]. RST gives a relative version of the internal approach by means of a *binary* relation of standardness. Roughly speaking, instead of only considering standard and nonstandard (ideal) sets, in the universe of RST there is a whole hierarchy of sets which are arranged according to linearly ordered levels of standardness. This makes it possible to say that a given set is more standard (or less ideal) than another one, thus formalizing an intuitive notion sometimes used in informal mathematical language.

Besides the membership relation symbol $\in$, a binary relation symbol `st` is in the language of RST. The formula $x\,\mathtt{st}\,y$ is to be read *"x is standard relatively to y"* or *"x is y-standard"*. [69] The axioms of RST are the following.

- (P1) *ZFC.*
  All axioms of Zermelo-Fraenkel set theory with choice are assumed.

- (P2) *The binary relation* `st` *is a total pre-order.*

For any formula φ, we will write $\forall^{[a]} x\, \varphi$ to mean $\forall x\,(x\,\mathtt{st}\,a \to \varphi)$ and $\exists^{[a]} x\, \varphi$ to mean $\exists x\,(x\,\mathtt{st}\,a \wedge \varphi)$.

- (P3) *Transfer Principle.*
  For every $\in$-formula φ whose free variables are $x_1, \dots, x_n, y$, and for every a, the following is an axiom:

[68] It follows from constructions in [KR1]. See also [KR2] part 2, §10.
[69] A similar concept was also considered by E. Gordon in [G1], where a notion of relative standardness was defined and studied within the framework of Nelson's IST.

$$\forall^{[a]} x_1 \cdots \forall^{[a]} x_n \ (\forall^{[a]} y\, \varphi \leftrightarrow \forall y\, \varphi)$$

- (P4) *Bounded Idealization.*
For every $\in$-formula φ, and for every $a_1, \ldots, a_n, b$ such that each a_i is not b-standard, the following is an axiom: [70]

$$\forall^{[a_1] fin} x'_1 \cdots \forall^{[a_n] fin} x'_n \, \exists^{[b]} y \, \forall x_1 \in x'_1 \cdots \forall x_n \in x'_n \, \varphi \ \leftrightarrow \\ \exists^{[b]} y \, \forall^{[a_1]} x_1 \cdots \forall^{[a_n]} x_n \, \varphi$$

- (P5) *Unbounded Idealization.*
For every $\in$-formula φ, and for every $a_1, \ldots, a_n$ the following is an axiom:

$$\forall^{[a_1] fin} x'_1 \cdots \forall^{[a_n] fin} x'_n \, \exists y \, \forall x_1 \in x'_1 \cdots \forall x_n \in x'_n \, \varphi \ \leftrightarrow \\ \exists y \, \forall^{[a_1]} x_1 \cdots \forall^{[a_n]} x_n \, \varphi$$

Let a be given. A formula φ is called *a-external* if it is built by means of the membership predicate $\in$, logic connectives, quantifiers $\forall$, $\exists$ and external quantifiers of the form $\forall^{[b]}, \exists^{[b]}$ where a is b-standard.

- (P6) *Standardization Property.*
For every a and for every a-external formula φ, the following is an axiom:

$$\forall^{[a]} x \, \exists^{[a]} y \, \forall^{[a]} z \ [\, z \in y \ \leftrightarrow \ (z \in x \wedge \varphi)\,]$$

The foundational strength of RST is given by the following result. [71]

Theorem 11.1.
RST is a conservative extension of ZFC. That is, for every $\in$-sentence σ:

$$\text{ZFC} \vdash \sigma \ \Leftrightarrow \ \text{RST} \vdash \sigma$$

§12. Gordon-Andreyev's Nonstandard Class Theory NCT.

Another axiomatic system aimed to improve on the internal approach, has been recently proposed by E.I Gordon [G2] and further developed jointly with P.V. Andreyev [AG]. Roughly speaking, their *Nonstandard Class Theory NCT* extends Gödel-Bernays class theory GB is a similar way as the Internal Set Theory (in its bounded form BST) extends ZFC. The availability of classes allows for formalization, within the framework of NCT, of various constructions employing external sets which can be carried out in IST with difficulties. With respect to IST, axioms of NCT are given simpler formalizations. In fact, transfer, idealization and standardization are formulated as single axioms rather than axiom schemata. The language of NCT is obtained by adding the symbol `st` for *standard classes* to the usual language of class theory. Its axioms are the following.

[70] We write "$\forall^{[a] fin} x$" to mean "for all finite a-standard x".
[71] Metatheorem in §5 of [P1].

- (GA1) *All axioms of Gödel-Bernays class theory GB are assumed, where choice is postulated for sets, and replacement takes the form of the collection axiom:*

$$\forall V\,\forall x\,\exists y\,\forall u \in x\ (\exists v\,\langle u,v\rangle \in V) \to (\exists v \in y\,\langle u,v\rangle \in V)$$

  To each axiom of existence of classes, its modification with quantifiers over class variables bounded by `st` predicate ($\forall^{st}, \exists^{st}$) is added.

- (GA2) *There exists the class of all standard sets.*

$$\exists S\,\forall x\ (\mathtt{st}(x) \leftrightarrow x \in S)$$

- (GA3) *Boundedness.*

$$\forall x\,\exists^{st} y\ x \in y$$

- (GA4) *Transfer Principle.*

$$\forall^{st} X\,(X \neq \emptyset \to \exists^{st} x\,(x \in X))$$

- (GA5) *Standardization Property.*

$$\forall X\,\exists^{st} Y\,\forall^{st} y\,(y \in Y \leftrightarrow y \in X)$$

  By definition, a class X is *internal* if

$$\exists^{st} Y\,\exists p\,\forall x\,(x \in X \leftrightarrow \langle x,p\rangle \in Y)$$

In this case, we denote X by $Y''p$ and we say that X is *p-standard*.

- (GA6) *Separation for Internal Classes.*

$$\forall^{int} X\,\forall x\,\exists y\,(y = x \cap X)$$

- (GA7) *Idealization Principle.*

$$\begin{aligned}\forall^{int} X\,\forall^{st} a_o\,[\forall^{stfin} c \subseteq a_o\,\exists x\,\forall a \in c\,(\langle x,a\rangle \in X) \leftrightarrow \\ \exists x\,\forall^{st} a \in a_o\,(\langle x,a\rangle \in X)\,]\end{aligned}$$

For any set x, let $\mu_p(x)$ denote its *p-monad*, defined as the intersection of all p-standard sets containing x. We say that a class X is *p-saturated* if $\forall x \in X\,(\mu_p(x) \subseteq X)$.

- (GA8) *Saturation Property.*

Any class X is p-saturated for some set p.

From a foundational point of view, NCT is justified by the following.

Theorem 12.1
(i) Every model of ZFC is embedded as the class of standard sets into a model of NCT.
(ii) Every model of BST is embedded as the class of all sets into a model of NCT, in such a way that the standardness of sets is preserved. (iii) Every model of BG (with the axiom of choice for sets) is embedded as the collection of all standard classes into a model of NCT.

§13. The *Approach of *ZFC.

Recently, the author presented the system *ZFC which axiomatizes the nonstandard embedding $*$ (see [D2],[D3]). It is an *external* approach, in the style of ZFBC, where the fundamental notions of standard, internal and external set are defined and their requisite properties proved, rather than postulated. Particular attention is paid to the practitioner of the widely popular superstructure approach. In particular, the average working mathematician in nonstandard analysis willing to adopt *ZFC as an axiomatic framework, needs to change nothing with respect to his familiar notions and notations. Roughly speaking, *ZFC expands the world of ordinary mathematics – as formalized by ZFC – by introducing a nonstandard version $^*A$ for each standard object A, and allowing the techniques of nonstandard analysis in its full generality. Besides the membership symbol $\in$, a new symbol $*$ for the nonstandard embedding map is considered in the language. The four groups of axioms of *ZFC are the following.

- (D1) *ZFC^- where the separation and replacement schemata are also assumed for formulas containing the $*$ symbol. Weak regularity is also assumed:*

$$\forall x \neq \emptyset \; \exists y \in x \; x \cap y \subseteq I$$

Thus all usual arguments of mathematics can be formalized within *ZFC with no restrictions. The proper class S of *standard sets* is defined as the class $\mathbf{WF} = \bigcup\{V_\alpha : \alpha \in \mathbf{ON}\}$ of wellfounded sets. In particular, elements of standard sets are standard and the class of standard sets is a model of ZFC. In the "standard" set theory every set is wellfounded because regularity is assumed. Thus it seems appropriate to consider $S = \mathbf{WF}$ as the universe of "standard" mathematics.

- (D2) *$*$ is a mapping with domain S.* [72]

[72] Formally, $*$ is a binary relation symbol, but in the following we will abuse notation and consider $*$ as a function symbol. Thus, for example, we shall write "$y = {}^*x$" instead of "$\forall z * (x,z) \to z = y$" and "$y \in {}^*x$" instead of "$\forall z * (x,z) \to y \in z$". When writing $^*x$, it is implicitly assumed that $x \in S$.

$$\{\forall x \forall y \forall z\ [*(x,y) \wedge *(x,z) \to (y = z \wedge x \in S)]\} \wedge (\forall x \in S\ \exists y * (x,y))$$

Thus one can put *'s on every object of ordinary mathematics.

- (D3) *The nonstandard embedding $*$ preserves all Gödel operations.* That is, for any A and B standard sets, the following equalities hold: [73]

$\mathcal{G}_1$. $^*(A \cup B) = {}^*A \cup {}^*B$
$\mathcal{G}_2$. $^*(A \cap B) = {}^*A \cap {}^*B$
$\mathcal{G}_3$. $^*(A \setminus B) = {}^*A \setminus {}^*B$
$\mathcal{G}_4$. $^*\{A, B\} = \{{}^*A, {}^*B\}$
$\mathcal{G}_5$. $^*(A \times B) = {}^*A \times {}^*B$
$\mathcal{G}_6$. $^*(\bigcup A) = \bigcup {}^*A$
$\mathcal{G}_7$. $^*\{\langle x, x\rangle : x \in A\} = \{\langle \xi, \xi\rangle : \xi \in {}^*A\}$
$\mathcal{G}_8$. $^*\{\langle x, y\rangle : x \in y \in A\} = \{\langle \xi, \eta\rangle : \xi \in \eta \in {}^*A\}$
$\mathcal{G}_9$. $^*\{x : \exists y\ \langle x, y\rangle \in A\} = \{\xi : \exists \eta\ \langle \xi, \eta\rangle \in {}^*A\}$, i.e. $^*\mathrm{dom}(A) = \mathrm{dom}({}^*A)$
$\mathcal{G}_{10}$. $^*\{y : \exists x\ \langle x, y\rangle \in A\} = \{\eta : \exists \xi\ \langle \xi, \eta\rangle \in {}^*A\}$, i.e. $^*\mathrm{range}(A) = \mathrm{range}({}^*A)$
$\mathcal{G}_{11}$. $^*\{\langle x, y\rangle : \langle y, x\rangle \in A\} = \{\langle \xi, \eta\rangle : \langle \eta, \xi\rangle \in {}^*A\}$
$\mathcal{G}_{12}$. $^*\{\langle x, y, z\rangle : \langle x, z, y\rangle \in A\} = \{\langle \xi, \eta, \zeta\rangle : \langle \xi, \zeta, \eta\rangle \in {}^*A\}$

As a straightforward consequence, the nonstandard embedding $*$ is one-to-one. Exactly as in the superstructure approach, the class I of *internal sets* is defined as the collection $I = \{a : a \in {}^*b \text{ for some } b \in S\}$. By applying axiom (D3), it is easily proved that I is a transitive class, i.e. elements of internal sets are internal.

- (D4) *Saturation Schema.*
  If a cardinal κ is defined by an $\in$-formula, then the κ-saturation property holds. More formally, for every $\in$-formula $\varphi(x)$ having exactly one free variable, the following is an axiom.

$$\begin{gathered}[\forall x, y \in S\ \varphi^S(x) \wedge \varphi^S(y) \to (x = y \wedge \text{“}x \text{ is a cardinal”})] \\ \Downarrow \\ \forall \kappa\ \varphi^S(\kappa) \to \text{“}\kappa\text{-saturation property”}\end{gathered}$$

The fundamental principles of nonstandard analysis are theorems of *ZFC.

Theorem 13.1. [74]
(i) Transfer Principle. For every $\in$-formula $\varphi(x_1, \ldots, x_n)$ and standard elements $a_1, \ldots, a_n$:

[73] Recall that this axiom can be formalized by a single formula, because the above operations are expressed by (bounded) $\in$-formulas. We remark that the given list of Gödel operations is redundant. For instance, $A \cap B = (A \cup B) \setminus [(A \setminus B) \cup (B \setminus A)]$, thus $\mathcal{G}_2$ is a composition of $\mathcal{G}_1$ and $\mathcal{G}_3$, etc. However, all the above 12 operations are mentioned in order to give a more clear picture of the basic properties of $*$.

[74] [D3] §2.

$$\varphi^S(a_1, \ldots, a_n) \;\Leftrightarrow \varphi^I({}^*a_1, \ldots, {}^*a_n)$$

(ii) Standardization Property. For every set A, the following collections are sets: ${}^oA \doteq \{b \in S : {}^*b \in A\} \in S$ and ${}^\sigma A \doteq \{a \in A : a = {}^*b$ for some $b \in S\}$.

Roughly speaking, axiom (D4) states that any level of saturation is available, provided one can first name it. The next result proved in *ZFC shows that, in typical situations occurring in practice, the saturation schema allows the full strength of saturation, without restrictions on cardinalities. Let $\varphi(x)$ be any $\in$-formula having exactly one free variable.

Theorem 13.2. [75]
Assume the following. For every cardinal κ, κ-saturation $\Rightarrow \varphi^S(x)$ for all standard x of cardinality less that κ. Then $\varphi^S(x)$ holds for all standard x.

Let us consider a typical example to show how the above theorem can be used in practice. Recall the following characterization in nonstandard analysis.

> Assume κ-saturation. If X is a standard topological Hausdorff space with $|X| < \kappa$, then X is compact if and only if for every $\xi \in {}^*X$, there is some $x \in X$ with $\xi \sim {}^*x$. [76]

Making use of the above characterization, a nice and short nonstandard proof of the Tychonoff theorem is obtained. [77] Thus, the following is proved

> For all families $\{X_i : i \in I\}$ of compact topological Hausdorff spaces with $|X_i| < \kappa$ for all $i \in I$ and $|I| < \kappa$, the topological product space $\prod_i X_i$ is compact.

As a consequence of the previous theorem, the above result actually proves the Tychonoff theorem for all standard topological spaces, without any restrictions on cardinalities. Notice that, since the full axiom of choice is assumed, every set is in bijection with some standard set. Thus one can apply the map $*$ to any (possibly external) mathematical structure, simply by considering an isomorphic standard copy of it. This allows layering nonstandard methods, because (up to isomorphisms) one can put $*$'s on everything, even on external sets. [78] Though nonwellfounded, from several points of view the structure of the universe of *ZFC is similar to the one of ZFC (see [D3] §2). For instance, the universal class is given by the cumulative hierarchy $\mathbf{V} = \bigcup_{\alpha \in ON} V_\alpha({}^*V_\alpha)$ and there is a linearly-ordered valued function (*pseudorank*) $\mathbf{R} : \mathbf{V} \to \boldsymbol{\lambda}$ such that $\mathbf{R}(x) = \sup\{\mathbf{R}(x') + 1 : x' \in x\}$, etc.

[75] [D3] §2.

[76] $\xi \sim {}^*x$ means that $\xi \in {}^*A$ for each standard neighborhood A of x.

[77] For instance, see [Li] §3.

[78] Actually, with some limitation in the use of saturation. In fact, for every infinite standard set A, the cardinality of its nonstandard extension ${}^*A$ is larger than any $\in$-definable cardinal.

The justification of *ZFC from a foundational point of view is given by the following result.

Theorem 13.3 [79]
Every model of ZFC is elementarily embedded into the standard universe of a model of *ZFC. In particular, *ZFC is a conservative extension of ZFC. That is, for every $\in$-sentence σ

$$\mathrm{ZFC} \vdash \sigma \;\Leftrightarrow\; {}^*\mathrm{ZFC} \vdash \sigma^S$$

§14. Nonstandard Regular Finite Set Theory.
The Nonstandard Regular Finite Set Theory NRFST, introduced by S. Baratella and R. Ferro in [BF2], gives a different foundational approach to nonstandard mathematics. The axiom of Infinity is replaced by its negation and a new notion of infinity is introduced by means of nonstandard methods. Roughly speaking, the underlying philosophical assumption is that there are natural numbers whose entire process of construction cannot be "recalled" (these will be the internal non-standard natural numbers).

Two unary predicates, `st` for "standard" and `int` for "internal", are added to the language of set theory. The intended interpretation of `st` is the collection of finite sets. The axioms of **NRFST** are: [80]

- (BF1) *A finite set theory for the standard universe.*
  Relativizations to the standard universe of the axioms of extensionality, pair, union, replacement schema, empty set, power set, regularity, and the negation of the axiom of infinity, are assumed. [81]
- (BF2) *The universe of internal sets is transitive.*

$$\forall x\,\forall^{int} y\ (x \in y \to \mathtt{int}(x))$$

- (BF3) *Transfer Principle.*
  For every $\in$-formula φ whose free variables are $x_1, \ldots, x_n$, the following is an axiom:

$$\forall^{st} x_1 \cdots \forall^{st} x_n\ (\,\varphi^{st}(x_1, \ldots, x_n) \;\leftrightarrow\; \varphi^{int}(x_1, \ldots, x_n)\,)$$

- (BF4) *Standardization Property.*

$$\forall x \forall^{st} y\, \exists^{st} z\, (z = x \cap y)$$

[79] [D3] §3. Models of *ZFC are constructed by using *pseudo-superstructures* [D1].

[80] We will use the same notation for standard-bounded and internal-bounded quantifiers $\forall^{st}, \exists^{st}, \forall^{int}, \exists^{int}$ as we already did in previous sections.

[81] Precisely, all relativizations φ^{st} of axioms φ of the Regular Finite Set Theory RFST as presented in [BF1], are considered. In particular, the regularity axiom is assumed in the form that every set has a rank. We refer the reader to [BF1] for precise formulations.

- (BF5) *Idealization Principle.*

$$\forall^{st} y_1, \cdots, \forall^{st} y_n \, [\forall^{st} z \, \exists^{st} y \, \forall x \in z \, \varphi^{st}(x, y, y_1, \ldots, y_n)] \rightarrow$$
$$\exists^{int} y \, \forall^{st} x \, \varphi^{int}(x, y, y_1, \ldots, y_n)$$

The above axiom implies that there is a proper inclusion among the standard, internal and external universes.

- (BF6) *A fragment of ZFC for the external universe.*
  The axioms of extensionality, pair, union, separation schema, and the axiom of choice are assumed.

NRFST is strong enough to prove the axiom of infinity for external sets, and some other basic properties. [82] As a consequence, this nonstandard finite theory should be able to support ordinary mathematics. An interesting discussion about the underlying philosophy of NRFST can be found in the last section of [BF2].

By suitably interpreting the predicates `st` and `int` in a model constructed by Kawai [Kw], it is proved the following consistency result.

Theorem 14.1.
NRFST is a conservative extension of ZFC. That is, for every $\in$-sentence σ:

$$\text{ZFC} \vdash \sigma \;\Leftrightarrow\; \text{NRFST} \vdash \sigma$$

Historically, we remark that a similar task of developing a theory without formally infinite sets, was carried out by Vopenka's Alternative Set Theory AST. A large amount of mathematics has been developed on the basis of AST, mainly by a group of chech and slovak mathematicians. Even if it can be considered as a formalization of nonstandard phenomena, strictly speaking AST is not a nonstandard set theory in that it is not concerned in transfering results to conventional mathematics. A treatment of AST is outside the scope of this survey, thus we refer the reader to [Vo] for the underlying philosophy of AST, and to [VH] (and subsequent papers) for mathematics developed within AST.

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[82] See [BF2] Theorem 1.7. A stronger theory NRFST* is also presented in that paper, which includes a stronger idealization principle and the replacement schema for external sets.

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