I Can't Believe It's Not Random!

Joel Moreira's math blog

Sets of nice recurrence

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— 1. Introduction —

Let (X,μ) be a probability space and $T:X\to X$ be a (measurable) map such that the set $T^{-1}A:=\{x\in X:Tx\in A\}$ has the same measure as the set A for all (measurable) sets $A\subset X$. We call the triple (X,μ,T) a measure preserving system. All sets and maps from now on are measurable. Given a set $A\subset X$ and some $\epsilon>0$ we define

$$R_\epsilon(A):=\{n\in\mathbb{N}:\mu(T^{-n}A\cap A)>\mu(A)^2-\epsilon\}$$

In a previous post I talked about recurrence theorems, and in particular about how large the set $R_\epsilon(A)$ is in general. Note that by the mean ergodic theorem we have $\lim\mu(T^{-n}A\cap A)\geq\mu(A)^2$ and in particular this limit exists, so the set $R_\epsilon(A)$ is non-empty.

> ***Definition 1*** *A set* $R\subset\mathbb{N}$ *is a* set of nice recurrence *if for every measure preserving system* (X,μ,T)*, every* $A\subset X$ *and every* $\epsilon>0$ *the intersection* $R\cap R_\epsilon(A)$ *is non-empty.*

We already observed that $\mathbb{N}$ itself is a set of nice recurrence, but there are much more rarified sets of nice recurrence. For instance it follows from the result explored in this post that for any polynomial $f\in\mathbb{Z}[x]$ with integer coefficients and $f(0)=0$ that the set $f(\mathbb{N})=\{f(n):n\in\mathbb{Z}\}$ is a set of nice recurrence.

The property of nice recurrence is a strengthening of the property of recurrence. In this previous post (cf. Theorem 10 in there) I showed that recurrent sets are exactly the intersective sets. In this post I will prove an analog of this result regarding sets of nice recurrence.

— 2. A more general setup —

For the readers convenience, I will give first a proof that a set is recurrent if and only if it is an intersective set. There is a subtlety here regarding the notion of upper density, as for each Følner sequence there is a different notion of upper density. I find that, when playing with different Følner sequences, things become more clear by working with a general countable, abelian group, and so that's what we will do. Actually this generalizes to any amenable group, but for the sake of presentation I will restrict to abelian groups.

The reader uncomfortable with Følner sequences and general abelian groups can use the model case $G=\mathbb{Z}$ and $F_N=\{1,...,N\}$.

Recall that in a group G, a (left) Følner sequence is a sequence (F_N) of finite subsets of G such that for each $n\in G$ we have

$$\lim_{N\to\infty}\frac{|F_N\cap(n+F_N)|}{|F_N|}=1$$

A countable group is amenable if and only if it has a Følner sequence, and it is known that every abelian group is amenable and hence has a Følner sequence. For each Følner sequence $(F_N)_N$ we define the upper density of a set $E\subset G$ by

$$\bar{d}_{(F_N)}(E):=\limsup_{N\to\infty}\frac{|E\cap F_N|}{|F_N|}$$

Moreover we define the Banach upper density $d^*(E)$ of a set $E\subset G$ as the supremum over all Følner sequences of the upper density $\bar{d}_{(F_N)}(E)$. A measure preserving action of a group in a probability space (X,μ) is an action of G on X such that each $n\in G$ induces a measure preserving transformation. We represent that measure preserving transformation by T_n. The action is ergodic if any set $A\subset X$ invariant under G (in other words such that $T_n^{-1}A=A$ for all $n\in G$) has either measure 1 or measure 0.

> ***Definition 2*** *Let G be a countable abelian group, let $R\subset G$ and let $(F_N)_N$ be a Følner sequence on G.*
>
> - *R is a set of recurrence if for every measure preserving action of G on (X,μ) and every $A\subset X$ with $\mu(A)>0$ there is some $n\in R$ such that $\mu(A\cap T_n^{-1}A)>0$.*
> - *R is (F_N)-intersective if for each $E\subset G$ with $\bar{d}_{(F_N)}(E)>0$ there is some $n\in R$ such that $\bar{d}_{(F_N)}[E\cap(E-n)]>0$.*
> - *R is Banach intersective if it is (F_N)-intersective for all Følner sequences (F_N) on G.*
> - *R is a set of nice recurrence if for every measure preserving action of G on (X,μ), every $\epsilon>0$ and every $A\subset X$ there is some $n\in R$ such that $\mu(A\cap T_n^{-1}A)>\mu(A)^2-\epsilon$.*
> - *R is a set of ergodic nice recurrence if for every ergodic measure preserving action of G on (X,μ), every $\epsilon>0$ and every $A\subset X$ there is some $n\in R$ such that $\mu(A\cap T_n^{-1}A)>\mu(A)^2-\epsilon$.*
> - *R is (F_N)-nicely intersective if for each $E\subset G$ and each $\epsilon>0$ there is some $n\in R$ such that $\bar{d}_{(F_N)}[E\cap(E-n)]>\bar{d}_{(F_N)}(E)^2-\epsilon$.*
> - *R is Banach nicely intersective if it is (F_N)-nicely intersective for all Følner sequences (F_N) on G.*

Note that in the definition of sets of recurrence and sets of nice recurrence there is no reference to Følner sequences.

— 3. Sets of recurrence —

For sets of recurrence we have the following characterization:

> ***Theorem 3*** *Let G be a countable abelian group and let $R\subset G$. Let $(F_N)_{N\in\mathbb{N}}$ be a Følner sequence on G. Then R is a set of recurrence if and only if R is (F_N)-intersective.*

Proof: To prove one direction we can use the Furstenberg's correspondence principle. Then assuming that R is recurrent we get that R is (F_N)-intersective.

To prove the other direction, assume that R is (F_N)-intersective and let $\{T_n\}_{n\in G}$ be a measure preserving action of G on a probability space (X,μ) and let $A\subset X$ have positive measure. For each $x\in X$ let $E(x)\subset G$ be the set

$E(x)=\{n\in G:T_nx\in A\}$. Let

$$f_N(x)=\frac{|E(x)\cap F_N|}{|F_N|}=\frac{1}{|F_N|}\sum_{n\in F_N}1_A(T_nx)$$

Note that $\int_X f_N d\mu=\mu(A)$ for all N, so by Fatou's lemma we get that

$$\int_X \bar{d}_{(F_N)}[E(x)]d\mu(x)\geq \limsup_{N\to\infty}\int_X f_N d\mu=\mu(A)>0$$

In particular we get that $d_{(F_N)}[E(x)]>0$ in a set of positive measure, call it C. For each $x\in C$ there is some $n=n(x)\in R$ such that $E(x)\cap(E(x)-n)$ is non-empty. Let $a=a(x)$ be in that intersection. Since there are only countably many choices for the pair $(n(x),a(x))\in R\times G$, we conclude that there is a subset $D\subset C$ with positive measure and a pair $(n,a)\in R\times G$ such that for each $x\in D$ we have $a\in E(x)\cap(E(x)-n)$.

Thus if $x\in D$ then both $T_ax\in A$ (because $a\in E(x)$) and $T_{a+n}x\in A$ (because $a+n\in E(x)$). Equivalently we have $T_a(x)\in A\cap T_n^{-1}A$ and so $T_a(D)\subset A\cap T_n^{-1}A$, so $\mu(A\cap T_n^{-1}A)>0$. $\square$

As a corollary we get the following:

> ***Corollary 4*** *Let G be a countable abelian group and let $R\subset G$. Then R is a set of ergodic recurrence if and only if R is Banach-intersective.*

Proof: If R is recurrent, then it is (F_N)-intersective for all Følner sequence $(F_N)_{N\in\mathbb{N}}$ and hence it is Banach intersective.

Reciprocally, if it is Banach intersective, then it is (F_N)-intersective for some (and actually all) Følner sequence and by the previous result we get that it is also recurrent. $\square$

Maybe even more surprising, at least a priori, is the following observation:

> ***Corollary 5*** *Let $(F_N)_{N\in\mathbb{N}}$ and $(F'_N)_{N\in\mathbb{N}}$ be two Følner sequence in G. Then a set $R\subset G$ is (F_N)-intersective if and only if it is (F'_N)-intersective.*

— 4. Sets of nice recurrence —

We now prove a weak analogue of the Theorem 3 for sets of nice recurrence.

> ***Theorem 6*** *Let G be a countable abelian group and let $R\subset G$ be a Banach nicely intersective set. Then R is a set of ergodic nice recurrence.*

Proof: Let $\{T_n\}_{n\in G}$ be an ergodic measure preserving action of G on the probability space (X,μ) and let $A\subset X$. For each $x\in X$ let $E(x)\subset G$ be the set $E(x):=\{n\in G:T_nx\in A\}$. Now fix an arbitrary Følner sequence $(F_N)_{N\in\mathbb{N}}$ in G, fix $m\in G$ and let

$$f_N(x)=\frac{|E(x)\cap(E(x)-m)\cap F_N|}{|F_N|}=\frac{1}{|F_N|}\sum_{n\in F_N}1_{E(x)\cap(E(x)-m)}(n)$$

Note that for $n, m \in G$ and $x \in X$ we have

$$n \in E(x) \cap (E(x) - m) \iff T_n x \in A \cap T_m^{-1} A$$

By the mean ergodic theorem and the previous equation we have

$$\mu(A \cap T_m^{-1} A) = \lim_{N\to\infty} \frac{1}{|F_N|} \sum_{n\in F_N} 1_{A\cap T_m^{-1}A}(T_n x) = \lim_{N\to\infty} f_N(x)$$

where the convergence holds in the L^2 norm. Since f_N are bounded functions we conclude that some subsequence of $\{f_N\}$ converges almost everywhere, hence for almost every $x \in X$ we have $\bar{d}[E(x) \cap (E(x) - m)] = \mu(A \cap T_m^{-1} A)$ where the density is with respect to a subsequence of the original Følner sequence. Applying this procedure for each of the countably many elements m of G we have that for some Følner sequence $(F_N')_{N\in\mathbb{N}}$, for all $m \in G$ and all x in a set of full measure we have

$$\mu(A \cap T_m^{-1} A) = \bar{d}_{\{F_N'\}}[E(x) \cap (E(x) - m)]$$

The result now follows from the definitions. □

Remark 1 *Note that, from the proof, it follows that we don't need R to be (F_N)-nicely intersective for all Følner sequences $(F_N)_{N\in\mathbb{N}}$ in G, but just for some family of Følner sequences $\mathcal{F}$ such that any Følner sequence in G has a subsequence in $\mathcal{F}$. If moreover we consider two Følner sequences to be the same if they give the same upper density to every set, is it true that there exists such a countable family $\mathcal{F}$? Assuming that this is the case (although this seems rather unlikely), then we can even replace the condition of R to be Banach with the condition that R is (F_N)-nicely intersective for some Følner sequence.*

Remark 2 *Note that a partial converse of the Theorem 6 (the fact that a set of ergodic nice recurrence is (F_N)-nicely intersective for some Følner sequence) follows from a strong form of Furstenberg's correspondence principle, where we get an ergodic system. The precise statement we are invoking here is the proposition 3.1 of this paper by Bergelson, Host and Kra.*

— 5. A negative result on recurrence —

In a previous post I stated the following theorem of Bergelson:

Theorem 7 *Let (X, μ) be a probability space, let $a > 0$ and let $\{A_n\}_{n\in\mathbb{N}}$ be a countable family of subsets of X, all with $\mu(A_n) > a$. Then there exists a set $E \subset \mathbb{N}$ with $\bar{d}(E) \geq a$ (where the density is with respect to the Følner sequence $F_N = [1, N]$) and such that for each $n, m \in E$ we have $\mu(A_n \cap A_m) > 0$*

On a different direction, it is a consequence of (the proof of) the Poincaré recurrence theorem that:

Theorem 8 *Let (X, μ) be a probability space, let $a > 0$ and let $\{A_n\}_{n\geq 0}$ be a countable family of subsets of X, all with $\mu(A_n) > a$ and such that for $m > n$ we have $\mu(A_n \cap A_m) = \mu(A_0 \cap A_{m-n})$. Then for each $\epsilon > 0$ there exists an infinite set $E \subset \mathbb{N}$ such that for each $n, m \in E$ we have*