Analysis based on large finite linear orderings (Abstract)

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The general project underlying this work is to develop geometry from the idea that space is not infinitely divisible but built up from a "large" but still finite number of *minimal parts*. The minimal parts of a space can be regarded as the nodes, and their nearest neighbor relations as determining the edges, of a *minimal parts graph* which completely determines that space. The aim is to show that conventional continuous geometry and analysis can be recovered (or reconstituted) in minimal parts geometry using nonstandard methods. Here I shall concentrate on the one dimensional case, since the central analytical difficulties are already present there, even though the geometry is trivial.

A *large* linear ordering $L = [0, 1, \cdots, \Omega]$ always has the natural numbers N as an initial segment with proper end extensions, ${}^*N$, of N such that $N < {}^*N < L$. From L we obtain a large finite system of rational numbers

$$Q_L = \{\pm p/q : 0 \leq p \leq \Omega,\, 1 \leq q \leq \Omega\}$$

which contains *proper* (standard) rationals Q (based on N) and *extended* (non-standard) rationals ${}^*Q \supseteq Q$ (based on ${}^*N$). The *L-reals*, R_L are the elements of ${}^*Q$ that lie within the boundaries of Q (so that reals in ${}^*Q - Q$ correspond to irrationals), and the L-infinitesimals, I_L, are the L-reals, ϵ, such that $|\epsilon| < 1/n$, for all $n \in N^+$.

The L-reals form a ring and the L-infinitesimals are a maximal ideal in that ring. The quotient ring R_L/I_L is thus a field, in fact an ordered subfield of the standard reals which is real closed, contains all the "known" transcendentals (e, π, the Liouville transcendentals such as $\Sigma_{n=0}^{\infty} 1/10^{n!}$, etc.), and is closed under the elementary analytic functions (the exponential, the natural logarithm, the trigonometric functions, etc.)

A one-dimensional minimal parts geometry can be identified with a large finite linear ordering $\mathcal{L} = [p_{-\Omega}, \ldots, p_{\Omega}]$, where the p_is are the minimal parts of the space and pairs $\{p_i, p_{i+1}\}$ are accounted nearest neighbors. To marry the geometry of $[p_{-\Omega}, \ldots, p_0, \ldots p_{\Omega}]$ to the analysis determined by $L = [0, 1, \cdots, \Omega]$, choose ${}^*N < a < 2aa! < \omega$, and define a line segment of *real unit length* to consist of any $a! + 1$ adjacent minimal parts (e.g., $[p_0, \ldots, p_{a!}]$). We assign coordinate $x \in R_L^+$ to the minimal part p_n ($n \geq 0$) if $|n - xa!| < a!/k$ for all $k \in N$. This means that the *minimal parts* of the geometry $\mathcal{L}$ must be distinguished from its *points*, each of which is composed of many minimal parts: if P is a point and p_i is any of its constituent minimal parts, then

$$P = \{p_j : (\forall k \in N^+)[|j - i| < a!/k]\}$$

A natural theory of differentiation and integration can be developed in the theory.