

Exponential patterns in arithmetic Ramsey theory

by

JULIAN SAHASRABUDHE (Memphis, TN)

1. Introduction. The study of arithmetic Ramsey theory, broadly speaking, looks to better understand what arithmetic structure is guaranteed within a colour class of an arbitrary finite colouring of the natural numbers. This area has its origins in the 1892 work of Hilbert [10] and has received considerable attention over the last century. For example, a classical result of Schur [21], now over one hundred years old, says that if one colours the integers with finitely many colours, one can find elements x, y, z satisfying $x + y = z$ that all receive the same colour. Another cornerstone result, published in 1927, is due to van der Waerden [26] who showed that every finite colouring of the integers contains arbitrarily long monochromatic arithmetic progressions. The theory of finite systems of equations that may be solved in a colour class has come to be well understood: Rado [18], in 1933, classified the finite systems of homogeneous linear equations that admit a monochromatic solution in an arbitrary colouring of the natural numbers, and subsequently Hindman and Leader [14] extended these results to a complete theory for finite linear systems. For the details of this development and many other related results, see the survey of Hindman [13].

Progress on finding collections of non-linear patterns that admit monochromatic solutions has proven to be more difficult. Various results on polynomial patterns have been obtained, including the celebrated “polynomial van der Waerden theorem” of Bergelson and Leibman [3] (see also the elementary proof of Walters [27]), which states that for any polynomials f_1, \dots, f_d , $d \in \mathbb{N}$, with integer coefficients and constant term zero, one can find a monochromatic pattern of the form $\{a + f_1(b), \dots, a + f_d(b)\}$, in a given finite colouring, with $a, b \in \mathbb{N}$. However, many open questions remain, even in the domain of polynomial patterns [5]. For example, it is not

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known if a finite colouring of the integers admits a monochromatic solution to $x^2 + y^2 = z^2$ [7]. Perhaps even more surprisingly, it is unknown whether a finite colouring of \mathbb{N} must contain a monochromatic quadruple of the form $\{a, b, a + b, ab\}$ [15]. However, a recent breakthrough on this problem is due to Moreira [17], who has shown that every finite colouring of \mathbb{N} admits a monochromatic pattern of the form $\{a, a + b, ab\}$. In what follows, we drop the set braces around the monochromatic patterns.

Infinite linear systems of equations have also been studied. Notably, Hindman [12] proved that every finite colouring of the natural numbers admits positive integers x_1, x_2, \dots such that all the finite sums of distinct x_i receive the same colour. Other examples of infinite linear systems that admit monochromatic solutions in an arbitrary finite colouring are known [2], [16], [25]; however, it appears that we are far from a classification of such infinite systems of equations [15].

The starting point of this section is a surprising observation made by Sisto in 2011 [22], who showed that an arbitrary 2-colouring of \mathbb{N} admits infinitely many monochromatic triples of the form a, b, a^b (henceforth: *exponential triples*). In his paper, Sisto went on to ask if one can find monochromatic exponential triples in an arbitrary *finite* colouring of \mathbb{N} . Brown [4], simplifying and extending the proof of Sisto, gave further examples of exponential, monochromatic patterns that are present in an arbitrary 2-colouring of the positive integers and proved some weaker results for more colours.

Our first task will be to answer the original question of Sisto by showing that any *finite* colouring of the positive integers admits $a, b > 1$ such that a, b, a^b is monochromatic.

THEOREM 1. *For every finite colouring of \mathbb{N} there exist integers $a, b > 1$ such that a, b, a^b is monochromatic.*

As the proof of this is much simpler than our main theorem, we devote Section 2 to giving a self-contained proof of Theorem 1. We should remark that there is no natural “density-type” result for exponential triples, in the sense of Szemerédi’s theorem [23] (see [24]). In fact, for $N \in \mathbb{N}$ the set $[N] \setminus \{a^i : a \in \mathbb{N}, i \geq 2\}$ is of size $(1 - o(1))N$ and contains no exponential triple.

The main result that is presented in this section is Theorem 4, which says that a certain class of patterns defined by exponentiation and multiplication (which we call FEP-sets) is partition regular. As the statement of the result itself is a little technical, we postpone the presentation of Theorem 4 and start, instead, by mentioning two more palatable corollaries of our master theorem. The first such corollary of Theorem 4 actually has many (but not all) of the difficulties of the full theorem.

THEOREM 2. *For every finite colouring of \mathbb{N} there exist $a, b > 1$ such that a, b, ab, a^b is monochromatic.*

To go further, we need some terminology. If X is a finite set, call a function $f: \mathbb{N} \rightarrow X$ a *finite colouring* of \mathbb{N} and refer to the elements of X as *colours*. For a finite colouring f of \mathbb{N} , we say that a set $A \subseteq \mathbb{N}$ is *monochromatic* if f is constant on A . If $\mathcal{A} \subseteq \mathcal{P}(\mathbb{N})$, we say that \mathcal{A} is *partition regular* if for every finite colouring of \mathbb{N} , one can find a monochromatic set contained in \mathcal{A} . For $m \in \mathbb{N}$ and integers x_1, \dots, x_m , we define the *finite sum set* and *finite product set generated by x_1, \dots, x_m* as

$$\text{FS}(x_1, \dots, x_m) = \left\{ \sum_{i \in I} x_i : \emptyset \neq I \subseteq [m] \right\},$$

$$\text{FP}(x_1, \dots, x_m) = \left\{ \prod_{i \in I} x_i : \emptyset \neq I \subseteq [m] \right\},$$

respectively. Define the *finite exponential set generated by x_1, \dots, x_m* by first setting $\text{FE}(x_m) = \{x_m\}$ and then, inductively assuming that $\text{FE}(x_m), \dots, \text{FE}(x_2, \dots, x_m)$ have been defined, we define $\text{FE}(x_1, \dots, x_m)$ to be the set of integers

$$\{x_i^{e_{i+1}e_{i+2}\cdots e_m}\},$$

where $i \in [m]$ and

$$e_{i+1} \in \text{FE}(x_{i+1}, \dots, x_m) \cup \{1\}, \dots, e_m \in \text{FE}(x_m) \cup \{1\}.$$

Informally speaking, $\text{FE}(x_1, \dots, x_m)$ is the set of all exponential compositions of the integers x_1, \dots, x_m that “respect the order” of x_1, \dots, x_m . The analogy with finite sum and product sets is made more salient by first considering exponentiation as a binary operation \star , defined by

$$a \star b = a^b,$$

and then noting that the set $\text{FE}(x_1, \dots, x_m)$ contains all elements of the form (although not exclusively)

$$x_{i_1} \star \cdots \star x_{i_l},$$

where $l \in [m]$, $1 \leq i_1 < \cdots < i_l \leq m$, and the brackets may be placed in any syntactically valid manner.

Using this notation, we state our second, “easily digestible” corollary of Theorem 4, before we go on to give it in full generality.

THEOREM 3. *For every $m \in \mathbb{N}$ and every finite colouring of \mathbb{N} there exist integers $x_1, \dots, x_m > 1$ for which*

$$\text{FP}(x_1, \dots, x_m) \cup \text{FE}(x_1, \dots, x_m)$$

is monochromatic.

Before coming to our main theorem, we augment our definition of a finite product set. Given a set $S \subseteq [m]$ and an arbitrary “weight” function $W: \mathbb{N}^{(\leq m)} \rightarrow \mathbb{N}$, on subsets of \mathbb{N} of cardinality at most m , we define

$$\begin{aligned} \text{FP}_{S,W}(x_1, \dots, x_m) \\ = \left\{ \prod_{i \in S} x_i^{p_i} : p_i \in [0, W(\{x_{i+1}, \dots, x_m\})] \text{ for each } i \in S \right\}, \end{aligned}$$

and refer to an element of $\text{FP}_{S,W}(x_1, \dots, x_m)$ as a W -weighted product of x_1, \dots, x_m supported on $\{x_i\}_{i \in S}$. To illustrate, set $m = 4$, $S = [4]$ and define the weight function on subsets of \mathbb{N} of size at most 4 by $W(\{x_1, \dots, x_j\}) = \sum_{i=1}^j x_i$ for $j \in [4]$. Also set $W(\emptyset) = 10$. In this case, $\text{FP}_{S,W}(x_1, \dots, x_m)$ is precisely the set of integers $x_1^a x_2^b x_3^c x_4^d$, where $a \in [0, x_2 + x_3 + x_4]$, $b \in [0, x_3 + x_4]$, $c \in [0, x_4]$ and $d \in [0, 10]$. If instead $S = \{2, 3\}$, we would have $\text{FP}_{S,W}(x_1, x_2, x_3, x_4) = \{x_2^a x_3^b : a \in [0, x_3 + x_4], b \in [0, x_4]\}$.

We define the *finite exponential-product set generated by x_1, \dots, x_m with weight W* to be the set of integers

$$\begin{aligned} \text{FEP}_W(x_1, \dots, x_m) \\ = \left\{ \prod_{i \in B} x_i^{e_i} : \emptyset \neq B \subseteq [m], e_i \in \text{FP}_{[m] \setminus B, W}(x_{i+1}, \dots, x_m) \text{ for each } i \in B \right\}. \end{aligned}$$

One can think of the above FEP-set as the collection of integers that one can construct from x_1, \dots, x_m according to the following rule: “First choose some non-empty $B \subseteq [m]$ as the indices of the ‘bases’ in the product; then, for each $i \in B$, choose an integer e_i to be a W -weighted product of x_1, \dots, x_m that is supported on $\{x_{i+1}, \dots, x_m\} \setminus \{x_i : i \in B\}$; then form $\prod_{i \in B} x_i^{e_i}$ ”.

So, for appropriately chosen W , the set $\text{FEP}_W(a, b, c)$ contains a, b, c ; the “height-one” exponential compositions a^b, b^c, a^c ; the products ab, ac, bc, abc ; various “tower-type” expressions such as $a^{b^c}, a^{b^{c^2}}, a^{b^c c}, a^{b^{c^c}}, b^{c^2}$; along with many patterns which mix exponentiation and multiplication, for example, $a^b c, a^{b^c} c, (ab)^c$, and ab^c . It is also worth pointing out that for every $k \in \mathbb{N}$, one may choose W so that $\text{FEP}_W(a, b, c)$ contains the “exponential progression” $a, b, a^b, a^{b^2}, \dots, a^{b^k}$.

The main result of this section, to whose proof we dedicate Sections 3 and 4, is that the collection $\{\text{FEP}_W(x_1, \dots, x_m)\}_{x_1, \dots, x_m \in \mathbb{N}}$ of FEP_W -sets is partition regular for any $m \in \mathbb{N}$ and any weight function W .

THEOREM 4. *Let $m \in \mathbb{N}$ and $W: \mathbb{N}^{(\leq m)} \rightarrow \mathbb{N}$ be an arbitrary function. For every finite colouring of \mathbb{N} there exist integers $x_1, \dots, x_m > 1$ such that $\text{FEP}_W(x_1, \dots, x_m)$ is monochromatic.*

From Theorem 4 we easily deduce Theorem 3 (and hence Theorem 2) by simply choosing W to be a function that is growing sufficiently quickly.

Towards proving the non-partition regularity of certain exponential patterns, in Section 5 we give examples of colourings which we shall use to show the non-partition regularity of several natural exponential patterns. For example, we show that a, b, a^b, b^a is *not* partition regular. Indeed, using these colourings along with Theorem 4, we shall deduce the following classification of “height-one” exponential systems.

Suppose that $m \in \mathbb{N}$, and that R is a binary relation on $[m] \times [m]$. If $x_1, \dots, x_m \in \mathbb{N}$, we define the set of integers

$$R\{x_1, \dots, x_m\} = \{x_1, \dots, x_m\} \cup \{x_i^{x_j} : (i, j) \in R\},$$

and then define the *exponential pattern with shape R* to be the collection of all such $R\{x_1, \dots, x_m\}$ with $x_1, \dots, x_m > 1$. For example, if $m = 4$ and $R = \{(1, 2), (2, 3), (2, 4)\}$, the exponential pattern with shape R is the collection

$$\{\{x_1, x_2, x_3, x_4, x_1^{x_2}, x_2^{x_3}, x_2^{x_4}\} : x_1, x_2, x_3, x_4 \in \mathbb{N} \setminus \{1\}\}.$$

We classify the relations that give rise to partition regular exponential patterns.

THEOREM 5. *Let $m \in \mathbb{N}$ and let $R \subseteq [m] \times [m]$ be a binary relation. The exponential pattern with shape R is partition regular if and only if R does not contain a directed cycle.*

In Section 6 we turn to the relationship between exponentiation and addition in the context of partition regularity. In sharp contrast to the situation with *multiplication* and exponentiation, we show that partition regular patterns involving addition and exponentiation are very limited. In particular, we show that the partition regular patterns $x, y, x + y$ and a, b, a^b are *inconsistent*. This means that an arbitrary colouring of \mathbb{N} need not admit a monochromatic set of the form $x, y, x + y, a, b, a^b$, despite the fact it must have a monochromatic triple $x, y, x + y$ (by Schur’s Theorem) and a monochromatic triple of the form a, b, a^b (Theorem 1).

THEOREM 6. *There exists a finite colouring of \mathbb{N} for which there is no monochromatic set of the form $x, y, x + y, a, b, a^b$ with $x, y, a, b \in \mathbb{N}$.*

More generally, we say that two partition regular patterns $\mathcal{A}, \mathcal{B} \subseteq \mathcal{P}(\mathbb{N})$ are *consistent* if for every finite colouring of \mathbb{N} , one can find $A \in \mathcal{A}$ and $B \in \mathcal{B}$ for which f is constant on both A, B and $f(A) = f(B)$.

So we have the following situation: while $x, y, x + y$ and $u, v, u \cdot v$ are known to be consistent (by a result of Hindman ⁽¹⁾ [11]) and our main

⁽¹⁾ Hindman actually showed much more: in a given finite colouring, one can find x_1, x_2, \dots and y_1, y_2, \dots such that $\text{FS}(x_1, x_2, \dots)$ and $\text{FP}(y_1, y_2, \dots)$ lie in the same colour class.

theorem certainly implies that a, b, a^b and $u, v, u \cdot v$ are consistent, it is *not* true that $x, y, x + y$ and a, b, a^b are consistent.

In Section 7, we outline some further questions and conjectures in this area.

2. Monochromatic exponential triples. The aim of this section is to give a short, self-contained proof that for every finite colouring of \mathbb{N} one can find $a, b > 1$ such that the triple a, b, a^b is monochromatic. The purpose of this is two-fold: first, to present a short solution to Sisto's question and, second, to give a flavour for the style of proof that we adopt here.

We use an infinitary approach throughout this section despite the fact that we could have used finitary arguments and obtained explicit bounds on the various parameters. This approach streamlines the presentation and makes it more readable.

We shall require van der Waerden's classical theorem: for every $l, k \in \mathbb{N}$, there exists an integer $W_k(l)$ such that every k -colouring of $[W_k(l)]$ admits a monochromatic arithmetic progression of length l .

Let $k \in \mathbb{N}$, X be a countable set and $f: X \rightarrow [k]$ be a k -colouring. Also, for each $M \in \mathbb{N}$, let $f_M: X \rightarrow [k]$ be a k -colouring. We say that the sequence $\{f_M\}$ converges to f and write $f_M \rightarrow f$ if for every finite set $Y \subseteq X$ there exists some $M^* \in \mathbb{N}$ such that for all $M \geq M^*$ we have $f_M(x) = f(x)$ for all $x \in Y$. The following basic fact on sequences of colourings is often referred to as the *compactness property*.

FACT 7. *Let X be a countable set, $k \in \mathbb{N}$, and let $f_M: X \rightarrow [k]$ be a k -colouring for each $M \in \mathbb{N}$. There exists an increasing sequence $\{M(n)\}_n$ and a colouring $f: X \rightarrow [k]$ such that $f_{M(n)} \rightarrow f$ as $n \rightarrow \infty$. ■*

Given colourings $f_1, \dots, f_r: \mathbb{N} \rightarrow [k]$, we call a sequence of colours $c_1, \dots, c_r \in [k]$ large with respect to f_1, \dots, f_r if, for each $M \in \mathbb{N}$, we can find a progression $P(M)$ of length M such that $f_i(P(M)) = c_i$ for each $i \in [r]$. The following is an easy consequence of van der Waerden's theorem.

LEMMA 8. *Let $r, k \in \mathbb{N}$, $f_1, \dots, f_r, f_{r+1}: \mathbb{N} \rightarrow [k]$ be k -colourings, and let $c_1, \dots, c_r \in [k]$. If c_1, \dots, c_r is large with respect to f_1, \dots, f_r then there exists a colour $c_{r+1} \in [k]$ such that c_1, \dots, c_{r+1} is large with respect to f_1, \dots, f_{r+1} .*

Proof. Let $M \in \mathbb{N}$ be a parameter. Use the fact that c_1, \dots, c_r is large with respect to f_1, \dots, f_r to find a progression $P(M)$ of length $W_k(M)$ (the k -colour, length- M van der Waerden number) such that $f_i(P(M)) = c_i$ for all $i \in [r]$. We apply van der Waerden's theorem to the progression $P(M)$ coloured with f_{r+1} to obtain a progression $P'(M) \subseteq P(M)$ of length M which is monochromatic with respect to f_{r+1} . Therefore $P'(M)$ is monochromatic with respect to each colouring f_1, \dots, f_{r+1} . Now apply this argument

for each $M \in \mathbb{N}$. Some colour is obtained infinitely often as the colour $f_{r+1}(P(M))$. Choose c_{r+1} to be this colour. Thus c_1, \dots, c_{r+1} is large with respect to f_1, \dots, f_{r+1} . ■

For $d \in \mathbb{N}$ and a k -colouring $f: \mathbb{N} \rightarrow [k]$, we define the colouring $f(d; \cdot): \mathbb{N} \rightarrow [k]$ by $f(d; x) = f(2^{d2^x})$ for $x \in \mathbb{N}$. The following lemma is the core of the proof of Theorem 1. Notice that from conclusion (3) in Lemma 9, we immediately obtain a contradiction if we choose $r > k$.

LEMMA 9. *For $k \in \mathbb{N}$, suppose that $f: \mathbb{N} \rightarrow [k]$ is a k -colouring admitting no monochromatic exponential triple. Then, for any $r \in \mathbb{N}$, we may find colours $c_1, \dots, c_r \in [k]$ and colourings $\tilde{f}_1, \dots, \tilde{f}_r: \mathbb{N} \rightarrow [k]$ such that the following hold:*

- (1) *The sequence of colours c_1, \dots, c_r is large with respect to $\tilde{f}_1, \dots, \tilde{f}_r$.*
- (2) *For each $i \in [r]$ there exists a sequence $\{d_i(N)\}_N$ of positive integers for which*

$$f(d_i(N); \cdot) \rightarrow \tilde{f}_i \quad \text{as } N \rightarrow \infty.$$

- (3) *The colours c_1, \dots, c_r are distinct.*

Proof. We apply induction on r . If $r = 1$ we choose $d_1(N) = 1$ for all $N \in \mathbb{N}$, and set $\tilde{f}_1 = f_1(1; \cdot)$. We therefore have the trivial convergence $f(1; \cdot) \rightarrow \tilde{f}_1$. Now, by van der Waerden's theorem, there exists $c_1 \in [k]$ that is large with respect to \tilde{f}_1 . This proves the lemma for $r = 1$.

For the induction step, assume we have found colourings $\tilde{f}_1, \dots, \tilde{f}_r$ with associated colours c_1, \dots, c_r , which satisfy the statement of the lemma. For $M \in \mathbb{N}$, we use the fact that the sequence c_1, \dots, c_r is large with respect to $\tilde{f}_1, \dots, \tilde{f}_r$ to find a progression $P(M)$ of length M for which $f_i(P(M)) = c_i$ for all $i \in [r]$. Letting $d(M)$ be the common difference of $P(M)$, we pass to a subsequence of $\{d(M)\}_M$ such that the colouring $f(d(M); \cdot)$ tends to a limit. We take this subsequence as our sequence $\{d_{r+1}(M)\}$ and take the resulting limit point to be \tilde{f}_{r+1} . Thus we have the convergence $f(d_{r+1}(N); \cdot) \rightarrow \tilde{f}_{r+1}$ as $N \rightarrow \infty$, by construction.

We now claim that $\tilde{f}_{r+1}(x) \notin \{c_1, \dots, c_r\}$ for all $x \in \mathbb{N}$. Suppose that $\tilde{f}_{r+1}(x_0) = c_t$ for some $x_0 \in \mathbb{N}$ and $t \in [r]$. Since we have the convergence $f(d_{r+1}(N); \cdot) \rightarrow \tilde{f}_{r+1}$, we may choose some sufficiently large N so that $N \geq 2^{x_0}$ and $\tilde{f}_{r+1}(x) = f(d_{r+1}(N); x)$ for all $x \in [x_0]$. Therefore

$$c_t = \tilde{f}_{r+1}(x_0) = f(d_{r+1}(N); x_0) = f(2^{d_{r+1}(N)2^{x_0}}).$$

Now recall that $d_{r+1}(N)$ is the common difference of a progression $P(N) = \{a(N) + xd_{r+1}(N) : x \in [0, N - 1]\}$ of length $N \geq 2^{x_0}$ with $\tilde{f}_t(P(N)) = c_t$. With this in mind, we choose a sufficiently large integer N' so that $f(d_t(N'); x) = \tilde{f}_t(x)$ for all $x \in [a(N) + d_{r+1}(N)2^{x_0}]$. We therefore have

$$c_t = \tilde{f}_t(a(N)) = f(d_t(N'); a(N)) = f(2^{d_t(N')2^{a(N)}})$$

and

$$\begin{aligned} c_t &= \tilde{f}_t(a(N) + d_{r+1}(N)2^{x_0}) = f(d_t(N'); a(N) + d_{r+1}(N)2^{x_0}) \\ &= f(2 \star (d_t(N')2 \star (a(N) + d_{r+1}(N)2^{x_0}))). \end{aligned}$$

However, this yields a contradiction, as we may find an exponential triple, monochromatic in c_t , by choosing

$$a = 2^{d_t(N')2^{a(N)}}, \quad b = 2^{d_{r+1}(N)2^{x_0}}$$

and then noting that

$$a^b = 2 \star (d_t(N')2 \star (a(N) + d_{r+1}(N)2^{x_0})).$$

Therefore, $\tilde{f}_{r+1}(x) \notin \{c_1, \dots, c_r\}$.

We complete the proof by applying Lemma 8 to find a colour c_{r+1} such that the sequence of colours c_1, \dots, c_{r+1} is large with respect to $\tilde{f}_1, \dots, \tilde{f}_{r+1}$. As $\tilde{f}_{r+1}(x) \notin \{c_1, \dots, c_r\}$ it follows that c_{r+1} is distinct from c_1, \dots, c_r . ■

Theorem 1 now easily follows by assuming that a given k -colouring, $k \in \mathbb{N}$, admits no monochromatic exponential triple and then applying the above lemma with the choice of $r > k$.

3. Definitions and preliminaries. Given integers $0 \leq x \leq y$, define $[x, y] = \{x, x+1, \dots, y\}$ and $[y] = \{1, \dots, y\}$. We denote by \mathbb{N}_0 the set $\mathbb{N} \cup \{0\}$. If X is a set and $k \in \mathbb{N}$, we let $X^{(k)}$ denote the collection of all subsets of X of cardinality k and let $X^{(\leq k)}$ denote the collection of subsets of cardinality *at most* k . Suppose that $n, d \in \mathbb{N}$, $A, B \subseteq \mathbb{Z}^n$, and $t \in \mathbb{Z}^n$. We define $A + t = \{a + t : a \in A\}$, $dA = \{da : a \in A\}$, and $A + B = \{a + b : a \in A, b \in B\}$.

For positive integers n, T , define an n -dimensional T -box to be a set of the form $[0, T]^d + s \subseteq \mathbb{N}_0^n$, where $s \in \mathbb{N}_0^n$. We refer to s as the *corner* of the T -box (it is clear that the corner is uniquely defined). For $d, L \in \mathbb{N}$, define a *grid of common difference d , length L , and dimension n* to be a set of the form $s + d[0, L]^n$, where $s \in \mathbb{N}_0^n$.

We also define a *fat grid of thickness T , common difference d , length L , and dimension n* to be a subset of \mathbb{N}_0^n of the form $s + [0, T]^n + d[0, L]^n$, where $s \in \mathbb{N}_0^n$ and $d > T$. If G is a fat grid as above, we denote by $B(G)$ the box $s + [0, T]^n$ ($B(G)$ is uniquely determined as $d > T + 1$). Finally, we define the *corner* of the fat grid G to be s . Note that the corner of a grid G coincides with the corner of its box $B(G)$.

In what follows, we shall make repeated use of the following theorem of Gallai [8], [18], which gives a natural generalization of van der Waerden's theorem [26] to higher dimensions.

THEOREM 10. *Let $n \in \mathbb{N}$. Every finite colouring of \mathbb{N}^n admits arbitrary long monochromatic grids of dimension n .*

We shall need another well-known theorem that lies in the canon of classical arithmetic Ramsey theory: the Folkman–Sanders “Finite Sums Theorem” [8], [20], [19].

THEOREM 11. *For every $k, m \in \mathbb{N}$ there exists an integer $N(k, m)$ such that every k -colouring of $[N(k, m)]$ admits $x_1, \dots, x_m \in [N(k, m)]$ for which $\text{FS}(x_1, \dots, x_m)$ is monochromatic.*

We slightly refine some of the notation introduced in the previous section. Let X be a countable set, $k \in \mathbb{N}$, and let $f_M: X \rightarrow [k]$ be a k -colouring, for each $M \in \mathbb{N}$. If f_M tends to a limit $f: X \rightarrow [k]$ we write

$$\lim_M f_M = f,$$

and if $S = \{x_1 < x_2 < \dots\} \subseteq \mathbb{N}$ we define

$$\lim_{M \in S} f_M = \lim_n f_{x_n},$$

provided the limit on the right-hand side exists.

The following is a basic fact about such sequences of colourings. The first conclusion is a restatement of the *compactness principle* for sequences of finite colourings, as we have seen before, while the latter conclusion is merely an iterated version of the compactness principle, where the colourings are indexed by a multidimensional array.

FACT 12. *Let $n, k \in \mathbb{N}$. The following statements hold:*

- (1) *Every sequence $\{f_M\}_M$ of finite colourings $f_M: \mathbb{N}^n \rightarrow [k]$ admits a convergent subsequence. That is, there is an infinite set $S \subseteq \mathbb{N}$ for which the limit $\lim_{M \in S} f_M$ exists.*
- (2) *Let $m \in \mathbb{N}$, let $C_1, \dots, C_m \subseteq \mathbb{N}$ be infinite sets, and suppose that, for each $M_1 \in C_1, \dots, M_m \in C_m$, we have a k -colouring $f_{M_1, \dots, M_m}: \mathbb{N}^n \rightarrow [k]$. Then there exist infinite sets $C'_1 \subseteq C_1, \dots, C'_m \subseteq C_m$ such that the limits*

$$\lim_{M_t \in C'_t} \lim_{M_{t-1} \in C'_{t-1}} \dots \lim_{M_1 \in C'_1} f_{M_1, \dots, M_m}$$

exist for each $t \in [m]$ and every choice of $M_{t+1} \in C_{t+1}, \dots, M_m \in C_m$, if $t < m$.

Proof. We prove the second conclusion by an iterative application of the first conclusion along with a “diagonalization” argument. We show, applying induction on $t \in [0, m]$, that we may find sets $C'_1, \dots, C'_t \subseteq \mathbb{N}$ such that each of the limits

$$\lim_{M_t \in C'_t} \dots \lim_{M_1 \in C'_1} f_{M_1, \dots, M_m}$$

exist for any $t \in [0, m]$ and $(M_{t+1}, \dots, M_m) \in C_{t+1} \times \dots \times C_m$.

The base step $t = 0$ is trivial, so we assume the lemma holds for $t - 1 \geq 0$ and show that it holds for t . Set

$$\tilde{f}_{M_t, \dots, M_m} = \lim_{M_{t-1} \in C'_{t-1}} \cdots \lim_{M_1 \in C'_1} f_{M_1, \dots, M_m},$$

to emphasize that the limiting colouring depends only on M_t, \dots, M_m . If $t = m$, we simply apply part (1) to the colouring \tilde{f}_{M_m} to find a set C'_m for which the limit exists. Otherwise assume that $t < m$ and enumerate the set $C_{t+1} \times \cdots \times C_m$ as $\{x_n\}_{n \in \mathbb{N}}$. Apply part (1), iteratively, to construct infinite subsets of C_t , $C(x_1) \supseteq C(x_2) \supseteq \cdots$, such that

$$\lim_{M_t \in C(x_n)} \tilde{f}_{M_t, x_n}$$

exists for each tuple x_n in the enumeration. Now define a “diagonal” set $C'_t = \{y_n\}$ by choosing $y_n \in C(x_n) \neq \emptyset$ to be such that $y_n > y_{n-1}$. This set C'_t satisfies the induction hypothesis and thus we are done, by induction. ■

Throughout the rest of the section, we shall be interested in colourings $f: \mathbb{N}_0^n \times \mathbb{N}_0 \rightarrow [k]$, where we think of the first n coordinates as grouped together. These coordinates will receive most of the attention, while the $(n + 1)$ st coordinate will enjoy a somewhat different role. We refer to such colourings as *partitioned colourings*, and write $f: \mathbb{N}_0^n \times \mathbb{N}_0 \rightarrow [k]$ and $f(x; y)$, where $x \in \mathbb{N}_0^n, y \in \mathbb{N}_0$, to emphasize the separation of the coordinates. As the role of these colourings is ubiquitous in our proof of Theorem 4, we shall not always explicitly call these colourings “partitioned”.

Given a finite colouring $f: \mathbb{N}_0^n \rightarrow [k]$, we say that a fat grid $G = s + [0, T]^n + d[0, L]^n$ (as above) is *consistent with respect to f* (or merely *consistent*, if f is clear from context) if the colouring f is identical along each translate of $B(G) = s + [0, T]^n$ that comprises G . That is,

$$f(x) = f(x + vd) \quad \text{for all } x \in B(G) \text{ and } v \in [0, L]^n.$$

Note that it is easy to see that every finite colouring $f: \mathbb{N}_0^n \rightarrow [k]$ admits arbitrarily long and arbitrarily thick, fat grids of dimension n that are consistent with respect to f . Simply apply Gallai’s theorem to the product colouring

$$F_T(x) = (f(x + v) : v \in [0, T]^n)$$

to obtain a monochromatic grid of length T for all $T \in \mathbb{N}$.

For $N \in \mathbb{N}$ and a partitioned colouring $f: \mathbb{N}_0^n \times \mathbb{N}_0 \rightarrow [k]$, we say that an n -dimensional, fat grid of length and thickness N is *consistent with respect to f* if it is consistent with respect to each colouring $f(\cdot; y)$ for $y \in [0, N]$. Similarly to our observation above, one can see that a finite colouring f admits arbitrarily long fat grids that are consistent with respect to f .

We now introduce an object that will help us keep track of certain large-scale information about a colouring in the course of our arguments.

Suppose that $f: \mathbb{N}_0^n \times \mathbb{N}_0 \rightarrow [k]$ is a partitioned k -colouring. We say that $\partial f: \mathbb{N}_0^n \times \mathbb{N}_0 \rightarrow [k]$ is a *derived colouring* of f if for every $N \in \mathbb{N}$, there exists an n -dimensional fat grid of length and thickness N that is consistent with respect to f and, for each $y \in [0, N]$, the colouring $f(\cdot; y)$ along $B(G)$ is identical to the colouring $\partial f(\cdot; y)$ on $[0, N]^n$. In other words, if we write $G(N) = s + [0, N]^n + d[0, N]^n$ ($s \in \mathbb{N}_0^n$, $d \in \mathbb{N}$), we have

$$f(s + x + vd; y) = \partial f(x; y)$$

for all $y \in [0, N]$ and $v, x \in [0, N]^n$. Additionally, we define $\partial^0 f = f$ and, for $l \in \mathbb{N}$, we use the notation $\partial^l f$ to represent a derived colouring of $\partial^{l-1} f$.

Another important notion for us is that of a *synchronized array* of colourings. Suppose that $n, r, l \in \mathbb{N}$, that $f_1, \dots, f_r: \mathbb{N}_0^n \times \mathbb{N}_0 \rightarrow [k]$ are partitioned k -colourings, and that $\partial^i f_j: \mathbb{N}^n \times \mathbb{N} \rightarrow [k]$, $i \in [l]$, $j \in [r]$, are derived colourings. Further assume that $d(M) \in \mathbb{N}$ and $s(M) \in \mathbb{N}_0^n$ for each $M \in \mathbb{N}$. We refer to the array of colourings

$$\begin{aligned} &f_1, \partial f_1, \dots, \partial^l f_1 \\ &\quad \vdots \\ &f_r, \partial f_r, \dots, \partial^l f_r \end{aligned}$$

as a *synchronized array* with *profile* $(d(M), s(M))$ if, for every $M \in \mathbb{N}$, there exists a fat grid $G(M)$ such that the following conditions hold:

- (1) $G(M)$ is consistent with respect to each of the partitioned colourings $\partial^i f_j$, $i \in [l]$, $j \in [r]$;
- (2) $G(M)$ has dimension n ;
- (3) $G(M)$ has length and thickness M ;
- (4) $G(M)$ has common difference $d(M)$ and corner $s(M)$;
- (5) for each $i \in [l-1]$ and $j \in [r]$, the colouring $\partial^i f_j(\cdot; y)$ along the M -box $B(G(M))$ agrees with the colouring $\partial^{i+1} f_j(\cdot, y)$ along $[0, M]^n$, that is,

$$\partial^i f_j(s(M) + x + d(M)v; y) = \partial^{i+1} f_j(x; y)$$

for all $y \in [0, M]$ and $v, x \in [0, M]^n$.

We call a grid $G(M)$ that satisfies (1)–(5) above *M -synchronizing for the array $(\partial^i f_i)_{i,j}$* . We record some key facts about synchronized arrays. The following lemma allows us to incrementally construct synchronized arrays. This will be crucial in our arguments as we build large collections of subsequences of \mathbb{N} that interact in a favourable way.

For the proof of the following lemma, it is convenient to introduce another fragment of notation. For a partitioned colouring $f: \mathbb{N}_0^n \times \mathbb{N}_0 \rightarrow [k]$ and $s \in \mathbb{N}_0^n$, let $T_s f: \mathbb{N}_0^n \times \mathbb{N}_0 \rightarrow [k]$ denote the partitioned colouring defined by $g(x; y) = f(x + s; y)$.

LEMMA 13. *Let $n, m, l, r, k \in \mathbb{N}$. We have the following properties of synchronized arrays.*

- (1) *If $f: \mathbb{N}_0^n \times \mathbb{N}_0 \rightarrow [k]$ is a partitioned k -colouring then there exist derived, partitioned colourings $\partial f, \dots, \partial^l f: \mathbb{N}_0^n \times \mathbb{N}_0 \rightarrow [k]$ such that $(f, \partial f, \partial^2 f, \dots, \partial^l f)$ is a synchronized array.*
- (2) *Let f_1, \dots, f_r be partitioned k -colourings of $\mathbb{N}_0^n \times \mathbb{N}_0$ and let $\partial f_i, \dots, \partial^l f_i$, $i \in [r]$, represent derived colourings such that*

$$(f_i, \partial f_i, \dots, \partial^l f_i)_{i \in [r]}$$

is a synchronized array. Then, given $f_{r+1}: \mathbb{N}_0^n \times \mathbb{N}_0 \rightarrow [k]$, we may find derived colourings $\partial f_{r+1}, \dots, \partial^l f_{r+1}$ such that the array

$$(f_i, \partial f_i, \dots, \partial^l f_i)_{i \in [r+1]}$$

is synchronized.

- (3) *If $(f_i, \partial f_i, \dots, \partial^l f_i)_{i \in [r]}$ is a synchronized array with profile $(d(M), s(M))_{M \in \mathbb{N}}$, and if $(M(n))_{n \in \mathbb{N}}$ is an increasing sequence of integers, then $(f_i, \partial f_i, \dots, \partial^l f_i)_{i \in [r]}$ also has profile $(d(M(n)), s(M(n)))_{n \in \mathbb{N}}$.*

Proof. Note that the proof of (3) follows immediately from the definition of a synchronized array. To prove the first two parts of the lemma we iteratively apply Gallai’s theorem (Theorem 10) to an appropriate product colouring and then “take limits”. Recall that the finitary form of Gallai’s theorem tells us that if $l, d, c \in \mathbb{N}$, then there exists a smallest integer $N = N(l, d, c)$ such that every c -colouring of a grid of length N and dimension d admits a monochromatic subgrid of length l and dimension d . We only prove (1) here, as the proof of (2) is very similar.

We work by induction on l . For the base case $l = 0$, let $f: \mathbb{N}_0^n \times \mathbb{N}_0 \rightarrow [k]$ be a partitioned colouring. We need to construct, for each $M \in \mathbb{N}$, a sequence of fat grids of length and thickness M that are consistent with respect to each of the colourings $f(\cdot; 0), f(\cdot; 1), \dots, f(\cdot; M)$. To this end, let $M \in \mathbb{N}$ and define the “product colouring” $F_M: \mathbb{N}_0^n \rightarrow [k]^{(M+1)^{n+1}}$ by

$$F_M(z) = (f(z + x; y) : y \in [0, M], x \in [0, M]^n).$$

Now apply Gallai’s theorem to this colouring to obtain $s(M) \in \mathbb{N}_0^n$ and $d(M) \in \mathbb{N}$ such that F_M is monochromatic on $s(M) + d(M)[0, M]^n$. Then we are done, as this is equivalent to the fat grid $G(M) = s(M) + [0, M]^n + d(M)[0, M]^n$ being consistent with respect to each $f(\cdot; 0), f(\cdot; 1), \dots, f(\cdot; M)$. Hence, for $l = 0$, we have constructed a synchronized array.

Now, moving to the induction step, assume that for $l \geq 0$ we have a synchronized array $f, \partial f, \dots, \partial^l f$. This means that for each $M \in \mathbb{N}$, there exists some fat grid $G(M) = s(M) + [0, M]^n + d(M)[0, M]^n$ ($s(M) \in \mathbb{N}_0^n$ and $d(M) \in \mathbb{N}$) with length and thickness M that is M -synchronizing for this

array. We consider the set $\{T_{s(M)}\partial^l f\}_{M \in \mathbb{N}}$ of “shifted” partitioned colourings. By the compactness principle, there exists an infinite set $S \subseteq \mathbb{N}$ for which the limit

$$\lim_{M \in S} T_{s(M)}\partial^l f$$

exists. Note that this limit point is a derived colouring of $\partial^l f$, and therefore we may denote it by $\partial^{l+1} f$.

Now let us redefine $G'(M) = G(S(M))$, where $S = \{S(1) < S(2) < \dots\}$, and note that $G'(M)$ can be thought of as a fat grid with length and thickness M , by possibly disregarding some terms, as we must have $S(M) \geq M$. Therefore the grids $G'(M)$ are synchronizing with respect to $f, \partial f, \dots, \partial^l f$, by conclusion (3). Also, the grids $G'(M)$ witness conditions (1)–(3) in the definition of a synchronized array with respect to the array $f, \partial f, \dots, \partial^{l+1} f$. All that remains is to refine these grids $G'(M)$ so that they are consistent with respect to $\partial^{l+1} f$. This is achieved in a manner similar to the basis step. For each $M \in \mathbb{N}$, set $N = N(M, n, k^{(M+1)^{n+1}})$ and consider the grid $G'(N)$. We use Gallai’s theorem to pass to a subgrid of $G'(N)$ that is consistent with respect to $\partial^{l+1} f$. We write $G'(N) = d[0, N]^n + [0, N]^n + s$ and define the product colouring of the n -dimensional grid $F: [0, N]^n \rightarrow [k]^{(M+1)^{n+1}}$ by setting

$$F(z) = (\partial^{l+1} f(dz + x + s; y) : x \in [0, M]^n, y \in [0, M]),$$

for each $z \in [0, N]^n$. Applying Gallai’s theorem to the colouring F , we obtain a subgrid $G'' = G''(M) \subseteq G'$ of length M for which $F(z)$ is monochromatic. Hence, $G''(M)$ is consistent with respect to all colourings $f, \partial f, \dots, \partial^{l+1} f$. Applying this argument for all $M \in \mathbb{N}$ proves the first conclusion of the lemma.

The proof of the second conclusion is of a very similar nature and so we do not include it here. ■

The following is a technical fact that makes explicit the observation that nothing essential changes when we append a new “dead” variable to the functions $f_i(x_1, \dots, x_n; y)$ in a synchronized array.

FACT 14. *For $r \in \mathbb{N}$, let $(f_i, \partial f_i, \dots, \partial^l f_i)_{i \in [r]}$ be a synchronized array with profile $(d(M), s(M))_{M \in \mathbb{N}}$. Define new functions $\tilde{f}_i, \widetilde{\partial^j f}_i$ for $i \in [r]$, $j \in [l]$ as follows:*

$$\begin{aligned} \tilde{f}_i(x_1, \dots, x_n, x_{n+1}; y) &= f_i(x_1, \dots, x_n; y), \\ \widetilde{\partial^j f}_i(x_1, \dots, x_n, x_{n+1}; y) &= \partial^j f_i(x_1, \dots, x_n; y), \end{aligned}$$

for each $(x_1, \dots, x_{n+1}; y) \in \mathbb{N}_0^{n+1}$. Then the following assertions hold:

- (1) $\widetilde{\partial f}_i$ is a derived colouring of \widetilde{f}_i and hence we may write $\partial \widetilde{f}_j = \widetilde{\partial f}_j$.
- (2) $(\widetilde{f}_i, \widetilde{\partial f}_i, \dots, \widetilde{\partial^l f}_i)_{i \in [r]}$ is a synchronized array with the same profile, $(d(M), s(M))_{M \in \mathbb{N}}$, as the original array. ■

4. Proof of Theorem 4. The proof of Theorem 4 is broken into three main parts: Lemma 15, Lemma 16 and the “Proof of Theorem 4”. The core of the argument lies in Lemma 16, where we build up a large collection X of integers with the property that “many” pairs $a, b \in X$ are such that the pair a, a^b is monochromatic ⁽²⁾. This set of “good” pairs will have a sufficiently rich structure that we can apply a Ramsey-type theorem (Lemma 15) to find a monochromatic finite product set among these “good” pairs. We then observe that the rich structure of these pairs will allow us to extend our finite product set to a finite exponential-product set. In the final, rather technical stages of the proof, we simply need to check that all of our choices work. So, while the “Proof of Theorem 4” is rather drawn out, there is nothing new happening here. We are only combining the ingredients, taking apart our system and checking that our previous choices yield the desired results.

Given $m, n \in \mathbb{N}$ and $A_1, \dots, A_m \subseteq [n]$, set

$$\text{FU}(A_1, \dots, A_m) = \left\{ \bigcup_{i \in I} A_i : \emptyset \neq I \subseteq [m] \right\}.$$

It is a well known fact (for example, as a consequence of the Hales–Jewett theorem [9]) that, for sufficiently large N , every finite colouring of $\mathcal{P}([N])$ admits a monochromatic set of the form $\text{FU}(A_1, \dots, A_m)$. We require a slight strengthening of this well-known result. As usual, we define the sum $x_i + x_{i+1} + \dots + x_r$ to be 0 if $i > r$.

LEMMA 15. *Let $m, k \in \mathbb{N}$. Then there exists a positive integer $N = N(k, m)$ so that every k -colouring of $\mathcal{P}([N])$ admits $A_1, \dots, A_m \subseteq [N]$ such that $\text{FU}(A_1, \dots, A_m)$ is monochromatic and $\max A_i < \min A_{i+1}$ for each $i \in [m-1]$.*

Proof. For $r, t, k \in \mathbb{N}$, let $R_r(t) = R_r(k; t)$ denote the minimum integer n such that every k -colouring of $[n]^{(r)}$ admits a set $X \subseteq [n]$ of cardinality t with $X^{(r)}$ monochromatic. Let $S = S(m, k)$ denote the minimum integer n for which every k -colouring of $[n]$ admits a monochromatic set of the form $\text{FS}(a_1, \dots, a_m)$. We claim that the choice of $N = R_1(R_2(\dots R_S(mS) \dots))$ suffices for the lemma.

To see this, let $F: \mathcal{P}([N]) \rightarrow [k]$ be a k -colouring and define

$$N_i = R_i(R_{i+1}(\dots R_S(mS) \dots)),$$

⁽²⁾ We actually need more complicated exponential compositions, but this is the sort of thing we are looking for.

for $i = 1, \dots, S$. By simply applying Ramsey's theorem at each step, it is clear that we can inductively construct sets $[N] \supset X_1 \supset \dots \supset X_S$ with $|X_i| = N_{i+1}$ for $i \in [0, S-1]$ and $|X_S| = mS$ so that F is constant on each of $X_i^{(1)}, \dots, X_i^{(i)}$ for each $i \in [S]$.

Now define a k -colouring g of $[S]$ by $g(x) = F(A)$ for some $A \in X_S^{(x)}$. Since F is constant on $X_S^{(x)}$, for $x \in [S]$, the value of $g(x)$ does not depend on the choice of A , and hence is well defined. Now apply the Finite Sums Theorem (Theorem 11) to find $\text{FS}(a_1, \dots, a_m) \subseteq [S]$ monochromatic with respect to the colouring g . To finish the proof, write $X_S = \{x(1) < \dots < x(mS)\}$ and choose A_1 to be the first a_1 elements of X_S , choose A_2 to be the next a_2 elements, and so on. More formally, for $i \in [m]$ set

$$A_i = \{x(a_1 + \dots + a_{i-1} + 1), x(a_1 + \dots + a_{i-1} + 2), \dots, x(a_1 + \dots + a_i)\}.$$

By construction, we have $\max A_i < \min A_{i+1}$. To see that $\text{FU}(A_1, \dots, A_m)$ is monochromatic, simply note that $|A_{i_1} \cup \dots \cup A_{i_l}| = a_{i_1} + \dots + a_{i_l}$ for each $l \in [m]$, $1 \leq i_1 < \dots < i_l$, and hence

$$F(A_{i_1} \cup \dots \cup A_{i_l}) = g(a_{i_1} + \dots + a_{i_l}) = g(a_1).$$

This completes the proof. ■

We are now in a position to take a large step towards the proof of Theorem 4. In what follows, we take our given colouring and construct a tremendous number of subsequences which interact in a favourable way. For the moment, we are not interested in explicitly constructing our desired pattern; we are only setting the scene. Later, in the proof of Theorem 4, we shall carefully extract our FEP-set from the collection of sequences that we build here.

In the proof below, it will be convenient to index the coordinates of \mathbb{N}^{r+1} by elements of $[0, r]$ instead of elements of $[1, r+1]$, as is usual.

LEMMA 16. *If $f: \mathbb{N} \rightarrow [k]$ is a k -colouring, then for all $l, r \in \mathbb{N}_0$ with $l \geq r$, we may construct*

- *partitioned colourings $f_S: \mathbb{N}_0^{[0,r]} \times \mathbb{N}_0 \rightarrow [k]$ for each $S \in \mathcal{P}([0, r]) \setminus \{\emptyset\}$;*
- *derived colourings $\partial^1 f_S, \dots, \partial^l f_S: \mathbb{N}_0^{[0,r]} \times \mathbb{N}_0 \rightarrow [k]$ for each $S \in \mathcal{P}([0, r]) \setminus \{\emptyset\}$;*
- *sequences $(d_0(M))_{M \in \mathbb{N}}, \dots, (d_{r+1}(M))_{M \in \mathbb{N}}$ of integers, where we set $d_0(M) = 1$ for all $M \in \mathbb{N}$;*
- *sequences $(s_1(M))_{M \in \mathbb{N}}, \dots, (s_{r+1}(M))_{M \in \mathbb{N}}$ of integer vectors, where $s_j(M) \in \mathbb{N}_0^{[0,r]}$, $j \in [r+1]$, $M \in \mathbb{N}$, and we write*

$$s_i(M) = (s_{i,0}(M), \dots, s_{i,r}(M)) \quad \text{for } i \in [r+1];$$

- *infinite sets C_0, \dots, C_r of positive integers;*

- for each $i \in [0, r]$, and non-empty $S \subseteq [0, i]$, we have partitioned colourings

$$f_{S,i,M_0,\dots,M_i} : \mathbb{N}_0^{[0,r]} \times \mathbb{N}_0 \rightarrow [k],$$

defined by

$$\begin{aligned} & f_{S,i,M_0,\dots,M_i}(x_0, \dots, x_r; y) \\ &= f\left(2 \star \left(\sum_{a \in S} d_a(M_a) 2^{s_{a+1,a}(M_{a+1}) + s_{a+2,a}(M_{a+2}) + \dots + s_{i,a}(M_i) + x_a} + y\right)\right), \end{aligned}$$

such that the following conditions hold:

- (1) for each $i \in [0, r]$, the array $(f_S, \partial f_S, \dots, \partial^l f_S)_{\emptyset \neq S \subseteq [i]}$ is synchronized with profile $(d_{i+1}(M), s_{i+1}(M))_{M \in \mathbb{N}}$;
- (2) for each $i \in [0, r]$, $S \subseteq [0, i]$, $t \in [0, i]$, and $M_{t+1}, \dots, M_i \in \mathbb{N}$, the limit

$$\lim_{M_t \in C_t} \lim_{M_{t-1} \in C_{t-1}} \dots \lim_{M_0 \in C_0} f_{S,i,M_0,\dots,M_i}$$

exists and

$$\lim_{M_i \in C_i} \lim_{M_{i-1} \in C_{i-1}} \dots \lim_{M_0 \in C_0} f_{S,i,M_0,\dots,M_i} = \partial^{i-\max S} f_S.$$

Proof. We fix $l \in \mathbb{N}$ arbitrarily and apply induction on $r \leq l$. In the base case, $r = 0$, we set $d_0(M_0) = 1$ for all $M_0 \in \mathbb{N}$ and consider the colouring

$$f_{\{0\},0,M_0}(x_0; y) = f(2^{d_0(M_0)2^{x_0}+y}) = f(2^{2^{x_0}+y}),$$

which (trivially) tends to some limiting colouring as $M_0 \rightarrow \infty$, which we denote by $f_{\{0\}}$, in accordance with condition (2) of the induction hypothesis. Hence we have satisfied (2) with the choice of $C_0 = \mathbb{N}$. To satisfy (1), we apply Lemma 13 to construct a sequence of derived colourings

$$f_{\{0\}}, \partial f_{\{0\}}, \dots, \partial^l f_{\{0\}}$$

such that the above is a synchronized array with profile $(d_1(M), s_1(M))_{M \in \mathbb{N}}$.

Turning to the induction step, assume that the lemma holds for some $r - 1 \leq l - 1$: our aim is to show that it holds for r .

Construction of the new colourings $\partial^i f_S$. We construct $\partial^i f_S$ in two cases. In one case, if $r \notin S$, we are essentially done: the induction hypothesis gives us what we need. In the other case, if $r \in S$, we find new sequences of integers such that the limiting colourings converge to something appropriate. To start with the easy case, $r \notin S$, we use the induction hypothesis to construct $f_S, \partial f_S, \dots, \partial^l f_S$ for all sets $\emptyset \neq S \subseteq [r - 1]$. We define the new colourings $\partial^i f_S$ for $\emptyset \neq S \subseteq [r - 1]$ by simply joining a new “dead” variable x_r to the $\partial^i f_S$ that are granted by the induction hypothesis, that is, define

$$\widetilde{\partial^i f_S}(x_0, \dots, x_{r-1}, x_r; y) = \partial^i f_S(x_0, \dots, x_{r-1}; y)$$

for all $x_0, \dots, x_r, y \in \mathbb{N}$. From Fact 14, we know that nothing is altered in adjoining this new variable, and so we take $\widetilde{\partial^i f_S}$ as our $\partial^i f_S$ for all

$\emptyset \neq S \subseteq [r-1]$, as in the statement of the lemma. As nothing essential is changed, we drop the tilde in the notation.

This leaves us to construct $\partial^i f_S$ where $r \in S$ and $i \in [0, l]$. To achieve this, we consider the following (totally new) collection of sequences, indexed by the non-empty subsets $S \subseteq [r]$ with $r \in S$:

$$(4.1) \quad 2 \star \left(\left(\sum_{a \in S \setminus \{r\}} d_a(M_a) 2 \star (s_{a+1,a}(M_{a+1}) + \cdots + s_{r,a}(M_r) + x_a) \right) + d_r(M_r) 2^{x_r} + y \right),$$

where $x_0, \dots, x_r \in \mathbb{N}_0$. We shall take f_S to be an appropriate limit point of f along this sequence. More carefully, recall that the auxiliary colouring $f_{S,r,M_0,\dots,M_r}(x; y)$ records the colouring f along the sequence in (4.1). Now apply Fact 12 to an appropriate product colouring (or otherwise) to obtain the following.

CLAIM 17. *There exist infinite sets C'_0, \dots, C'_r such that $C'_0 \subseteq C_0, \dots, C'_{r-1} \subseteq C_{r-1}, C'_r \subseteq \mathbb{N}$ and each of the limits*

$$\lim_{M_t \in C'_t} \cdots \lim_{M_0 \in C'_0} f_{S,r,M_0,\dots,M_r}$$

exists for $t \in [0, r]$, $S \subseteq [r]$, and $M_{t+1}, \dots, M_r \in \mathbb{N}$. ■

So we choose C'_0, \dots, C'_r to be our “infinite sets of integers” as in the statement of Lemma 16 and note that these sets guarantee the existence of the limits in conclusion (2) of the lemma. Moreover, observe that each of the synchronized arrays $(f_S, \partial f_S, \dots, \partial^l f_S: \emptyset \neq S \subseteq [i])$, $i \in [r-1]$, will still be synchronized after restricting our indices to the $C'_i \subseteq C_i$, by Lemma 13(3).

We now define our new colourings f_S for each $\emptyset \neq S \subseteq [0, r]$ with $r \in S$. We set

$$f_S = \lim_{M_r \in C'_r} \cdots \lim_{M_0 \in C'_0} f_{S,r,M_0,\dots,M_r}$$

for each $\emptyset \neq S \subseteq [0, r]$ with $r \in S$.

Finally, we apply Lemma 13 to extend the synchronized array

$$(f_S, \partial f_S, \dots, \partial^l f_S)_{\emptyset \neq S \subseteq [0, r-1]}$$

to include all of the (newly constructed) colourings f_S for all $\emptyset \neq S \subseteq [0, r]$ with $r \in S$. As a result, we obtain a synchronized array

$$(f_S, \partial f_S, \dots, \partial^l f_S)_{\emptyset \neq S \subseteq [r]}$$

with some (new) profile, which we denote by $(d_{r+1}(M), s_{r+1}(M))_{M \in \mathbb{N}}$. We have thus constructed all the objects required by the lemma and have checked that they satisfy (1). In what follows, we check that these objects satisfy (2).

Check of conclusion (2). First note that we have already ensured that each of the limits appearing in (2) exists. All that remains is to check that the identities hold. Now if $i \in [0, r-1]$, we have the desired identity by induction, and this identity remains unaltered after we pass to a subsequence of the M_0, \dots, M_r , as we have done above. Hence only the identities

$$(4.2) \quad \lim_{M_r \in C'_r} \cdots \lim_{M_0 \in C'_0} f_{S,r,M_0,\dots,M_r} = \partial^{r-\max S} f_S$$

remain to be checked for $\emptyset \neq S \subseteq [0, r]$.

As we shall only consider $M_i \in C'_i$ for $i \in [r]$, we shall drop the extra notation in the subscript of the limits. First, dealing with a trivial case, note that if $r \in S$, we have

$$\lim_{M_r} \cdots \lim_{M_0} f_{S,r,M_0,\dots,M_r} = f_S = \partial^0 f_S,$$

by construction of the colouring f_S . Proceeding to the non-degenerate case, we fix some $S \subseteq [r-1]$ and observe that, by induction,

$$\lim_{M_{r-1}} \cdots \lim_{M_0} f_{S,r-1,M_0,\dots,M_{r-1}} = \partial^{r-1-\max S} f_S.$$

So for any fixed value of $M_r \in C'_r$,

$$(4.3) \quad \begin{aligned} \lim_{M_{r-1}} \cdots \lim_{M_0} f_{S,r-1,M_0,\dots,M_{r-1}}(x + s_r(M_r); y) &= \partial^{r-1-\max S} f_S(x + s_r(M_r); y) \\ &= \partial^{r-\max S} f_S(x; y), \end{aligned}$$

where the last equation holds for any $x \in \mathbb{N}^{[0,r]}$, $y \in \mathbb{N}$ with $y, |x|_\infty \leq M_r$. Now note that

$$f_{S,r,M_0,\dots,M_r}(x; y) = f_{S,r-1,M_0,\dots,M_{r-1}}(x + s_r(M_r); y)$$

for $y, |x|_\infty \leq M_r$. Substituting this into (4.3) and taking the limit as $M_r \in C'_r$ tends to infinity yields

$$\lim_{M_r} \cdots \lim_{M_0} f_{S,r,M_0,\dots,M_r} = \partial^{r-\max S} f_S,$$

as desired.

This completes the proof of (2) in the induction hypothesis and hence concludes the proof, by induction. ■

In what follows, we put the pieces together to prove Theorem 4.

Proof of Theorem 4. Let f be a k -colouring of \mathbb{N} . By Lemma 15, there exists some integer $N = N(k, m)$ such that every k -colouring of $\mathcal{P}([N])$ admits $A_1, \dots, A_m \subseteq [N]$ such that $\max A_i < \min A_{i+1}$, $i \in [m-1]$, and $\text{FU}(A_1, \dots, A_m)$ is monochromatic. Now, apply Lemma 16 with $r = l = N$. We define a colouring $F: \mathcal{P}([N]) \rightarrow [k]$ by

$$F(S) = \partial^{N-\max S} f_S(0; 0).$$

In other words, we define $F(S)$ to be the colour attained by

$$(4.4) \quad \lim_{M_N \in C_N} \lim_{M_{N-1} \in C_{N-1}} \cdots \lim_{M_0 \in C_0} f \left(2 \star \sum_{a \in S} d_a(M_a) 2^{s_{a+1,a}(M_{a+1}) + \cdots + s_{N,a}(M_N)} \right).$$

Here, the values $s_{i,j}(M_k), d_i(M_i)$ are as in the statement of Lemma 16.

Now apply Lemma 15 to find sets $A_1, \dots, A_m \subseteq [N]$ such that $\text{FU}(A_1, \dots, A_m)$ is monochromatic with respect to the colouring F , and $\max A_i < \min A_{i+1}$ for each $i \in [m-1]$.

To understand what we have gained, for each $S \subseteq [N]$, let us define the numbers $a(S) = a(S; (M_a)_{\min S \leq a \leq N})$ by

$$a(S) = 2 \star \sum_{a \in S} d_a(M_a) 2^{s_{a+1,a}(M_{a+1}) + \cdots + s_{N,a}(M_N)}.$$

In this notation, the $a(A_1), \dots, a(A_m)$ generate a monochromatic finite product set for sufficiently large values of $M_0 \in C_0, \dots, M_N \in C_N$ (in the sense of (4.4)). This is because $a(S \cup S') = a(S)a(S')$ for any disjoint pair of sets $S, S' \subseteq [m]$. In what follows, we show that we can pick the parameters M_0, \dots, M_N so that $a(A_1), \dots, a(A_m)$ generate a monochromatic FEP_W -set.

Before we turn to our main task, note that we may assume $W(A) \geq W(B)$ for all $A, B \in \mathbb{N}^{(\leq m)}$ with $A \supseteq B$ by possibly replacing W with $W'(A) = \max\{W(B) : B \subseteq A\}$. The following claim simply makes explicit the various dependencies that we need on M_N, \dots, M_1 . We note that the proof of the claim below is trivial, as each integer M_t depends only on M_{t+1}, \dots, M_N and U_t, \dots, U_N , while each U_t depends only on $U_{t+1}, \dots, U_N, M_{t+1}, \dots, M_N$.

CLAIM 18. *There exists $(M_0, \dots, M_N) \in C_0 \times \cdots \times C_N$ and auxiliary integers $U_0, \dots, U_N \in \mathbb{N}$ such that, for each $t \in [0, N]$, U_t satisfies the following conditions:*

- (1) $U_t \geq \max \text{FEP}_W(\{a(S) : S \subseteq [t+1, N]\})$;
- (2) $U_t \geq M_{t+1} + \sum_{N \geq i \geq t+1} |s_i(M_i)|_\infty + d_i(M_i)M_i$;
- (3) $U_t \geq \sum_{N \geq i \geq t+1} |s_i(M_i)|_\infty + 1d_i(M_i)W(\{a(S) : S \subseteq [t+1, N]\})2 \star (s_{i+1,i}(M_i) + \cdots + s_{N,i}(M_N))$;
- (4) $U_t \geq U_{t+1}$,

and M_t satisfies the following conditions:

- (1) for each non-empty $S \subseteq [0, N]$ and $l \in [\max S, N]$, we have

$$\lim_{M_{t-1}} \cdots \lim_{M_0} f_{S,l,M_0,\dots,M_l}(x; y) = \lim_{M_t} \lim_{M_{t-1}} \cdots \lim_{M_0} f_{S,l,M_0,\dots,M_l}(x; y)$$

for all x, y satisfying $|x|_\infty, y \leq U_t$;

- (2) for each M_t we have $M_t \geq U_t$.

Actually, conditions (2), (3) on U_t are redundant, but we include them as they come at no added cost and they make explicit the precise conditions that we shall require later. Also notice that condition (1) on the choice of M_t implies that for each non-empty set $S \subseteq [0, N]$ and $l \in [\max S, N]$, we have

$$(4.5) \quad f_{S,l,M_0,\dots,M_l}(x; y) = \partial^{l-\max S} f_S(x; y)$$

for all x, y satisfying $|x|_\infty, y \leq U_l$.

We now turn to check that the choice of M_0, \dots, M_N granted by Claim 18 ensures that the integers

$$a(A_1) = a(A_1; M_0, \dots, M_N), \dots, a(A_m) = a(A_m; M_0, \dots, M_N)$$

generate a monochromatic FEP-set with weight W . As we have fixed the choices of M_0, \dots, M_N , for $i \in [0, N]$ set

$$\begin{aligned} s_i &= (s_{i,0}(M_i), \dots, s_{i,N}(M_i)), & d_i &= d_i(M_i), \\ \tilde{s}_i &= s_{i+1,i}(M_{i+1}) + s_{i+2,i}(M_{i+2}) + \dots + s_{N,i}(M_N), \\ a_i &= a(A_i) = 2 \star \sum_{a \in A_i} d_a 2^{\tilde{s}_a}. \end{aligned}$$

We begin checking that the FEP-set generated by a_1, \dots, a_m is monochromatic with the following observation, which recalls the relationship between the colourings $\partial^{j+1} f_S(x)$ and $\partial^j f_S(x)$ in the context of the proof. Formally, it follows from Lemma 16(1).

OBSERVATION 19. *Let $S \subseteq [N]$. If t is an integer such that $N \geq t \geq \max S + 1$ and $j \in [0, N - 1]$, then*

$$\partial^{j+1} f_S(x; y) = \partial^j f_S(x + s_t + v d_t; y)$$

for all $y \in \mathbb{N}_0$, $x, v \in \mathbb{N}_0^{[0,N]}$ with $y, |x|_\infty, |v|_\infty \leq M_t$.

From this observation, we quickly arrive at the following claim.

CLAIM 20. *Suppose that $S \subseteq [N]$ and u, t are integers such that $u \geq 0$, $t \in [N]$ and $t - u \geq \max S$. Then*

$$\partial^{t-\max S} f_S(x; y) = \partial^{t-u-\max S} f_S\left(x + \sum_{i \in [t-u+1, t]} s_i + v_i d_i; y\right)$$

for all $y \in \mathbb{N}_0$ and $x, v_{t-u+1}, \dots, v_t \in \mathbb{N}_0^{[0,N]}$ with $y, |x|_\infty \leq M_t$ and $|v_i|_\infty \leq M_i$ for each $i \in [t - u + 1, t]$.

Proof of Claim. We use induction on u . If $u = 0$, there is nothing to prove—we have an identity. So suppose we have shown

$$(4.6) \quad \partial^{t-\max S} f_S(x; y) = \partial^{t-u-\max S} f_S\left(x + \sum_{i \in [t-u+1, t]} s_i + v_i d_i; y\right)$$

for some $u \geq 0$ and all $y \in \mathbb{N}$, $x, v_i \in \mathbb{N}_0^n$, $i \in [t - u + 1, t]$, satisfying $y, |x|_\infty \leq M_t$ and $|v_i|_\infty \leq M_i$. Assume that $t - u - 1 \geq \max S$, otherwise there is nothing more to prove. Then apply Observation 19 to show that the above equals

$$\partial^{t-u-1-\max S} f_S \left(\left(x + \sum_{i \in [t-u+1, t]} s_i + v_i d_i \right) + s_{t-u} + v_{t-u} d_{t-u} ; y \right)$$

for $y, |x|_\infty \leq M_t, |v_i|_\infty \leq M_i, i \in [t - u - 1, t]$. This holds by Observation 19, as $y \leq M_t \leq M_{t-u}$ and each coordinate of the left argument of $\partial^{t-u-\max S} f_S$ at (4.6) is bounded by

$$\left(|x|_\infty + \sum_{i \in [t-u+1, t]} |s_i|_\infty + |v_i|_\infty d_i \right) \leq M_t + \sum_{i \in [t-u+1, t]} |s_i|_\infty + M_i d_i \leq M_{t-u},$$

where the last inequality holds by condition (2) on U_t and condition (2) on M_t in Claim 18. Thus we are done, by induction. ■

To show that $\text{FEP}_W(a_1, \dots, a_m)$ is monochromatic, we again apply induction: we show that for each $\alpha \in \text{FEP}_W(a_1, \dots, a_m) \setminus \text{FP}(a_1, \dots, a_m)$ we can find some $\beta < \alpha$, $\beta \in \text{FEP}_W(a_1, \dots, a_m)$ that has the same colour as α . Hence, by induction, each element of $\text{FEP}_W(a_1, \dots, a_m)$ is given the same colour as some element in $\text{FP}(a_1, \dots, a_m)$, which we already know to be monochromatic. For example, if we first know that $\text{FP}(a, b, c, d)$ is monochromatic in colour RED, say, we show that $a^b c^d \in \text{FEP}_W(a, d, c, d)$ is RED by showing that $a^b c^d$ has the same colour as $a c^d$, which we then show has the same colour as $a c \in \text{FP}(a, b, c, d)$.

CLAIM 21. *For every $\alpha \in \text{FEP}(a_1, \dots, a_m) \setminus \text{FP}(a_1, \dots, a_m)$ there is some $\beta \in \text{FEP}(a_1, \dots, a_m)$ such that $\beta < \alpha$ and $f(\alpha) = f(\beta)$.*

Proof of Claim. Let $\alpha \in \text{FEP}_W(a_1, \dots, a_m) \setminus \text{FP}(a_1, \dots, a_m)$ and write

$$(4.7) \quad \alpha = \prod_{i \in B} a_i^{e_i},$$

where $B \subseteq [m]$ and $e_i \in \text{FP}_{[m] \setminus B, W}(a_{i+1}, \dots, a_m)$. For each e_i , we fix some representation of e_i as product of powers of the elements $\{a_j\}_{j \notin B, j > i}$, and define C_i to be the support of this product. That is, for $i \in B$, define $C_i \subseteq [m]$ to be the set for which

$$e_i = \prod_{j \in C_i} a_j^{p_{i,j}},$$

where $p_{i,j} \in [W(\{a_{j+1}, \dots, a_m\})]$ for each $j \in C_i$. Note that if $p_{i,j} \neq 0$ we have

$$(4.8) \quad j \geq i + 1.$$

Now since $\alpha \notin \text{FP}(a_1, \dots, a_m)$, we must have $\bigcup_{i \in B} C_i \neq \emptyset$. So we may set $j^* = \min \bigcup_{i \in B} C_i$. We define β to be α “with a_{j^*} removed from the exponents”. More formally, if $i \in B$, let $e'_i \in \mathbb{N}$ satisfy $e'_i \cdot a_{j^*}^{p_{i,j^*}} = e_i$. Then define

$$\beta = \prod_{i \in B} a_i^{e'_i}.$$

Notice that $\beta \in \text{FEP}_W(a_1, \dots, a_m)$ and $\beta < \alpha$. We show that $f(\alpha) = f(\beta)$.

CLAIM 22. *Let α, β be as above. Then $f(\alpha) = f(\beta)$.*

Proof of Claim. We partition B as $B = B^- \cup B^+$, where B^- consists of all indices $< j^*$, while B^+ consists of all indices $> j^*$. Of course, j^* cannot appear in B , due to the form of FEP-sets. Note that B^- is the collection of indices i where a_{j^*} *could* appear (in the sense of being a “legal” pattern in the FEP-set) as an exponent of a_i in α . We partition B^- into the indices of the bases where a_{j^*} *does* appear as an exponent and the indices where a_{j^*} *does not* appear as an exponent. That is, write $B^- = B_0^- \cup B_1^-$, where $B_0^- = \{i : j^* \notin C_i\}$ and $B_1^- = \{i : j^* \in C_i\}$. For simplicity, we also set $S = \bigcup_{i \in B^-} A_i$ and $A^* = A_{j^*}$.

We now expand α, β in terms of the parameters d_a, \tilde{s}_a . To do this recall that, for $j \in [0, N]$,

$$a_j = 2 \star \sum_{a \in A_j} d_a 2^{\tilde{s}_a},$$

and therefore, for any i with $j \in C_i$,

$$a_j^{p_{i,j}} = 2 \star \sum_{a \in A_j} d_a 2^{\tilde{s}_a} p_{i,j},$$

so that

$$e'_i = \prod_{j \in C_i \setminus \{j^*\}} a_j^{p_{i,j}} = 2 \star \sum_{j \in C_i \setminus \{j^*\}} \sum_{a \in A_j} d_a 2^{\tilde{s}_a} p_{i,j}$$

and

$$\begin{aligned} e_i &= \prod_{j \in C_i} a_j^{p_{i,j}} = 2 \star \sum_{j \in C_i} \sum_{a \in A_j} d_a 2^{\tilde{s}_a} p_{i,j} \\ &= 2 \star \left(\left(\sum_{j \in C_i \setminus \{j^*\}} p_{i,j} \sum_{a \in A_j} d_a 2^{\tilde{s}_a} \right) + p_{i,j^*} \sum_{a \in A^*} d_a 2^{\tilde{s}_a} \right). \end{aligned}$$

With this in mind, we define numbers ρ_a for each $a \in [0, N]$ by $\rho_a = 0$ if $a \notin \bigcup_{i \in B} A_i$, and if $a \in A_i$ for some $i \in B$ (indeed, this i is unique as the A_i are disjoint), we set

$$(4.9) \quad \rho_a = \sum_{j \in C_i \setminus \{j^*\}} p_{i,j} \sum_{b \in A_j} d_b 2^{\tilde{s}_b}.$$

We define the numbers $\tilde{\rho}_a$ for $a \in [0, N]$ by setting $\tilde{\rho}_a = \rho_a$ if $a \notin \bigcup_{i \in B_1^-} A_i$, and if $a \in A_i$ for some $i \in B_1^-$, we set

$$(4.10) \quad \tilde{\rho}_a = \rho_a + p_{i,j^*} \sum_{b \in A^*} d_b 2^{\tilde{s}_b}.$$

Note that $p_{i,j^*} \leq W(\{a_{j^*+1}, \dots, a_m\})$ by the structure of FEP-sets. We write

$$(4.11) \quad \begin{aligned} \beta &= \prod_{i \in B} a_i^{e'_i} = 2 \star \left(\sum_{i \in B} e'_i \sum_{a \in A_i} d_a 2^{\tilde{s}_a} \right) \\ &= 2 \star \left(\sum_{i \in B} \sum_{a \in A_i} d_a 2^{\tilde{s}_a + \rho_a} \right), \end{aligned}$$

and in a similar way we have

$$(4.12) \quad \alpha = 2 \star \left(\sum_{i \in B} \sum_{a \in A_i} d_a 2^{\tilde{s}_a + \tilde{\rho}_a} \right).$$

We now make note of some properties of the numbers $\rho_a, \tilde{\rho}_a$. Let $\rho = (\rho_0, \dots, \rho_N)$, and $\tilde{\rho} = (\tilde{\rho}_0, \dots, \tilde{\rho}_N)$. By rearranging (4.10), we may express

$$(4.13) \quad \tilde{\rho} = \rho + \sum_{b \in A^*} d_b w^{(b)},$$

for some $w^{(b)} \in \mathbb{N}_0^{[0, N]}$ and $b \in A^*$ that are supported on $\bigcup_{i \in B_1^-} A_i \subseteq [0, N]$.

We now note that

$$(4.14) \quad |w^{(b)}|_\infty \leq M_b$$

for each $b \in A^*$. To see this, simply use (4.10) to obtain

$$\begin{aligned} |w^{(b)}| &\leq \max_{i \leq j^*-1} p_{i,j^*} 2^{\tilde{s}_b} \leq W(\{a_{j^*+1}, \dots, a_N\}) 2^{s_{b+1,b}(M_{b+1}) + \dots + s_{N,b}(M_N)} \\ &\leq M_b, \end{aligned}$$

where the first inequality follows from (4.8) and the last inequality holds by condition (3) on U_1, \dots, U_N and condition (2) on M_0, \dots, M_N in Claim 18.

It is also important to observe that

$$(4.15) \quad |\rho|_\infty + \sum_{i=\max A^*+1}^N |s_i|_\infty \leq U_{\max A^*},$$

$$(4.16) \quad |\tilde{\rho}|_\infty + \sum_{i=\max A^*+1}^N |s_i|_\infty \leq U_{\max S}.$$

The first inequality holds by the minimality of j^* among $\bigcup_{i \in B} C_i$ and Claim 18, while the second inequality follows from (4.10), the fact that $\min A^* > \max S$ and condition (2) on U_t in Claim 18.

Now write $\rho = \rho^- + \rho^+$, where ρ^- is supported on the coordinates with indices in $S = \bigcup_{i \in B^-} A_i$ and ρ^+ is supported on coordinates with indices in $\bigcup_{i \in B^+} A_i$. We also write $\tilde{\rho} = \tilde{\rho}^- + \tilde{\rho}^+ (= \tilde{\rho}^- + \rho^+)$, in a similar way.

We now proceed to show $f(\alpha) = f(\beta)$. We have

$$(4.17) \quad f(\beta) = f\left(2 \star \left(\sum_{i \in B^-} \sum_{a \in A_i} d_a 2^{\tilde{s}_a + \rho_a} + \sum_{i \in B^+} \sum_{a \in A_i} d_a 2^{\tilde{s}_a + \rho_a} \right)\right) \\ = f_{S, \max A^*, M_0, \dots, M_{\max A^*}} \left(\rho^- + \sum_{\max A^* + 1 \leq i \leq N} s_i ; \sum_{i \in B^+} \sum_{a \in A_i} d_a 2^{\tilde{s}_a + \rho_a} \right).$$

Now set

$$\gamma = \sum_{i \in B^+} \sum_{a \in A_i} d_a 2^{\tilde{s}_a + \rho_a}$$

and note that

$$(4.18) \quad \gamma \leq \max \text{FEP}_W(\{a(S') : S' \subseteq [\max A^* + 1, N]\})$$

$$(4.19) \quad \leq U_{\max A^*} \leq M_{\max A^*},$$

where the second inequality holds by condition (1) on U_t in Claim 18, and the third inequality holds by condition (2) on M_t , in the same claim. This, along with the fact that the left argument in (4.17) is bounded by $U_{\max A^*}$ (as noted at (4.15)), implies that (4.17) equals

$$(4.20) \quad \partial^{\max A^* - \max S} f_S \left(\rho^- + \sum_{\max A^* + 1 \leq i \leq N} s_i ; \gamma \right),$$

by our selection of M_0, \dots, M_N at Claim 18. Now, apply Claim 20 with the choice of $t = \max A^*$ and $u = \max A^* - \max S$ to learn that (4.20) equals

$$(4.21) \quad \partial^0 f_S \left(\rho^- + \sum_{\max A^* + 1 \leq i \leq N} s_i + \sum_{\max S + 1 \leq i \leq \max A^*} s_i + v_i d_i ; \gamma \right),$$

for any choice of the $v_i \in \mathbb{N}_0^{[0, N]}$, $i \in [\max S + 1, \max A^*]$, satisfying $|v_i|_\infty \leq M_i$. We choose $v_i = w^{(i)}$ as in (4.13); then absorb the $w^{(i)} d_i$ terms into the ρ , in order to make $\tilde{\rho}$, as in (4.13). That is, we learn that (4.21) is

$$f_S \left(\tilde{\rho}^- + \sum_{\max S + 1 \leq i \leq N} s_i ; \gamma \right) = f_{S, \max S, M_0, \dots, M_{\max S}} \left(\tilde{\rho}^- + \sum_{\max S + 1 \leq i \leq N} s_i ; \gamma \right) \\ = f \left(2 \star \left(\sum_{i \in B} \sum_{a \in A_i} d_a 2^{s_a + \tilde{\rho}_a} \right) \right) = f(\alpha).$$

To obtain the first equality, we have used $\gamma \leq M_{\max A^*} \leq M_{\max S}$, by (4.18), and $\tilde{\rho}^- + \sum_{\max S + 1 \leq i \leq N} |s_i|_\infty \leq U_{\max S}$, as noted at (4.16), and then applied (4.5). The second to last equation follows by recalling the definition of γ ; the last equation follows by recalling the expansion of α , given above at (4.12). Thus, we conclude that $f(\alpha) = f(\beta)$. ■

To finish the proof of the theorem, we simply apply induction on the elements of the FEP-set, as noted above. This completes the proof of the claim and the proof of the Theorem. ■

5. Non-partition regularity and a classification of first order systems. In this section we introduce our “log*-colourings”. These colourings will allow us to give many natural examples of exponential patterns which are not partition regular. They also form the missing piece in the proof of our classification result for “height-one” systems (Theorem 5). For an integer $n \geq 4$, we define the n th log*-colouring to be the colouring (essentially) defined by “colour x with the least positive residue of $\log^* x$, when taken modulo n ”.

To be more precise, we declare all logarithms to be in base 2 and, for $k \in \mathbb{N}$, let $\log_{(k)}$ denote the k th iterated logarithm. Define the log* function $L: \mathbb{N} \rightarrow \mathbb{N} \cup \{0\}$ as $L(x) = \min\{k : \log_{(k)} x \leq 1\}$. It is easy to check that

- L is increasing;
- if $x \geq 2$, then $L(\log x) + 1 = L(x)$;
- if $a, b \in \mathbb{N}$ and $a \leq b$ then $L(a + b) \leq L(b) + 1$.

The restriction that these log*-colourings impose on partition regular patterns is made precise by the following result.

LEMMA 23. *For $r \in \mathbb{N}$, there exists a finite colouring $f: \mathbb{N} \rightarrow [r + 3]$ admitting no monochromatic pair b, a^b where $\log_{(r)} a \leq b$ and $a, b \neq 1$.*

Proof. Define $f: \mathbb{N} \rightarrow [r + 3]$ by $f(1) = r + 3$ and, for all $x > 1$, $f(x)$ is the least positive residue of $L(x)$ modulo $r + 2$.

To check the desired property of f , assume that b, a^b are monochromatic with respect to f and $\log_{(r)} a \leq b$. If $f(b)$ or $f(a^b)$ is $r + 3$ then they are both $r + 3$ and $a = b = 1$. So we may assume that $a, b \geq 2$, and in particular $a^b \geq 4$. We now show that $L(a^b) \geq L(b) + 1$. To do this, write

$$(5.1) \quad L(a^b) = L(\log_{(2)} a^b) + 2.$$

This is possible, as $a^b \geq 4$. Now the quantity at (5.1) is equal to

$$L(\log(b \log a)) + 2 = L(\log b + \log_{(2)} a) + 2 \geq L(\log b) + 2 = L(b) + 1,$$

as desired. To bound $L(a^b)$ from above, we again write

$$L(a^b) = L(\log b + \log_{(2)} a) + 2 \leq \max\{L(b) - 1, L(\log_{(r)} a) + (r - 2)\} + 3.$$

Since $\log_{(r)} a \leq b$, the above is at most $L(b) + (r + 1)$. Putting these bounds together yields $L(a^b) \in \{L(b) + 1, L(b) + 2, \dots, L(b) + (r + 1)\}$. Therefore $L(a^b) \equiv L(b) \pmod{r + 2}$ is impossible. This completes the proof. ■

We now readily draw several conclusions. First, if we take $r = 1$ in the above lemma, we obtain a 4-colouring which admits no monochro-

matic quadruple of the form a, b, a^b, b^a with $a, b > 1$. Going further, we see that this colouring, along with our main theorem (Theorem 4), grants us a classification of “height-one” exponential systems (Theorem 5). Recall that this theorem says that one can find a monochromatic pattern of the form $\{x_1, \dots, x_m\} \cup \{x_i^{x_j} : (i, j) \in R\}$, where $R \subseteq [m] \times [m]$, in an arbitrary colouring if and only if R does not contain a directed cycle, a set of the form $(y_1, y_2), \dots, (y_{l-1}, y_l), (y_l, y_1)$, where $y_1, \dots, y_l, l \in [m]$.

Proof of Theorem 5. The positive direction follows immediately from the partition regularity of FEP-sets. Conversely, suppose that R is a binary relation on $[m]$ with a directed cycle. Without loss of generality, we may assume this cycle is $(1, 2), \dots, (l-1, l), (l, 1)$, $l \in [m]$. Apply Lemma 23 to obtain a 4-colouring f admitting no monochromatic pair a, a^b with $a < b$. If R is partition regular, we can find x_1, \dots, x_l such that $x_1, \dots, x_l, x_1^{x_2}, x_2^{x_3}, \dots, x_l^{x_1}$ is monochromatic with respect to f . Hence $x_1 > x_2 > \dots > x_l > x_1$, a contradiction. This completes the proof. ■

Lemma 23 can also be interpreted as giving a lower bound on the minimum integer $N(k)$, $k \in \mathbb{N}$, for which every k -colouring of $[N(k)]$ admits a monochromatic exponential triple. In particular, it shows that $N(k)$ is at least a tower of 2s of height $k - 3$.

Finally, we remark that the colouring given by Lemma 23 and $r = 1$ also forbids the natural infinite extension of finite exponential sets. Indeed, we can even forbid patterns such as $\{x_1, x_2, \dots\} \cup \{x_1^{x_2}, x_2^{x_3}, x_3^{x_4}, \dots\}$, where $x_i > 1$ for all $i \in \mathbb{N}$.

6. Inconsistency of exponential triples and Schur triples. In this short section we note the peculiar relationship between exponentiation and addition in the context of partition regularity. We give an example of a 16-colouring of \mathbb{N} for which there is no monochromatic set of the form $x, y, x + y, a, b, a^b$, where $x, y, a, b \in \mathbb{N}$. Curiously, we require a single case of Fermat’s Last Theorem for our result. As the case $n = 4$ has an elementary proof (which is actually due to Pierre de Fermat [1]), we appeal only to this case. A proof of this result appears in [6], among many other sources.

THEOREM 24. *There are no integer solutions to*

$$X^4 + Y^4 = Z^4$$

with $X, Y, Z \neq 0$.

We now prove the main result of this section, that $x, y, x + y, a, b, a^b$ are inconsistent partition regular patterns.

Proof of Theorem 6. We define our colouring f by first defining two auxiliary colourings $f_1, f_2: \mathbb{N} \rightarrow \{0, 1, 2, 3\}$ and then defining $f(x) = (f_1(x), f_2(x))$. We define $f_1(x)$ to simply be the remainder of x modulo 4. Now, for $x \in$

$\mathbb{N} \setminus \{1\}$, we define

$$l(x) = \max\{b : x = a^b \text{ with } a, b \in \mathbb{N}\},$$

and $l(1) = 0$. Note that $l(x^y) = l(x)y$ for all $x, y \in \mathbb{N}$. We define $f_2(x)$ to be the remainder of $l(x)$ modulo 4 for all $x \in \mathbb{N}$.

Now suppose that $a, b, x, y \in \mathbb{N}$ are such that $x, y, x + y, a, b, a^b$, all receive the same colour. We learn that $x + y \equiv y \pmod{4}$ and thus $x \equiv 0 \pmod{4}$. It follows that $b \equiv 0 \pmod{4}$ and hence $l(a^b) = bl(a) \equiv 0 \pmod{4}$. Thus $l(x) \equiv l(y) \equiv l(x + y) \equiv 0 \pmod{4}$. So we may write $x, y, x + y$ in the form

$$x = u^4, \quad y = v^4, \quad x + y = w^4,$$

where u, v, w are positive integers. But this is impossible as we have obtained a non-trivial solution to the equation $u^4 + v^4 = w^4$. ■

7. Open problems. The \log^* -colourings, given in Section 5, impose a strong ordering on the elements of an exponential partition regular pattern and can actually forbid any pattern where a base appears in its own exponent. For example, patterns such as $a, b, a^{b^a}, a, b, a^{ba}$, or (trivially) a, a^a are not partition regular. On the other hand, a quick glance at Theorem 4 reveals that a “height-two exponent” has a great deal of freedom. For example, the elements $a \star (b^{c^c}), a \star (b \star (c^{c^c}))$ are included in the $\text{FEP}_W(a, b, c)$ -set, for an appropriately chosen W . What we know much less about is the repetition of “height-one” exponents. The following question arises naturally.

QUESTION 25. *Is the pattern $a, b, a^b, a^{b^b}, a, b > 1$, partition regular?*

We conjecture the answer to this question is “no” and support our guess with two observations. Firstly, we show we cannot affirmatively answer Question 25 by only considering numbers of the form 2^x , as we have done in Theorem 4. Secondly, we show that $a, b, a^{b^{b^b}}$ is not partition regular. The main trick that we use to resolve these questions is captured in the following lemma.

LEMMA 26. *Let $a_1 < a_2 < \dots$ be a lacunary sequence of positive real numbers, that is, $\liminf a_{n+1}/a_n > 1$. Then there exists a finite colouring f of \mathbb{R} such that there is no monochromatic pair $x, x + a_n$ with $x \in \mathbb{R}, n \in \mathbb{N}$.*

Proof. In the context of this lemma only, we let $\{x\}$ stand for the fractional part of a real number x ; for a prime p and integer $n > 0$, we let $\nu_p(n)$ be the largest integer k for which p^k divides n . We shall also use the braces $\{ \}$ to denote sets, but the usage will always be clear from context.

Let $\varepsilon > 0$ be such that $a_n > (1 + \varepsilon)a_n$ for all $n \in \mathbb{N}$, and choose $l \in \mathbb{N}$ such that $(1 + \varepsilon)^l > 4$. Notice that if we partition $\{a_n : n \in \mathbb{N}\} = \bigcup_{i=0}^{l-1} \{a_n : n = lm + i\}$, it is sufficient to find, for each individual part of the partition, a colouring that forbids the differences in that part; we then

simply take the product of all of these colourings to forbid all differences in the union. Notice that if we arrange each set in the above partition as an increasing sequence, each sequence satisfies $b_{n+1} > (1 + \varepsilon)^l b_n > 4b_n$ for $n \in \mathbb{N}$. So it is sufficient to show that if $\{b_n\}$ is a sequence satisfying $b_n > 4b_{n-1}$, $n \in \mathbb{N}$, then we can find a colouring of \mathbb{R} forbidding differences in $\{b_n\}$.

To do this, we first find $\alpha \in (0, 1)$ such that $\{\alpha b_n\} \not\subseteq [-1/4, 1/4]$ for all n . Set

$$S_n = \{\alpha \in [0, 1] : \{\alpha b_n\} \not\subseteq [-1/4, 1/4]\}.$$

We show that $\bigcap_n S_n \neq \emptyset$, by way of the following simple claim. If $I \subseteq \mathbb{R}/\mathbb{Z}$ is an interval of length $1/(2b_n)$, then I intersects S_{n+1} in an interval of size $1/(2b_{n+1})$. To see this, simply note that as α varies over I , $\{\alpha b_{n+1}\}$ twice ranges over all of $[0, 1]$. Hence $S_{n+1} \cap I$ contains an interval of length $1/(2b_{n+1})$.

We now see that $\bigcap_n S_n \neq \emptyset$; choose $I_1 \subseteq S_1$ to be an interval of length $1/(2b_1)$, and then iteratively apply the lemma to construct $I_1 \supseteq I_2 \supseteq \dots$ so that $\emptyset \neq \bigcap_{n < N} I \subseteq \bigcup_{n < N} S_n$ for all N . As the I_i are closed, it follows that $\bigcap_n S_n \neq \emptyset$. We choose α to be a number in this intersection.

For the construction of the colouring, partition $[0, 1] = [0, 1/4) \cup [1/4, 1/2) \cup [3/4, 1)$. Now define a colouring $f: \mathbb{R} \rightarrow [4]$ by setting $f(x)$ to be $i \in [4]$ if $\{\alpha x\} \in [(i-1)/4, i/4)$. Finally, note that it is impossible for $x, x + a_n$ to be in the same colour class, as this would imply that $\{\alpha x\}, \{\alpha x + \alpha a_n\}$ lie in the same interval of length $1/4$, which is forbidden by the choice of α . ■

We now use the above lemma to show that we can colour the subsequence $\{2^n : n \in \mathbb{N}\}$ so that there is no monochromatic triple of the form a, a^{b^b} .

PROPOSITION 27. *One can finitely colour \mathbb{N} so that there is no monochromatic pair a, a^{b^b} with $a = 2^s, b = 2^t$ for some $s, t \in \mathbb{N}$.*

Proof. Notice that the sequence $\{n2^n : n \in \mathbb{N}\}$ satisfies the conditions of Lemma 26, and so we may find a finite colouring f of \mathbb{N} with no monochromatic pair of the form $m, m + n2^n$ with $m, n \in \mathbb{N}$. We then define the colouring c by $c(x) = f(\nu_2(\nu_2(x)))$ for $x \in \mathbb{N} \setminus \{1\}$, and define $c(1)$ to be some colour which is distinct from all other colours. Now if $c(a) = c(a^{b^b})$, where a, b are of the above form, then $f(\nu_2(s)) = c(a) = c(a^{b^b}) = f(\nu_2(s) + t2^t)$, which contradicts our choice of f . ■

We also show that in a finite colouring of \mathbb{N} we may forbid monochromatic pairs of the form $a, a \star (b \star (b \star b))$.

PROPOSITION 28. *There exists a finite colouring of \mathbb{N} such that there is no monochromatic pair of the form $a, a \star (b \star (b \star b))$, $a, b > 1$.*

Proof. We use Lemma 26 as in the previous proposition. For $n \in \mathbb{N}$, consider the sequence of integers $\{n^n \log_2 n : n \in \mathbb{N}\}$. Clearly this se-

quence satisfies the hypothesis of Lemma 26 and so we may apply the lemma to obtain a colouring f of \mathbb{R} which forbids a monochromatic pair with a difference of this type. We then define a finite colouring of $\mathbb{N} \setminus \{1\}$ by $c(x) = f(\log_2 \log_2(x))$ for $x \in \mathbb{N} \setminus \{1, 2\}$, and by defining $c(2)$ to be a colour distinct from all other colours. Now if we have a monochromatic pair $a, a \star (b \star (b \star b))$ where $a, b > 1$, it is clear that $a > 2$. So we must have $f(\log_2 \log_2(a)) = f(\log_2 \log_2(a) + b^b \log_2(b))$, contradicting the choice of f . ■

Another key property of FEP-sets is that they do not allow “height-one exponents” to also appear as a base in the same expression. For example, $a^b \cdot b$ is not, in general, contained in the set $\text{FEP}(a, b)$. However, we are unable to show that this is a necessary restriction.

QUESTION 29. *Is $a, b, ab, a^b, a^b \cdot b$ partition regular for $a, b > 1$?*

Again, we conjecture the answer to the above is negative. Curiously, “height-two exponents” may reappear as a base in the same FEP-expression. Indeed, we have $a^{b^c} \cdot c \in \text{FEP}(a, b, c)$. We close with an incredibly simple-looking problem that we are unable to resolve.

QUESTION 30. *Is a, b, a^{b+1} with $a, b > 1$ partition regular?*

This question has a natural multiplicative analogue.

QUESTION 31. *Is $a, b, a(b+1)$ partition regular?*

We conjecture the answers to both questions are “no”, but we are unable to resolve either.

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Julian Sahasrabudhe
 Department of Mathematics
 Dunn Hall
 University of Memphis
 Memphis, TN 38152, U.S.A.
 E-mail: julian.sahasra@gmail.com