Arithmetic as the theory of finite sets. (Abstract)

J.P. Mayberry
University of Bristol

In classical Greek arithmetic a number (*arithmos*) was defined to be a finite pluality of units – what we now call a finite set. We get a modernised version of classical arithmetic by reformulating it as the theory of finite sets. The resulting *Euclidean arithmetic* is essentially conventional (i.e., Zermelo-Fraenkel) set theory (with *urelements*), but formulated, in the first instance, as a free variable theory, using function symbols for the basic operations of pair set, power set, union, and transitive closure, and variable binding operators for subset selection ($\{x \in S : \Phi(x)\}$) and replacement ($\{\sigma(x) : x \in S\}$). We must also replace the Axiom of Infinity by an axiom asserting that every set is finite. (We can express the latter proposition since the presence of the subset selection operator makes it possible to define the bounded quantifiers ($\forall x \in S$) and ($\exists x \in S$).) This free variable theory can be conservatively extended to first order and (predicative) second order versions, and we can prove that all of these theories are proof-theoretically equivalent to the weak arithmetic $I\Delta_0 + \mathrm{Exp}$.

In Euclidean arithmetic we can give mathematically precise and natural definitions of *natural number system*, of the *closure* of such a system under particular arithmetical functions, and of the comparison of such systems as to *length*. Using nonstandard methods, we can show that Euclidean arithmetic contains non-isomorphic natural number systems and natural number systems that fail to be closed under various simple arithmetical functions (including even addition). Indeed, it contains a hierarchy of number systems corresponding to the Ω_n hierarchy in natural number arithmetic, in which each successive system of this hierarchy is (strictly) longer than its predecessor.