A Topological Characterization of Enlargements - Marco Forti[1]

Among various "elementary" approaches to nonstandard analysis given in [1], the "topological approach" endows every nonstandard model $^*X$ with the least T_1 topology which makes all functions $^*f$ continuous (the *Star topology*).[2] In this topology, the closure in $^*X$ of any $A \subseteq X$ is $^*A$, and these sets are precisely the clopen subsets of $^*X$; they generate a 0-dimensional topology (the *S-topology* of $^*X$) which agrees with the Star topology when the latter is Hausdorff, and is coarser otherwise.

Conversely, call *topological extension* of X any T_1 space $^*X$ such that X is a dense subspace of $^*X$, and every $f : X \to X$ has a distinguished continuous extension $^*f : {^*X} \to {^*X}$ preserving compositions (i.e. $^*(f \circ g) = {^*f} \circ {^*g}$). The main result of [2] isolates two simple necessary and sufficient conditions for a topological extension to be a true *nonstandard model* of X, namely

Theorem ([2], Thms. 3.2 and 5.5) *Let $^*X$ be a topological extension of X. Then $^*X$ is isomorphic to a limit ultrapower $X^I/\mathcal{D}|\mathcal{E}$ (i.e. it is a nonstandard extension) if and only if $*$ preserves equalizers and $^*X$ is accessible, i.e.*
1. $\{\xi \in {^*X} \mid {^*f}(\xi) = {^*g}(\xi)\} = {^*\{x \in X \mid f(x) = g(x)\}} = \overline{\{x \in X \mid f(x) = g(x)\}}$;
2. *for all $\xi, \eta \in {^*X}$ there are $f, g : X \to X$ and $\zeta \in {^*X}$ s.t. $^*f(\zeta) = \xi$, $^*g(\zeta) = \eta$.*

Moreover $^*X$ is isomorphic to an ultrapower $X^X/\mathcal{U}$ if and only if there exists $\zeta \in {^*X}$ such that any $\xi \in {^*X}$ is equal to $^*f(\zeta)$ for suitable $f : X \to X$.

In this topological context, the enlargement and saturation properties are related to weak compactness properties of the S- and Star topologies: [3]

Theorem ([2], Thm. 6.5) *Let $^*X$ be a topological extension of X.*
(i) $^*X$ is a $(2^{|X|})^+$-enlargement if and only if the S-topology is quasi-compact.[4]
(ii) If $^*X$ is $(2^{|X|})^+$-saturated, then the Star topology is Bolzano.
(iii) $^*X$ cannot be simultaneously nonstandard, $(2^{\aleph_0})^+$-enlarging and Hausdorff. In particular there exist no countably compact nonstandard extensions.

Sufficiently saturated hyper-extensions are also Weierstraß, by Theorem 6.4 of [2]. Therefore, in our context, Bolzano-Weierstraß together do not yield even *countable compactness.* Point (*iii*) of the above theorem states the perhaps surprising fact that the most important classes of topological extensions, namely *Nonstandard, Hausdorff,* and *Bolzano* extensions, which have pairwise nonempty intersection, cannot have any common element.

References

[1] V. Benci, M. Di Nasso, M. Forti - The Eightfold Path to Nonstandard Analysis (2004, submitted).

[2] M. Di Nasso, M. Forti - Topological and nonstandard extensions, *Monatsh. f. Math.*, to appear.

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[2] A *subbasis of closed sets* is given by the family $\{{^*f^{-1}}(\eta) \mid f \in X^X, \eta \in {^*X}\}$.

[3] In principle, we should isolate a *topological counterpart* of the notion of internal set (while the equation *standard* = *clopen* should be clear from the above facts). However all what we need in the following is only the trivial assumption that the basic closed sets $^*f^{-1}(\eta)$ are *internal.*

[4] A topological space S is *quasi-compact* [*Bolzano*] if every [countable] open cover has a finite subcover (so [countably] *compact* means quasi-compact [Bolzano] and Hausdorff). S is *Weierstraß* if every real valued continuous function on S is bounded.