HANDBOOK of SET-THEORETIC TOPOLOGY

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CHAPTER 11

# An Introduction to $\beta\omega$


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## Introduction

The aim of this paper is to give an introduction to the space $\beta\omega$, i.e. the Stone space of the Boolean algebra $\mathcal{P}(\omega)$ of subsets of $\omega$. There are several arguments in favour of writing such a paper. Firstly, in the last five years several important questions concerning the structure of $\beta\omega$ were solved. We have a good picture of $\beta\omega$ now. Secondly, results about $\beta\omega$ usually have wide applications in various parts of mathematics. The space $\beta\omega$ is an exciting place where topologists, set theorists, infinite combinatorists, Boolean algebraists, and sometimes even number theorists and analysts, meet.

Since this is a chapter in the Handbook of Set Theoretic Topology, I have written this paper from the perspective of a topologist. Our language is topological but at several places it was more natural to use Boolean algebras instead of their Stone spaces, so we freely did this. We mention our perspective at this early stage of the introduction since this gives the reader an idea about what types of results are to be expected in this paper. In addition, we do not aim to be complete. Several important results will not be proven in detail, or will not even be mentioned. For this reason we have called this paper "An introduction to $\beta\omega$". Also, we will not give lengthy historical comments giving proper credit to everybody, but we will usually only refer to the paper giving the final solution of the problem we are discussing.

It is probably true that the following facts are the most important results obtained in $\beta\omega$ in recent years:

(1) it is consistent that $P$-points do not exist in $\beta\omega\backslash\omega$ (Shelah; see MILLS [1978] or WIMMERS [1978]).

(2) some but not all points in $\beta\omega\backslash\omega$ are weak $P$-points (KUNEN [1978]),

(3) every point in $\beta\omega\backslash\omega$ is a $c$-point (BALCAR & VOJTÁŠ [1980]).

(1) will interest set theorists most, (2) fascinates topologists and (3) is connected with and important in Boolean algebras as well as topology. Due to our perspective, we will discuss (2), but we leave (1) and (3) untouched.

The space $\beta\omega$ is a monster having three heads. If one works in a model in which the Continuum Hypothesis (abbreviated CH) holds, then one will see only the first head. This head is smiling, friendly, and makes you feel comfortable working with $\beta\omega$. I do not know many open problems on $\beta\omega$ the answers of which are unknown under CH. In fact, one usually does not work with $\beta\omega$ or with $\beta\omega\backslash\omega$, but with a Boolean algebra satisfying a certain completeness property which characterizes the Boolean algebra $\mathcal{P}(\omega)$/fin under CH. This is the theme in Section 1. Here we discuss the spaces $\beta\omega$ and $\beta\omega\backslash\omega$ under CH. We begin by identifying the completeness property which characterizes $\mathcal{P}(\omega)$/fin and then work in Boolean algebras satisfying this completeness property. Because of the presence of the CH, transfinite inductions have length $\omega_1$ and because of the special properties of the Boolean algebras under consideration, we can always continue the transfinite inductions until stage $\omega_1$. The reader should observe that nowhere in Section 1 do we use the special structure of $\beta\omega$, with the exception, of course, of the completeness property of $\mathcal{P}(\omega)$/fin. If one works in a model in which CH does not hold, then one will see the second head of $\beta\omega$. This head constantly tries to confuse you and you will never be able to decide whether it speaks the truth. This head of $\beta\omega$ will be discussed in Section 2. It turns out that all but three of the CH results derived in Section 1 are consistently false. After reading the first two sections, the reader might feel that $\beta\omega$ is a horrible creature since it seems that all statements about it depend on special set theoretic assumptions. What can there 'really' (=in ZFC) be said about $\beta\omega$? The answer to this question is: quite a bit. The third head of $\beta\omega$ is its head in ZFC. Because of the first two heads, this head is rather vague, but some parts of it are very clear. If one wants to see the clear part, one will have to work like a slave, inventing ingeneous combinatorial arguments. One will have to use special properties of $\beta\omega$ and not only global properties. Some ZFC results on $\beta\omega$ are discussed in Sections 3 and 4.

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## 0. Preliminaries

In order to be able to understand the arguments in this paper, one should know some elementary facts about Boolean algebras and Čech–Stone compactifications. All one needs to know can be found in Comfort & Negrepontis [1974, §2]. A Boolean algebra is usually denoted by $\mathcal{B}$, its universal bounds by 0 and 1, concepts such as homomorphism, embedding, isomorphism, etc., should be familiar. Cardinals are initial (von Neumann) ordinals, and get the discrete topology. If $\alpha$ is an ordinal, then $W(\alpha)$ denotes the topological space with underlying set $\alpha$ equipped with the order topology. What should one know about Čech–Stone compactifications? Well, one should know that $\beta X$ is the unique compactification of the (completely regular Hausdorff) space $X$ with the property that disjoint zero-sets in $X$ have disjoint closures. This easily implies that given a map $f: X \to K$, where $K$ is compact, there exists a unique map $\beta f: \beta X \to K$ extending $f$. This map is called the *Stone extension* of $f$. I often hear the remark that $\beta\omega$ is clear, since it is the Stone space of $\mathcal{P}(\omega)$, but $\beta X$, for arbitrary $X$, is not clear, partly because it is not the Stone space of a Boolean algebra. For this reason in this paper we almost exclusively work with *strongly zero-dimensional* spaces, i.e. those spaces $X$ for which $\beta X$ is zero-dimensional, or equivalently, those spaces $X$ for which $\beta X$ is equivalent to the Stone space of the Boolean algebra $\mathcal{B}(X)$ consisting of all clopen (=both closed and open) subsets of X. Observe that in this case the existence of the Stone extension $\beta f$ discussed above is clear, since the existence of $f$ implies that $\mathcal{B}(K)$ can be embedded in $\mathcal{B}(X)$. *Henceforth, all topological spaces under discussion are assumed to be completely regular and Hausdorff.* The *Stone space* of a Boolean algebra $\mathcal{B}$ is denoted by st($\mathcal{B}$). Recall that a subset $U$ of a space $X$ is called *regular open* provided that $U = \text{int}_X \text{cl}_X U$. Let $\text{RO}(X) = \{U \subseteq X\colon U \text{ is regular open}\}$. Then RO($X$) becomes a complete Boolean algebra under the following operations:

$$U \wedge V = U \cap V,$$
$$U \vee V = \text{int}_X \text{cl}_X(U \cup V),$$
$$U' = \text{int}_X(X \setminus U).$$

If $X$ is compact, then the Stone space of RO($X$) will be denoted by $EX$. Since RO($X$) is complete, $EX$ is *extremally disconnected* (=closure of an open set is open). It is easily seen that topologically, $EX$ is characterized as follows: $EX$ is the unique extremally disconnected space which admits an *irreducible* (a continuous surjection $f: S \to T$ is called irreducible provided that $f(A) \neq T$ for all closed $A \subseteq S$ with $A \neq S$) perfect map $\pi: EX \to X$. The space $EX$ is called the *projective cover* of $X$. For a recent survey on projective covers, see Woods [1979]. A space $X$ is called *basically disconnected* if the closure of each open $F_\sigma$ is again open. Observe that, trivially, each extremally disconnected space is basically disconnected, but not conversely. As usual, fin denotes the ideal of finite subsets of $\omega$, and $\mathcal{P}(\omega)$/fin is the Boolean algebra we obtain from $\mathcal{P}(\omega)$ by calling $A$, $B \in \mathcal{P}(\omega)$ equivalent iff $A \,\Delta\, B \in \text{fin}$ ($A \,\Delta\, B = (A \setminus B) \cup (B \setminus A)$). As remarked above, $\beta\omega$ denotes st($\mathcal{P}(\omega)$). If $n > \omega$, then we identify $n$ with the point

$$\{A \in \mathcal{P}(\omega) : n \in A\}$$

from st($\mathcal{P}(\omega)$). Points from $\beta\omega \setminus \omega$ are called *free ultrafilters*. Obviously, $\beta\omega \setminus \omega \approx$ st($\mathcal{P}(\omega)$/fin). If $A \subseteq \omega$, we put

$$A^* = \{x \in \beta\omega \setminus \omega : A \in x\}.$$

It is clear that the collection $\{A^* : A \in \mathcal{P}(\omega)\}$ is a base for $\beta\omega \setminus \omega$. Also observe that $\beta\omega \setminus \omega = \omega^*$.

**0.1.** Lemma. (a) *If $V \subseteq \omega$ is infinite, then $\bar{V}$ is homeomorphic to $\beta\omega$.*
(b) *If $V$, $W \subseteq \omega$ are infinite, then $V^* \cap W^* = \emptyset$ iff $|V \cap W| < \omega$.*

Proof. (a) Is clear since $\mathcal{P}(V)$ is isomorphic to $\mathcal{P}(\omega)$. We leave the proof of (b) as an exercise to the reader. □

A point $x$ of a space $X$ is called a *P-point* if the intersection of countably many neighborhoods of $x$ is again a neighborhood of $x$.

Whenever $X$ is a set and $\kappa$ is a cardinal we define (as usual)

$$[X]^\kappa = \{A \subseteq X : |A| = \kappa\},$$
$$[X]^{\leq\kappa} = \{A \subseteq X : |A| \leq \kappa\},$$
$$[X]^{<\kappa} = \{A \subseteq X : |A| < \kappa\},$$

respectively. We also let $\subset$ denote *proper* inclusion.

If $X$ is a space, then $X^*$ denotes $\beta X\backslash X$ and if $U\subseteq X$ is open, then

$$\mathrm{Ex}(U)=\beta X\backslash \mathrm{cl}_{\beta X}(X\backslash U).$$

Observe that $\mathrm{Ex}(U)$ is open and that $\mathrm{Ex}(U)\cap X=U$. The reader can easily verify that the collection

$$\{\mathrm{Ex}(U): U\subseteq X \text{ is open}\}$$

is a base for the topology of $\beta X$. If $U\subseteq X$ is open, let

$$U'=\mathrm{Ex}(U)\cap X^*.$$

If $X$ is normal and $Y$ is closed in $X$, then $\mathrm{cl}_{\beta X} Y=\beta Y$. We identify $\mathrm{cl}_{\beta X} Y\backslash Y$ and $Y^*$ in this case. For definitions such as character, $\pi$-weight, cellularity etc., see JUHÁSZ [1980], or HODEL [1983]. By "$X$ is ccc" we mean that $X$ satisfies the countable chain condition. We say that a family of sets $\mathscr{F}$ has the *n-intersection property* ($n<\omega$) provided that $\bigcap\mathscr{G}\neq\emptyset$ for all $\mathscr{G}\in[\mathscr{F}]^{\leq n}$. To indicate that two spaces $X$ and $Y$ are homeomorphic, we write $X\approx Y$.

A *zero-set* of a space $X$ is any set of the form $f^{-1}(\{0\})$, where $f: X\to I$ is continuous. A *cozero-set* is the complement of a zero-set.

Let $\alpha$ and $\kappa$ be cardinals. We define, as usual,

$$\alpha^{\underline{\kappa}}=\sum\{\alpha^\lambda:(\lambda \text{ is a cardinal and}) \ \lambda<\kappa\}.$$

## 1. The spaces $\beta\omega$ and $\beta\omega\backslash\omega$ under CH

In this section we will see how $\beta\omega$ and $\omega^*$ behave under CH.

### *1.1. A characterization of $\mathscr{P}(\omega)$/fin*

Let $\mathscr{B}$ be a Boolean algebra and let $F, G\subseteq\mathscr{B}$. We say that $F<G$ provided that for all $F'\in[F]^{<\omega}$, $G'\in[G]^{<\omega}$ we have that $\vee F'<\wedge G'$.

**1.1.1.** DEFINITION. Let $\mathscr{B}$ be a Boolean algebra. We say that $\mathscr{B}$ satisfies *condition* $H_\omega$ provided that for all $F\in[\mathscr{B}\backslash\{1\}]^{\leq\omega}$ and $G\in[\mathscr{B}\backslash\{0\}]^{\leq\omega}$ such that $F<G$ there is an element $x\in\mathscr{B}$ such that $F<\{x\}<G$.

**1.1.2.** LEMMA. *$\mathscr{P}(\omega)$/fin satisfies condition $H_\omega$.*

PROOF. We will begin by proving the following assertion: if $A\in[\mathscr{P}(\omega)/\mathrm{fin}]^{\leq\omega}$ and $\{0\}<A$, then there is a $y\in\mathscr{P}(\omega)$/fin such that $\{0\}<\{y\}<A$. Indeed, enumerate $A$



as $\{a_n:n<\omega\}$, and for all $n<\omega$, let $C_n\in[\omega]^\omega$ be a representative of $a_n$. By induction, pick points $y_n$ for all $n<\omega$, such that

$$y_n\in\bigcap_{0\leq i\leq n} C_i\backslash\{y_0,y_1,\ldots,y_{n-1}\}$$

Let $Y=\{y_n:n<\omega\}$ and let $y$ be the element of $\mathscr{P}(\omega)$/fin corresponding to $y$. It is a good exercise to show that $\{0\}<\{y\}<A$.

Now let us return to the proof of the lemma. Suppose that $F, G\in[\mathscr{P}(\omega)/\mathrm{fin}]^{\leq\omega}$, $1\notin F$, $0\notin G$ and $F<G$. If $\vee F$ or $\wedge G$ exists, then using the above assertion, it is easy to find the required $x$. So assume that this is not the case. Enumerate $F$ as $\{f_n:n<\omega\}$ and $G$ as $\{g_n:n<\omega\}$. It is clear that without loss of generality we may assume that $f_0<f_1<\cdots$ and $g_0>g_1>\cdots$. For each $n<\omega$ take representatives $A_n$, $B_n\in[\omega]^\omega$ of $f_n$, resp. $g_n$. By induction on $k<\omega$, pick a point $d_k<\omega$ such that

$$(1)\quad d_k\in\bigcap_{0\leq i\leq k} B_i\backslash\Big(\bigcup_{0\leq i\leq k} A_i\cup\{d_0,\ldots,d_{k-1}\}\Big)$$

and put $D=\{d_k:k<\omega\}$. In addition, define

$$A'=\bigcup_{k<\omega}\Big(A_k\cap\bigcap_{0\leq i\leq k} B_i\Big).$$

Put $C=A'\cup D$. Then $C\in[\omega]^\omega$ while moreover

(2) if $n<\omega$, then $|A_n\backslash C|<\omega$, and

(3) if $m<\omega$, then $|C\backslash B_m|<\omega$.

Let $x$ be the element of $\mathscr{P}(\omega)$/fin corresponding to $C$. It is easy to see that (2) and (3) imply that $F<\{x\}<G$. □

**1.1.3.** DEFINITION. Let $\mathscr{B}$ be a Boolean algebra. We say that $\mathscr{B}$ satisfies *condition* $R_\omega$ provided that for all nonempty $F\in[\mathscr{B}\backslash\{1\}]^{\leq\omega}$, $G\in[\mathscr{B}\backslash\{0\}]^{\leq\omega}$ and $H\in[\mathscr{B}]^{\leq\omega}$ such that

(1) $F<G$, and

(2) $\forall\tilde F\in[F]^{<\omega}\forall\tilde G\in[G]^{<\omega}\forall h\in H: h\not\leq\vee\tilde F$ and $\wedge\tilde G\not\leq h$,

there is an element $x\in\mathscr{B}$ such that

(3) $F<\{x\}<G$, and

(4) $\forall h\in H: h\not\leq x$ and $x\not\leq h$.

The main reason that $\omega^*$ is relatively easy to deal with under CH is, as we will see later, because of the following lemma.

**1.1.4.** LEMMA. *If a Boolean algebra $\mathscr{B}$ satisfies condition $H_\omega$, then it satisfies condition $R_\omega$.*

PROOF. Let $F \in [\mathscr{B}\backslash\{1\}]^{<\omega}$, $G \in [\mathscr{B}\backslash\{0\}]^{<\omega}$ and $H \in [\mathscr{B}]^{<\omega}$ be as in 1.1.3 (1) and (2). Enumerate $F$ as $\{f_n : n < \omega\}$, $G$ as $\{g_n : n < \omega\}$ and $H$ as $\{h_n : n < \omega\}$. For each $h \in H$ and finite $\tilde{F} \in [F]^{<\omega}$ we have that $(\vee \tilde{F})' \wedge h \neq 0$, consequently there exists, by applying condition $H_\omega$ for all $n < \omega$, an element $d_n \in \mathscr{B}\backslash\{0\}$ such that

(1) $d_n < h_n$ and $\forall f \in F, f \wedge d_n = 0$.

Similarly, we can find $e_n \in \mathscr{B}\backslash\{0\}$ such that

(2) $\{e_n\} < G$ and $e_n \wedge h_n = 0$.

If the $d_n$'s and $e_n$'s are chosen with a little extra care, we can assure that $e_n \wedge d_m = 0$ for all $n$, $m < \omega$. Now define for all $n < \omega$,

$$\tilde{f}_n = f_n \vee e_n, \quad \text{and} \quad \tilde{g}_n = g_n \wedge d_n'.$$

Notice if $n$, $m < \omega$ then $\vee_{0 \leq i \leq n} \tilde{f}_i \leq \wedge_{0 \leq j \leq m} \tilde{g}_j$. By $H_\omega$, we can therefore find an element $x \in \mathscr{B}$ such that for all $n$, $m < \omega$,

$$\vee_{0 \leq i \leq n} \tilde{f}_i \leq x \leq \wedge_{0 \leq j \leq m} \tilde{g}_j.$$

An easy check shows that $x$ is as required. □

**1.1.5.** COROLLARY. *$\mathscr{P}(\omega)$/fin satisfies condition $R_\omega$.*

We now come to the main result of this section. The proof we give is slightly incomplete. The reader is encouraged to fill in all missing details (in case of problems, see COMFORT & NEGREPONTIS [1974, Lemma 6.10]). If $\mathscr{B}$ is a Boolean algebra (abbreviated: BA) and if $A \subseteq \mathscr{B}$, then $\langle\langle A\rangle\rangle \subseteq \mathscr{B}$ denotes the subalgebra of $\mathscr{B}$ generated by $A$.

**1.1.6.** THEOREM (CH). *If $\mathscr{B}$ is a Boolean algebra of cardinality at most $\mathfrak{c}$ satisfying condition $H_\omega$, then $\mathscr{B}$ is isomorphic to $\mathscr{P}(\omega)$/fin.*

PROOF. Let $\mathscr{B}$ and $\mathscr{E}$ be BA's satisfying condition $H_\omega$ such that $|\mathscr{B}|$, $|\mathscr{E}| \leq \mathfrak{c}$. By CH list $\mathscr{B}$ as $\{b_\alpha : \alpha < \omega_1\}$ and $\mathscr{E}$ as $\{e_\alpha : \alpha < \omega_1\}$.

Without loss of generality we may assume that $e_0 = 0$ and $b_0 = 0$. By transfinite induction, for $\alpha < \omega_1$ we will construct countable subalgebras $\mathscr{B}_\alpha \subseteq \mathscr{B}$ and $\mathscr{E}_\alpha \subseteq \mathscr{E}$ and an isomorphism $\sigma_\alpha : \mathscr{B}_\alpha \to \mathscr{E}_\alpha$ such that

(1) $b_\alpha \in \mathscr{B}_\alpha$ and $e_\alpha \in \mathscr{E}_\alpha$,

(2) if $\beta < \alpha$, then $\mathscr{B}_\beta \subseteq \mathscr{B}_\alpha$, $\mathscr{E}_\beta \subseteq \mathscr{E}_\alpha$ and $\sigma_\alpha \restriction \mathscr{B}_\beta = \sigma_\beta$.

Let $\mathscr{B}_0 = \{0, 1\}$ and $\mathscr{E}_0 = \{0, 1\}$ and let $\sigma_0 : \mathscr{B}_0 \to \mathscr{E}_0$ be defined in the obvious way. Suppose that $\mathscr{B}_\beta$, $\mathscr{E}_\beta$ and $\sigma_\beta$ are defined for all $\beta < \alpha < \omega_1$ satisfying (1) and (2). If $b_\alpha \in \bigcup_{\beta<\alpha} \mathscr{B}_\beta$ and $e_\alpha \in \bigcup_{\beta<\alpha} \mathscr{E}_\beta$, then define $\mathscr{B}_\alpha = \bigcup_{\beta<\alpha} \mathscr{B}_\beta$, $\mathscr{E}_\alpha = \bigcup_{\beta<\alpha} \mathscr{E}_\beta$ and



$\sigma_\alpha = \bigcup_{\beta<\alpha} \sigma_\beta$. Suppose next that e.g. $b_\alpha \notin \bigcup_{\beta<\alpha} \mathscr{B}_\beta = \mathscr{F}$. Let $\sigma = \bigcup_{\beta<\alpha} \sigma_\beta$. Put

$$\mathscr{F}_0 = \{f \in \mathscr{F} : f < b_\alpha\}, \quad \mathscr{F}_1 = \{f \in \mathscr{F} : b_\alpha < f\}, \quad \text{and} \quad \mathscr{F}_2 = \mathscr{F}\backslash(\mathscr{F}_0 \cup \mathscr{F}_1).$$

By Lemma 1.1.4 there is an element $e \in \mathscr{E}$ such that $\sigma(\mathscr{F}_0) < \{e\}$, $\{e\} < \sigma(\mathscr{F}_1)$, and for all $\tilde{e} \in \sigma(\mathscr{F}_2)$, $\tilde{e} \not\leq e$ and $e \not\leq \tilde{e}$. If we put $\sigma(b_\alpha) = e$ and $\sigma(b_\alpha') = e'$, then $\sigma$ can be extended to an isomorphism $\tilde{\sigma} : \langle\langle \mathscr{F} \cup \{b_\alpha\}\rangle\rangle \to \langle\langle \sigma(\mathscr{F}) \cup \{e\}\rangle\rangle$. If $e_\alpha \notin \langle\langle \sigma(\mathscr{F}) \cup \{e\}\rangle\rangle$, then we do the same thing as above with $\sigma$ replaced by $\sigma^{-1}$. This shows how to construct $\mathscr{B}_\alpha$ and $\mathscr{E}_\alpha$.

We conclude that $\mathscr{B}$ and $\mathscr{E}$ are isomorphic and by Corollary 1.1.5, this also shows that both $\mathscr{B}$ and $\mathscr{E}$ are isomorphic to $\mathscr{P}(\omega)$/fin. □

**1.1.7.** REMARK. Observe that each BA satisfying condition $H_\omega$ has cardinality at least $\mathfrak{c}$.

### *1.2. A topological translation*

In this section we will translate the results of Section 1.1 in topological language.

Let $X$ be a space. A subset $A \subseteq X$ is called *$C^*$-embedded* in $X$ provided that each map $f : A \to [0, 1]$ can be extended to a map $f : X \to [0, 1]$.

**1.2.1.** DEFINITION. A space $X$ is called an *F-space* if each cozero-set in $X$ is $C^*$-embedded in $X$.

The following lemma summarizes some relevant information on $F$-spaces.

**1.2.2.** LEMMA. (a) *$X$ is an F-space iff $\beta X$ is F-space.*

(b) *A normal space $X$ is an F-space iff any two disjoint open $F_\sigma$ subsets of $X$ have disjoint closures in $X$.*

(c) *Each basically disconnected space is an F-space.*

(d) *Any closed subspace of a normal F-space is again an F-space.*

(e) *If an F-space $X$ satisfies the countable chain condition, then it is extremally disconnected.*

PROOF. For (a), use that $X$ is $C^*$-embedded in $\beta X$. The proof of (b) is routine and (c) is trivial. The proof of (d) is easy if one uses the characterization of normal $F$-spaces stated in (b). For (e), first observe that it suffices to show that disjoint open subsets of $X$ have disjoint closures. Let $U$, $V \subseteq X$ be open and disjoint. Use the fact that $X$ is ccc to find dense cozero-sets $U' \subseteq U$ and $V' \subseteq V$. The function $f : U' \cup V' \to [0, 1]$ defined by $f(x) = 0$ if $x \in U'$ and $f(x) = 1$ if $x \in V'$ can be extended to a map $\bar{f} : X \to [0, 1]$. Since $U \subseteq \bar{f}^{-1}(\{0\})$ and $V \subseteq \bar{f}^{-1}(\{1\})$, we conclude that $\bar{U} \cap \bar{V} = \emptyset$. □

The following result gives a topological translation of condition $H_\omega$.

**1.2.3.** Lemma. *Let $X$ be a compact zero-dimensional space. The following statements are equivalent:*

(1) *$\mathscr{B}(X)$ satisfies condition $H_\omega$,*
(2) *$X$ is an $F$-space and each nonempty $G_\delta$ in $X$ has infinite interior.*

Proof. (1)$\Rightarrow$(2) follows directly from Lemma 1.2.2(b) and the fact that in a compact zero-dimensional space every open $F_\sigma$ is a countable union of clopen sets. That (2) implies (1) is routine. □

**1.2.4.** Corollary (CH). *Let $X$ be a space. The following statements are equivalent:*

(1) *$X \approx \omega^*$,*
(2) *$X$ is a compact zero-dimensional $F$-space of weight $\mathfrak{c}$ in which each nonempty $G_\delta$ has infinite interior.*

Proof. Follows directly from Theorem 1.1.6 and Lemma 1.2.3. □

A compact zero-dimensional $F$-space of weight $\mathfrak{c}$ in which each non-empty $G_\delta$ has infinite interior, will be called a *Parovičenko space* from now on. Corollary 1.2.4 says that, under CH, $\omega^*$ is, up to homeomorphism, the only Parovičenko space.

Corollary 1.2.4 is a very useful result since it turns out that the class of Parovičenko spaces is quite large. The following result, which is of independent interest, is the key in finding more Parovičenko spaces.

**1.2.5.** Theorem. *Let $X$ be a locally compact, $\sigma$-compact and noncompact space. Then $X^*$ is an $F$-space and each nonempty $G_\delta$ in $X^*$ has infinite interior.*

Proof. Let $F \subseteq X^*$ be any $F_\sigma$ and let $f: F \to [0,1]$ be continuous. Since $Y = X \cup F$ is $\sigma$-compact, it is normal and therefore, since $F$ is closed in $Y$, the Tietze Extension Theorem implies that $f$ can be extended to a map $\bar{f}: Y \to [0,1]$. Put $g = \bar{f} \restriction X$. Then $g$ can be extended to a map $\bar{g}: \beta X \to [0,1]$. Since clearly $\bar{g} \restriction F = f$, we see that $\bar{f} = \bar{g} \restriction X^*$ is the required extension of $f$.

Let $S \subseteq X^*$ be a nonempty $G_\delta$. Since the set $\{U' : U \text{ open in } X\}$ is a basis for $X^*$, it is clear that we can find open sets $U_n \subseteq X$ for all $n < \omega$, such that

$$\bar{U}_{n+1} \subseteq U_n \quad \text{and} \quad \emptyset \neq \bigcap_{n<\omega} U'_n \subseteq S$$

(since $U_n \subseteq X$ for all $n$, the bar means closure in $X$ of course). Since $X$ is locally compact and $\sigma$-compact, we can write $X$ as $\bigcup_{n<\omega} K_n$, where each $K_n$ is compact and moreover each compact $K \subseteq X$ is contained in some $K_n$. For each $n < \omega$ choose a nonempty open set $V_n \subseteq U_n$ such that

$$\bar{V}_n \text{ is compact and misses } K_n .$$



Put $V = \bigcup_{n<\omega} V_n$. If $n < \omega$, then $V \backslash U_n$ has compact closure in $X$, whence

$$V' \subseteq \bigcap_{n<\omega} U'_n \subseteq S .$$

In addition, $V' \neq \emptyset$ since $V$ does not have compact closure in $X$. The easy proof that $V'$ is infinite is left to the reader. □

We now present an interesting topological consequence of Theorem 1.1.6.

**1.2.6.** Theorem (CH). *Let $X$ be a zero-dimensional, locally compact, $\sigma$-compact, noncompact space of weight at most $\mathfrak{c}$. Then $X^*$ and $\omega^*$ are homeomorphic.*

Proof. Since $X$ is a zero-dimensional Lindelöf space, $X$ is strongly zero-dimensional and has at most $\mathfrak{c}^\omega = \mathfrak{c}$ clopen sets. It follows that $X^*$ is a zero-dimensional compact space of weight at most $\mathfrak{c}$. By Theorem 1.2.5 and Lemma 1.2.3 $\mathscr{B}(X^*)$ satisfies condition $H_\omega$. This implies that $\mathscr{B}(X^*)$ and $\mathscr{P}(\omega)/\text{fin}$ are isomorphic (Theorem 1.1.6), and consequently, by Stone duality, that $X^*$ and $\omega^*$ are homeomorphic. □

## 1.3. Continuous images of $\omega^*$

In this section we characterize the continuous images of $\omega^*$.

**1.3.1.** Theorem. *Let $\mathscr{B}$ be a Boolean algebra of cardinality at most $\omega_1$. Then $\mathscr{B}$ can be embedded in $\mathscr{P}(\omega)/\text{fin}$.*

Proof. Use the same technique as in the proof of Theorem 1.1.6. □

By Stone duality, Theorem 1.3.1 is equivalent to the statement that each compact and zero-dimensional space of weight at most $\omega_1$ is a continuous image of $\omega^*$. This result suggests the question whether the same result holds without the assumption on zero-dimensionality. This is indeed the case, see Theorem 1.3.3 below.

**1.3.2.** Lemma. *Let $X$ be a compact space of weight $\kappa$. Then there is a compact zero-dimensional space $Y$ of weight $\kappa$ which can be mapped onto $X$.*

Proof. Let $\mathscr{B} \in [\text{RO}(X)]^\kappa$ be such that $\mathscr{B}$ is a basis and put $\mathscr{E} = \langle\langle \mathscr{B} \rangle\rangle \subseteq \text{RO}(X)$. Observe that $|\mathscr{E}| = \kappa$. Let $Y$ be the Stone space of $\mathscr{E}$. □

**1.3.3.** Theorem. *Each compact space of weight at most $\omega_1$ is a continuous image of $\omega^*$.*

PROOF. Let $X$ be a compact space of weight $\omega_1$ and let $Y$ be as in Lemma 1.3.2. By Theorem 1.3.1, $\mathcal{B}(Y)$ embeds in $\mathcal{P}(\omega)$/fin. Consequently, by Stone duality, $\omega^*$ can be mapped onto $Y$. □

1.3.4. COROLLARY (CH). *Each compact space of weight at most $\mathfrak{c}$ is a continuous image of $\omega^*$.*

### 1.4. Closed subspaces of $\beta\omega$

In this section we characterize topologically the closed subspaces of $\beta\omega$. If $X$ is a closed subspace of $\beta\omega$, then $X$ must clearly be of weight at most $\mathfrak{c}$ and $X$ must be a zero-dimensional compact $F$-space by Lemma 1.2.2(d). It turns out that, under CH, these conditions are not only necessary but also sufficient.

Let $X$ be a space. A subset of $B \subseteq X$ is called a *$P$-set* provided that the intersection of countably many neighborhoods of $B$ is again a neighborhood of $B$.

Let $X$ and $Y$ be compact spaces. Let $A \subseteq X$ be closed and let $f: A \to Y$ be a continuous surjection. It is easily seen that the collection

$$\mathcal{B} = \{f^{-1}(y): y \in Y\} \cup \{\{x\}: x \in X \setminus A\}$$

is an upper-semicontinuous decomposition of $X$; the decomposition space $X/\mathcal{B}$ will be denoted by $X \cup_f Y$. If $\pi: X \to X \cup_f Y$ is the decomposition map, then we identify $Y$ and $\pi(A)$.

1.4.1. LEMMA. *Let $X$ and $Y$ be compact $F$-spaces, let $A \subseteq X$ be a closed $P$-set, and let $f: A \to Y$ be a continuous surjection. Then $X \cup_f Y$ is an $F$-space.*

PROOF. Let $U$ and $V$ be disjoint open $F_\sigma$ subsets of $X \cup_f Y$. Since $Y$ is an $F$-space, $(U \cap Y)^- \cap (V \cap Y)^- = \emptyset$. Let $E$ and $F$ be closed $G_\delta$ neighborhoods of $(U \cap Y)^-$ and $(V \cap Y)^-$ such that $E \cap F = \emptyset$. Then $U \setminus E$ and $V \setminus F$ are disjoint open $F_\sigma$ subsets of $X$ which both do not meet $A$. Since $X$ is an $F$-space and $A$ is a $P$-set,

(1) $(U \setminus E)^- \cap (V \setminus F)^- = \emptyset$, and

(2) $((U \setminus E)^- \cup (V \setminus F)^-) \cap A = \emptyset$,

This easily implies that $\bar{U} \cap \bar{V} = \emptyset$.

By Lemma 1.2.2(b), we may now conclude that $X \cup_f Y$ is an $F$-space. □

1.4.2. LEMMA. *Let $X$ be a compact space with the property that each nonempty $G_\delta$ has infinite interior. If $A \subseteq X$ is closed and nowhere dense and if $f: A \to Y$ is a continuous surjection, then the space $X \cup_f Y$ has also the property that each nonempty $G_\delta$ has infinite interior.*

PROOF. Obvious. □



1.4.3. LEMMA (CH). *$\omega^*$ contains a nowhere dense closed $P$-set $A$ which is homeomorphic to $\omega^*$.*

PROOF. By Theorem 1.2.6, we can represent $\omega^*$ by

$$Z = (\omega \times W(\omega_1 + 1))^* .$$

Let $A = (\omega \times \{\omega_1\})^*$. Trivially, $A \approx \omega^*$ and that $A$ is a $P$-set follows easily from the fact that $\omega_1$ is a $P$-point in $W(\omega_1 + 1)$. That $A$ is nowhere dense is clear. □

We now come to the main result of this section.

1.4.4. THEOREM (CH). *Let $X$ be a space. The following statements are equivalent:*

(1) *$X$ is a compact zero-dimensional $F$-space of weight at most $\mathfrak{c}$,*

(2) *$X$ can be embedded in $\beta\omega$ as a closed subspace,*

(3) *$X$ can be embedded as a nowhere dense closed $P$-set in $\omega^*$.*

PROOF. The implications (2)⇒(1) and (3)⇒(2) are trivial, so it suffices to prove that (1)⇒(3). To this end, let $X$ be a compact zero-dimensional $F$-space of weight at most $\mathfrak{c}$. By Lemma 1.4.3 we can find a closed nowhere dense $P$-set $A$ of $\omega^*$ such that $A \approx \omega^*$. In addition, by Corollary 1.3.4, there is a continuous surjection $f: A \to X$. Put $Z = \omega^* \cup_f X$. It is routine to verify that

(a) $Z$ is zero-dimensional,

(b) $Z$ is of weight $\mathfrak{c}$,

(c) $X$ is a nowhere dense closed $P$-set of $Z$.

Lemma 1.4.1 followed by Lemma 1.4.2 imply that $Z$ is a compact $F$-space in which each nonempty $G_\delta$ has infinite interior. Consequently, by Corollary 1.2.4, $Z \approx \omega^*$. □

Let $\mathcal{B}$ be a Boolean algebra. We say that $\mathcal{B}$ is *weakly countably complete* (abbreviated: WCC) iff the Stone space of $\mathcal{B}$ is an $F$-space. In Boolean algebraic language, $\mathcal{B}$ is a WCC BA iff $\mathcal{B}$ is a BA and

$$\forall B, C \in [\mathcal{B}]^{\leq\omega} \text{ such that } \forall b \in B\ \forall c \in C: b \wedge c = 0,$$

$$\text{there is an } a \in \mathcal{B} \text{ with } b \leq a \leq c' \text{ for all } b \in B,\ c \in C.$$

The following result is a purely Boolean algebraic consequence of Theorem 1.4.4.

1.4.5. THEOREM (CH). *Let $\mathcal{B}$ be a Boolean algebra. The following statements are equivalent:*

(1) *$\mathcal{B}$ is WCC and $|\mathcal{B}| \leq \mathfrak{c}$,*

(2) *$\mathcal{B}$ is a homomorphic image of $\mathcal{P}(\omega)$.*

1.4.6. COROLLARY (CH). *Each WCC Boolean algebra of cardinality at most $\mathfrak{c}$ is a homomorphic image of a complete Boolean algebra.*

We will now prove an interesting result without the aid of the CH.

1.4.7. THEOREM. *Let $X$ be a compact extremally disconnected space of weight at most $\mathfrak{c}$. Then $X$ can be embedded in $\beta\omega$.*

PROOF. We may assume that $X \subseteq I^{\mathfrak{c}}$, where as usual, $I = [0,1]$. Since $I^{\mathfrak{c}}$ is separable, ENGELKING [1977, 2.3.16], there is a continuous surjection $f: \beta\omega \to I^{\mathfrak{c}}$. Let $g = f \restriction f^{-1}(X)$ and take a closed $Z \subseteq f^{-1}(X)$ such that $g \restriction Z: Z \to X$ is irreducible. The existence of $Z$ easily follows from Zorn's Lemma (order all closed sets of $f^{-1}(X)$ that map onto $X$ by reverse inclusion). We claim that $h = g \restriction Z$ is a homeomorphism. For this it suffices to show that $h$ is one to one. To this end, take distinct points $x, y \in Z$. Find disjoint clopen neighborhoods $U$ and $V$ of, respectively, $x$ and $y$ (in $Z$). Since $h$ is irreducible,

$$h(x) \in \overline{\operatorname{int} h(U)}, \qquad h(y) \in \overline{\operatorname{int} h(V)}, \quad \text{and} \quad \operatorname{int} h(U) \cap \operatorname{int} h(V) = \emptyset .$$

Consequently, by the extremal disconnectivity of $X$,

$$\overline{\operatorname{int} h(U)} \cap \overline{\operatorname{int} h(V)} = \emptyset ,$$

and we conclude that $h(x) \neq h(y)$. □

*1.5. $C^*$-embedded subspaces of $\beta\omega$*

In this section we will characterize those subspaces of $\beta\omega$ that are $C^*$-embedded in $\beta\omega$. It is interesting that being $C^*$-embedded in $\beta\omega$ turns out to be a topological property and does not depend on how a given set is placed in $\beta\omega$.

1.5.1. DEFINITION. A space $X$ is called *weakly Lindelöf* provided that for any open cover $\mathscr{U}$ of $X$ there is a countable subfamily $\mathscr{E} \subseteq \mathscr{U}$ such that $(\bigcup \mathscr{E})^- = X$.

Observe that each space satisfying the countable chain condition is weakly Lindelöf.

The following important result shows that $\beta\omega$ has 'many' $C^*$-embedded subspaces.

1.5.2. THEOREM. *Let $X \subseteq \beta\omega$ be weakly Lindelöf. Then $X$ is $C^*$-embedded in $\beta\omega$.*

PROOF. It clearly suffices to show that disjoint zero-sets in $X$ have disjoint closures in $\beta\omega$, COMFORT & NEGREPONTIS [1974, Theorem 2.6]. To prove this, let $Z_0$, $Z_1 \subseteq X$ be a disjoint zero-sets. There are disjoint open neighborhoods $U$ and $V$ of $Z_0$ and $Z_1$ such that $\mathrm{cl}_X U \cap \mathrm{cl}_X V = \emptyset$. For each $x \in X$ let $C_x \subseteq \beta\omega$ be a clopen neighborhood of $x$ in $\beta\omega$ such that

(1) if $x \in \mathrm{cl}_X U$, then $C_x \cap \mathrm{cl}_X V = \emptyset$,
(2) if $x \in \mathrm{cl}_X V$, then $C_x \cap \mathrm{cl}_X U = \emptyset$,
(3) if $x \notin (\mathrm{cl}_X U \cup \mathrm{cl}_X V)$, then $C_x \cap (\mathrm{cl}_X U \cup \mathrm{cl}_X V) = \emptyset$.

Since $X$ is weakly Lindelöf, there is a sequence $x_n$ $(n < \omega)$ in $X$ such that $\bigcup_{n<\omega} (C_{x_n} \cap X)$ is dense in $X$. For each $n < \omega$, put

$$E_n = C_{x_n} \setminus \bigcup_{i<n} C_{x_i} .$$

Then the family $\{E_n : n < \omega\}$ is a pairwise disjoint collection clopen subsets of $\beta\omega$ such that

(1) $\bigcup_{n<\omega} (E_n \cap X)$ is dense in $X$, and
(2) each $E_n$ meets at most one of the $\mathrm{cl}_X U$ and $\mathrm{cl}_X V$.

Put $E = \bigcup\{E_n : n < \omega \ \&\ E_n \cap U \neq \emptyset\}$ and $F = \bigcup\{E_n : n < \omega \ \&\ E_n \cap V \neq \emptyset\}$. Then $E \cap F = \emptyset$ and therefore, since $\beta\omega$ is an $F$-space, $\bar{E} \cap \bar{F} = \emptyset$. Since obviously $U \subseteq \bar{E}$ and $V \subseteq \bar{F}$, we conclude that $Z_0$ and $Z_1$ have disjoint closures in $\beta\omega$. □

We will now show that, under CH, the converse of Theorem 1.5.2 is true, thus giving a *topological* characterization of those subspaces of $\beta\omega$ that are $C^*$-embedded in $\beta\omega$.

1.5.3. THEOREM (CH). *Let $X \subseteq \beta\omega$. The following statements are equivalent*:
(1) *$X$ is weakly Lindelöf,*
(2) *$X$ is $C^*$-embedded in $\beta\omega$,*
(3) $|C^*(X)| = \mathfrak{c}$.

PROOF. (1)⇒(2) is shown in Theorem 1.5.2 and (2)⇒(3) is clear since $|C^*(\beta\omega)| = \mathfrak{c}$. It therefore suffices to prove that the negation of (1) imples the negation of (3).

If $X$ is not weaky Lindelöf then, by CH, there is a family $\{C_\alpha : \alpha < \omega_1\}$ of clopen subsets of $\beta\omega$ such that

(1) $X \subseteq \bigcup_{\alpha<\omega_1} C_\alpha$,
(2) for each $\alpha < \omega_1$, $X \setminus (\bigcup_{\beta<\alpha} C_\beta \cap X)^- \neq \emptyset$.

We can therefore find a strictly increasing sequence of ordinals $\kappa_\alpha < \omega_1$ $(\alpha < \omega_1)$ and for each $\alpha < \omega_1$ a clopen set $D_\alpha \subseteq C_{\kappa_\alpha}$ such that

(3) $D_\alpha \cap X \neq \emptyset$,
(4) $D_\alpha \cap (\bigcup_{\beta<\alpha} (C_\beta \cap X) \cup \bigcup_{\beta<\alpha} (D_{\kappa_\beta} \cap X))^- = \emptyset$.

For each $\alpha < \omega_1$, put $\tilde{C}_\alpha = C_\alpha \cap \bar{X}$ and let $D = \bigcup_{\alpha<\omega_1} D_\alpha \cap \bar{X}$.

*Claim. $D$ is $C^*$-embedded in $\bigcup_{\alpha<\omega_1} \tilde{C}_\alpha$.*

Let $f: D \to [0, 1]$ be given. For each $\alpha < \omega_1$ put

$$f_\alpha = f \restriction D \cap \bigcup_{\beta \leq \alpha} \tilde{C}_\beta .$$

Observe that (4) implies that $\mathrm{dom}(f_\alpha)$ is an open $F_\sigma$-subset of $\bar{X}$ for all $\alpha < \omega_1$. We will construct, for each $\alpha < \omega_1$, an extension $g_\alpha : \bigcup_{\beta \leq \alpha} \tilde{C}_\beta \to I$ of $f_\alpha$ such that for all $\beta < \alpha$,

$$g_\beta \subseteq g_\alpha .$$

Suppose this is done for all $\beta < \alpha$. The function $\bigcup_{\beta<\alpha} g_\beta \cup f_\alpha$ is continuous on $\bigcup_{\beta<\alpha} \tilde{C}_\beta \cup (D \cap \tilde{C}_\alpha)$, and this set is an open $F_\sigma$-subset of $\bar{X}$. Therefore, since $\bar{X}$ is an $F$-space (Lemma 1.2.2(d)), this function can be extended to get the required $g_\alpha$.

Finally put $g = \bigcup_{\alpha<\omega_1} g_\alpha$. It is clear that $g$ is as required.

Since $D$ is a union of $\omega_1$ pairwise disjoint nonempty clopen sets, $|C^*(D)| \leq 2^{\omega_1}$ and consequently, by the Claim, $|C^*(\bigcup_{\alpha<\omega_1} \tilde{C}_\alpha)| \geq 2^{\omega_1}$. Since $X$ is dense in $\bigcup_{\alpha<\omega_1} \tilde{C}_\alpha$ this imples that

$$|C^*(X)| \geq 2^{\omega_1} > \mathfrak{c} ,$$

which is a contradiction. □

**1.5.4.** COROLLARY (CH). *If $x \in \omega^*$, then $\omega^* \backslash \{x\}$ is not $C^*$-embedded in $\omega^*$.*

PROOF. If $\omega^*\backslash\{x\}$ is $C^*$-embedded in $\omega^*$, then $\omega^*\backslash\{x\}$ is $C^*$-embedded in $\beta\omega$ since $\omega^*$ is $C^*$-embedded in $\beta\omega$. By Theorem 1.5.3 it therefore suffices to prove the following easy

*Fact. If $x \in \omega^*$, then $\omega^*\backslash\{x\}$ is not weakly Lindelöf.*

Assume, to the contrary, that $\omega^*\backslash\{x\}$ is weakly Lindelöf. Put $\mathscr{U} = \{C \subseteq \omega^* : C$ is clopen and $x \notin C\}$. Since by assumption $\omega^*\backslash\{x\}$ is weakly Lindelöf, there are $C_n \in \mathscr{U}$ $(n < \omega)$ such that $\bigcup_{n<\omega} C_n$ is dense in $\omega^*$. By Lemma 1.1.2 or by Theorem 1.2.5, there is a nonempty clopen $E \subseteq \omega^*$ such that $E \cap (\bigcup_{n<\omega} C_n) = \emptyset$. It is clear that without loss of generality we may assume that $x \notin E$. Then $E$ must meet $\bigcup_{n<\omega} C_n$, which is not the case and therefore we have obtained the desired contradiction. □

### 1.6. *Autohomeomorphisms of $\omega^*$*

In this section we will concentrate on autohomeomorphisms of $\omega^*$. Our main results are Theorems 1.6.4 and 1.6.5.

If $\pi: \omega \to \omega$ is a permutation, then $\beta\pi \restriction \omega^*$ is an autohomeomorphism of $\omega^*$.



Let $\pi_0$ and $\pi_1$ be two permutations of $\omega$. We claim that if $\beta\pi_0 \restriction \omega^* = \beta\pi_1 \restriction \omega^*$, then $|\{n < \omega : \pi_0(n) \neq \pi_1(n)\}| < \omega$. If not, then we can find an infinite set $E \subseteq \omega$ such that $\pi_0(E) \cap \pi_1(E) = \emptyset$. Take $x \in \omega^*$ such that $E \in x$. Since $\pi_i(E) \in \beta\pi_i(x)$ for $i < 2$, we conclude that $\beta\pi_0(x) \neq \beta\pi_1(x)$, which contradicts our assumptions. Since it is clear that we can find a family $\{\pi_\xi : \xi < \mathfrak{c}\}$ of permutations of $\omega$ such that for all $\eta < \xi < \mathfrak{c}$ we have that $\{n : \pi_\eta(n) \neq \pi_\xi(n)\}$ is infinite, this shows that $\omega^*$ has at least $\mathfrak{c}$ autohomeomorphisms which are induced from permutations on $\omega$. Are there others? Under CH, there are.

**1.6.1.** LEMMA (CH). *$\omega^*$ has precisely $2^{\mathfrak{c}}$ autohomeomorphisms.*

PROOF. Since by Theorem 1.2.6, $\omega^* \approx (\omega \times 2^{\mathfrak{c}})^*$ (here $2^{\mathfrak{c}}$ denotes the Cantor cube of weight $\mathfrak{c}$) and since $2^{\mathfrak{c}}$ has $2^{\mathfrak{c}}$ autohomeomorphisms, being a topological group of cardinality $2^{\mathfrak{c}}$, it easily follows that $\omega^*$ has at least $2^{\mathfrak{c}}$ autohomeomorphisms. Since $\omega^*$ has weight $\mathfrak{c}$, it cannot have more than $2^{\mathfrak{c}}$ autohomeomorphisms. □

We will now prove two results which are steps in the proof of Theorem 1.6.4.

**1.6.2.** LEMMA. *Let $U$ and $V$ be noncompact open $F_\sigma$-subsets of $\omega^*$. Then there is an autohomeomorphism $h: \omega^* \to \omega^*$ with $h(U) = V$.*

PROOF. Find partitions $\{A_n : n < \omega\}$ and $\{B_n : n < \omega\}$ of $\omega$ in infinite sets such that

$$U = \bigcup_{n<\omega} A_n^* \quad \text{and} \quad V = \bigcup_{n<\omega} B_n^* .$$

Let $\pi: \omega \to \omega$ be a permutation such that $\pi(A_n) = B_n$ for all $n < \omega$. Then $h = \beta\pi \restriction \omega^*$ is clearly as required. □

**1.6.3.** COROLLARY (CH). *Let $S$ and $T$ be nowhere dense $P$-sets in $\omega^*$ such that $S \approx T \approx \omega^*$. Then there is an autohomeomorphism*

$$h: \omega^* \to \omega^* \quad \textit{with } h(S) = T .$$

PROOF. Let $X$ be a homeomorph of $\omega \times \omega^*$ disjoint from $\omega^*$. Since, by Theorem 1.2.6, $X^* \approx \omega^* \approx S$, we can identify $X^*$ and $S$. In other words, we assume that $\beta X \cap \omega^* = S$. We topologize $Z_0 = \omega^* \cup X$ by pasting $\beta X$ and $\omega^*$ together. Formally,

$$U \subseteq Z_0 \text{ is open iff } U \cap \beta X \text{ is open in } \beta X \text{ and } U \cap \omega^* \text{ is open in } \omega^* .$$

By using similar arguments as in the proofs of Lemmas 1.4.1 and 1.4.2, the reader can easily verify that $Z_0$ is a Parovičenko space, consequently $Z_0 \approx \omega^*$. Similarly, take a homeomorph $Y$ of $\omega \times \omega^*$ disjoint from $\omega^*$ such that $\beta Y \cap \omega^* = T$ and form the Parovičenko space $Z_1 = \omega^* \cup Y$. By Lemma 1.6.2 there is a homeomorphism $\tilde{h}: Z_0 \to Z_1$ with $\tilde{h}(X) = Y$. Then $h = \tilde{h} \restriction \omega^*$ is clearly as required. □

**1.6.4.** THEOREM (CH). *Let $S, T \subseteq \omega^*$ be nowhere dense $P$-sets such that $S \approx T \approx \omega^*$ and let $h: T \to S$ be a homeomorphism. Then $h$ can be extended to a homeomorphism $\bar{h}: \omega^* \to \omega^*$.*

PROOF. Let $f: \omega^* \to \omega^*$ be a homeomorphism such that $f(S) \cap T = \emptyset$. It is clear that such homeomorphism exists since all clopen subsets of $\omega^*$ are homeomorphic to $\omega^*$ and $S \cup T$ is nowhere dense. Put $Z = f(S) \cup T$ and define $\varphi: Z \to Z$ by

$$\begin{cases} \varphi(t) = f(h(t)) & \text{if } t \in T, \\ \varphi(t) = h^{-1}(f^{-1}(t)) & \text{if } t \in f(S). \end{cases}$$

Now if we can extend $\varphi: Z \to Z$ to a homeomorphism $\bar{\varphi}: \omega^* \to \omega^*$ then $\bar{h} = f^{-1} \circ \bar{\varphi}$ is a homeomorphism of $\omega^*$ extending $h$. Since $Z \approx \omega^*$, in view of Corollary 1.6.3, it therefore suffices to prove the following

*Fact. There is a nowhere dense $P$-set $A \subseteq \omega^*$ such that $A \approx \omega^*$ and each autohomeomorphism of $A$ extends to an autohomeomorphism of $\omega^*$.*

Put $X = \omega \times W(\omega_1 + 1) \times \omega^*$ and $Y = \omega \times \{\omega_1\} \times \omega^*$. It is easy to see that $Y^* \subseteq X^*$ is a nowhere dense $P$-set. The projection $\pi: Y \to \omega^*$ extends to a map $\beta\pi: \beta Y \to \omega^*$. Let $f = \beta\pi \restriction Y^*$ and define $B = X^* \cup_f Y^*$. By Lemmas 1.4.1 and 1.4.2, $B$ is a Parovičenko space. Obviously, $A = \omega^*$ is a nowhere dense $P$-set in $B$. Let $h: A \to A$ be any homeomorphism. The homeomorphism $\tilde{h} = \mathrm{id} \times \mathrm{id} \times h$ of $X$ extends to a homeomorphism $\beta\tilde{h}: \beta X \to \beta X$. Define $\bar{h}: B \to B$ by

$$\begin{cases} \bar{h}(x) = h(x) & \text{if } x \in A, \\ \bar{h}(x) = \beta\tilde{h}(x) & \text{if } x \notin A. \end{cases}$$

An easy check shows that $\bar{h}$ is an autohomeomorphism of $B$ extending $h$. Since $B \approx \omega^*$ (Corollary 1.2.4) this is as required. □ .

The following result is, in a sense, also a result on extending homeomorphisms.

**1.6.5.** THEOREM (CH). *Let $p, q \in \omega^*$ be $P$-points. Then there is an autohomeomorphism $h: \omega^* \to \omega^*$ with $h(p) = q$.*

PROOF. Adapt the proof of Theorem 1.1.6. □

Observe that Theorem 1.4.4 implies that $P$-points in $\omega^*$ exist.



**1.6.6.** REMARK. Theorem 1.6.4 and 1.6.5 have a common generalization. In VAN DOUWEN & VAN MILL [1981b] it will be shown that any homeomorphism between (arbitrary) nowhere dense closed $P$-sets extends to an autohomeomorphism of $\omega^*$. The proof which van Douwen and I have of this result is conceptually simple, but technically complicated. Since we believe that the proof is not in its final form yet, in this section we have only worked out some special cases which have simpler proofs.

### 1.7. *P-points and nonhomogeneity of $\omega^*$*

Since $\omega$ is homogeneous, $\omega^*$ looks homogeneous and the question naturally arises whether $\omega^*$ is homogeneous. We will prove that, under CH, this is not the case. We will show later that $\omega^*$ is not homogeneous in ZFC. Observe that Theorem 1.4.4 implies that, under CH, $\omega^*$ contains a $P$-point. If all points of $\omega^*$ are $P$-points, then the compactness of $\omega^*$ implies that $\omega^*$ is finite, which is clearly not the case. Therefore, $\omega^*$ contains both $P$-points and non $P$-points and we conclude that $\omega^*$ is not homogeneous under CH. The proof, just given here that there are $P$-points in $\omega^*$, is not very economical. We will therefore give an easier proof of this fact.

**1.7.1.** LEMMA. *$\omega^*$ cannot be covered by $\omega_1$ nowhere dense sets.*

PROOF. Let $\{D_\alpha : \alpha < \omega_1\}$ be a family of $\omega_1$ nowhere dense subsets of $\omega^*$. By using Lemma 1.1.2 or Theorem 1.2.5, find a family $\{C_\alpha : \alpha < \omega_1\}$ of nonempty clopen subsets of $\omega^*$ such that for all $\alpha < \omega_1$,

(1) $C_\alpha \cap D_\alpha = \emptyset$,

(2) if $\beta < \alpha$, then $C_\alpha \subseteq C_\beta$.

Consequently, any point of $\bigcap_{\alpha<\omega_1} C_\alpha$ misses $\bigcup_{\alpha<\omega_1} D_\alpha$. □

**1.7.2.** COROLLARY (CH). *$\omega^*$ contains $P$-points.*

PROOF. Let $\mathscr{A} = \{\bar{U}\backslash U : U \subseteq \omega^*$ is an open $F_\sigma\}$. By CH, $|\mathscr{A}| \leq \omega_1$. By Lemma 1.7.1, $\omega^* \backslash \bigcup \mathscr{A} \neq \emptyset$ and each point of this set is a $P$-point. □

Since by Theorem 1.6.5, for any two $P$-points $x, y \in \omega^*$, under CH there is an autohomeomorphism $h: \omega^* \to \omega^*$ with $h(x) = y$, all $P$-points are topologically the same. In view of the above results, one therefore naturally wonders whether $P$-points and non $P$-points are the only types of points in $\omega^*$. This is not true, as the next result shows.

A point $x$ of a space $X$ is called a *weak $P$-point* provided that $x \notin \bar{F}$ for all countable $F \subseteq X \backslash \{x\}$.

1.7.3. THEOREM (CH). (1) *There is a weak P-point in $\omega^*$ which is not a P-point,*
(2) *There is a point $x \in \omega^*$ such that*
(a) *for some countable $F \subseteq \omega^* \setminus \{x\}$ we have that $x \in \bar{F}$,*
(b) *for all countable discrete $D \subseteq \omega^* \setminus \{x\}$ we have that $x \notin \bar{D}$.*

PROOF. Let $\mathcal{M}$ be the BA of Lebesgue measurable subsets of $[0, 1]$ and let $\mathcal{N}$ be the ideal of null-sets. We put $\mathcal{B} = \mathcal{M}/\mathcal{N}$. It is well-known, and easy to prove, that $|\mathcal{B}| = \mathfrak{c}$ and that $\mathcal{B}$ is complete. Consequently, $X = \mathrm{st}(\mathcal{B})$ is an extremally disconnected compactum of weight $\mathfrak{c}$. If $M \in \mathcal{M}$, then the $\mathcal{N}$-equivalence class of $M$ is denoted by $[M]$. $\lambda$ denotes Lebesgue measure.

*Fact* 1. *If $D \subseteq X$ is countable, then $D$ is nowhere dense.*

Take $M \in \mathcal{M}$ and list $D$ as $\{d_n : n < \omega\}$. Since $d_n$ is an ultrafilter in the BA $\mathcal{B}$, there exists an element $M_n \in \mathcal{B}$ such that
(1) $[M_n] \in d_n$, and
(2) $\lambda(M_n) < 2^{-2-n} \cdot \lambda(M)$.
Then $\{x \in X : [M \setminus \bigcup_{n<\omega} M_n] \in x\}$ is a nonempty open subset of $[M]$ which misses $\{d_n : n < \omega\}$.

*Fact* 2. *X is* ccc.

Let $\mathcal{A} \subseteq \mathcal{M}$ be uncountable such that $\lambda(A) > 0$ for all $A \in \mathcal{A}$ while moreover the family

$$\{\{x \in X : [A] \in x\} : A \in \mathcal{A}\}$$

is pairwise disjoint. Let $\mathcal{U}$ be a countable open basis for $[0, 1]$ which is closed under finite unions and for all $U \in \mathcal{U}$, put

$$\mathcal{A}(U) = \{A \in \mathcal{A} : \lambda(A \cap U) > \tfrac{1}{2}\lambda(U)\}.$$

If $A \in \mathcal{A}$, then there is a compact $K \subseteq A$ with $\lambda(K) > 0$. For this $K$ there is an element $U \in \mathcal{U}$ with $K \subseteq U$ and $\lambda(K) > \frac{1}{2}\lambda(U)$. We conclude that $A \in \mathcal{A}(U)$ and since $A$ is arbitrarily chosen, this implies that

$$\bigcup_{U \in \mathcal{U}} \mathcal{A}(U) = \mathcal{A}.$$

Hence there must be an element $U \in \mathcal{U}$ such that $\mathcal{A}(U)$ is uncountable. But this contradicts the definition of $\mathcal{A}(U)$.

*Fact* 3. *There is a family $\mathcal{D}$ of $\mathfrak{c}$ nowhere dense subsets of X such that each nowhere dense subset of X is contained in an element of $\mathcal{D}$.*

Since $X$ has weight $\mathfrak{c}$, by Fact 2 we can take $\mathcal{D}$ to be the collection of all nowhere dense closed $G_\delta$'s.

*Fact* 4 (CH). *There is a point $x \in Y^*$, where $Y = \omega \times X$, such that*
(1) *x is a P-point of $Y^*$,*
(2) *if $D \subseteq \omega \times X$ is any nowhere dense set, then $x \notin \bar{D}$.*



By Fact 3 and by CH there is a family $\mathcal{E}$ *of* $\omega_1$ nowhere dense subsets of $Y$ such that each nowhere dense subset of $Y$ is contained in an element of $\mathcal{E}$. By Theorem 1.2.5, $Y^* \approx \omega^*$, and therefore by Lemma 1.7.1, there is a point

$$x \in Y^* \setminus (\bigcup\{E^* : E \in \mathcal{E}\} \cup \bigcup\{\bar{U} \setminus U : U \subseteq Y^* \text{ is an open } F_\sigma\})$$

(it is easily seen that if $E \in \mathcal{E}$, then $E^* \subseteq Y^*$ is nowhere dense). It is clear that $x$ is as required.

Now, since $\beta Y$ is an extremally disconnected compactum of weight $\mathfrak{c}$, by Theorem 1.4.4 (3), $\beta Y$ can be embedded in $\omega^*$ as a closed $P$-set. If we take $x \in \beta Y$ such as in Fact 4, then Facts 1 and 2 imply (if we identify $\beta Y$ with a $P$-set in $\omega^*$) that $x$ is a weak $P$-point which is not a $P$-point (in $\omega^*$ as well as in $\beta Y$). This proves (1). To prove (2), substitute $X$ by the projective cover of the Cantor set and proceed similarly. The details of checking this out are left to the reader. □

1.7.4. REMARK. As we will see later, Theorem 1.7.3 is true in ZFC by a more complicated argument. We have included the above proof since this way of constructing points will be used frequently in the remaining part of this paper. If the reader understands the proof of Theorem 1.7.3, she or he will have less trouble understanding the more complicated forthcoming constructions.

### *1.8. Retracts of $\beta\omega$ and $\omega^*$*

In this section we study subspaces of $\beta\omega$ or $\omega^*$ on which $\beta\omega$ or $\omega^*$ can be retracted.

1.8.1. THEOREM (CH). *Let X be a closed P-set of $\omega^*$. Then X is a retract of $\omega^*$.*

PROOF. By CH, we can enumerate $\mathcal{B}(X)$ by $\{C_\alpha : \alpha < \omega_1\}$. It is easy, using the fact that $X$ is a $P$-set, to construct for each $\alpha < \omega_1$ a countable subalgebra $\mathcal{B}_\alpha \subseteq \mathcal{B}(X)$ and an embedding $\rho_\alpha : \mathcal{B}_\alpha \to \mathcal{B}(\omega^*)$ such that
(1) if $\beta < \alpha$, then $\mathcal{B}_\beta \subseteq \mathcal{B}_\alpha$,
(2) if $\beta < \alpha$, then $\rho_\alpha \upharpoonright \mathcal{B}_\beta = \rho_\beta$,
(3) $C_\alpha \in \mathcal{B}_\alpha$,
(4) if $C \in \mathcal{B}_\alpha$, then $\rho_\alpha(C) \cap X = C$.
Define $\rho : \mathcal{B}(X) \to \mathcal{B}(\omega^*)$ by $\rho(C) = \rho_\alpha(C)$ if $C \in \mathcal{B}_\alpha$.

Now define $r : \omega^* \to X$ by

$$\{r(x)\} = \bigcap\{C \in \mathcal{B}(X) : x \in \rho(C)\}.$$

An easy check shows that $r$ is a retraction. □

We will now prove a result on retracts of $\beta\omega$ which does not use CH.

**1.8.2.** THEOREM. *Let $X \subseteq \beta\omega$ be a closed subspace of countable $\pi$-weight. Then $X$ is a retract of $\beta\omega$.*

PROOF. Let $\mathscr{C}$ be a countable subalgebra of $\mathscr{B}(X)$ which forms a $\pi$-basis for $X$ (for the definition of $\pi$-basis, see JUHÁSZ [1980] or HODEL [1983]). It is trivial to find a function $\rho : \mathscr{C} \to \mathscr{B}(\beta\omega)$ such that

(1) $\rho$ is an embedding,

(2) if $C \in \mathscr{C}$, then $\rho(C) \cap X = C$.

Define a function $\kappa : \mathscr{B}(X) \to \mathscr{B}(\beta\omega)$ by

$$\kappa(A) = (\bigcup\{\rho(C) : C \in \mathscr{C}, C \subseteq A\})^{-} .$$

Since $\mathscr{P}(\omega)$ is complete or, in topological language, since $\beta\omega$ is extremally disconnected, $\kappa$ is well-defined.

*Fact* 1. *If $A \in \mathscr{B}(X)$, then $\kappa(A) \cap X = A$.*

Since $\bigcup\{\rho(C) : C \in \mathscr{C},\ C \subseteq A\} \cap \bigcup\{\rho(C) : C \in \mathscr{C},\ C \subseteq X \backslash A\} = \emptyset$, this is immediate.

*Fact* 2. *$\kappa$ is an embedding.*

This follows easily from the fact that $\rho$ is an embedding.

Now, as in the proof of the previous theorem, define $r : \beta\omega \to X$ by

$$\{r(x)\} = \bigcap\{A \in \mathscr{B}(X) : x \in \kappa(A)\} .$$

By Stone dualty, $r$ is continuous, and by Fact 1, $r \restriction X =$ identity. $\square$

Observe that the above theorem is interesting since it shows that a certain class of subspaces of $\beta\omega$ is always a retract of $\beta\omega$ no matter how these sets are placed in $\beta\omega$. However, topologically, there are not many closed subspaces of $\beta\omega$ which have countable $\pi$-weight, so in this sense the theorem is quite restrictive. M. TALAGRAND [1981] has given recently a quite complicated example of a separable closed subspace of $\beta\omega$ which is not a retract of $\beta\omega$ (his construction is under CH; it is desirable to find such an example in ZFC only). Therefore, the above theorem cannot be generalized. The following result shows precisely how far one can go, and it also illustrates the complexity of Talagrand's Example.

**1.8.3.** THEOREM. *Any separable, extremally disconnected compact space can be embedded in $\omega^*$ in such a way that it is a retract of $\beta\omega$.*

PROOF. Let $X$ be a separable, extremally disconnected compact space. Since $\beta\omega$ maps onto each separable compact space, there is a continuous surjection $f : \beta\omega \to X$. Let $Z \subseteq \beta\omega$ be such that $f \restriction Z$ is an irreducible surjection from $Z$ onto $X$. Since $X$ is extremally disconnected, as in the proof of Theorem 1.4.7, $f \restriction Z$ is a homeomorphism.



Since $\omega^*$ is infinite, it contains a countable relatively discrete subspace $D$. By Theorem 1.5.2, $\bar{D} \approx \beta\omega$, and by Theorem 1.8.2, $\bar{D}$ is a retract of $\beta\omega$. Since, as was shown above, $X$ embeds in $\bar{D}$ as a retract, the desired result follows. $\square$

*1.9. Nowhere dense P-sets in $\omega^*$*

Nowhere dense $P$-sets have played an important role in this section. The question therefore naturally arises whether each point $x \in \omega^*$ is contained in a nowhere dense $P$-set. Under CH, we will show that this is not the case.

**1.9.1.** DEFINITION. Let $\kappa \geqslant \omega$. A subset $A$ of a space $X$ is called a *$P_\kappa$-set* provided that the intersection of fewer than $\kappa$ neighborhoods of $A$ is again a neighborhood of $A$.

$P_{\omega_1}$-sets are precisely the $P$-sets of course, and any subset of any space is a $P_\omega$-set.

**1.9.2.** LEMMA. *Let $X$ be a space of $\pi$-weight $\leqslant\kappa$, where $\kappa$ is regular and $\kappa \geqslant \omega$. For each $1 \leqslant n < \omega$ there is a family $\mathscr{F}_n$ of closed subsets of $X$ such that*

*(1) $\mathscr{F}_n$ has the $n$-intersection property,*

*(2) if $K \subseteq X$ is any nowhere dense $P_\kappa$-set, then for some $F \in \mathscr{F}_n$ we have that $F \cap K = \emptyset$.*

PROOF. Let $\mathscr{U} = \{U_\alpha : \alpha < \kappa\}$ be a $\pi$-basis for $X$ and let $\mathscr{D}$ be the family of all nowhere dense subsets of $X$. For each $D \in \mathscr{D}$, put

$$H(D) = \{\alpha < \kappa : \bar{U}_\alpha \cap D = \emptyset\} .$$

Define ordinals $\mu(D, m) < \kappa$ $(1 \leqslant m < \omega)$ as follows

$$\mu(D, 1) = \min H(D) ,$$

$$\mu(D, m) = \min\{\alpha < \kappa : \forall\beta \leqslant \mu(D, m-1)\ \exists\xi \leqslant \alpha\ \bar{U}_\xi \subseteq U_\beta \backslash \bar{D}\}$$

(observe that $\mu(D, 2)$ need not be defined if $\kappa$ is singular).

Define $F(D, n) = \bigcup\{\bar{U}_\alpha : \alpha \in H(D) \text{ and } \alpha \leqslant \mu(D, n)\}$.

*Fact. $\{F(D, n) : D \in \mathscr{D}\}$ has the $n$-intersection property. In fact, if $\mathscr{E} \in [\mathscr{D}]^n$ then there is an $\alpha \leqslant \max\{\mu(D, n) : D \in \mathscr{E}\}$ such that $U_\alpha \subseteq \bigcap\{F(D, n) : D \in \mathscr{E}\}$.*

Induction on $n$. The case $n = 1$ is trivial. Suppose the fact to be true for all $i \leqslant n$ and take $D_1, D_2, \ldots, D_{n+1} \in \mathscr{D}$. We may assume that for all $i \leqslant n+1$ we have that $\mu(D_i, n) \leqslant \mu(D_{n+1}, n)$.

By induction hypothesis, there is an $\alpha \leqslant \max\{\mu(D_i, n) : 1 \leqslant i \leqslant n\}$ such that $U_\alpha \subseteq \bigcap\{F(D_i, n) : 1 \leqslant i \leqslant n\} \subseteq \bigcap\{F(D_i, n+1) : 1 \leqslant i \leqslant n\}$. Since $\alpha \leqslant \mu(D_{n+1}, n)$, there is a $\beta \leqslant \mu(D_{n+1}, n+1)$ with $\bar{U}_\beta \subseteq U_\alpha \backslash \bar{D}_{n+1}$.

Now let $\mathscr{F}_n = \{\overline{F(K,n)} : K \subseteq X$ is a nowhere dense $P_\kappa$-set$\}$. Since $F(K, n)$ is a union of less than $\kappa$ closed sets each of which do not intersect $K$, clearly $\overline{F(K,n)} \cap K = \emptyset$. Consequently, $\mathscr{F}_n$ is as required. □

We now come to the main result of this section.

**1.9.3.** THEOREM. *Let $X$ be a compact space of $\pi$-weight $\leq \kappa$ ($\kappa > \omega$). Then there is an $x \in X$ such that $x \notin K$ for all nowhere dense $P_\kappa$-sets $K \subseteq X$.*

PROOF. Suppose first that $\kappa$ is regular. Let $\{V_n : n < \omega\}$ be a sequence of countably many nonempty pairwise disjoint open subsets of $X$. By Lemma 1.9.2, there is a family $\mathscr{F}_n$ of closed subsets of $\bar{V}_n$ such that

(1) $\mathscr{F}_n$ has the $n$-intersection property,

(2) if $K \subseteq X$ is a nowhere dense $P_\kappa$-set, then there is an $F \in \mathscr{F}_n$ with $F \cap K = \emptyset$. (Observe that if $K \subseteq X$ is a nowhere dense $P_\kappa$-set, then $K \cap \bar{V}_n$ is a nowhere dense $P_\kappa$-set of $\bar{V}_n$). Take any point $x$ in the intersection

$$\bigcap \left\{ \overline{\bigcup_{n<\omega} g(n)} : g \in \prod_{n<\omega} \mathscr{F}_n \right\}.$$

Since $\kappa > \omega$, $x$ is as required.

Now observe that if $\kappa$ is singular, then any $P_\kappa$-set of $X$ is a $P_{\kappa^+}$-set. Therefore, the theorem for singular $\kappa$ follows from the theorem for regular $\kappa$. □

**1.9.4.** COROLLARY (CH). *There is a point $x \in \omega^*$ such that $x \notin K$ for all nowhere dense $P$-sets $K \subseteq \omega^*$.*

### Notes for Section 1

Theorem 1.1.6 is due to PAROVIČENKO [1963]. Lemma 1.2.2 is well known. For other results of this type see COMFORT & NEGREPONTIS [1974]. Corollary 1.2.4 is due to PAROVIČENKO [1963]. Theorem 1.2.5 can be found in GILLMAN & JERISON [1960]. The argument given here is due to NEGREPONTIS [1967]. That Theorem 1.2.6 is a consequence of Parovičenko's characterization of $\omega^*$, was observed by many people. Theorem 1.3.1 is due to PAROVIČENKO [1963]. For another proof of this result see BLASZCZYK & SZYMAŃSKI [1980b]. Lemmas 1.4.1 and 1.4.2 are implicit in BALCAR, FRANKIEWICZ & MILLS [1980]. Theorem 1.4.4 (1)⇔(2) is due to LOUVEAU [1973]; the equivalence (1)⇔(3) can be found in BALCAR, FRANKIEWICZ & MILLS [1980]. Theorem 1.4.7 is due to EFIMOV [1970]. A result stronger than Theorem 1.5.2 is due to COMFORT, HINDMAN & NEGREPONTIS [1969]. A result stronger than Theorem 1.5.3 is due to WOODS [1976a]. For a related result, see WOODS [1976b]. An important step in the proof of Theorem 1.5.3 is due to FINE & GILLMAN [1960]. Corollary 1.5.4 is due to GILLMAN [1966]. Lemma 1.6.1 is due to W. RUDIN [1956] but the proof we give is due to van DOUWEN & VAN MILL [1981d]. Lemma 1.6.2 is well-known and Theorem 1.6.4 is a special case of a result in VAN DOUWEN & VAN MILL [1981b]. Theorem 1.6.5 is due to W. RUDIN [1956]. Corollary 1.7.2 is also due to W. RUDIN [1956]. Theorem 1.7.3 is due to KUNEN [1976]. Theorem 1.8.1 can be found in VAN DOUWEN & VAN MILL [1981b]. The easy Theorem 1.8.2 seems to be new. The proof is in the spirit of VAN MILL [1979b]. For a related result see VAN DOUWEN & VAN MILL [1980]. Theorem 1.8.3 is well-known. I don't know who proved this for the first time. Corollary 1.9.4 is due to Kunen and Theorem 1.9.3 is due to KUNEN, VAN MILL & MILLS [1980]. The proof presented here, which was suggested to me by Alan Dow, is different from the proof given in KUNEN, VAN MILL & MILLS [1980]. It is in the spirit of CHAE & SMITH [1980] and VAN DOUWEN [1981].



## 2. The spaces $\beta\omega$ and $\beta\omega\backslash\omega$ under $\neg$CH

In this section we will see how $\beta\omega$ and $\omega^*$ behave in various models in which CH is not true. All the CH results derived in Section 1 are consistently false, except for Theorem 1.7.3, which is true in ZFC (see Theorems 4.3.3 and 4.4.1), and Lemma 1.4.3 and Theorem 1.7.3 of which we do not know whether they can be false.

### *2.1. A characterization of $\mathscr{P}(\omega)$/fin, II*

The main result in Section 1.1, namely Theorem 1.1.6, is false under $\neg$CH. In fact, Theorem 1.1.6 is equivalent to CH.

**2.1.1.** THEOREM. CH *is equivalent to the statement that all Boolean algebras of cardinality $\mathfrak{c}$ which satisfy condition $H_\omega$ are isomorphic.*

PROOF. Our proof is in topological language, since this is more convenient here. The Boolean algebraic reader can easily translate this proof in Boolean algebraic language.

We will construct two Parovičenko spaces that cannot be homeomorphic under $\neg$CH.

*Example 1. A Parovičenko space $S$ having a point $p$ such that $\chi(p, X) = \omega_1$.*

By Lemma 1.2.3 there is an $\omega_1$-sequence $\langle C_\alpha : \alpha < \omega_1 \rangle$ of clopen subsets in $\omega^*$ with $C_\alpha \subset C_\beta$ if $\beta < \alpha < \omega_1$. Let $P = \bigcap_{\alpha<\omega_1} C_\alpha$, and let $S = X/P$, the quotient space obtained from $X$ by collapsing $P$ to one point. By Lemma 1.4.1, $S$ is an $F$-space. The other properties of $S$ required in the definition of a Parovičenko space are easily checked. If we put $p = \{P\}$, then, obviously, $\chi(p, S) = \omega_1$.

*Example 2. A Parovičenko space $T$ with $\pi(x, T) = \mathfrak{c}$ for all $x \in T$.*

Put $T = (\omega \times 2^{\mathfrak{c}})^*$. Since $\omega \times 2^{\mathfrak{c}}$ is a zero-dimensional Lindelöf space of weight $\mathfrak{c}$, $T$ is zero-dimensional and has weight $\mathfrak{c}$. Consequently, Theorem 1.2.5 implies that $T$ is a Parovičenko space.

For $\alpha < \mathfrak{c}$ denote the $\alpha$-th projection $2^{\mathfrak{c}} \to 2$ by $\pi_\alpha$. For $\alpha < \mathfrak{c}$ and $i < 2$ define

$$K(\alpha, i) = T \cap (\omega \times \pi_\alpha^{-1}(\{i\}))^- .$$

Note that $K(\alpha, i)$ is a nonempty clopen subset of $T$ and that $K(\alpha, i) = K(\alpha', i')$ iff $\alpha = \alpha'$ and $i = i'$. Define

$$\mathcal{K} = \{K(\alpha, i) : \alpha < \mathfrak{c}, i < 2\} .$$

*Claim. Any intersection of $\omega_1$ distinct members of $\mathcal{K}$ has empty interior.*

For symmetry reasons it suffices to prove that $I = \bigcap_{\alpha<\omega_1} K(\alpha, 0)$ has empty interior. Suppose that this is not true. Then there is a clopen $U \subseteq \beta(\omega \times 2^{\mathfrak{c}})$ such that $\emptyset \neq U \cap T \subseteq I$. For every $\alpha < \omega_1$ the set $U \backslash (\omega \times \pi_\alpha^{-1}(\{0\}))$ is a compact subset of $\omega \times 2^{\mathfrak{c}}$, and since $U \cap (\omega \times 2^{\mathfrak{c}})$ is not compact, there is an integer $n_\alpha$ such that $\emptyset \neq U \cap (\{n_\alpha\} \times 2^{\mathfrak{c}}) \subseteq \{n_\alpha\} \times \pi_\alpha^{-1}(\{0\})$. There is an integer $n$ such that $A = \{\alpha < \omega_1 : n_\alpha = n\}$ is infinite. But then $\{n\} \times \bigcap_{\alpha \in A} \pi_\alpha^{-1}(\{0\})$ is a subset of $\{n\} \times 2^{\mathfrak{c}}$ with nonempty interior, which is absurd.

Let $x \in T$ be arbitrary, and let $\mathcal{U}$ be a $\pi$-base for $x$. The family $\mathcal{F} = \{K \in \mathcal{K} : x \in K\}$ has cardinality $\mathfrak{c}$. For each $K \in \mathcal{F}$ there is a $U(K) \in \mathcal{U}$ with $U(K) \subseteq K$, hence $|\mathcal{U}| \geq |\mathcal{F}| = \mathfrak{c}$ since the Claim implies that $|\{K \in \mathcal{K} : U(K) = U\}| \leq \omega$ for all $U \in \mathcal{U}$. It follows that $\pi(x, T) = \mathfrak{c}$ since we know already that $T$ has weight at most $\mathfrak{c}$. □

**2.1.2.** Remark. The above result suggests the following interesting question: is CH equivalent to the statement that (*) all Boolean algebras of cardinality $\mathfrak{c}$ which satisfy condition $H_\omega$ have isomorphic completions? This question was first considered by Broverman & Weiss [1981], who showed that (*) is not a theorem of ZFC. Subsequently, van Mill & Williams [1983] proved that (*) implies that $\mathfrak{c} < 2^{\omega_1}$. Whether (*) iff CH is still unknown. (See the remarks on p. 564.)

### *2.2. A topological translation, II*

Section 2.1 of course implies that Corollary 1.2.4 is equivalent to CH. Whether Theorem 1.2.6 is equivalent to CH is unknown, although it is easy to show it is not a Theorem of ZFC. In the proof of Theorem 2.1.1 we showed that, among others, $(\omega \times 2^{\mathfrak{c}})^*$ contains a nonempty intersection of $\omega_1$ clopen sets with empty interior. However, MA + ¬CH implies that each nonempty intersection of $\omega_1$ clopen subsets of $\omega^*$ has nonempty interior, see 2.3. Consequently, MA + ¬CH implies that $(\omega \times 2^{\mathfrak{c}})^*$ is not homeomorphic to $\omega^*$. This argument does not apply to prove that a space such as $(\omega \times 2^{\omega})^*$ is not homeomorphic to $\omega^*$, since MA + ¬CH easily implies that if $X$ is a locally compact, $\sigma$-compact, noncompact space of countable $\pi$-weight, then each nonempty intersection of fewer than $\mathfrak{c}$ open subsets of $X^*$ has nonempty interior. One might therefore hope that if $X$ is topologically very 'close' to $\omega$, then $X^*$ and $\omega^*$ are homeomorphic. Even this is not true.



**2.2.1.** Theorem. *It is consistent that $\omega^*$ and $(\omega \times W(\omega + 1))^*$ are not homeomorphic.*

Proof. Shelah [1978] has recently shown that it is consistent that all homeomorphisms of $\omega^*$ are induced, i.e. that for each autohomeomorphism $h : \omega^* \to \omega^*$ there is a permutation $\pi : \omega \to \omega$ such that $h = \beta\pi \restriction \omega^*$. We will show that for any permutation $\pi$ of $\omega$, the set of fixed points of $\beta\pi$ is a clopen subset of $\beta\omega$, consequently, the set of fixed points of $\beta\pi \restriction \omega^*$ is a clopen subset of $\omega^*$. Let $\pi : \omega \to \omega$ be a permutation, and let $p \in \omega^*$ be a fixed point of $\beta\pi$. Let $E = \{n < \omega : \pi(n) = n\}$. If $E \in p$, then $p$ has clearly a clopen neighborhood consisting of fixed points of $\beta\pi$, namely, the closure of $E$ in $\beta\omega$. So assume that $E \notin p$. Define $F = \omega \backslash E$. Since for all $n \in F$ we have that $\pi(n) \neq n$, it is easy to split $F$ in two sets $F_0$ and $F_1$ such that $\pi(F_0) \cap F_0 = \emptyset$ and $\pi(F_1) \cap F_1 = \emptyset$. Without loss of generality, $F_0 \in p$. Then $\pi(F_0) \in \beta\pi(p)$, whence $p \neq \beta\pi(p)$, a contradiction. We conclude that the set of fixed points of $\beta\pi$ is open, whence clopen. To prove that in Shelah's model, $\omega^*$ and $(\omega \times W(\omega + 1))^*$ are not homeomorphic, it therefore suffices to produce an autohomeomorphism $h$ of $(\omega \times W(\omega + 1))^*$ such that the set $\mathrm{Fix}(h)$ of fixed points of $h$ is not clopen. To this end, let $E, F \subseteq \omega$ be two complementary infinite sets and let $\pi : \omega \to \omega$ be a permutation such that $\pi(E) = F$ (which implies that $\pi(F) = E$). Define $f : \omega \times W(\omega + 1) \to \omega \times W(\omega + 1)$ by

$$\begin{cases} f(\langle n, m \rangle) = \langle n, \pi(m) \rangle \quad (m \in \omega), \\ f(\langle n, \omega \rangle) = \langle n, \omega \rangle . \end{cases}$$

Put $h = \beta f \restriction (\omega \times W(\omega + 1))^*$. It is easily seen that

$$\mathrm{Fix}(h) = (\omega \times \{\omega\})^* ,$$

which implies that $\mathrm{Fix}(h)$ is not clopen. □

**2.2.2.** Remark. Observe that in the proof of Theorem 2.2.1 we found an easily described topological property that distinguishes between $\omega^*$ and $(\omega \times W(\omega + 1))^*$.

### *2.3. Continuous images of $\omega^*$, II*

It is well known that Corollary 1.3.4 is not a result of ZFC. Kunen [1968, 12.7 and 12.3] proved that in a model formed by adding $\omega_2$ Cohen reals to a model of CH, there is no $\omega_2$ sequence of subsets of $\omega$ which is strictly decreasing (mod fin).

Therefore, in this model $\omega^*$ cannot be mapped onto $W(\mathfrak{c}+1)$. VAN DOUWEN & PRZYMUSIŃSKI [1980] have used results of ROTHBERGER [1952] and PRZYMUSIŃSKI [1978] to show that Corollary 1.3.4 is not true under the following hypothesis:

$$(*) \qquad \omega_2 \leq \mathfrak{c} < 2^{\omega_1} = \omega_{\omega_2}.$$

This is interesting since (*) only involves cardinals.

For a discussion of Martin's Axiom (abbreviated MA), see M.E. RUDIN [1977]. The following statements are consequences of MA:

$P(\mathfrak{c})$: *If $\mathcal{A}$ is a family of less than $\mathfrak{c}$ subsets of $\omega$ such that for all $\mathcal{B} \in [\mathcal{A}]^{<\omega}$ we have that $|\bigcap \mathcal{B}| = \omega$, then there is an infinite $B \subseteq \omega$ such that $|B \backslash A| < \omega$ for all $A \in \mathcal{A}$,*

$S(\mathfrak{c})$: *Suppose that $\mathcal{A}$ and $\mathcal{B}$ are families of less than $\mathfrak{c}$ subsets of $\omega$ such that for all $A \in \mathcal{A}$ and $\mathcal{F} \in [\mathcal{B}]^{<\omega}$ we have that $|A \cap \bigcap \mathcal{F}| = \omega$. Then there is an infinite $C \subseteq \omega$ such that $|C \cap A| = \omega$ for each $A \in \mathcal{A}$, and $|C \backslash B| < \omega$ for all $B \in \mathcal{B}$.*

We are now in a position to generalize Theorem 1.3.3.

**2.3.1.** THEOREM (MA). *Each compact space of weight less than $\mathfrak{c}$ is a continuous image of $\omega^*$.*

PROOF. Let $Y$ be a compact space of weight $\kappa$, where $\kappa < \mathfrak{c}$. We may assume that $Y$ is a nowhere dense subspace of $[0,1]^\kappa$. Since $\kappa \leq \mathfrak{c}$, $[0,1]^\kappa$ is separable, and we can therefore find a countable dense set $D$ of $[0,1]^\kappa$ which misses $Y$. Let $\mathcal{E}$ be an open base for $[0,1]^\kappa$ which is closed under finite unions and such that $|\mathcal{E}| = \kappa$. Put

$$\mathcal{A} = \{E \cap D : E \in \mathcal{E} \ \& \ E \cap Y \neq \emptyset\},$$

and

$$\mathcal{B} = \{E \cap D : E \in \mathcal{E} \ \& \ Y \subseteq E\},$$

respectively. It is easily seen that $\mathcal{A}$ and $\mathcal{B}$ satisfy the hypotheses of $S(\mathfrak{c})$. Consequently, we can find a subset $J \subseteq D$ such that

(1) $|A \cap J| = \omega$ for all $A \in \mathcal{A}$,

(2) $|J \backslash B| < \omega$ for all $B \in \mathcal{B}$.

Let $Z = Y \cup J$. We claim that $Z$ is compact and that $J$ is a dense set of isolated points of $Z$. If this is true, then $Z$ is a compactification of $\omega$, which implies that $Y$ is a continuous image of $\omega^*$.

If $E \in \mathcal{E}$ and $E \cap Y \neq \emptyset$, then $E \cap D \in \mathcal{A}$ which implies that $E \cap J$ is infinite. Hence $J$ is dense in $Z$. Let $x$ be a limit point of $J$ which does not belong to $Y$. Since $\mathcal{E}$ is closed under finite unions and since $Y$ is compact, there are disjoint $E_0$, $E_1 \in \mathcal{E}$ with $Y \subseteq E_0$ and $x \in E_1$. Then $E_0 \cap D \in \mathcal{B}$ which implies, by (2), that $J \backslash E_0$ is finite. But $E_1$ contains infinitely many points of $J$, contradiction. We conclude that $Z$ is compact and that $J$ is relatively discrete. □



Of course, the above theorem suggests the question, due to VAN DOUWEN & PRZYMUSIŃSKI [1980, 2.8], whether MA implies that each compact space of weight $\mathfrak{c}$ is a continuous image of $\omega^*$. In the remaining part of this section we will show that this is not the case.

Let $\kappa$ and $\lambda$ be infinite cardinals and consider the following statement:

$G(\kappa, \lambda)$: there are a $\kappa$-sequence $\langle U_\xi : \xi < \kappa \rangle$ of clopen sets in $\omega^*$ and a $\lambda$-sequence $\langle V_\xi : \xi < \lambda \rangle$ of clopen sets in $\omega^*$ such that

(1) $U_\xi \subset U_\eta$ if $\xi < \eta < \kappa$,

(2) $V_\xi \subset V_\eta$ if $\xi < \eta < \lambda$,

(3) $(\bigcup_{\xi<\kappa} U_\xi) \cap (\bigcup_{\xi<\lambda} V_\xi) = \emptyset$, but

(4) $(\bigcup_{\xi<\kappa} U_\xi)^- \cap (\bigcup_{\xi<\lambda} V_\xi)^- \neq \emptyset$.

This has a straightforward translation in terms of the existence of certain families of subsets of $\omega$ which we leave to the reader.

By Lemma 1.1.2, $G(\omega, \omega)$ is false, but interestingly, $G(\omega_1, \omega_1)$ is true, HAUSDORFF [1936].

**2.3.2.** THEOREM. *There is a compact space $X$ and a continuous surjection $f : X \to \omega^*$ such that, under MA + $\neg$CH + $\neg G(\mathfrak{c}, \mathfrak{c})$, $X$ has weight $\mathfrak{c}$, $f$ is irreducible, and $\omega^*$ cannot be mapped onto $X$.*

PROOF. Let $Y = \omega^*$ with the $G_{<\mathfrak{c}}$-topology, i.e the underlying set of $Y$ is $\omega^*$ and the intersections of fewer than $\mathfrak{c}$ clopen subsets of $\omega^*$ form an open basis for $Y$. Let $\mathcal{E}$ be a basis for $Y$ of cardinality $w(Y)$ consisting of clopen sets. By transfinite induction, for each $\alpha < \mathfrak{c}$ we will construct subalgebras $\mathcal{E}_\alpha \subseteq \mathcal{B}(Y)$ such that

(1) $\mathcal{E}_0 = \langle\langle \mathcal{E} \rangle\rangle$,

(2) $|\mathcal{E}_\alpha| \leq |\bigcup_{\beta<\alpha} \mathcal{E}_\beta|^\xi$,

(3) if $\beta < \alpha < \mathfrak{c}$ and if $\mathcal{F} \in [\mathcal{E}_\beta]^{<\mathfrak{c}}$, then $\bigcup \mathcal{F} \in \mathcal{E}_\alpha$.

It is straightforward to construct these algebras since the union of fewer than $\mathfrak{c}$ clopen subsets of $Y$ is clopen.

Put $\mathcal{B} = \bigcup_{\alpha<\mathfrak{c}} \mathcal{E}_\alpha$ and observe that if $\mathfrak{c}$ is regular, then $\mathcal{B}$ is a $<\mathfrak{c}$-closed subalgebra of $\mathcal{B}(Y)$. Let $X = \mathrm{st}(\mathcal{B})$. We claim that $X$ is as required. It is clear that the function $f : X \to \omega^*$ defined by

$$\{f(x)\} = \bigcap \{\bar{B} : B \in x\}$$

is a continuous surjection.

From now on, assume MA + $\neg$CH + $\neg G(\mathfrak{c}, \mathfrak{c})$. We also identify $Y$ and the subspace of $X$ consisting of the fixed ultrafilters on $\mathcal{B}$.

First observe that (2) implies that $X$ has weight $\mathfrak{c}$, since MA implies that $2^\kappa = \mathfrak{c}$ for all $\omega \leq \kappa < \mathfrak{c}$, see M.E. RUDIN [1977]. We first claim that $f$ is irreducible. To this end, let $A \subseteq X$ be a proper closed subset. Since $Y$ is dense in $X$, we can find a point $y \in Y \backslash A$. Choose $E \in \mathscr{E}$ such that $\bar{E} \cap A = \emptyset$ and $y \in \bar{E}$ (the closure is taken in $X$). Since $E$ is an intersection of fewer than $\mathfrak{c}$ clopen subsets of $\omega^*$, by $P(\mathfrak{c})$ we can find a nonempty clopen $C \subsetneq \omega^*$ such that $C \subseteq E$. It is clear that $C \cap f(A) = \emptyset$.

We will now show that $\omega^*$ cannot be mapped onto $X$. Fix $y \in Y$. We will construct a family $\{B(y, \alpha) : \alpha < \mathfrak{c}\}$ and a family $\{E(y, \alpha) : \alpha < \mathfrak{c}\}$ of clopen subsets of $X$ such that

(4) $\alpha < \beta < \mathfrak{c} \to B(y, \alpha) \subset B(y, \beta) \subset X \backslash \{y\}$,

(5) $\alpha < \beta < \mathfrak{c} \to E(y, \alpha) \subset E(y, \beta) \subset X \backslash \{y\}$,

(6) $\alpha < \beta < \mathfrak{c} \to B(y, \alpha) \cap E(y, \beta) = \emptyset$,

(7) $(\bigcup\{B(y, \alpha) : \alpha < \mathfrak{c}\})^- \cap (\bigcup\{E(y, \alpha) : \alpha < \mathfrak{c}\})^- = \{y\}$,

(8) $\bigcup\{B(y, \alpha) : \alpha < \mathfrak{c}\} \cup \bigcup\{E(y, \alpha) : \alpha < \mathfrak{c}\} = X \backslash \{y\}$.

(This construction is a triviality of course). Let $\{Z_\alpha : \alpha < \mathfrak{c}\}$ enumerate the family of all clopen subsets of $X$ containing $y$. To achieve (7) and (8), we will make the required families of clopen sets such that

(9) $Z_\alpha \cap B(y, \alpha) \neq \emptyset$, $Z_\alpha \cap E(y, \alpha) \neq \emptyset$ and $X \backslash Z_\alpha \subseteq B(y, \alpha) \cup E(y, \alpha)$.

So our induction hypotheses are (4), (5), (6) and (9). Suppose that we have completed the construction for all $\alpha < \beta < \mathfrak{c}$. Put

$$B = \bigcup_{\alpha<\beta} B(y, \alpha) \quad \text{and} \quad E = \bigcup_{\alpha<\beta} E(y, \alpha).$$

Then $\bar{B}$ and $\bar{E}$ are both open, since $\mathscr{B}$ is $<\mathfrak{c}$-closed, which implies, by (6), that $\bar{B} \cap \bar{E} = \emptyset$. Since $y$ is a $P_\mathfrak{c}$-point of $X$, i.e. the intersection of fewer than $\mathfrak{c}$ neighborhoods of $y$ is again a neighborhood of $y$, $y \notin \bar{B} \cup \bar{E}$. Let $F \subseteq Z_\alpha$ be a clopen neighborhood of $y$ which misses $\bar{B} \cup \bar{E}$. Take two disjoint clopen nonempty subsets $G, H \subseteq F$ which do not contain $y$. Define

$$B(y, \alpha) = \bar{B} \cup G \cup (X \backslash (\bar{E} \cup Z_\alpha)) \quad \text{and} \quad E(y, \alpha) = \bar{E} \cup H.$$

It is clear that our inductive hypotheses are satisfied.

Now suppose that there is a continuous surjection $g : \omega^* \to X$. Put

$$B_y = (\bigcup\{g^{-1}(B(y, \alpha)) : \alpha < \mathfrak{c}\})^- \quad \text{and} \quad E_y = (\bigcup\{g^{-1}(E(y, \alpha)) : \alpha < \mathfrak{c}\})^-.$$

By $\neg G(\mathfrak{c}, \mathfrak{c})$, $B_y \cap E_y = \emptyset$. Observe that $y$ is the unique point of $Y$ with the property that $g^{-1}(y)$ meets both $B_y$ and $E_y$.

Let $Y_0 = \{y \in Y : B_y \cup E_y = \omega^*\}$. Then $B_y$ and $E_y$ are both clopen and since if $y_0, y_1 \in Y$ are distinct, then $B_{y_0} \neq B_{y_1}$, we have that $|Y_0| \leq \mathfrak{c}$.

Let $Y_1 = \{y \in Y : B_y \cup E_y \neq \omega^*\}$. If $y \in Y_1$, then $g^{-1}(y)$ has nonempty interior in $\omega^*$, which implies that $|Y_1| \leq \mathfrak{c}$.



Since by 3.1.2 (c), $|Y| = 2^\mathfrak{c}$ and $Y = Y_0 \cup Y_1$, we have the desired contradiction. □

**2.3.3.** REMARK. An inspection of the proof of Theorem 2.3.2 will show that we 'only' need the following hypotheses:

(1) $2^\kappa = \mathfrak{c}$ if $\omega \leq \kappa < \mathfrak{c}$, and

(2) $\neg G(\mathfrak{c}, \mathfrak{c})$.

**2.3.4.** REMARK. If $\mathscr{B}$ is the BA of clopen subsets of the space of Theorem 2.3.2, then $\mathscr{P}(\omega)/\text{fin}$ can be embedded in $\mathscr{B}$, $\mathscr{P}(\omega)/\text{fin}$ and $\mathscr{B}$ have isomorphic completions, but $|\mathscr{B}| = \mathfrak{c}$ and $\mathscr{B}$ cannot be embedded in $\mathscr{P}(\omega)/\text{fin}$.

**2.3.5.** REMARK. Let $X$ be the space of Theorem 2.3.2. Observe that $Z = (\omega \times X)^*$ is a Parovičenko space which is not a continuous image of $\omega^*$, since $Z$ can be mapped onto $X$.

Until now it is not clear yet that Theorem 2.3.2 has some use, for it is not obvious at all that MA + $\neg$CH + $\neg G(\mathfrak{c}, \mathfrak{c})$ can be true. Fortunately, KUNEN [1981] has shown the following.

**2.3.6.** THEOREM. (A) *It is consistent with* MA + $\neg$CH *that* $G(\omega_1, \mathfrak{c})$ *and* $G(\mathfrak{c}, \mathfrak{c})$ *both are false,*

(B) *it is consistent with* MA + $\neg$CH *that* $G(\omega_1, \mathfrak{c})$ *and* $G(\mathfrak{c}, \mathfrak{c})$ *both are true.*

Let $X$ be the space constructed in the proof of Theorem 1.7.3, i.e. $X$ is the Stone space of the reduced measure algebra of $[0, 1]$. It is unknown whether it is consistent that $X$ is not a continuous image of $\omega^*$. It will not be possible to deduce this from rather global properties of $X$, since BELL [1980] has constructed in ZFC examples of spaces which are very similar to $X$ and which are continuous images of $\omega^*$.

Let us finally notice that PRZYMUSIŃSKI [1982] has shown that each perfectly normal compact space is a continuous image of $\omega^*$. The big open question in this area is whether every first countable compactum is a continuous image of $\omega^*$.

## 2.4. *Closed subspaces of* $\beta\omega$, *II*

In Section 1.4 we showed that every compact zero-dimensional $F$-space of weight $\mathfrak{c}$ embeds, under CH, in $\beta\omega$. This suggests to consider the following statement:

FE: *Every compact zero-dimensional F-space can be embedded in an Extremally disconnected space.*

Observe that in Boolean algebraic language FE is the statement that each WCC BA is a homomorphic image of some complete BA.

It is convenient to factor FE as FB + BE, where

FB: *Every compact zero-dimensional F-space can be embedded in a Basically disconnected space,*

and

BE: *Every Basically disconnected compact space can be embedded in an Extremally disconnected space.*

Of course, both FB and BE have straightforward Boolean algebraic translations. In Section 1.4 we showed, in particular, that the restriction of FE to spaces of weight $\mathfrak{c}$ holds under CH. VAN DOUWEN & VAN MILL [1980] construct, under MA + $\mathfrak{c} = \omega_2$, an example of a compact zero-dimensional $F$-space $V$ of weight $\mathfrak{c}$ that cannot be embedded in any basically disconnected space. As a consequence, neither FE nor FB are theorems of ZFC. Very little is known about BE, it is known however that the Čech–Stone compactification of any $P$-space embeds in an extremally disconnected space, see Section 4.4. We conclude that Theorems 1.4.4, 1.4.5 and Corollary 1.4.6 are false under MA + $\mathfrak{c} = \omega_2$.

## 2.5. $C^*$-embedded subspaces of $\beta\omega$, II

It is easy to see that Theorem 1.5.3 need not be true. If $2^{\omega_1} = \mathfrak{c}$, then by Theorem 1.4.7, $\beta\omega_1$ embeds in $\omega^*$, say by the embedding $h$. It is clear that $h(\omega_1)$ is $C^*$-embedded in $\beta\omega$, but $h(\omega_1)$ is not weakly Lindelöf. It is not so clear that Corollary 1.5.4 need not be true.

**2.5.1.** LEMMA [$\forall\kappa < \mathfrak{c},\ \neg G(\kappa, \omega) + \neg G(\mathfrak{c}, \mathfrak{c})$]. *If $A \subseteq \omega^*$ is a closed $P_\mathfrak{c}$-set, then $\omega^*\backslash A$ is $C^*$-embedded in $\omega^*$.*

PROOF. Striving for a contradiction, assume there are disjoint, nonempty, closed $G_\delta$-subsets $Z_0$, $Z_1 \subseteq \omega^*\backslash A$ such that $\bar{Z}_0 \cap \bar{Z}_1 \neq \emptyset$. Pick a point $a \in \bar{Z}_0 \cap \bar{Z}_1$ and let $\{C_\alpha : \alpha < \mathfrak{c}\}$ enumerate the family of all clopen subsets of $\omega^*$ containing $a$. By transfinite induction on $\alpha < \mathfrak{c}$, we will construct clopen subsets $G^i_\alpha$ $(i < 2)$ of $\omega^*$ such that

(1) $G^i_\alpha \subseteq Z_i$ and $G^i_\alpha \cap C_\alpha \neq \emptyset$,

(2) if $\beta < \alpha$, then $G^i_\beta \subset G^i_\alpha$.

If we can complete the induction, then we contradict $\neg G(\mathfrak{c}, \mathfrak{c})$. Suppose that the sets $G^i_\beta$ are defined for all $\beta < \alpha$, $i < 2$. Since $A$ is a $P_\mathfrak{c}$-set, there is a clopen $C \subseteq \omega^*\backslash A$ such that

$$\bigcup_{\beta<\alpha} G^0_\beta \cup \bigcup_{\beta<\alpha} G^1_\beta \subseteq C.$$

By $\neg G(\alpha, \omega)$, we can find clopen sets $C_i \subseteq C \cap Z_i$ such that

$$\bigcup_{\beta<\alpha} G^i_\beta \subseteq C_i.$$



Put $C'_\alpha = C_\alpha \cap (\omega^*\backslash C)$. Take $x_i \in Z_i \cap C'_\alpha$. Let $E_i \subseteq \omega^*$ be clopen neighborhoods of $x_i$ not meeting $A$. By Theorem 1.2.5, $E_i \cap Z_i$ contains a non-empty clopen set, say $F_i$. Define $G^i_\alpha = C_i \cup F_i$. This completes the induction, which gives us the required contradiction. □

**2.5.2.** COROLLARY [$\forall\kappa < \mathfrak{c},\ \neg G(\kappa, \omega) + \neg G(\mathfrak{c}, \mathfrak{c})$]. *Let $A = \{x \in \omega^* : \exists$ closed nowhere dense $P_\mathfrak{c}$-set $B \subseteq \omega^*$ containing $x\}$. If $x \in A$, then $\omega^*\backslash\{x\}$ is $C^*$-embedded in $\omega^*$.*

PROOF. By Lemma 2.5.1, if $x \in A$ and if $B$ is a nowhere dense closed $P_\mathfrak{c}$-set containing $x$, then $\beta(\omega^*\backslash B) = \omega^*$. But this easily implies that $\beta(\omega^*\backslash\{x\}) = \omega^*$. □

The question arises of course whether Corollary 2.5.2 is of any use, i.e. is it possible that the set $A$ is nonempty, while moreover the combinatorial hypotheses required for the proof of Corollary 2.5.2 hold. The answer is yes of course. By Theorem 2.3.6, there is a model in which MA + $\neg$CH + $\neg G(\mathfrak{c}, \mathfrak{c})$ is true. It is easy to show that MA implies there are $P_\mathfrak{c}$-points and that MA implies $\neg G(\omega, \kappa)$ for all $\kappa < \mathfrak{c}$.

Consequently, we obtain

**2.5.3.** COROLLARY TO COROLLARY. *It is consistent that for some $x \in \omega^*$ we have that $\beta(\omega^*\backslash\{x\}) = \omega^*$.*

It is unpleasant that the point $x$ of Corollary 2.5.3 does not 'really' exist, since it is a $P_\mathfrak{c}$-point. We will show that there are many points which 'really' exist for which it is consistent that their complements in $\omega^*$ are $C^*$-embedded in $\omega^*$.

**2.5.4.** THEOREM [$\forall\kappa < \mathfrak{c},\ \neg G(\kappa, \omega) + \neg G(\mathfrak{c}, \mathfrak{c})$]. *If $x \in \omega^*$ is not a $P$-point, then $\omega^*\backslash\{x\}$ is $C^*$-embedded in $\omega^*$.*

PROOF. Let $x \in \omega^*$ be not a $P$-point and let $U \subseteq \omega^*$ be an open $F_\sigma$ such that $x \in \bar{U}\backslash U$. Let $f : \omega^*\backslash\{x\} \to [0, 1]$ be continuous. Let $f_0 = f \restriction \bar{U}\backslash\{x\}$ and $f_1 = f \restriction \omega^*\backslash\bar{U}$. By $\forall\kappa < \mathfrak{c}$, $\neg G(\kappa, \omega)$, we have that $\bar{U}$ is a $P_\mathfrak{c}$-set in $\omega^*$. Consequently, by Lemma 2.5.1, we can extend $f_1$ to a continuous map $g_1 : \omega^*\backslash\bar{U} \to [0, 1]$. By Theorem 1.2.5, $\omega^*\backslash\bar{U} = (\omega^*\backslash\bar{U}) \cup (\bar{U}\backslash U)$. This implies that $g_1(t) = f(t)$ for all $t \in \bar{U}\backslash(U \cup \{x\})$. By Theorem 1.5.2, $U$ is $C^*$-embedded in $\bar{U}$, consequently, $\bar{U}\backslash\{x\}$ is $C^*$-embedded in $\bar{U}$. We therefore conclude that we can extend $f_0$ to a continuous map $g_0 : \bar{U} \to [0, 1]$. Since $g_0 \restriction \bar{U}\backslash(U \cup \{x\}) = g_1 \restriction \bar{U}\backslash(U \cup \{x\})$, we conclude that $g_0 \restriction \bar{U}\backslash U = g_1 \restriction \bar{U}\backslash U$ since $x$ is not isolated (Theorem 1.2.5). Define $g : \omega^* \to [0, 1]$ by

$$\begin{cases} g(t) = g_0(t) & \text{if } t \in \bar{U}, \\ g(t) = g_1(t) & \text{if } t \notin \bar{U}. \end{cases}$$

It is obvious that $g$ is continuous and that $g$ extends $f$. □