Since P-points need not exist in ω^*, one might hope that the following statements can be simultaneously true:

(1) $\forall\kappa < \mathfrak{c}$, $\neg G(\kappa, \omega)$,

(2) $\neg G(\mathfrak{c}, \mathfrak{c})$,

(3) no P-points,

for then, there would be a model in which $\beta(\omega^*\setminus\{x\}) = \omega^*$ for *all* $x \in \omega^*$. Unfortunately, as was pointed out to me by Ken Kunen, (1) implies $\neg$(3). Define an order $<^*$ on ω^ω by

$$f <^* g \quad \text{iff} \quad |\{n < \omega : f(n) \geq g(n)\}| < \omega .$$

A subset $A \subseteq \omega^\omega$ is called *dominating* if for each $f \in \omega^\omega$ there is a $g \in A$ with $f <^* g$.

2.5.5. LEMMA. *Suppose that no subset of ω^ω of cardinality less than $\mathfrak{c}$ dominates. Then ω^* contains a P-point.*

PROOF. Let $\{f_\alpha : \alpha < \mathfrak{c}\}$ enumerate ω^ω. By transfinite induction on $\alpha < \mathfrak{c}$ we will construct a filter $\mathscr{F}_\alpha \subseteq \mathscr{P}(\omega)$ such that

(1) finite intersections of elements of $\mathscr{F}_\alpha$ have infinite intersections,

(2) there is an element $F \in \mathscr{F}_\alpha$ such that either $|f_\alpha^{-1}(n) \cap F| < \omega$ for all $n < \omega$, or $F \subseteq f_\alpha^{-1}(\{0, 1, \ldots, n\})$ for certain $n < \omega$,

(3) if $\kappa < \alpha$ then $\mathscr{F}_\kappa \subseteq \mathscr{F}_\alpha$ and $|\mathscr{F}_\alpha| \leq |\alpha| \cdot \omega$.

Suppose we have constructed everything for all $\kappa < \alpha$ and define $\mathscr{F} = \bigcup_{\kappa<\alpha} \mathscr{F}_\kappa$. Observe that $|\mathscr{F}| \leq |\alpha| \cdot \omega < \mathfrak{c}$. For each $F \in \mathscr{F}$ define a function $g(F) : \omega \to \omega$ by

$$g(F)(n) = \begin{cases} \min(F \cap f_\alpha^{-1}(n)) & \text{if } F \cap f_\alpha^{-1}(n) \neq \emptyset , \\ 0 & \text{otherwise} . \end{cases}$$

Since $|\mathscr{F}| < \mathfrak{c}$, we can find a function $f \in \omega^\omega$ such that $f \not<^* g(F)$ for all $F \in \mathscr{F}$. Define

$$X = \bigcup_{n<\omega} f_\alpha^{-1}(n) \cap \{j < \omega : j \leq f(n)\}$$

and define $\mathscr{F}_\alpha$ to be the filter generated by $\mathscr{F} \cup \{X\}$ if $|F \cap X| = \omega$ for all $F \in \mathscr{F}$. Otherwise, define $\mathscr{F}_\alpha = \mathscr{F}$.

It is clear that any ultrafilter extending $\bigcup_{\alpha<\mathfrak{c}} \mathscr{F}_\alpha$ is a P-point. □

In view of the above Lemma it therefore suffices to prove the following.

2.5.6. LEMMA. $(\forall\kappa < \mathfrak{c}, \neg G(\kappa, \omega)) \to$ *(no subset of ω^ω of cardinality less than $\mathfrak{c}$ dominates).*

PROOF. Let $\kappa = \min\{\lambda : \exists F \subseteq \omega^\omega$ such that F dominates and $|F| = \lambda\}$. Choose $F \subseteq \omega^\omega$ of cardinality κ such that F dominates. We may assume that $F = \{f_\alpha : \alpha < \kappa\}$, where $\alpha < \beta$ implies that $f_\alpha <^* f_\beta$. For each $\alpha < \kappa$, let

$$S_\alpha = \{\langle m, n\rangle : n < f(m)\} .$$

If $T_n = (\omega\setminus\{0, 1, \ldots, n\}) \times \omega$ for all $n < \omega$, then the families $\{S_\alpha : \alpha < \kappa\}$ and $\{T_n : n < \omega\}$ form a (κ, ω) gap (defined on $\omega \times \omega$). □

2.6. *Autohomeomorphisms of ω^*. II*

As remarked in Section 2.2, SHELAH [1978] has shown it to be consistent that all autohomeomorphisms of ω^* are induced by a permutation of ω. Consequently, in this model ω^* has precisely $\mathfrak{c}$ autohomeomorphisms and we conclude that Lemma 1.6.1 can be false. I do not know whether Theorem 1.6.4 is a result of ZFC. This is caused by the fact that I do not know whether in ZFC there is a nowhere dense P-set in ω^* which is homeomorphic to ω^*. Theorem 1.6.5 is false under MA + $\neg$CH, since this axiom easily implies that there are $P_\mathfrak{c}$-points in ω^* and P-points which are not $P_\mathfrak{c}$-points.

2.7. *P-points and nonhomogeneity of ω^*, II*

It was an open problem for many years whether P-points in ω^* could be constructed without using additional set theoretic hypotheses. Finally, Shelah, see MILLS [1980] or WIMMERS [1980], proved it to be consistent that P-points in ω^* do not exist. Therefore, Corollary 1.7.2 cannot be established in ZFC. In Sections 3 and 4 we will give several proofs that ω^* is not homogeneous.

Theorem 1.7.3 is true in ZFC, see 4.3.3 and 4.4.1.

2.8. *Retracts of $\beta\omega$ and ω^*, II*

Theorem 1.8.1 has to be consistently false of course. This can be seen in various ways, one of which we give below. As usual,

$$U(\omega_1) = \{p \in \beta\omega_1 : \forall P \in p, |P| = \omega_1\} .$$

2.8.1. THEOREM (MA + $\neg$CH). *There is a nowhere dense closed P-set X in ω^* which is not a retract of ω^*.*

PROOF. By KUNEN [1976, 1.2], the space $\beta\omega_1$ embeds in ω^* as a $P_\mathfrak{c}$-set under MA + $\neg$CH. Let $h : \beta\omega_1 \to \omega^*$ be an embedding such that $h(\beta\omega_1)$ is a P-set and put $X = h(U(\omega_1))$. Since $U(\omega_1)$ is a P-set in $\beta\omega_1$, X is a P-set in ω^*. By COMFORT & NEGREPONTIS [1974, 12.2], it follows that there is a family of $\mathscr{A}$ of ω_2 uncountable subsets of ω_1 such that for distinct $A, B \in \mathscr{A}$ the set $A \cap B$ is countable. This immediately implies that the cellularity of $U(\omega_1)$ is at least ω_2. Since ω_1 is dense in $\beta\omega_1$, there cannot be a retraction from $\beta\omega_1$ onto $U(\omega_1)$, which immediately implies that there cannot be a retraction from ω^* onto X. □

2.9. Nowhere dense P-sets in ω^*, II

Balcar, Frankiewicz & Mills [1980] prove it to be consistent that ω^* can be covered by nowhere dense closed P-sets. Consequently, Corollary 1.9.4 is not a result of ZFC.

Dow & van Mill [1981] show that no compact space can be covered by nowhere dense ccc P-sets i.e. P-sets satisfying the countable chain condition. It is not known whether there is a compact space that can be covered by nowhere dense P-sets of cellularity at most ω_1, however it is known that ω^* is not a consistent example.

2.9.1. Proposition. *There is a point $x \in \omega^*$ such that $x \notin K$ for any nowhere dense P-set $K \subseteq \omega^*$ of cellularity at most ω_1.*

Proof. Under CH, this is a consequence of Theorem 1.9.3. So assume $\neg$CH. It is left to the reader to prove that the R-points constructed in the proof of Lemma 3.3.4 have the required property. □

It is unknown whether in ZFC there is a nowhere dense P-set in ω^* of cellularity at most ω_1.

The cover of ω^* constructed by Balcar, Frankiewicz and Mills consists of P-sets of different 'cofinalities'. Interestingly, Nyikos [1982] has recently shown that it is consistent that ω^* can be covered by nowhere dense closed P-sets which are all an intersection of a chain of ω_1 clopen subsets of ω^*.

Notes for Section 2

Theorem 2.1.1 is due to van Douwen & van Mill [1978]. Theorem 2.2.1 is due to van Douwen & van Mill [1981d]. That Theorem 2.3.1 holds was established in van Douwen & Przymusiński [1980]. It was known from the work of Baumgartner that MA does not imply that each compact space of weight $\mathfrak{c}$ is a continuous image of ω^*. Theorem 2.3.2, which is due to the author, gives another proof of this result. As noticed in Section 2.3, the interesting Theorem 2.3.6 is due to Kunen [1980].

Section 2.5, with the exception of 2.5.5 and 2.5.6, was taken from van Douwen & van Mill [1981c]. Lemma 2.5.5 is due to Ketonen [1976]. That $\forall\kappa<\mathfrak{c}$, $\neg G(\kappa,\omega)$ implies that P-points in ω^* exist, was pointed out to me by Ken Kunen. Lemma 2.5.6 is due to Kunen and Hechler [1975]. Theorem 2.8.1 is well-known. Proposition 2.9.1 is due to Dow & van Mill [1980].

3. Partial orderings on $\beta\omega$

In this section we will concentrate on various partial orderings on $\beta\omega$, which can be used to prove that certain spaces are not homogeneous.

3.1. The Rudin–Keisler order on $\beta\omega$

Let $f:\omega\to\omega$ be a function and let $\beta f:\beta\omega\to\beta\omega$ be its Stone extension. It is easily verified that

$$(*)\qquad \beta f(p)=q \quad\text{iff}\quad \forall P\in p: f(P)\in q \quad\text{iff}\quad \forall Q\in q: f^{-1}(Q)\in p.$$

Define an equivalence relation $\sim$ on $\beta\omega$ by

$$p\sim q \quad\text{iff}\quad \exists \text{ permutation } \pi:\omega\to\omega \text{ with } \beta\pi(p)=q.$$

It is clear that $\sim$ is indeed an equivalence relation.

Let $p, q\in\beta\omega$ and write

$$p\leqslant q \quad\text{iff}\quad \exists f\in\omega^\omega \text{ with } \beta f(q)=p.$$

The following theorem, which we will not prove in detail, summarizes relevant information about $\sim$ and $\leqslant$.

3.1.1. Theorem. *Let $p, q, r\in\beta\omega$. Then*
(a) *$p\leqslant p$,*
(b) *if $p\leqslant q$ and $q\leqslant r$, then $p\leqslant r$,*
(c) *if $p\leqslant q$ and $q\leqslant p$, then $p\sim q$.*

Observe that only 3.1.1(c) requires proof. For information concerning the proof of Theorem 3.1.1(c) and many related things, see Comfort & Negrepontis [1974, section 9].

Observe that Theorem 3.1.1 shows that the quotient relation defined by $\leqslant$ on $\beta\omega/\!\sim$ is a partial ordering.

The relation $\leqslant$ on $\beta\omega$ is called the *Rudin–Keisler* order on $\beta\omega$. If $p\in\beta\omega$, then the set $\{q\in\beta\omega: q\leqslant p\}$ is equal to

$$\{\beta f(p): f\in\omega^\omega\}$$

and therefore has cardinality at most $\mathfrak{c}$, since $|\omega^\omega|=\mathfrak{c}$. Is there for all $p\in\beta\omega$ a point $q\in\beta\omega$ such that $q\not\leqslant p$? It seems strange, but at this moment we do not have the tools yet to answer this question, since we have almost not deduced any results about $\beta\omega$ in ZFC alone. In fact, we did not even find the cardinality of $\beta\omega$. Let us quickly compute $|\beta\omega|$, in order to answer the above question.

3.1.2. LEMMA. (a) *There is a family* $\{A_\alpha : \alpha < \mathfrak{c}\}$ *of infinite subsets of* ω *such that if* $\alpha < \beta$, *then* $A_\alpha \cap A_\beta$ *is finite.*

(b) *There is a family* $\{\langle A^0_\alpha, A^1_\alpha\rangle : \alpha < \mathfrak{c}\}$ *of pairs of disjoint subsets of* ω *such that for all finite* $F \subseteq \mathfrak{c}$ *and for each* $f : F \to 2$ *we have that* $\bigcap_{\alpha\in F} A^{f(\alpha)}_\alpha$ *is infinite.*

(c) $|\beta\omega| = 2^{\mathfrak{c}}$; *in fact, if* $A \subseteq \beta\omega$ *is countably infinite, then* $|\bar{A}| = 2^{\mathfrak{c}}$.

PROOF. For each irrational number $r \in \mathbb{R}$ choose a sequence $S(r)$ of rational numbers converging to r. The family $\{S(r) : r \text{ irrational}\}$ is obviously as required in (a), except that it does not consist of subsets of ω, but of the countable set $\mathbb{Q}$. But this causes no problems of course.

Let $\{A_\alpha : \alpha < \mathfrak{c}\}$ be a family of subsets of ω as in (a). For each $\alpha < \mathfrak{c}$, define

$$B^0_\alpha = \{F \in [\omega]^{<\omega} : F \cap A_\alpha \neq \emptyset\}, \quad \text{and} \quad B^1_\alpha = [\omega]^{<\omega}\setminus B^0_\alpha .$$

An easy check shows that the family $\{\langle B^0_\alpha, B^1_\alpha\rangle : \alpha < \mathfrak{c}\}$ has the properties of the required family in (b), except that it is not defined on ω, but on the countable set $[\omega]^{<\omega}$. But this again causes no problems of course.

Let $\{\langle A^0_\alpha, A^1_\alpha\rangle : \alpha < \mathfrak{c}\}$ be a family as in (b).

If $f \in 2^{\mathfrak{c}}$, take a point $p_f \in \beta\omega$ such that $\{A^{f(\alpha)}_\alpha : \alpha < \mathfrak{c}\} \subseteq p_f$. It is clear that $|\beta\omega| \geq |\{p_f : f \in 2^{\mathfrak{c}}\}| = 2^{\mathfrak{c}}$. Since $|\mathcal{P}(\omega)| = \mathfrak{c}$, $|\beta\omega| \leq 2^{\mathfrak{c}}$ which proves that $|\beta\omega| = 2^{\mathfrak{c}}$. Statement (c) now follows from Theorem 1.5.2 and from the fact that each countably infinite space contains a countably infinite relatively discrete subspace. □

The proof of the above lemma tells us two important facts, namely that combinatorial arguments are important if one wishes to study $\beta\omega$ without extra hypotheses, and that for obtaining certain families of subsets of ω, one should not try to define them directly on ω but rather on a suitable countable set which is, in the given situation, easier to handle than ω.

Let us now return to our question: given $p \in \beta\omega$, is there a point $q \in \beta\omega$ such that $q \not\leq p$? The answer is now easy of course, since $|\{q \in \beta\omega : q \leq p\}| \leq \mathfrak{c}$ and $|\beta\omega| = 2^{\mathfrak{c}}$, by Lemma 3.1.2(c). Let us specify the question a little bit: given $p \in \omega^*$, is there a point $q \in \beta\omega$ such that $p \not\leq q$ and $q \not\leq p$? It may come as a shock, but the answer to this question is not known. Under CH it is easy to show that the answer is yes, but in ZFC the answer is unknown. It is known, however, that at least there are points $p, q \in \beta\omega$ with $p \not\leq q$ and $q \not\leq p$ and in the remaining part of this section we will construct such points.

3.1.3. DEFINITION. Let $\mathcal{F} \subseteq \mathcal{P}(\omega)$ be a filter no element of which is finite. An indexed family $\{\langle A^0_i, A^1_i\rangle : i \in I\}$ of pairs of disjoint subsets of ω is called an *independent family with respect to* $\mathcal{F}$ provided that for all $\sigma \in [I]^{<\omega}$, $f \in 2^\sigma$ and $F \in \mathcal{F}$ the set $F \cap \bigcap_{i\in\sigma} A^{f(i)}_i$ is infinite.

Let $\mathcal{CF}$ denote the filter of cofinite subset of ω.

3.1.4. LEMMA. *There is an independent family* $\{\langle A^0_\alpha, A^1_\alpha\rangle : \alpha < \mathfrak{c}\} \subseteq \mathcal{P}(\omega)$ *w.r.t.* $\mathcal{CF}$.

PROOF. Lemma 3.1.2 (b). □

If $\mathcal{A} \subseteq \mathcal{P}(\omega)$ we denote by $\langle\mathcal{A}\rangle$ the (possibly improper) filter on ω generated by $\mathcal{A}$.

3.1.5. LEMMA. *Let* $\mathcal{F}, \mathcal{G} \subseteq \mathcal{P}(\omega)$ *be filters and assume that* $\{\langle A^0_i, A^1_i\rangle : i \in I\}$ *is independent w.r.t.* $\mathcal{F}$ *as well as* $\mathcal{G}$. *For each* $f \in \omega^\omega$ *there is a finite* $J \subseteq I$ *and a subset* $A \subseteq \omega$ *such that* $\{\langle A^0_i, A^1_i\rangle : i \in I\setminus J\}$ *is independent w.r.t.* $\langle \mathcal{F} \cup \{A\}\rangle$ *as well as* $\langle \mathcal{G} \cup \{\omega\setminus f^{-1}(A)\}\rangle$.

PROOF. Fix $a \in I$ arbitrarily.

Case 1: $\{\langle A^0_i, A^1_i\rangle : i \in I\setminus\{a\}\}$ *is independent w.r.t.* $\langle \mathcal{G} \cup \{\omega\setminus f^{-1}(A^0_a)\}\rangle$.

We then put $A = A^0_a$ and $J = \{a\}$. An easy check shows that A and J are as required.

Case 2: *Not Case* 1.

Then there are a finite $K \subseteq I\setminus\{a\}$ and a function $\bar{f} \in 2^K$ and an element $G \in \mathcal{G}$ such that

$$(*) \qquad \Big|\bigcap_{i\in K} A^{\bar{f}(i)}_i \cap G \cap (\omega\setminus f^{-1}(A^0_a))\Big| < \omega .$$

Now put $A = \omega\setminus A^0_a$ and $J = K \cup \{a\}$. It is clear that $\{\langle A^0_i, A^1_i\rangle : i \in I\setminus J\}$ is independent w.r.t. $\langle \mathcal{F} \cup \{A\}\rangle$, so it remains to be shown that $\{\langle A^0_i, A^1_i\rangle : i \in I\setminus J\}$ is independent w.r.t. $\langle \mathcal{G} \cup \{\omega\setminus f^{-1}(A)\}\rangle$. To this end, let $L \subseteq I\setminus J$ be finite and take $g \in 2^L$. Choose $G_0 \in \mathcal{G}$ arbitrarily. Then

$$\bigcap_{i\in L} A^{g(i)}_i \cap G_0 \cap (\omega\setminus f^{-1}(A)) = \bigcap_{i\in L} A^{g(i)}_i \cap G_0 \cap f^{-1}(A^0_\alpha)$$

$$\supseteq \bigcap_{i\in L} A^{g(i)}_i \cap \bigcap_{i\in K} A^{\bar{f}(i)}_i \cap (G_0 \cap G) \cap f^{-1}(A^0_a),$$

which is infinite by (*) and by our assumption that $\{\langle A^0_i, A^1_i\rangle : i \in I\}$ is independent w.r.t. $\mathcal{G}$. □

We now come to the main result of this section.

3.1.6. THEOREM. *There are points* $p, q \in \beta\omega$ *such that* $p \not\leq q$ *and* $q \not\leq p$.

PROOF. By Lemma 3.1.4 there is an independent family $\{\langle A^0_\alpha, A^1_\alpha\rangle : \alpha < \mathfrak{c}\} \subseteq \mathcal{P}(\omega)$ w.r.t. $\mathcal{CF}$. Let $\{f_\alpha : 1 \leq \alpha < \mathfrak{c}\}$ enumerate ω^ω. By transfinite induction on α we will construct $\mathcal{F}_\alpha$, $\mathcal{G}_\alpha$ and $K_\alpha \subseteq 2^\omega$ so that

(1) $\mathscr{F}_\alpha$ and $\mathscr{G}_\alpha$ are filters on ω and $\{\langle A_\xi^0, A_\xi^1\rangle : \xi \in K_\alpha\}$ is independent w.r.t. $\mathscr{F}_\alpha$ as well as $\mathscr{G}_\alpha$,

(2) $K_0 = 2^\omega$ and $\mathscr{F}_0 = \mathscr{G}_0 = \mathscr{CF}$,

(3) $\kappa < \alpha$ implies $\mathscr{F}_\kappa \subseteq \mathscr{F}_\alpha$, $\mathscr{G}_\kappa \subseteq \mathscr{G}_\alpha$ and $K_\alpha \subseteq K_\kappa$,

(4) for each α, $|2^\alpha \backslash K_\alpha| \leq |\alpha| \cdot \omega$,

(5) for each $\alpha \geq 1$ there are sets A, $B \subseteq \omega$ with $\{A, \omega \backslash f_\alpha^{-1}(B)\} \subseteq \mathscr{F}_\alpha$ and $\{B, \omega\backslash f_\alpha^{-1}(A)\} \subseteq \mathscr{G}_\alpha$.

Suppose that we have completed the construction for all $\kappa < \alpha$, $\alpha < \mathfrak{c}$. Put $K = \bigcap_{\kappa<\alpha} K_\kappa$, $\mathscr{G} = \bigcup_{\kappa<\alpha} \mathscr{G}_\kappa$ and $\mathscr{F} = \bigcup_{\kappa<\alpha}\mathscr{F}_\kappa$. Observe that, by (4), $|K| = \mathfrak{c}$ and that, by (1), $\{\langle A_\xi^0, A_\xi^1\rangle : \xi \in K\}$ is independent w.r.t. $\mathscr{F}$ as well as $\mathscr{G}$. Using Lemma 3.1.5 twice, it is easy to construct $\mathscr{F}_\alpha$, $\mathscr{G}_\alpha$ and K_α satisfying (1) through (5).

Now let $p \in \beta\omega$ extend $\bigcup_{\alpha<\mathfrak{c}} \mathscr{F}_\alpha$ and let $q \in \beta\omega$ extend $\bigcup_{\alpha<\mathfrak{c}} \mathscr{G}_\alpha$. By (5) it easily follows that $p \not\leq q$ and $q \not\leq p$. □

3.1.7. REMARK. Points p and q as in the above Theorem 3.1.6 are called $\leq$-*incomparable*. Observe that if p, $q \in \beta\omega$ are $\leq$-incomparable, then both p and q belong to ω^*.

3.1.8. REMARK. The technique of proof used in Theorem 3.1.6 is quite important. At each stage of the construction we give up a negligible number of the initial independent family in order to obtain in return a required property of the filter(s) we wish to construct. Under CH (or MA) such a delicate process is not necessary, since then at each stage of the construction we are at a countable level and one can then construct by hand enough sets to continue the induction. Under —CH, in a transfinite induction of length $\mathfrak{c}$ one has to pass level ω_1, and if one then for example in the previous steps constructed a family of sets which constitute a Hausdorff gap (i.e. a family of sets which witnesses the fact that $G(\omega_1, \omega_1)$ holds), then there is usually no way to continue the induction. In the proof of Theorem 3.1.6 this cannot happen, since before starting the induction enough sets were identified which ensure that one can always pick new sets to continue the induction.

3.1.9. REMARK. If $\mathfrak{c}^+ = 2^\mathfrak{c}$, then ω^* has a $\leq$-cofinal well-ordered subset (of cardinality $2^\mathfrak{c}$); and the condition $\mathfrak{c}^+ < 2^\mathfrak{c}$ is equivalent to the statement that any subset of ω^* of cardinality $2^\mathfrak{c}$ has a pairwise $\leq$-incomparable subset of cardinality $2^\mathfrak{c}$. For details, see COMFORT & NEGREPONTIS [1974, Corollaries 10.11 and 10.15].

3.2. *The Rudin–Frolík order on ω^**

The *Rudin–Frolík (pre-)order $\sqsubseteq$ on ω^** is defined as follows:

$$p \sqsubseteq q \quad \text{iff} \quad \text{there is an embedding } h : \beta\omega \to \omega^* \text{ with } h(p) = q .$$

This order and the Rudin–Keisler order $\leq$ of Section 3.1 are related by the following lemma.

3.2.1. LEMMA. *If p, $q \in \omega^*$ and if $p \sqsubseteq q$, then $p \leq q$.*

PROOF. Let $h : \beta\omega \to \omega^*$ be an embedding with $h(p) = q$. Since $h(\omega)$ is a relatively discrete, there is a sequence C_n of subsets of ω such that for all $n < \omega$,

(1) $h(n) \in \bar{C}_n$, and

(2) if $n < m$, then $C_n \cap C_m = \emptyset$.

By adding $\omega \backslash \bigcup_{n<\omega} C_n$ to C_0 we may assume that the sequence $\{C_n : n < \omega\}$ is a partition of ω. Define $g : \omega \to \omega$ by

$$g(k) = n \quad \text{if } k \in C_n .$$

and let $\beta g : \beta\omega \to \beta\omega$ be its Stone extension. We claim that $\beta g(q) = p$. Take $Q \in q$ arbitrarily. The set $\{n : h(n) \in \bar{Q}\}$ must belong to p, since $h(p) = q$, but

$$\{n : h(n) \in \bar{Q}\} \subseteq g(Q) ,$$

and we therefore may conclude that $g(Q) \in p$. Consequently, $\beta g(q) = p$, which is as required. □

3.2.2. COROLLARY. *If p, $q \in \omega^*$ and if $p \sqsubseteq q$ and $q \sqsubseteq p$, then $p \sim q$.*

PROOF. Apply Lemma 3.2.1 and Theorem 3.1.1(c). □

3.2.3. LEMMA. *If p, q, $r \in \omega^*$ and if $p \sqsubseteq q$ and $q \sqsubseteq r$, then $p \sqsubseteq r$.*

PROOF. Let $f : \beta\omega \to \omega^*$ be an embedding with $f(p) = q$ and, similarly, let $g : \beta\omega \to \omega^*$ be an embedding with $g(q) = r$. Let $h = (g \restriction f(\beta\omega)) \circ f$. Then h is an embedding with $h(p) = r$. □

Observe that Corollary 3.2.2, Lemma 3.2.3 and Theorem 3.1.1(c) show that the quotient relation defined by $\sqsubseteq$ on $\beta\omega/\sim$ is a partial ordering.

We will now show that the orders $\leq$ and $\sqsubseteq$ are powerful tools if one wishes to study $\beta\omega$. First a preliminary lemma.

3.2.4. LEMMA. *Let $f : \omega^* \to \omega^*$ be a homeomorphism and let $q \in \omega^*$. Then*

$$\{p \in \omega^* : p \sqsubseteq q\} = \{p \in \omega^* : p \sqsubseteq f(q)\} .$$

PROOF. Obvious. □

This enables us to give our first 'real' proof that ω^* is not homogeneous.

3.2.5. THEOREM *ω^* is not homogeneous.*

PROOF. Let $D=\{d_n : n<\omega\}$ be a relatively discrete subset of ω^* and take a point $x\in \bar{D}\backslash D$. Observe that D is C^*-embedded in ω^* (Theorem 1.5.2), hence $\bar{D}=\beta D$. Put $A=\{y\in \bar{D}\backslash D : \exists$ homeomorphism $f:\omega^*\to\omega^*$ with $f(x)=y\}$. Let $h:\omega\to D$ be defined by $h(n)=d_n$ and let βh be its Stone extension. If $y\in A$, then clearly $(\beta h)^{-1}(y)\sqsubseteq y$ and consequently, by Lemma 3.2.3, $(\beta h)^{-1}(y)\sqsubseteq x$. Since βh is one to one, and since by Lemma 3.2.1 $|\{q\in\beta\omega : q\sqsubseteq x\}|\leqslant \mathfrak{c}$, we conclude that $|A|\leqslant\mathfrak{c}$. In Lemma 3.1.2 (c) we proved that $|\beta\omega|=2^{\mathfrak{c}}$. Since $\beta D\approx\beta\omega$, we therefore can find $2^{\mathfrak{c}}$ points in $\bar{D}\backslash A$. □

3.3. *Another order on $\beta\omega$*

Define an order $\leqslant$ on $\beta\omega$ by

$$p\leqslant q \quad \text{if} \quad \text{there is a finite to one } f\in\omega^\omega \text{ with } \beta f(q)=p\,.$$

This order is obviously quite similar to the Rudin–Keisler order. One might hope that at least on ω^*, the orders $\leqslant$ and $\leqslant$ are the same. The aim of this section is to show that this is not true. We will construct points $p, q\in\omega^*$ such that $p\leqslant q$, but p and q are $\leqslant$-incomparable. We first need a generalization of the concept of an independent family of subsets of ω. We will directly translate the new concept in terms of clopen subsets of ω^*.

3.3.1. DEFINITION. An indexed family $\{A^i_j : I\in I,\ j\in J\}$ of clopen subsets of ω^* is called a *J by I independent matrix* if

(1) for all distinct $j_0, j_1\in J$ and $i\in I$ we have that $A^i_{j_0}\cap A^i_{j_1}=\emptyset$,
(2) if $F\in[I]^{<\omega}$ and $f\in J^F$, then

$$\bigcap\{A^\alpha_{f(\alpha)} : \alpha\in F\}\neq\emptyset\,.$$

We will first show that large families of this type exist.

3.3.2. LEMMA. *There is a $\mathfrak{c}$ by $\mathfrak{c}$ independent matrix of clopen subsets of ω^*.*

PROOF. Let $S=\{\langle k,f\rangle : k<\omega\ \&\ f\in\mathscr{P}(k)^{\mathscr{P}(k)}\}$. For each $X, Y\in\mathscr{P}(\omega)$, put

$$A^Y_X=\{\langle k,f\rangle\in S : f(Y\cap k)=X\cap k\}\,.$$

An easy check shows that the family $\{A^Y_X : X, Y\in\mathscr{P}(\omega)\}$, defined on the countable set S, gives us a $\mathfrak{c}$ by $\mathfrak{c}$ independent matrix of clopen subsets of ω^*. □

3.3.3. DEFINITION. A closed subset $A\subseteq\omega^*$ is called an *R-set* if there is an open F_σ $U\subseteq\omega^*$ such that $A\subseteq\bar{U}\backslash U$ and $A\cap\bar{F}=\emptyset$ for all $F\cap[U]^{<\mathfrak{c}}$. An R-set consisting of precisely one point is called an *R-point*.

3.3.4. LEMMA. *There exists an R-point in ω^*.*

PROOF. Let $\{C_n : n<\omega\}$ be a sequence of pairwise disjoint nonempty clopen subsets of ω^*. Put $C=\bigcup_{n<\omega}C_n$. For each $n<\omega$, let $\{A^i_\alpha(n) : i<\omega,\ \alpha<\mathfrak{c}\}$ be a $\mathfrak{c}$ by ω independent matrix of clopen subsets of C_n (Lemma 3.3.2). Put

$$\mathscr{F}=\{F\subseteq C : \forall n<\omega\ \forall i\leqslant n\ \exists\alpha<\mathfrak{c} \text{ such that } A^i_\alpha(n)\subseteq F\}\,.$$

Notice that if $\mathscr{G}\in[\mathscr{F}]^n$, then $\bigcap\mathscr{G}\cap C_i\neq\emptyset$ for all $i\leqslant n-1$. Let $D\in[C]^{<\mathfrak{c}}$. For each $n<\omega$ and $i\leqslant n$ choose $\alpha(n,i)<\mathfrak{c}$ such that $A^i_{\alpha(n,i)}\cap D=\emptyset$ and put

$$F=\bigcup_{n<\omega}\bigcup_{i\leqslant n}A^i_{\alpha(n,i)}(n)\,.$$

Then $F\in\mathscr{F}$ and $F\cap D=\emptyset$. Since F is clopen (in C) and since disjoint clopen subsets of C have disjoint closures in ω^* (recall that ω^* is an F-space), we conclude that $\bar{F}\cap\bar{D}=\emptyset$. (Observe that $D\subseteq C\backslash F$). Consequently, each point of $\bigcap_{F\in\mathscr{F}}\bar{F}$ is an R-point of ω^*. □

The following result is the key in deriving our main result of this section.

3.3.5. THEOREM. *Let $\mathscr{A}$ be a family of $\mathfrak{c}$ R-sets in ω^*. If $\{C_n : n<\omega\}$ is a family of countably many nonempty clopen subsets of ω^*, then for each $n<\omega$ there is a point $x_n\in C_n$ such that*

$$\bigcup\mathscr{A}\cap\{x_n : n<\omega\}^-=\emptyset\,.$$

PROOF. List $\mathscr{A}$ as $\{A_\alpha : \alpha<\mathfrak{c}\}$. By induction, for each $\alpha<\mathfrak{c}$ we will construct for each $n<\omega$ a nonempty closed subset $F^n_\alpha\subseteq C_n$ such that

(1) $(\bigcup_{n<\omega}F^n_\alpha)^-\cap A_\alpha=\emptyset$.
(2) $\chi(F^n_\alpha,\omega^*)\leqslant|\alpha|\cdot\omega$ for each $n<\omega$,
(3) if $\kappa<\alpha$ and $n<\omega$, then $F^n_\alpha\subseteq F^n_\kappa$.

Let $U\subseteq\omega^*$ be an open F_σ which witnesses the fact that A_0 is an R-set. Define $E=\{n<\omega : C_n\backslash\bar{U}\neq\emptyset\}$ and for each $n\in E$ choose a nonempty clopen $E_n\subseteq C_n\backslash\bar{U}$. For all $n\notin E$, pick a point $t_n\in C_n\cap U$. Since A_0 is an R-set, $A_0\cap\{t_n : n\notin E\}^-=\emptyset$. Consequently, we can find for any $n\notin E$ a clopen neighborhood E_n of t_n such that $E_n\subseteq C_n$ and $(\bigcup_{n\notin E}E_n)^-\cap A_0=\emptyset$. For each $n<\omega$ define $F^n_0=E_n$. By construction, $(\bigcup_{n\notin E}E_n)^-\cap A_0=\emptyset$ and since ω^* is an F-space, $(\bigcup_{n\in E}E_n)^-\cap A_0\subseteq(\bigcup_{n\in E}E_n)^-\cap\bar{U}=\emptyset$. Consequently, the F^n_0's are as required.

Suppose that we have completed the construction for all $\mu<\alpha<\mathfrak{c}$. Put $G_n=\bigcap_{\mu<\alpha}F^n_\mu$ for all $n<\omega$ and observe that $\chi(G_n,\omega^*)\leqslant|\alpha|\cdot\omega$. Let $U\subseteq\omega^*$ be an open F_σ which witnesses the fact that A_α is an R-set. Put $E=\{n<\omega : G_n\cap\bar{U}\neq\emptyset\}$ and for each $n\in E$ let $\{V^n_\rho : \rho<|\alpha|\cdot\omega\}$ be a neighborhood basis for G_n. For each $n \in E$ and $\rho < |\alpha| \cdot \omega$ pick a point in $V_\rho^n \cap U$ and let Z be the set of points obtained in this way. Then $|Z| < \mathfrak{c}$ and therefore there is a clopen neighborhood C of A_α which misses $\bar{Z}$. Define $F_\alpha^n = G_n$ if $n \notin E$ and $F_\alpha^n = G_n \setminus C$ if $n \in E$. An easy check, again using the fact that ω^* is an F-space, shows that everything is defined properly.

For each $n < \omega$ take a point $x_n \in \bigcap_{\alpha<\mathfrak{c}} F_\alpha^n$. Then $\{x_n : n < \omega\}$ is as required. □

We need one more lemma.

3.3.6. LEMMA. *Let $f \in \omega^\omega$ be finite to one. If $p \in \omega^*$ is an R-point, then $\beta f^{-1}(\{p\})$ is an R-set.*

PROOF. Observe that $\beta f : \beta\omega \to \beta\omega$ is open and that $\beta f(\omega^*) \subseteq \omega^*$. If $p \notin \beta f(\beta\omega)$, then there is nothing to prove, so, without loss of generality, f is onto. Let $U \subseteq \omega^*$ be an open F_σ which witnesses the fact that p is an R-point. Since βf is open, $Z = \beta f^{-1}(\{p\}) \subseteq \bar{V} \setminus V$, where $V = \beta f^{-1}(U)$. Notice that V is an open F_σ of ω^* since $\beta f^{-1}(\omega^*) = \omega^*$. It is clear that if $F \in [V]^{<\mathfrak{c}}$, then $\bar{F} \cap Z = \emptyset$, whence Z is an R-set. □

We now come to the main result of this section.

3.3.7. THEOREM. *For each R-point $x \in \omega^*$ there is a point $y \in \omega^*$ such that $x \leqslant y$ but x and y are $\leqslant$-incomparable.*

PROOF. Let $f : \omega \to \omega$ be such that $|f^{-1}(\{n\})| = \omega$ for all $n < \omega$. For each $n < \omega$, let $\{E_n^i : i < \omega\}$ be a family of countably many pairwise disjoint (faithfully indexed) nonempty clopen subsets of $\beta f^{-1}(\{n\}) \cap \omega^*$. By Theorem 3.3.5 and Lemma 3.3.6 we may pick for all i, $n < \omega$ a point $x_n^i \in E_n^i$ such that $\{x_n^i : i, n < \omega\}^- \cap \bigcup\{\beta g^{-1}(\{x\}) : g \in \omega^\omega$ is finite to one$\} = \emptyset$. For each $i < \omega$, let $S_i = \{x_n^i : n < \omega\}$. Observe that $\beta f(S_i) = \omega$ which implies that $\bar{S}_i \cap \beta f^{-1}(\{x\}) \neq \emptyset$. If $i \neq j$ then, since ω^* is an F-space, $\bar{S}_i \cap \bar{S}_j = \emptyset$ and this implies that

$$|\{x_n^i : i, n < \omega\}^- \cap \beta f^{-1}(\{x\})| \geqslant \omega ,$$

Therefore $\{x_n^i : i, n < \omega\}^- \cap \beta f^{-1}(\{x\})$ contains a countably infinite relatively discrete set, which is C^*-embedded in ω^* by Theorem 1.5.2, and which therefore has the property that its closure has cardinality $2^\mathfrak{c}$ (Lemma 3.1.2(c)). Since $|\{p \in \omega^* : p \leqslant x\}| \leqslant \mathfrak{c}$, we can therefore find a point $y \in \beta f^{-1}(\{x\}) \setminus (\{p \in \omega^* : p \leqslant x\} \cup \bigcup\{\beta g^{-1}(\{x\}) : g \in \omega^\omega$ is finite to one$\})$. It is clear that y is as required. □

Since, by Lemma 3.3.4, R-points in ω^* exist, we have therefore obtained, the following.

3.3.8. COROLLARY. *There are points p, $q \in \omega^*$ such that $p \leqslant q$ but p and q are $\leqslant$-incomparable.*

3.3.9. REMARK. It might come as a surprise that the proof in this section has not very much in common with the proof in Section 3.1 that $\leqslant$-incomparable points in $\beta\omega$ exist. I do not know whether Corollary 3.3.8 can also be obtained by the method of Section 3.1. Notice however that both methods have an important fact in common, namely that beforehand things have been arranged so that a transfinite induction of length $\mathfrak{c}$ was possible.

3.3.10. REMARK. In Section 4.5 we will compare the orders $\leqslant$, $\sqsubseteq$ and $\leqslant$ with one another.

3.4. *Applications of the Rudin–Keisler order*

In this section we will give a surprisingly general nonhomogeneity result. This will allow us to give another proof that ω^* is not homogenous.

3.4.1. THEOREM. *Let X be an infinite compact space in which all countable discrete subspaces are C^*-embedded. Then X is not homogeneous.*

PROOF. Since X contains a countable discrete subspace, for convenience assume that $\omega \subseteq X$. The assumptions on X then imply that $\bar{\omega} = \beta\omega$. By Theorem 3.1.6 there are points p, $q \in \beta\omega$ which are $\leqslant$-incomparable. We claim there is no homeomorphism $h : X \to X$ with $h(p) = q$. Striving for a contradiction, assume that such an h exists.

Let $\{U_n : n < \omega\}$ be a family of open subsets of X such that

(1) $n \in U_n \subseteq \bar{U}_n \subseteq X \setminus \omega^*$,

(2) if $n \neq m$, then $\bar{U}_n \cap \bar{U}_m = \emptyset$.

Put $E = \{n < \omega : h(n) \notin \bigcup_{m<\omega} U_m \cup \bar{\omega}\}$.

Case 1: $q \in \overline{h(E)}$. Since $h(E) \cup \omega$ is clearly a discrete subset of X, and since $h(E) \cap \omega = \emptyset$, the assumptions on X imply that $\overline{h(E)} \cap \bar{\omega} = \emptyset$, which is impossible since $q \in \bar{\omega}$.

Put $F = \{n < \omega : h(n) \in \omega^*\}$.

Case 2: $q \in \overline{h(F)}$. Then $p \in \bar{F}$ and we may conclude that $p \sqsubseteq q$ and consequently, by Lemma 3.2.1, $p \leqslant q$. This is a contradiction.

Put $G = \{n < \omega : h(n) \in \bigcup_{m<\omega} U_m\}$.

Case 3: $q \in \overline{h(G)}$. Define a function $f : \omega \to \omega$ by

$$\begin{cases} f(k) = 0 & \text{if } k \notin G, \\ f(k) = n & \text{if } k \in G \text{ and } h(k) \in U_n . \end{cases}$$

An easy check shows that $\beta f(p) = q$, i.e. $q \leqslant p$, which is also a contradiction.

Since $E \cup F \cup G = \omega$ and $p \in \bar{\omega}$, $q = h(p) \in \overline{h(E)} \cup \overline{h(F)} \cup \overline{h(G)}$. We therefore have derived a contradiction. □

3.4.2. COROLLARY. *No compact infinite F-space is homogeneous. In particular, ω^* is not homogeneous.*

PROOF. By using the same technique as in the proof of Theorem 1.5.2, the reader can easily check that every countable subspace of an F-space is C^*-embedded. □

Notes for Section 3

The Rudin–Keisler order on $\beta\omega$ was defined by KATĚTOV [1961] and independently, by M.E. RUDIN [1966] and KEISLER [1967]. Theorem 3.1.1 is due to Katětov, M.E. Rudin and Keisler. Lemma 3.1.2(a) and (b) are well-known. The proof of Lemma 3.1.2(a) is due to SIERPIŃSKI [1928]. The proof of Lemma 3.1.2(b) from 3.1.2(a) is new and was suggested to me by Charley Mills. Lemma 3.1.2(c) is due to HAUSDORFF [1936]. Lemma 3.1.4 is well-known and also follows from the fact that $2^{\mathfrak{c}}$ is separable. Lemma 3.1.5 and Theorem 3.1.6 are due to KUNEN [1972]. In fact, Kunen proves that there are $\mathfrak{c}$ pairwise $\leq$-incomparable points. Recently, SHELAH and R.E. RUDIN [1978] showed that there even exist $2^{\mathfrak{c}}$ pairwise $\leq$-incomparable points. The Rudin–Frolík order on ω^* was defined by M.E. RUDIN [1966] and FROLÍK [1967a]. See also M.E. RUDIN [1971]. Theorem 3.2.3 is due to FROLÍK [1967a]. Under CH, it was earlier shown by W. RUDIN [1956]. Lemma 3.3.2 is due to KUNEN [1978]. All other results in section 3.3 were taken from VAN MILL [1981a]. Theorem 3.4.1 was formulated in COMFORT [1977]. The method of proof used in Theorem 3.4.1 is due to FROLÍK [1967b]. Much of the material presented in this chapter can also be found in COMFORT and NEGREPONTIS [1974].

For some recent information concerning the Rudin–Frolík order, see BUKOVSKÝ & BUTKOVICOVÁ [1981].

4. Weak P-points and other points in ω^*

We have seen that, under CH, there are P-points and non P-points in ω^*, whence ω^* is not homogeneous, see section 1.7. However, in section 2.7 we saw that this nonhomogeneity proof in ZFC did not work. In sections 3.2 and 3.4 we gave proofs in ZFC that ω^* is not homogeneous, but these proofs 'only' showed *that* ω^* is not homogeneous but not *why* it is not homogeneous. The aim of this section is to present several 'special' points in ω^*, thus giving a 'real' proof that ω^* is not homogeneous.

4.1. A technical result

The aim of this section is to prove a technical result which enables us later to construct several special points in ω^*.

4.1.1. DEFINITION. Let X be a compact extremally disconnected space, and let $\mathscr{C} = \{C_n : n < \omega\}$ be a sequence of nonempty, faithfully indexed, pairwise disjoint, clopen subsets of X and put $Z = X \setminus \bigcup \mathscr{C}$. In addition, let $f : Z \to Y$ be a continuous surjection and let $B \subseteq Z$ be closed.

If $1 \leq n < \omega$, an indexed family $\{A_i : i \in I\}$ of clopen subsets of X is *precisely n-linked* w.r.t. $\langle B, f\rangle$ if for all $\sigma \in [I]^n$,

$$f\Big(\bigcap_{i \in \sigma} A_i \cap B\Big) = Y,$$

but for all $\sigma \in [I]^{n+1}$, $\bigcap_{i\in\sigma} A_i \cap Z = \emptyset$.

An indexed family $\{A_{in} : i \in I,\ 1 \leq n < \omega\}$ of clopen subsets of X is a *linked system* w.r.t. $\langle B, f\rangle$, if for each n, $\{A_{in} : i \in I\}$ is precisely n-linked w.r.t. $\langle B, f\rangle$, and for each n and i, $A_{in} \subseteq A_{i,n+1}$. An indexed family $\{A^j_{in} : i \in I,\ 1 \leq n < \omega,\ j \in J\}$ is an I by J *independent linked family* w.r.t. $\langle B, f\rangle$ if for each $j \in J$, $\{A^j_{in} : i \in I,\ 1 \leq n < \omega\}$ is a linked system w.r.t. $\langle B, f\rangle$, and:

$$f\Big(\bigcap_{j\in\tau}\Big(\bigcap_{i\in\sigma_j} A^j_{in_j}\Big) \cap B\Big) = Y,$$

whenever $\tau \in [J]^{<\omega}$, and for each $j \in \tau$, $1 \leq n_j < \omega$ and $\sigma_j \in [I]^{n_j}$.

If $f : \omega \to \omega$ is a function, let $\bar{f} = \beta f \restriction \omega^*$. In Definition 4.1.1, let $X = \beta\omega$ and, for all $n < \omega$, let $C_n = \{n\}$. We then have the following important lemma.

4.1.2. LEMMA. *There is a finite to one function $\pi : \omega \to \omega$ and a $\mathfrak{c}$ by $\mathfrak{c}$ independent linked family of clopen subsets of $\beta\omega$ w.r.t. $\langle \omega^*, \bar{\pi}\rangle$.*

PROOF. Let $S = \{\langle k, g\rangle : k \in \omega\ \&\ g \in \mathscr{P}\mathscr{P}(k)^{\mathscr{P}(k)}\}$. Identify S and ω and define $\pi : S \to \omega$ by $\pi(\langle k, g\rangle) = k$. It is clear that π is finite to one. For all $X, Y \in \mathscr{P}(\omega)$ and $n < \omega$, define

$$A^Y_{Xn} = \{\langle k, g\rangle \in S : |g(Y \cap k)| \leq n\ \&\ X \cap k \in g(Y \cap k)\}.$$

It is easily seen that the family

$$\{E^Y_{Xn} : X \in \mathscr{P}(\omega),\ 1 \leq n < \omega,\ Y \in \mathscr{P}(\omega)\},$$

where E^Y_{Xn} is the closure of A^Y_{Xn} in βS, is as required. □

4.1.3. DEFINITION. Let X be a space. A closed subspace $A \subseteq X$ is called κ-OK provided that for each sequence $\{U_n : n < \omega\}$ of neighborhoods of A, there is a sequence $\{V_\alpha : \alpha < \kappa\}$ of neighborhoods of A such that for each $n \geq 1$ and $\alpha_1 < \alpha_2 < \cdots < \alpha_n < \kappa$,

$$\bigcap_{1 \leq i \leq n} V_{\alpha_i} \subseteq U_n .$$

Observe that the property of being κ-OK gets stronger as κ gets bigger. A point $x \in X$ is called a κ-*OK point* if $\{x\}$ is a κ-OK set of X.

4.1.4. DEFINITION. Let X be a space and let $\mathscr{U} = \{U_n : n < \omega\}$ be a family of open subsets of X. A closed subset $Z \subseteq \overline{\bigcup \mathscr{U}} \backslash \bigcup \mathscr{U}$ is called *nice* w.r.t. $\mathscr{U}$ provided that for each neighborhood V of Z the set $\{n < \omega : V \cap U_n = \emptyset\}$ is finite.

We now come to the main result of this section.

4.1.5. THEOREM. *Let X be a compact extremally disconnected space of weight $\mathfrak{c}$ and let $\mathscr{C} = \{C_n : n < \omega\}$ be a sequence of nonempty, faithfully indexed, pairwise disjoint, clopen subsets of X and put $Z = \overline{\bigcup \mathscr{C}} \backslash \bigcup \mathscr{C}$. If $A \subseteq Z$ is nice w.r.t. $\mathscr{C}$ and if Y is a continuous image of ω^*, then there is a closed set $B \subseteq A$ which is a $\mathfrak{c}$-OK set of Z and which admits an irreducible surjection on Y.*

PROOF. Since $\overline{\bigcup \mathscr{C}}$ is clopen in X, without loss of generality, $\overline{\bigcup \mathscr{C}} = X$. Define $f : \bigcup \mathscr{C} \to \omega$ by $f(x) = n$ iff $x \in C_n$ and let $\beta f : X \to \beta\omega$ be its Stone extension. Since X is an F-space, $\overline{\bigcup \mathscr{C}} = \beta(\bigcup \mathscr{C})$. Let $\pi : \omega \to \omega$ be the finite to one function of Lemma 4.1.2 and let $\{E^\beta_{\alpha n} : \alpha < \mathfrak{c},\ 1 \leq n < \omega,\ \beta < \mathfrak{c}\}$ be the $\mathfrak{c}$ by $\mathfrak{c}$ independent linked family of clopen subsets of $\beta\omega$ w.r.t. $\langle \omega^*, \bar{\pi} \rangle$ of Lemma 4.1.2. In addition, let $g : \omega^* \to Y$ be a continuous surjection.

Define $h : Z \to Y$ by $h = g \circ \bar{\pi} \circ (\beta f \restriction Z)$ and observe that the family

$$\{A^\beta_{\alpha n} : \alpha < \mathfrak{c},\ 1 \leq n < \omega,\ \beta < \mathfrak{c}\},$$

where $A^\beta_{\alpha n} = \beta f^{-1}(E^\beta_{\alpha n})$, is an independent linked family w.r.t. $\langle A, h \rangle$. For this one only needs to verify that $\beta f(A) = \omega^*$, and this is easy. Let $\{Z_\mu : \mu < \mathfrak{c}\ \&\ \mu \text{ is even}\}$ enumerate the family of all clopen subsets of X and let $\{\langle S_{\mu n} : n < \omega \rangle : \mu < \mathfrak{c}\ \&\ \mu$ is odd$\}$ enumerate all sequences of nonempty clopen subsets of X satisfying

$$S_{\mu, n+1} \subseteq S_{\mu n} \backslash \bigcup_{i \leq n} C_i .$$

Furthermore, assume that each sequence is listed cofinally often. By induction on μ we construct F_μ and K_μ so that:

(1) $F_\mu \subseteq A$ is closed, $K_\mu \subseteq \mathfrak{c}$, and $\{A^\beta_{\alpha n} : \alpha < \mathfrak{c},\ 1 \leq n < \omega,\ \beta \in K_\mu\}$ is an independent linked family w.r.t. $\langle F_\mu, h \rangle$,

(2) $K_0 = 2^\omega$ and $F_0 = A$;

(3) $\nu < \mu$ implies $F_\nu \supseteq F_\mu$ and $K_\mu \subseteq K_\nu$,

(4) if μ is a limit ordinal, $F_\mu = \bigcap_{\nu < \mu} F_\nu$ and $K_\mu = \bigcap_{\nu < \mu} K_\nu$,

(5) for each μ, $K_\mu \backslash K_{\mu+1}$ is finite,

(6) if μ is even, either $F_{\mu+1} \subseteq Z_\mu$ or $h(F_{\mu+1} \cap Z_\mu) \neq Y$,

(7) if μ is odd and $F_\mu \subseteq \bigcap_{n<\omega} S_{\mu n}$, then there are clopen neighborhoods $D_{\mu\alpha}$ of $F_{\mu+1}$ for $\alpha < \mathfrak{c}$ such that for all $n \geq 1$ and $\alpha_1 < \alpha_2 < \cdots < \alpha_n < \mathfrak{c}$, there is an $m < \omega$ such that

$$(D_{\mu\alpha_1} \cap \cdots \cap D_{\mu\alpha_n}) \backslash S_{\mu n} \subseteq \bigcup_{i \leq m} C_i .$$

Fix $\mu < \mathfrak{c}$ and assume that F_ν, K_ν have been constructed for all $\nu \leq \mu$. We will construct $F_{\mu+1}$ and $K_{\mu+1}$.

Suppose first that μ is even and define $T = F_\mu \cap Z_\mu$. If

$$\{A^\beta_{\alpha n} : \alpha < \mathfrak{c},\ 1 \leq n < \omega,\ \beta \in K_\mu\}$$

is an independent linked family w.r.t. $\langle T, h \rangle$, we put $F_{\mu+1} = T$ and $K_{\mu+1} = K_\mu$. If not, then

$$h\Big(F_\mu \cap Z_\mu \cap \bigcap_{\beta \in \tau} \Big(\bigcap_{\alpha \in \sigma_\beta} A^\beta_{\alpha n_\beta} \Big)\Big) \neq Y$$

for some $\tau \in [K_\mu]^{<\omega}$, $n_\beta < \omega$ and $\sigma_\beta \in [\mathfrak{c}]^{n_\beta}$. Then let $K_{\mu+1} = K_\mu \backslash \tau$, and let

$$F_{\mu+1} = F_\mu \cap \bigcap_{\beta \in \tau} \Big(\bigcap_{\alpha \in \sigma_\beta} A^\beta_{\alpha n_\beta} \Big).$$

Clearly, $F_{\mu+1}$ and $K_{\mu+1}$ are as required.

If μ is odd and there is an $n < \omega$ such that $F_\mu \backslash S_{\mu n} \neq \emptyset$, put $F_{\mu+1} = F_\mu$ and $K_{\mu+1} = K_\mu$. In case $F_\mu \subseteq \bigcap_{n<\omega} S_{\mu n}$, fix $\beta \in K_\mu$ and let $K_{\mu+1} = K_\mu \backslash \{\beta\}$. For each $\alpha < \mathfrak{c}$, define

$$D_{\mu\alpha} = \Big(\bigcup_{1 \leq n < \omega} A^\beta_{\alpha n} \cap S_{\mu n} \Big)^{-},$$

and put $F_{\mu+1} = \bigcap_{\alpha < \mathfrak{c}} D_{\mu\alpha} \cap F_\mu$. We claim that $F_{\mu+1}$ and the sequence $\langle D_{\mu\alpha} : \alpha < \mathfrak{c} \rangle$ are as required.

First observe that each $D_{\mu\alpha}$ is clopen since it is the closure of an open set in X. To verify condition (7), let $\alpha_1 < \alpha_2 < \cdots < \alpha_n < \mathfrak{c}$ and put

$$T = (D_{\mu\alpha_1} \cap \cdots \cap D_{\mu\alpha_n}) \setminus S_{\mu n}.$$

If $n = 1$, then clearly $T = \emptyset$, since $D_{\mu\alpha_1} \subseteq S_{\mu 1}$. Therefore, assume that $n > 1$.

Claim. $T \subseteq A^{\beta}_{\alpha_1, n-1} \cap \cdots \cap A^{\beta}_{\alpha_n, n-1}$.

Take $x \in D_{\mu\alpha_1} \cap \cdots \cap D_{\mu\alpha_n}$ and assume that

$$x \in \bigcap_{1 \le i \le n} A^{\beta}_{\alpha_i k_i} \cap S_{\mu k_i},$$

where $k_{i_0} \ge n$ for some $1 \le i_0 \le n$. Since $S_{\mu k_{i_0}} \subseteq S_{\mu n}$ it follows that $x \notin T$. Next suppose that

$$x \in \bigcap_{1 \le i \le n} A^{\beta}_{\alpha_i k_i} \cap S_{\mu k_i}$$

where $k_i < n$ for all $1 \le i \le n$. Since $A^{\beta}_{\alpha_i k_i} \subseteq A^{\beta}_{\alpha_i, n-1}$ for all $1 \le i \le n$, this implies that $x \in \bigcap_{1 \le i \le n} A^{\beta}_{\alpha_i, n-1}$. We therefore conclude that

$$T = (D_{\mu\alpha_1} \cap \cdots \cap D_{\mu\alpha_n}) \cap (X \setminus S_{\mu n})$$

$$= \Big(\bigcap_{1 \le i \le n} \Big(\bigcup_{1 \le k < \omega} A^{\beta}_{\alpha_i k} \cap S_{\mu k} \Big) \cap (X \setminus S_{\mu n}) \Big)^{-}$$

$$\subseteq \bigcap_{1 \le i \le n} A^{\beta}_{\alpha_i, n-1}.$$

This implies that for some $m < \omega$ we have that $T \subseteq \bigcup_{i \le m} C_i$, since the $\{A^{\beta}_{\alpha_i, n-1} : 1 \le i \le n\}$ are precisely $(n-1)$-linked.

Finally, to verify condition (1), observe that $D_{\mu\alpha} \supseteq S_{\mu n} \cap A^{\beta}_{\alpha n}$ for each $n < \omega$.

Now put $B = \bigcap_{\mu < \mathfrak{c}} F_\mu$. We claim that B is as required. By (1), $h(F_\mu) = Y$ for all $\mu < \mathfrak{c}$ and therefore, by compactness and by (3), $h(B) = Y$. By (7), B is a $\mathfrak{c}$-OK set of Z and it therefore suffices to prove that $h \restriction B$ is irreducible. If $B' \subseteq B$ is a proper closed set, then for some even $\mu < \mathfrak{c}$, $B' \subseteq Z_\mu$ and $B \setminus Z_\mu \ne \emptyset$. Then, by (6), $h(F_{\mu+1} \cap Z_\mu) \ne Y$. Since $B' \subseteq F_{\mu+1} \cap Z_\mu$, we conclude that $h(B') \ne Y$, consequently, $h \restriction B$ is irreducible. □

4.2. *A compactification of* ω

We will show that there is a compactification $\gamma\omega$ of ω such that $\gamma\omega \setminus \omega$ is not separable but yet satisfies the countable chain condition. This compactification we need in the next section to construct several special points in ω^*.

Let $P = \{f \in \omega^\omega : 0 \le f(n) \le n+1$ for each $n < \omega\}$ and $N = \{f \restriction n : f \in P$ and $n < \omega\}$. Define $T = \{\pi \in N^\omega : \mathrm{dom}(\pi(n)) = n+1$ for each $n < \omega\}$. For each $s \in N$, let $C_s = \{t \in N : s \subseteq t\}$ and for each $\pi \in T$ put

$$C_\pi = \bigcup_{n < \omega} C_{\pi(n)}.$$

Observe that $N \setminus C_\pi$ is infinite for each π. Let $\mathscr{B}$ be the smallest Boolean subalgebra of $\mathscr{P}(N)$ containing $\mathscr{A} = \{C_\pi : \pi \in T\} \cup \{N \setminus C_\pi : \pi \in T\}$. Notice that $\{\{s\} : s \in N\} \cup \{C_s : s \in N\} \subseteq \mathscr{B}$. Let $\gamma\omega$ denote the Stone space of $\mathscr{B}$. Clearly, $\gamma\omega$ is a compactification of the countable discrete space $\{\{B \in \mathscr{B} : s \in B\} : s \in N\}$ which we identify with ω. Put $X = \gamma\omega \setminus \omega$.

4.2.1. LEMMA. *X is not separable.*

PROOF. Let $\{p_n : n < \omega\}$ be countably many free ultrafilters on $\mathscr{B}$. For each $n < \omega$, there exists $\pi(n)$ with $\mathrm{dom}(\pi(n)) = n+1$ such that $C_{\pi(n)} \in p_n$. Simply observe that $N = \{s \in N : \mathrm{dom}(s) \le n\} \cup \bigcup\{C_s : \mathrm{dom}(s) = n+1\}$ for each $n < \omega$. Consequently, $\{p \in X : N \setminus C_\pi \in p\}$ is a nonempty open set of X disjoint from $\{p_n : n < \omega\}$. □

A family of sets is called *linked* provided that each subfamily of cardinality at most 2 has nonempty intersection. Call a family of sets *σ-linked* provided that it is the union of countably many linked subfamilies. It is obvious that a space having a σ-linked base is ccc.

4.2.2. LEMMA. *X has a σ-linked base.*

PROOF. It suffices to show that $\{B \in \mathscr{B} : |B| = \omega\} = \bigcup_{n \in \omega} \mathscr{B}_n$ such that for each n every two members of $\mathscr{B}_n$ have infinite intersection. To this end, for each $j \in \omega$ and for each $s \in N$ with $2j - 1 \le \mathrm{dom}\, s$, define

$$\mathscr{B}(j, s) = \Big\{ B \in \mathscr{B} : \exists K \in [T]^{<\omega} \text{ and } L \in [T]^j \text{ with } s \in \bigcap_{\pi \in K} C_\pi \cap \bigcap_{\pi \in L} N \setminus C_\pi \in [B]^\omega \Big\}.$$

Since for each $B \in \mathscr{B}$ with $|B| = \omega$, there exists a set D which is a finite intersection of elements of $\mathscr{A}$, with $D \in [B]^\omega$ and since any infinite subset of N contains elements of arbitrarily large domain, it follows that

$$\{B \in \mathscr{B} : |B| = \omega\} = \bigcup\{\mathscr{B}(j, s) : j \in \omega,\ s \in N, \text{ and } 2j - 1 \le \mathrm{dom}\, s\}.$$

Fix an index j and $s \in N$ with $2j - 1 \le \mathrm{dom}\, s$. If $\{B_0, B_1\} \subseteq \mathscr{B}(j, s)$, then there exist

$K_i \in [T]^{<\omega}$ and $L_i \in [T]^j$ such that for each $i<2$,

$$s \in D_i = \bigcap_{\pi\in K_i} C_\pi \cap \bigcap_{\pi\in L_i} N\backslash C_\pi \in [B_i]^\omega .$$

We now define, by induction on dom $s \leqslant n$, an $h \in P$ such that

$$\{h \upharpoonright n : \text{dom } s \leqslant n\} \subseteq D_0 \cap D_1 .$$

Stage dom s: Let $h\upharpoonright \text{dom } s = s$. Then $h \upharpoonright \text{dom } s \in D_0\cap D_1$. Assume we have defined $h\upharpoonright n$ for some dom $s \leqslant n$ such that $h\upharpoonright n \in D_0 \cap D_1$.

Stage $n+1$: Define $h\upharpoonright n+1$ to be some sequence in N of domain $n+1$ that extends $h\upharpoonright n$ and such that $h\upharpoonright n+1 \not\in \{\pi(n): \pi\in L_0\cup L_1\}$. This is possible because there are $n+2$ sequences in N of domain $n+1$ that extend $h\upharpoonright n$ and $|L_0\cup L_1| \leqslant 2j < \text{dom } s+2 \leqslant n+2$. Then $h\upharpoonright n+1 \in D_0 \cap D_1$. □

4.3. Weak P-points in ω^*

In this section we will show that ω^* contains at least two types of weak P-points. Let X be a space. A subset $F\subseteq X$ is called a *weak P-set* provided that $F\cap \bar{D} = \emptyset$ for any countable $D \subseteq X\backslash F$.

4.3.1. LEMMA. *Let X be a space and let $S\subseteq X$. Then*
(a) *if S is ω_1-OK then S is a weak P-set of X, and*
(b) *if S is κ-OK, where* cf$(\kappa) \geqslant \omega_1$, *and S is not a P-set, then* $c(X)\geqslant\kappa$.

PROOF. Let $F=\{t_n : n<\omega\}\subseteq X\backslash S$ be any sequence. Since S is ω_1-OK, we can find a collection $\{U_\xi : \xi<\omega_1\}$ of neighborhoods of S such that for all $\xi_1<\xi_2<\cdots<\xi_n<\omega_1$, we have that

$$(*)\qquad \bigcap_{1\leqslant i\leqslant n} U_{\xi_i} \subseteq X\backslash\{t_n\} .$$

If $U_\xi \cap F \neq \emptyset$ for all $\xi<\omega_1$, then there are an uncountable $A\subseteq\omega_1$ and an $n<\omega$ such that $t_n \in \bigcap_{\xi\in A} U_\xi$. But this obviously contradicts (*). For (b), let $F_n \subseteq \bar{F}_n \subseteq X\backslash S$ $(n<\omega)$ be a sequence of open sets in X such that

$$S \cap \left(\overline{\bigcup_{n<\omega} \bar{F}_n} \backslash \bigcup_{n<\omega} \bar{F}_n\right) \neq \emptyset .$$

Choose a family $\{U_\xi : \xi<\kappa\}$ of neighborhoods of S such that for all $\xi_1<\xi_2<\cdots<\xi_n<\kappa$, $\bigcap_{1\leqslant i\leqslant n} U_{\xi_i} \subseteq X\backslash \bar{F}_n$. Since cf$(\kappa)\geqslant\omega_1$, there have to be a set $A\in[\kappa]^\kappa$ and an $n<\omega$ such that $U_\xi\cap F_n\neq\emptyset$ for all $\xi\in A$. Then

$$\mathcal{B} = \{U_\xi \cap F_n : \xi \in A\}$$

is a family of κ open subsets of X such that any intersection of n of them has empty intersection. By transfinite induction, for each $\xi<\kappa$ we will define a maximal subfamily $\mathcal{G}_\xi \subseteq \mathcal{B}$ such that $\bigcap\mathcal{G}_\xi \neq\emptyset$ and $\mathcal{G}_\xi\neq\mathcal{G}_\eta$ for all $\eta<\xi<\kappa$. If $\mathcal{G}_\eta$ has been defined for all $\eta<\xi<\kappa$, then take

$$B \in \mathcal{B} \backslash \bigcup_{\eta<\xi} \mathcal{G}_\eta$$

Such a B exists since $|\mathcal{G}_\eta| \leqslant n-1$ for all $\eta<\xi$. Then let $\mathcal{G}_\xi$ be any maximal subfamily of $\mathcal{B}$ which contains B and has nonempty intersection. The family $\{\bigcap\mathcal{G}_\xi : \xi<\kappa\}$ consists of κ pairwise disjoint nonempty open subsets of X, whence $c(X)\geqslant\kappa$. □

4.3.2. COROLLARY. *Let $x\in X$ be ω_1-OK. Then x is a weak P-point of X.*

We now come to the main result of this section.

4.3.3. THEOREM. *Let $A=\{x\in\omega^* : x$ is $\mathfrak{c}$-OK$\}$ and $B=\{x\in\omega^*: x$ is a weak P-point and $x\in\bar{C}\backslash C$ for some $C\subseteq\omega^*$ satisfying the ccc$\}$. Then $A\neq\emptyset$, $B\neq\emptyset$ and $A\cap B=\emptyset$.*

PROOF. That $A\neq\emptyset$ follows directly from Theorem 4.1.4 and that $A\cap B=\emptyset$ is a consequence of Lemma 4.3.1(b). It remains to show that $B\neq\emptyset$. To this end, let X be the ccc nonseparable remainder of ω constructed in section 4.2. Observe that the one point compactification $a(\omega\times X)$ of $\omega\times X$, is a continuous image of ω^*, whence, by Theorem 4.1.5, there is a closed $\mathfrak{c}$-OK set $Z\subseteq\omega^*$ and irreducible surjection $f: Z\to a(\omega\times X)$. For each $n<\omega$, put $Z_n = f^{-1}(\{n\}\times X)$. Observe that by irreducibility of f, we have that $(\bigcup_{n<\omega} Z_n)^- = Z$. Let $\pi:\omega\times X\to X$ be the projection and let $\mathcal{U}$ be a maximal disjoint family of separable clopen subsets of X. Since X is ccc but not separable, $(\bigcup\mathcal{U})^-\neq X$ and consequently we can pick a nonempty clopen $C\subseteq X$ which misses $\bigcup\mathcal{U}$. Observe that C is nowhere separable. For each countable $D\subseteq\bigcup_{n<\omega} Z_n$, let $\{C_n(D): n<\omega\}$ be a maximal disjoint family of nonempty clopen subsets of C such that

$$\bigcup_{n<\omega} C_n(D) \cup (\overline{\pi(f(D))} \cap C) = \emptyset$$

and define

$$F(D) = \bigcup_{n<\omega} f^{-1}\Big(\bigcup_{i\leqslant n} \{n\}\times C_i(D)\Big) .$$

Define $F = \bigcap\{\overline{F(D)} : D \in [f^{-1}(\omega \times X)]^{\leq\omega}\}$. It is easily seen that F is nice w.r.t. $\{f^{-1}(\{n\} \times X) : n < \omega\}$. Since X is an F-space, being closed in ω^*, and since, by irreducibility of f, Z is ccc, we may conclude that Z is extremally disconnected (Lemma 1.2.2). By Theorem 4.1.4, there is a point $x \in S = Z \setminus \bigcup_{n<\omega} Z_n$ which belongs to F and which is a $\mathfrak{c}$-OK point of S.

Claim. If $D \in [\bigcup_{n<\omega} Z_n]^{\leq\omega}$, then $x \notin \bar{D}$.

By construction, $F(D)$ is a clopen subspace of $\bigcup_{n<\omega} Z_n$ which misses D. Since Z is extremally disconnected,

$$\overline{F(D)} \cap \left(\overline{\bigcup_{n<\omega} Z_n \setminus F(D)}\right) = \emptyset .$$

Since $x \in \overline{F(D)}$, we conclude that $x \notin \bar{D}$.

The claim now can be used to prove quite easily that x is a weak P-point of ω^*. □

4.3.4. Corollary. *ω^* contains weak P-points.*

Notice that Theorem 4.3.3 proves once again that ω^* is not homogeneous. This proof is totally different from the previous ones, since we found an easy to state topological property shared by some but not all points in ω^*.

A space X is called *first order homogeneous* provided that no property which can be expressed in first order language distinguishes points of X. It is clear that any homogeneous space is first order homogeneous, but not conversely. It was shown by van Douwen & van Mill [1981a] that $\mathfrak{c}$-OK points can be used to show that ω^* is not first order homogeneous.

4.4. Some other points of interest

In this section we will continue our search for 'special' points in ω^*. In Theorem 4.3.3 we constructed two types of weak P-points in ω^*. It is natural to ask whether every point $x \in \omega^*$ which is a limit point of some countable subset of $\omega^* \setminus \{x\}$ is also a limit point of some countable discrete subspace of $\omega^* \setminus \{x\}$. Our first result is that the answer to this question is in the negative.

4.4.1. Theorem. *There is a point $x \in \omega^*$ such that x is a limit point of some countable subset of $\omega^* \setminus \{x\}$, but not of any countable discrete subset of $\omega^* \setminus \{x\}$.*

Proof. It is clear that the one point compactification S of $\omega \times [0, 1]$ is a continuous image of ω^*. Therefore, by Theorem 4.1.4, there is a $\mathfrak{c}$-OK set $T \subseteq \omega^*$ which can be mapped by an irreducible map, say f, onto S. For all $n < \omega$, put $T_n = f^{-1}(\{n\} \times [0, 1])$.

Claim. For each $n \geq 1$ there is a family $\mathcal{F}_n$ of closed subsets of $[0, 1]$ such that

(1) *$\mathcal{F}_n$ has the n-intersection property,*

(2) *if $D \subseteq [0, 1]$ is nowhere dense, then there is an $F \in \mathcal{F}_n$ with $F \cap D = \emptyset$.*

This was shown in Lemma 1.9.2.

For each $n < \omega$, let $\mathcal{G}_n = \{f^{-1}(\{n\} \times F) : F \in \mathcal{F}_{n+1}\}$. Observe that G_n has the $(n+1)$-intersection property and that for each nowhere dense set $D \subseteq T_n$ there is a $G \in \mathcal{G}_n$ with $G \cap D = \emptyset$. (This is obvious since $f(D)$ is nowhere dense in $\{n\} \times [0, 1]$.) Put

$$G = \bigcap\{\bar{A} : A \subseteq f^{-1}(\omega \times [0, 1]) \text{ and } A \cap T_n \in \mathcal{G}_n \text{ for all } n < \omega\} .$$

It is easily seen that $G \subseteq T \setminus \bigcup_{n<\omega} T_n$ and that G is nice w.r.t. the sequence $\{T_n : n < \omega\}$.

By similar arguments as in the proof of Theorem 4.3.3 we may conclude that T is extremally disconnected and by an appeal to Theorem 4.1.5 we can find a point $x \in G$ which is a $\mathfrak{c}$-OK point of $T \setminus \bigcup_{n<\omega} T_n$. It is easily seen that the point x is as required. The details of checking this out are left to the reader. □

Let us now pose a rather innocent question. Does every point in ω^* have character $\mathfrak{c}$ in ω^*? Under CH, this is obviously true. However, under $\neg$CH there can be points in ω^* which have character ω_1, Kunen [1972]. The following now directly comes to mind: can all points in ω^* be of character less than $\mathfrak{c}$? The answer to this question is in the negative. In Lemma 3.3.4 we showed that there is an R-point in ω^*, and R-points obviously have character $\mathfrak{c}$ in ω^*. Let us give a somewhat easier proof than the one in Lemma 3.3.4, that points of character $\mathfrak{c}$ in ω^* exist.

4.4.2. Theorem. *There is a point $x \in \omega^*$ such that $\chi(x, \omega^*) = \mathfrak{c}$.*

Proof. Let $\{(A^0_\alpha, A^1_\alpha) : \alpha < \mathfrak{c}\}$ be a family of pairs of disjoint subsets of ω^* such as in Lemma 3.1.2(b). Take any point x in the intersection

$$\bigcap\{\overline{A^0_\alpha} \cap \omega^* : \alpha < \mathfrak{c}\} \cap \bigcap\Big\{C : C \subseteq \omega^* \text{ is clopen and } \exists D \in [\mathfrak{c}]^\omega \text{ such that } \omega^* \setminus C \subseteq \bigcap_{\alpha\in D} \overline{A^0_\alpha}\Big\}.$$

By using similar arguments as in the proof of Theorem 2.1.1, it follows that x has character $\mathfrak{c}$ in ω^*. □

In the proof of Theorem 2.1.1, we constructed a Parovičenko space T such that $\pi(x, T) = \mathfrak{c}$ for all $x \in T$. In view of the above result, is therefore quite natural to ask whether the above result can be strengthened to the statement that there is a point $x \in \omega^*$ such that $\pi(x, \omega^*) = \mathfrak{c}$. This is impossible. BELL & KUNEN [1980] show it to be consistent with $\mathfrak{c} = \omega_{\omega_1}$ that each point $x \in \omega^*$ has π-character $\omega_1 (< \mathfrak{c})$. However, the following is true in ZFC.

4.4.3. THEOREM. *There is a point $x \in \omega^*$ with $\pi(x, \omega^*) \geq \mathrm{cf}(\mathfrak{c})$.*

PROOF. Let $\{A^\alpha_\beta : \alpha, \beta < \mathfrak{c}\}$ be a $\mathfrak{c}$ by $\mathfrak{c}$ independent matrix of clopen subsets of ω^* (Lemma 3.3.2) and let $\{C_\alpha : \alpha < \mathfrak{c}\}$ enumerate the family of all clopen subsets of ω^*. For all $\alpha, \kappa < \mathfrak{c}$ there is at most one $\beta < \mathfrak{c}$ such that $C_\kappa \subseteq A^\alpha_\beta$. It is therefore easy to pick for each $\alpha < \mathfrak{c}$ an element $f(\alpha) < \mathfrak{c}$ such that for all $\beta < \alpha$,

$$C_\beta \not\subseteq A^\alpha_{f(\alpha)} .$$

Take any point $x \in \bigcap_{\alpha < \mathfrak{c}} A^\alpha_{f(\alpha)}$. We claim that $\pi(x, \omega^*) \geq \mathrm{cf}(\mathfrak{c})$. If $\pi(x, \omega^*) < \mathrm{cf}(\mathfrak{c})$, then there is an $\alpha < \mathfrak{c}$ such that the family $\{C_\beta : \beta < \alpha\}$ constitutes a π-basis for x. But $A^\alpha_{f(\alpha)}$ is a neighborhood of x which does not contain any C_β for all $\beta < \alpha$. □

Since ω^* is an F-space, each countable subspace of ω^* is C^*-embedded in ω^* (Theorem 1.5.2). If $2^{\omega_1} = \mathfrak{c}$, then $\beta\omega_1$ can be embedded in ω^* since $\beta\omega_1$ is extremally disconnected and has weight $\mathfrak{c}$ (Theorem 1.4.7). Therefore, under $2^{\omega_1} = \mathfrak{c}$, ω^* contains subspaces of cardinality ω_1 that are C^*-embedded. Having this in mind, it is quite natural to ask whether all subspaces of ω^* of cardinality ω_1 can be C^*-embedded. We will show that this is not the case. As usual, a P-space is a space in which all G_δ's are open.

4.4.4. THEOREM. *Let X be a P-space of weight at most $\mathfrak{c}$. Then X can be embedded in ω^*.*

PROOF. We may assume that $X \subseteq 2^\mathfrak{c}$ (here $2^\mathfrak{c}$ denotes the Cantor cube of weight $\mathfrak{c}$). Take $p \in 2^\mathfrak{c}$. The map $g_p : 2^\mathfrak{c} \to 2^\mathfrak{c}$ defined by $g_p(x) = x + p$ lifts to a map $eg_p : E(2^\mathfrak{c}) \to E(2^\mathfrak{c})$ ($E(2^\mathfrak{c})$ is the projective cover of $2^\mathfrak{c}$, see Section 0). The homeomorphism eg_p will be called h_p for short. Let π be the canonical irreducible surjection from $E(2^\mathfrak{c})$ onto $2^\mathfrak{c}$, i.e. π is defined by

$$\{\pi(u)\} = \bigcap\{\bar{U} : U \in u\} .$$

Take a point $u_0 \in \pi^{-1}(\mathbf{0})$, where $\mathbf{0}$ denotes the identity of $2^\mathfrak{c}$. If $p \in X$, let $u_p = h_p(u_0)$. Observe that

$$\pi(u_p) = \pi(h_p(u_0)) = g_p(\pi(u_0)) = g_p(\mathbf{0}) = p ,$$

whence $u_p \in \pi^{-1}(p)$.

Let U be a regular open subset of $2^\mathfrak{c}$. We can find a countable subset $D \subseteq \mathfrak{c}$ and

a regular open subset $U' \subseteq 2^D$ such that if $\pi_D : 2^\mathfrak{c} \to 2^D$ denotes the projection, then $\pi_D^{-1}(U') = U$ (uses the fact that $2^\mathfrak{c}$ is ccc, JUHÁSZ [1980]).

Claim. If $p \upharpoonright D = q \upharpoonright D$ for $p, q \in X$, then $U \in u_p$ iff $U \in u_q$.

Indeed, simply observe that $U \in u_p$ iff $U + p \in u_0$ iff $U + p + q \in u_q$ iff $U \in u_q$.

Now, let $P = \{u_p : p \in X\}$. We claim that $\pi \upharpoonright P : P \to X$ is a homeomorphism. For convenience, put $f = \pi \upharpoonright P$. Then f is clearly one to one, onto and continuous. It therefore suffices to show that f is open. Let U be a regular open subspace of $2^\mathfrak{c}$ and let U' and D be as above. The set $\tilde{U} = \{u \in E(2^\mathfrak{c}) : U \in u\}$ is a basic open subset of $E(2^\mathfrak{c})$, so we only need to show that $f(\tilde{U})$ is open in X. Take $p \in f(\tilde{U})$ arbitrarily. Define $Z = \{q \in X : p \upharpoonright D = q \upharpoonright D\}$. By the claim, $Z \subseteq f(\tilde{U})$. Observe that $Z = \pi_D^{-1}(p \upharpoonright D) \cap X$, whence Z is a G_δ in X. Since G_δ's in X are open, and since $p \in Z$, we conclude that $f(\tilde{U})$ is a neighborhood of p.

We conclude that X can be embedded in $E(2^\mathfrak{c})$. Since $E(2^\mathfrak{c})$ is separable, it has weight $\mathfrak{c}$, and therefore, by Theorem 1.4.7, it embeds in $\beta\omega$. Since $\beta\omega$ embeds in ω^*, we are done. □

4.4.5. COROLLARY. *There is a point $x \in \omega^*$ and a (relatively) discrete sequence $\{x_\alpha : \alpha < \omega_1\} \subseteq \omega^* \setminus \{x\}$, such that each neighborhood of x contains all but countably many of the x_α's.*

PROOF. There is clearly a P-space of cardinality ω_1 and containing precisely one nonisolated point. Now apply Theorem 4.4.4. □

4.4.6. REMARK. Observe that the proof of Theorem 4.4.4 actually shows that if X is a P-space, then βX can be embedded in the Čech–Stone compactification of some discrete space.

4.5. *Partial orderings on $\beta\omega$, II*

In Section 3 we defined three 'partial' orders on $\beta\omega$, namely $\preccurlyeq$, $\sqsubseteq$ and $\leqslant$. We observed that the following relations hold:

$$\begin{array}{ccc} p \sqsubseteq q & \searrow & \\ & & p \preccurlyeq q \\ p \leqslant q & \nearrow & \end{array}$$

(see Lemma 3.2.1 and the definition of $\leqslant$). In 3.3.8 we showed that there are points $p, q \in \omega^*$ with $p \preccurlyeq q$ but p and q are $\leqslant$-incomparable. We begin by establishing a similar result for the order $\sqsubseteq$.

4.5.1. THEOREM. *There is a finite to one function $\pi : \omega \to \omega$ such that for all $x \in \omega^*$ there is a $\mathfrak{c}$-OK point $y \in \omega^*$ with $\bar{\pi}(y) = x$.*

PROOF. We will be brief. Let $\pi : \omega \to \omega$ be the finite to one function of Lemma

4.1.2 and let $\mathscr{A}=\{A^{\beta}_{\alpha n}:\alpha<\mathfrak{c},\ 1\leq n<\omega,\ \beta<\mathfrak{c}\}$ be the $\mathfrak{c}$ by $\mathfrak{c}$ independent linked family of clopen subsets of $\beta\omega$ w.r.t. $\langle\omega^*,\bar{\pi}\rangle$ given by Lemma 4.1.2. Take $x\in\omega^*$ arbitrarily. Since $\mathscr{A}$ is independent w.r.t. $\langle\omega^*,\bar{\pi}\rangle$, $\mathscr{A}$ is also an independent linked family w.r.t. $\langle\bar{f}^{-1}(x),g\rangle$, where $g:\omega^*\to\{0\}$ maps ω^* onto 0. By using precisely the same technique as in the proof of Theorem 4.1.4, we can construct a $\mathfrak{c}$-OK point $y\in\omega^*$ which belongs to $\bar{f}^{-1}(x)$. □

4.5.2. COROLLARY. *ω^* contains $2^{\mathfrak{c}}$ $\mathfrak{c}$-OK points.*

PROOF. This is clear since $|\omega^*|=2^{\mathfrak{c}}$, Lemma 31.2(c). □

4.5.3. COROLLARY. *There are points $p,q\in\omega^*$ with $p\leqslant q$, and consequently $p\lessdot q$, but p and q are $\sqsubseteq$-incomparable.*

PROOF. By Theorem 4.3.3, there is a $\mathfrak{c}$-OK point $p\in\omega^*$. An application of Theorem 4.5.1 gives us a $\mathfrak{c}$-OK point $q\in\omega^*$ with $p\leqslant q$. Since p and q are weak P-points (Corollary 4.3.2), p and q are obviously $\sqsubseteq$-incomparable. □

Since $p\lessdot q$ whenever $p\sqsubseteq q$, the question naturally arises whether $p\sqsubseteq q$ implies that $p\leqslant q$. We will show that this is not the case.

4.5.4. THEOREM. *There are points $p,q\in\omega^*$ with $p\sqsubseteq q$, and consequently $p\lessdot q$, but p and a are $\leqslant$-incomparable.*

PROOF. Let $\{C_n:n<\omega\}$ be a sequence of pairwise disjoint nonempty clopen subsets of ω^*. For each $n<\omega$, let $\{E^n_m:m<\omega\}$ be a sequence of pairwise disjoint nonempty clopen subspaces of C_n. In addition, let $p\in\omega^*$ be an arbitrarily chosen R-point (Lemma 3.3.4). Let $G=\{f\in\omega^\omega:f$ is finite to one$\}$. For all $f\in G$ put $A_f=\bar{f}^{-1}(\{p\})$. By Lemma 3.3.6, each A_f is an R-set of ω^*. By Theorem 3.3.5 for all $n,m<\omega$ we can pick a point $x^n_m\in E^n_m$ such that

$$\{x^n_m:n,m<\omega\}^-\cap\bigcup_{f\in G}A_f=\emptyset\,.$$

For each $n<\omega$, put $Z_n=\{x^n_m:n,m<\omega\}^-$. Observe that $Z_n\approx\beta\omega$ since ω^* is an F-space and that consequently $|Z_n|=2^{\mathfrak{c}}$, Lemma 3.1.2(c). For each $n<\omega$, let $\langle q^n_\alpha:\alpha<2^{\mathfrak{c}}\rangle$ enumerate Z_n. We choose the enumeration to be most economical, i.e. each point of Z_n occurs precisely once in the sequence $\langle q^n_\alpha:\alpha<2^{\mathfrak{c}}\rangle$. Observe that this implies that if $\alpha<\beta<2^{\mathfrak{c}}$, then

$$(*)\qquad\{q^n_\alpha:n<\omega\}^-\cap\{q^n_\beta:n<\omega\}^-=\emptyset$$

(use that $\bigcup_{n<\omega}C_n$ is C^*-embedded in ω^*, Theorem 1.5.2). For each $\alpha<2^{\mathfrak{c}}$, define

$g_\alpha:\omega\to\omega^*$ by

$$g_\alpha(n)=q^n_\alpha\,.$$

Then $(*)$ implies that the set $\{\beta g_\alpha(p):\alpha<2^{\mathfrak{c}}\}$ has cardinality $2^{\mathfrak{c}}$. Since $|\{x\in\omega^*:x\leqslant p\}|\leqslant\mathfrak{c}$, we can therefore find an $\alpha<2^{\mathfrak{c}}$ such that $q=\beta g_\alpha(p)\notin\{x\in\omega^*:x\leqslant p\}$. Then q is as required. □

Notes for Section 4

The notion of a κ-OK point is due to KUNEN [1978]. Theorem 4.1.5 for the special case $X=\beta\omega$, $Z=\omega^*$, $A=\omega^*$ and $Y=\{0\}$ is due to KUNEN [1978]. Theorem 4.1.5 is implicit in VAN MILL [1981b] and was subsequently partly generalized in VAN MILL [1982]. The ccc nowhere separable remainder of ω described in Section 4.2 is due to BELL [1981]. Interestingly, this compactification is also an important step in the proof of the main result of VAN MILL [1982]. Lemma 4.3.1 is due to KUNEN [1978]. That the set A of Theorem 4.3.3 is nonempty is due to Kunen and that the set B of Theorem 4.3.3 is nonempty is due to VAN MILL [1981b]. Corollary 4.3.4 is due to KUNEN [1978] and for generalizations see VAN MILL [1979a], [1981b], [1982] and DOW [1982]. Theorem 4.4.1 is due to VAN MILL [1981b] and Theorem 4.4.2 to POSPÍŠIL [1939]. The proof of Theorem 4.4.2 presented here was taken from KUNEN [1974]. Theorem 4.4.3 is due to BELL & KUNEN [1981]. Corollary 4.4.5 is due independently to BALCAR, SIMON & VOJTÁS [1981], KUNEN and SHELAH. Theorem 4.4.4 is due to VAN DOUWEN (unpublished), but the proof presented here is due to DOW & VAN MILL [1982]. Our proof of Theorem 4.4.4 differs from van Douwen's proof, but both proofs have in common that they are based on the technique of Balcar, Simon, Vojtás, Kunen and Shelah. All other results in this chapter are new.

We have seen that there are many 'special' points in ω^*. In VAN MILL [1981b] it is shown that there are at least 16 definable types in ω^*. Call a space *π-homogeneous* provided that all nonempty open subspaces have the same π-weight. Define

$A_1=\{x\in\omega^*:\exists$ countable discrete $D\subseteq\omega^*\backslash\{x\}$ with $x\in\bar{D}\}$.

$A_2=\{x\in\omega^*:\exists$ countable, dense in itself, π-homogeneous subset $D\subseteq\omega^*\backslash\{x\}$ of countable π-weight such that $x\in\bar{D}\}$,

$A_3=\{x\in\omega^*:\exists$ countable, dense in itself, π-homogeneous subset $D\subseteq\omega^*\backslash\{x\}$ of π-weight ω_1 such that $x\in\bar{D}\}$.

$A_4=\{x\in\omega^*:\exists$ locally compact, ccc, nowhere separable $D\subseteq\omega^*\backslash\{x\}$ with $x\in\bar{D}\}$.

By using similar ideas as developed in this section it can be shown that for all subsets $F \subseteq \{1, 2, 3, 4\}$ the set

$$\bigcap_{i\in F} A_i \setminus \bigcup_{i\notin F} A_i$$

is nonempty. This gives 16 definable types of points in ω^*. For details, see VAN MILL [1981b].

5. Remarks

The reader will undoubtedly have noticed that we did not discuss several important facts about $\beta\omega$. For example, we did not say anything about normality in $\beta\omega$. There are several simple proofs that for any $p \in \omega^*$, the spaces $\beta\omega\setminus\{p\}$ and $\omega^*\setminus\{p\}$ are not normal under CH. However, for years there has not been made significant progress in this area of $\beta\omega$. We don't know that $\omega^*\setminus\{p\}$ is not normal for any $p \in \omega^*$ without the help of some set theoretic hypothesis. It is known however, that for some $x \in \omega^*$ the space $\omega^*\setminus\{x\}$ is not normal. The best result of this type is, as far as I know, due to BLASZCZYK & SZYMAŃSKI [1980a]. They showed that if $x \in \omega^*$ is a limit point of some countable discrete subset of ω^*, then $\omega^*\setminus\{x\}$ is not normal.

What else is there to say about $\beta\omega$? Consider the following question: is $\beta\omega$ homeomorphic to $(\beta\omega)^2$? The answer is of course: NO! It is easy to see that $(\beta\omega)^2$ is not extremally disconnected. Make the question a little bit less trivial: is $(\beta\omega)^2$ homeomorphic to $(\beta\omega)^3$? This question is easy to state, but the answer to the question is not simple at all. VAN DOUWEN [1982] showed that $(\beta\omega)^n \approx (\beta\omega)^m$ iff $n = m$, for all $n, m \geq 1$. The list of interesting results about $\beta\omega$ seems endless.

Let X be a space which is dense in itself. Define

$$n(X) = \min\{\kappa : X \text{ can be covered by } \kappa \text{ nowhere dense sets}\}.$$

This number is called the Novák number of X. It is clear that if $n(\omega^*) > \mathfrak{c}$, then ω^* contains P-points. Therefore, in Shelah's model in which there are no P-points, $n(\omega^*) \leq \mathfrak{c}$. Observe that $n(\omega^*) \geq \omega_2$. The Novák number $n(\omega^*)$ can be almost anything you want, for details, see BALCAR, PELANT & SIMON [1980] and also HECHLER [1978].

In this paper we have restricted our attention to the space $\beta\omega$. The reader should however realize that many of the results we obtained can be generalized to higher cardinals with proofs that are essentially identical. For example the Rudin–Keiler order can without any problem be defined for higher cardinals and the proof we gave for the existence of $\leq$-incomparable points in $\beta\omega$ can be copied

to prove without extra difficulty that there are $\leq$-incomparable uniform ultrafilters on any infinite cardinal κ. However, there are also results that exclusively only work for ω. For example, the Rudin–Frolík order on ω^* cannot even be defined for higher cardinals.

Open problems

The following problems are unsolved as far as I know. It is recognized that a few of the problems listed below may be inadequately worded, be trivial or be known. Of many of the problems it is unknown who asked the problem. For that reason we do not credit anybody for posing a certain problem. The following mathematicians (with addresses listed in the AMS-MAA Combined Membership List) are sources of continuing information on many of the problems and their background: B. Balcar, W.W. Comfort, E.K. van Douwen, N. Hindman, K. Kunen, J. van Mill, M.E. Rudin and R.G. Woods.

1. Is ω^* homeomorphic to ω_1^*? (No if MA.)
2. Are there points $p, q \in \omega^*$ such that if $f: \omega \to \omega$ is any finite to one map, then $\beta f(p) \neq \beta f(q)$? (Yes if MA.)
3. Are $\omega^*\setminus\{p\}$ and $\beta\omega\setminus\{p\}$ nonnormal for any $p \in \omega^*$? (Yes if MA.)
4. Is there a model in which there are no P-points *and* no Q-points in ω^*?
5. Is there a model in which every point in ω^* is an R-point?
6. Is there a ccc closed P-set in ω^*? (Yes if CH.)
7. Let X be a compact space that can be mapped onto ω^*. Is X non-homogeneous? (Yes if X has weight at most $\mathfrak{c}$.)
8. Is the autohomeomorphism group of ω^* algebraically simple? (Yes is consistent.)
9. Is there an extremally disconnected, normal, locally compact space that is not paracompact? (Yes if MA + $\neg$CH or if there is a weakly compact cardinal.)
10. Is every first countable compactum a continuous image of ω^*?
11. Which spaces can be embedded in $\beta\omega$?
12. Is there a separable closed subspace of ω^* which is not a retract of $\beta\omega$? (Yes if CH.)
13. Let (*) denote the statement that every Parovičenko space is coabsolute with ω^*. Is (*) equivalent to CH? (It is known that $\mathfrak{c} < 2^{\omega_1}$ implies $\neg$(*).)

14. Let X be the Stone space of the reduced measure algebra of $[0, 1]$. Is it consistent that X is not a continuous image of ω^*?

15. Let (**) denote the statement that every compact zero-dimensional F-space of weight $\mathfrak{c}$ can be embedded in ω^*. Is (**) equivalent to CH? (It is known that CH implies (**) but $\mathrm{MA}+\mathfrak{c}=\omega_2$ implies $\neg$(**).)

16. Is it consistent that there is a compact basically disconnected space of weight $\mathfrak{c}$ that cannot be embedded in $\beta\omega$? (Such an example cannot be the Čech-Stone compactification of a P-space.)

17. Is there a $p\in\omega^*$ such that $\omega^*\backslash\{p\}$ is not C^*-embedded in ω^*? (Yes if CH.)

18. Assume MA. Are there $P_\mathfrak{c}$-points p, $q\in\omega^*$ which are not of the same type, i.e. for which $h(p)\neq q$ for any autohomeomorphism h of ω^*?

19. Is it true that for all $p\in\omega^*$ there is a $q\in\omega^*$ such that p and q are $\leqslant$-incomparable?

20. Is every subspace of ω^* strongly zero-dimensional?

21. Is there a point $p\in\omega^*$ such that every compactification of $\omega\cup\{p\}$ contains a copy of $\beta\omega$?

22. Is there a point $p\in\omega^*$ for which there is a compactification of $\omega\cup\{p\}$ that does not contain a copy of $\beta\omega$? (Yes if MA.)

23. Let D be any nowhere dense subset of ω^*. Is D a $\mathfrak{c}$-set, i.e. is there a disjoint family $\mathscr{A}$ of $\mathfrak{c}$ open sets in ω^* such that $D\subseteq\bar{A}$ for all $A\in\mathscr{A}$? (Yes if $|D|=1$.)

24. Is there a point $p\in\omega^*$ such that if $f\colon\omega\to\omega$ is any map, then either $\beta f(p)\in\omega$ or $\beta f(p)$ has character $\mathfrak{c}$ in $\beta\omega$? (Yes if MA.)

Remarks added in August 1982. Murray Bell has constructed a consistent example of a compact space X of weight $\mathfrak{c}$ which is first countable in all but one point and which in addition is not a continuous image of ω^*. This gives a partial answer to Question 10. Alan Dow showed that if $\mathrm{cf}(\mathfrak{c})=\omega_1$, then all Parovičenko spaces are coabsolute. This solves Question 13 in the negative.

Remark added in May 1983. Andrzej Szymański has recently constructed, under MA, a separable closed subspace of ω^* which is not a retract of $\beta\omega$ (this concerns Question 12).

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