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Elemental Methods in Ergodic Ramsey Theory

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Contents

Introduction

These notes, which are based on a graduate course and seminars given at Wesleyan University in the fall of 1997, contain many of the main results of chromatic and density Ramsey theory in $\mathbf{Z}$. This subfield of combinatorial number theory deserves attention due in part to the fact that its principal theorems are (i) natural, (ii) easy to formulate, and (iii) non-trivial. These three features are hallmarks of the best in pure mathematics.

Most of the results we treat were originally proved combinatorially. We concentrate here on a variety of alternative approaches proceeding by way of topological dynamics and ergodic theory. Our goal is not to prove theorems of the greatest generality in the shortest amount of time, but to allow the reader to observe the methodology gradually unfolding.

The problems we will be attacking fall roughly into three classes, which are perhaps best introduced by example. As it happens, one result in each category stands out as characteristic.

Van der Waerden's theorem.

If the set of natural numbers $\mathbf{N} = \{1, 2, 3, \cdots\}$ is partitioned into finitely many cells (or finitely *colored*), then one of the cells contains arbitrarily long *arithmetic progressions.* This theorem, which settled a conjecture that had been open for some time, was proved by van der Waerden in 1927 ([vdW]; see also [GRS]). It is perhaps (some experts may disagree) the prototypical result of *Ramsey theory*, a field that takes its name from F. Ramsey, who in 1930 proved a similar theorem about finite colorings. (We describe it momentarily.) Loosely speaking, Ramsey theory pertains to the existence of monochromatic subsets for finite colorings of large structures. The results of this type we examine fall into two distinct categories, according to whether the sought after monochromatic configurations are finite (as in van der Waerden's theorem), or infinite.

Ramsey's theorem is of the latter type. Moreover, it is a very appealing problem which is easily formulated: suppose you finitely color all two element subsets of natural numbers. Then for some infinite $E \subset \mathbf{N}$, the set of all two element subsets of E is monochromatic. A finitistic version of the problem is often called the "party theorem" because it can be formulated as follows: let $k \in \mathbf{N}$. Then for any sufficiently large dinner party, you can find a group of k

people who are either (a) mutual acquaintences all, or (b) complete strangers to one another.

As a matter of fact, quite a bit of computer time has been devoted to the determination of exactly how large such a dinner party must be for various values of k. It may be the rate at which this determination becomes intractible which causes such fervent interest. For $k = 3$, a party of 6 is sufficient (you can check it by hand), for $k = 4$ it takes 18 (you'll want a PC for this), and for $k = 5$, amazingly, the best that is known (as of July 1997) is that it takes somewhere between 43 and 49. (For more "Ramsey numbers", as well as an exhaustive list of references on the subject, the reader is referred to [Ra].) Ramsey theory of this type is clearly wonderful sport, however it is not our purpose to explore such fine points of optimization; our sole concern is with which monochromatic configurations are guaranteed, and not with how long it takes for them to show up.

Hindman's theorem.

A theorem of N. Hindman proved in 1974 ([H1]) is in many ways a much better model than Ramsey's theorem for "infinitary" Ramsey theory. Take any sequence of natural numbers $(n_i)_{i=1}^{\infty}$. Let Γ be the set of all finite sums taken from the sequence, where each member may be summed only once, i.e.

$$\Gamma = \{n_{i_1} + n_{i_2} + \cdots + n_{i_k} : k \in \mathbf{N}, i_1 < i_2 < \cdots < i_k\}.$$

Γ is said to be an *IP-set.*

IP-sets are of mathematical interest in part because they are "almost" closed under addition, in the sense that if x and y are two members of an IP-set Γ which are formed by taking finite sums of two *disjoint* collections from the original generating sequence, then $x + y$ is in Γ as well. This seemingly weak attribute is sufficient to give IP-sets many of the properties one would expect might be reserved for "bigger" subsets of natural numbers, such as subsemigroups of $\mathbf{N}$ (which are properly closed under addition). The reason this is important is that IP-sets, which can be terribly thin, almost wafery objects (consider that there is no limit whatsoever on how fast the generating sequence $(n_i)_{i=1}^{\infty}$ increases), are in a sense cheaper to come by than these other, bulkier types of sets. Indeed, Hindman's theorem is a kind of proof of this "cheapness". It states that for any finite coloring of the natural numbers, there exist monochromatic IP-sets.

Hindman's theorem has definite aesthetic advantages over Ramsey's theorem, and a few practical ones as well. Most importantly, it is much less trivial. Furthermore, it is more relevant to the dynamical approaches to Ramsey theory we are advocating. Also, Hindman's theorem has a precursor in the classical literature which makes it a most natural question. Schur's theorem, one of the oldest Ramsey-type results, states that for any finite coloring of the natural numbers, some two numbers and their sum occur in one color, that is, there exists a monochromatic configuration of the form $\{x, y, x + y\}$. Later efforts established that for any k, one could find k numbers and all of their sums (taken one at a time) in a single color. Hindman's theorem is the natural infinitary extension of these classical results. Even so, it wasn't resolved until 58 years after Schur's theorem was proved.

Szemerédi's theorem.

At first sight, Szemerédi's theorem looks like a minor modification of van der Waerden's theorem. In terms of depth, however, this is hardly the case. When it was proved (in 1975), it settled a conjecture of Erdös and Turàn which had been made 38 years earlier. The motivation for the Erdös-Turàn conjecture (see [ET]) was the question "*why* is it that some cell of any finite partition of the natural numbers contains arbitrarily long arithmetic progressions?" It is obvious that many sufficiently "sparse" subsets of the natural numbers, such as the powers of 2, fail to contain arithmetic progressions even of length 3. Indeed, the sequence of squares $1, 4, 9, 16, \cdots$ fails to contain an arithmetic progression of length 4 (a theorem of Fermat). Erdös and Turàn wondered if there might exist any subsets of $\mathbf{N}$ that are not "sparse", and which yet fail to contain arbitrarily long arithmetic progressions.

To be more precise, consider the following notion of "sparseness": a subset E of the natural numbers is said to be of *zero density* if $\lim_{N\to\infty} \frac{|E\cap\{1,2,\cdots,N\}|}{N} = 0$. That is, if the "percentage" of longer and longer initial blocks of the natural numbers taken up by E tends to zero. Erdös and Turán's conjecture is: let $k \in \mathbf{N}$. Any subset of $\mathbf{N}$ which fails to contain an arithmetic progression of length k must be of zero density. If true, this would imply van der Waerden's theorem, for in every finite partition of the natural numbers there is at least one cell which fails to be of zero density.

The truth of the conjecture is obvious for $k = 1$ and $k = 2$. K. Roth established the $k = 3$ case in 1952. In 1969, Szemerédi proved it for $k = 4$. Finally in 1975 Szemerédi's proof for general k appeared.

Szemerédi's theorem is a very deep result whose original proof is not all that accessible. H. Furstenberg reproved the result in 1977 by recasting it as a multiple recurrence theorem in ergodic theory. Hence *ergodic Ramsey theory.* In the decade following this breakthrough, Furstenberg and his colleagues (especially Y. Katznelson and B. Weiss) developed this new field, proving many new results in density Ramsey theory and reproving the results of chromatic Ramsey theory via both recurrence theorems in ergodic theory (for density results) and topological dynamics (for chromatic results).

Two theorems of Furstenberg and Katznelson, in particular, marked successively more sizable jumps in non-triviality even from Szemerédi's theorem. They are, in turn, a recurrence theorem for commuting IP-systems of measure preserving systems (1985) and a density version of the Hales-Jewett theorem (1991). These results lie beyond the scope of the current exposition, as do many other recent advances in the field (see for example [L], [BL3], [BM2], [G1], and [G2]). However most of the major results (at least in $\mathbf{Z}$) up to and including a polynomial extension of Szemerédi's theorem due to Bergelson and Leibman

([BL1]) are proved here. The reader is referred to [F2] and [B4] for further exposition of the subject.

An effort is made in chapters 1-3 to be as self-contained as possible. (The graduate course they are taken from was given to first, second and third year students.) Accordingly, with only one exception all that is needed (besides some degree of mathematical sophistication) is a bit of knowledge of abstract algebra (the reader should know what groups, semigroups, fields and vector spaces are). Familiarity with some point set topology and measure theory would be convenient, although most of what is needed in these areas is developed. Chapters 4 and 5 will only be found to be self-contained by those readers well-acquainted with ergodic theory. Throughout the notes, many details of proofs and interesting facts are relegated to 213 exercises, most of which are located in the first three chapters. These exercises, which invite one to engage the subject matter in a more active manner, are a distinguishing feature of this manuscript.

I would finally like to acknowledge the considerable inspiration and support I have been given in this undertaking, for which I am chiefly indebted to four notable people. The first of these is H. Furstenberg. Most of the methods contained in chapters 3-5 are either due to Furstenberg directly or are adaptations of methods culled from the work of he and his co-workers. Without his monumental original contributions to the subject, it is senseless to imagine this volume ever coming to be. Most of my knowledge of finite sums systems and their combinatorial significance was obtained from N. Hindman, first by reading his papers, then in our subsequent collaboration. A. Fieldsteel was responsible for inviting me to Wesleyan University and for suggesting that I teach the course from which these notes are drawn, so in a moderately direct way he is responsible for their existence. Finally, it is my pleasure to thank V. Bergelson. Professor Bergelson guided my initiation into this subject and has pointed me in the direction of almost everything I know about it. If not for the infectious nature of his extraordinary enthusiasm for mathematics, I might well have taken up something different by now.

A web page associated with these notes will be maintained at www.math.umd.edu/ randall/ert, featuring solutions to selected exercises, additional exposition, and a (hopefully short) list of typographical errors, for which submissions are welcome.

Chapter 1

Ramsey Theory and Topological Dynamics

1.1 Preliminaries.

If X is a set then a *topology* on X is a subset of the power set of X, $\mathcal{T} \subset \mathcal{P}(X)$, that is closed under finite intersections, closed under arbitrary unions, and that contains both X and $\emptyset$. The members of $\mathcal{T}$ are called *open* sets and their complements are called *closed* sets. The collection of closed sets is therefore closed under finite unions and arbitrary intersections. The *closure* $\overline{E}$ of a set $E \subset X$ is the intersection of all closed sets containing E. The pair $(X, \mathcal{T})$ (or just X, if $\mathcal{T}$ is understood) is called a *topological space.* If $E \subset X$ then we can create a topology $\mathcal{S}$ on E by taking $\mathcal{S}$ to consist of all intersections of E with a member of $\mathcal{T}$. We call this the *induced* topology on E.

Example. (The cofinite topology.) Let X be any set and put $\mathcal{T} = \{U \subset X : |U^c| < \infty\}$. Then $(X, \mathcal{T})$ is a topological space.

A *metric* on a set X is a function $\rho : X \times X \to [0, \infty)$ with $\rho(x, y) = 0$ if and only if $x = y$ and with $\rho(x, z) \leq \rho(x, y) + \rho(y, z)$ for all $x, y, z \in X$. The pair (X, ρ) (again, just X if ρ is understood) is called a *metric space.*

Example. Let $\mathbf{R}$ be the set of real numbers and put $\rho(x, y) = |x - y|$, $x, y \in \mathbf{R}$. Then ρ is a metric.

A metric space has a natural topology consisting of those sets U having the property that for every $x \in U$, there exists $\epsilon > 0$ such that $B_\epsilon(x) = \{y \in X : \rho(x, y) < \epsilon\} \subset U$. For any topological space $(X, \mathcal{T})$, if there is a metric on X such that the topology induced by ρ in this fashion is just $\mathcal{T}$, then the space $(X, \mathcal{T})$ is called *metrizable.* (Not every topological space is metrizable.)

A sequence $(x_n)_{n=1}^{\infty}$ in a topological space is said to *converge* to $x \in X$, and we write either $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$, if for every open set U containing x (such sets are called *neighborhoods* of x), $x_n \in U$ but for finitely many n.

A topological space X is said to be *compact* if for every collection of open sets whose union is X (such a collection is called an *open cover*) there exists a finite subcollection still covering X. Equivalently, X is compact if its collection of closed sets has the *finite intersection property*; namely, if every sub-collection of closed sets, any finitely many of which have non-empty intersection, has non-empty intersection. For metric spaces, compactness is equivalent to *sequential compactness*: the property that every sequence in X have a convergent subsequence.

A necessary and sufficient condition for a metric space X to be compact is that it be *totally bounded* and *complete.* X is totally bounded if for every $\epsilon > 0$ there exists a finite set $\{x_1, \cdots, x_k\} \subset X$ having the property that for every $y \in X$ there exists i, $1 \leq i \leq k$, with $\rho(y, x_i) < \epsilon$. (Such a set $\{x_1, \cdots, x_k\}$ is called an ϵ-*net.*) X is complete if every Cauchy sequence in X converges ($(x_n)_{n=1}^{\infty}$ is Cauchy if for every $\epsilon > 0$ there exists $k \in \mathbf{N}$ such that $\rho(x_n, x_m) < \epsilon$ whenever $n, m > k$).

If X is a set and $\mathcal{G}$ is a family of its subsets, then the topology $\mathcal{T}$ *generated* by $\mathcal{G}$ consists of all unions of families of sets each of whose elements are finite intersections of the members of $\mathcal{G}$. We say that $\mathcal{G}$ is a *subbasis* for $\mathcal{T}$.

Example. Let $\mathcal{G}$ consist of all unbounded open intervals in $\mathbf{R}$ (namely those of the form either $(-\infty, a)$ or (a, ∞) for some $a \in \mathbf{R}$. The topology generated by $\mathcal{G}$ on $\mathbf{R}$ is the usual topology (namely, the open sets are precisely those which can be expressed as the union of some collection of open intervals).

If X and Y are topological spaces then a function $T : X \to Y$ is said to be *continuous* if for every open set U in Y, the pre-image $T^{-1}U = \{x \in X : Tx \in U\}$ is open in X. If the topologies are given by metrics, say ρ on X and η on Y, this is equivalent to the condition that for every $x \in X$ and every $\epsilon > 0$ there exists $\delta > 0$ such that $\rho(x, z) < \delta$ implies $\eta(Tx, Tz) < \epsilon$. Continuous maps T on a compact space X are *uniformly continuous*; that is, for every $\epsilon > 0$ there exists $\delta > 0$ such that $\rho(x, z) < \delta$ implies $\eta(Tx, Tz) < \epsilon$ (the difference between this condition and continuity is that δ doesn't depend on x). If $T : X \to Y$ is one-to-one and onto (i.e. a *bijection*) and both T and T^{-1} are continuous then T is said to be a *homeomorphism.* We call a continuous function $T : X \to X$ a *transformation.* If it is a homeomorphism, the transformation T is said to be *invertible.* We note that if $T : X \to Y$ is continuous and $B \subset X$ is compact (that is, in the induced topology), then the image of B under T, $TB = \{Tx : x \in B\}$, is also compact.

If I is a set and for every $i \in I$, X_i is a set, then we denote by $\prod_{i \in I} X_i$ the set of "I-tuples" $(x_i)_{i \in I}$ with $x_i \in X_i$, $i \in I$. (That this set has members in the event that all of the X_i's are non-empty is the axiom of choice. We use the axiom of choice and its equivalent formulations, Zorn's Lemma and the possibility of well-ordering any set, freely.) If for all $i \in I$, X_i is a topological space, then the *product topology* on $\prod_{i \in I} X_i$ is the topology generated by sets of the form $\{(x_i)_{i \in I} : x_j \in U_j\}$, where $j \in I$ is fixed and $U_j \subset X_j$ is open. Tychonoff's theorem (itself equivalent to the axiom of choice as well) states that if X_i is compact for all i then $\prod_{i \in I} X_i$ is compact in the product topology. If every

X_i is the same space, say X, then we will write X^I for $\prod_{i\in I} X_i$. In general if X is a compact metric space and I is countable then X^I will be compact and metrizable.

A subset $U \subset X$ is called *residual* if it contains a countable intersection of dense open sets. Equivalently, if U^c is a countable union of nowhere dense sets (a set is nowhere dense if its closure contains non-empty open sets). In complete metric spaces, residual sets are non-empty. (This is the Baire category theorem.) Indeed, in some sense, they are quite large. The complement of a residual set is called a set of *first category.*

A *relation* on a set $\mathcal{Z}$ is a subset of $\mathcal{Z} \times \mathcal{Z}$. A relation $\mathcal{E} \subset \mathcal{Z} \times \mathcal{Z}$ with the properties:

(a) if $(x, y) \in \mathcal{E}$ and $(y, x) \in \mathcal{E}$ then $x = y$,
(b) if $(x, y) \in \mathcal{E}$ and $(y, z) \in \mathcal{E}$ then $(x, z) \in \mathbf{Z}$, and
(c) $(x, x) \in \mathcal{E}$ for all $x \in \mathcal{Z}$.

will be called a *partial order* on $\mathcal{Z}$. If E is a relation then we may write $x \prec y$ if and only if $(x, y) \in \mathcal{E}$, and say simply that "$\prec$" is a partial order on $\mathcal{Z}$.

Example. Let X be a set and let $\mathcal{Z} = \mathcal{P}(X)$ be the power set of X. Then $\subset$ (inclusion) is a partial order on $\mathcal{Z}$.

Suppose that $\mathcal{Z}$ is a set, $\prec$ is a partial order on $\mathcal{Z}$, and $\mathcal{T} \subset \mathcal{Z}$ has the property that for every $x, y \in \mathcal{T}$, either $x \prec y$ or $y \prec z$. Then $\mathcal{T}$ will be called a *totally ordered set*, or a *chain.* If $z \in \mathcal{Z}$ has the property that $x \in \mathcal{Z}$ and $x \prec z$ implies that $x = z$, then z will be called a *minimal element* of $\mathcal{Z}$ relative to $\prec$. Zorn's Lemma (which is equivalent to the axiom of choice) states that if $\prec$ is a partial order on $\mathcal{Z}$ and every totally ordered subset $\mathcal{T} \subset \mathcal{Z}$ is bounded from below (that is, there exists $y \in \mathbf{Z}$ such that $y \prec x$ for all $x \in \mathcal{T}$), then there exists a minimal element $z \in Z$.

Example. (An application of Zorn's Lemma.) Let X be a compact topological space. We claim that there exists a minimal non-empty closed set B in X. (This is not obvious. There are topological spaces for which singletons are not closed sets.) Let $\mathcal{Z}$ be the family of non-empty closed subsets of X. $\mathcal{Z}$ is partially ordered by $\subset$. Let $\mathcal{T}$ be a totally ordered subset of $\mathcal{Z}$. Putting $C = \bigcap_{V\in\mathcal{T}} V$, C is non-empty by the finite intersection property. (The intersection of any finite subset of $\mathcal{T}$ is equal to the least element of that subset and hence non-empty.) Moreover $C \subset V$ for every $V \in \mathcal{T}$, so the hypotheses of Zorn's Lemma are met and there exists a non-empty, minimal closed subset of X relative to $\subset$.

1.2 Van der Waerden's theorem.

A (perhaps *the*) fundamental result of Ramsey theory is van der Waerden's theorem ([vdW]; see also [GRS]), which has many equivalent formulations. The most basic is:

vdW1. Let $k, r \in \mathbf{N}$. If $\mathbf{N} = \bigcup_{i=1}^{r} C_i$ then some C_i contains an arithmetic progression of length k.

Here is another formulation, which is *finitistic.*

vdW2. Let $k, r \in \mathbf{N}$. There exists $N = N(k,r) \in \mathbf{N}$ with the property that for any partition $\{1, 2, \cdots, N\} = \bigcup_{i=1}^{r} C_i$ some C_i contains an arithmetic progression of length k.

Our last purely combinatorial formulation of van der Waerden's theorem is the following *affine* form, which uses the following notion: If $F \subset \mathbf{N}$, then an *affine image* of F is a set of the form $a + bF = \{a + bf : f \in F\}$, where $a, b \in G$ with $b \neq 0$.

vdW3. For any finite partition of $\mathbf{N}$, one of the cells contains affine images of every finite set.

Exercise 1.1. Prove that **vdW1-3** are equivalent.

Exercise 1.2. Prove that **vdW1** and **vdW3** are equivalent to the statements which would result by changing $\mathbf{N}$ to $\mathbf{Z}$ in their formulation.

Exercise 1.3. Prove that if **vdW2** holds for $r = 2$ and all k then it holds in general.

The reader may wish to try and prove van der Waerden's theorem for specific values of r and k. For $k = 2$, things are easy. For $k = 3$, $r = 2$ (the first non-trivial case), things still aren't so bad. The following exercise may offer a fair amount of amusement, however.

Exercise 1.4. Find an upper bound for $N(3,3)$.

A few words about the terminology of partitions are in order: the sets of a partition (the cells, that is) are often termed *colors*, and by the color of a point we mean the cell to which it belongs. Hence, a partition into two cells is also called a *2-coloring.* A configuration (such as an arithmetic progression) which is contained in a single cell of the partition is called *monochromatic.* We are now ready to offer a loose definition of Ramsey theory.

> *Ramsey theory is a collection of results which, given a finite coloring of some structure, guarantee the existence of certain monochromatic configurations or substructures.*

In this chapter we will concentrate on monochromatic configurations, postponing the matter of monochromatic substructures to the next. Throughout, we will be utilizing the methods of *topological dynamics*; that is, we consider topological spaces and their continuous self-maps. This approach to Ramsey theory was pioneered by Furstenberg and Weiss (see [FW], [F2]). In order to accomplish this, we will need to formulate results in this setting which are equivalent to (or at least imply) the corresponding Ramsey-theoretic theorems under consideration. Here is our first example of such a result.

vdW4. ([FW].) Let $k \in \mathbf{N}$ and $\epsilon > 0$. For any compact metric space X and continuous map $T : X \to X$ there exist $x \in X$ and $n \in \mathbf{N}$ such that $\rho(x, T^{in}x) < \epsilon$, $1 \leq i < k$.

The label **vdW4** we have given this theorem suggests that it is another formulation of van der Waerden's theorem. This is in fact the case. We will now argue that **vdW4** implies **vdW1**, leaving the reverse implication as an exercise.

Let $k, r \in \mathbf{N}$ and suppose that $\mathbf{N} = \bigcup_{i=1}^{r} C_i$. In order to apply **vdW4** we need to use our partition to construct a topological space X and a continuous self-map T of X. Let

$$\Omega = \{1, \cdots, r\}^{\mathbf{N}} = \{\gamma : \mathbf{N} \to \{1, \cdots, r\}\}$$

be the set of all sequences taking values in $\{1, \cdots, r\}$. For $\gamma \in \Omega$, define $T\gamma \in \Omega$ by $T\gamma(n) = \gamma(n+1)$. T is called the *shift* on Ω.

Exercise 1.5. Show that:

(a) Ω is a compact metric space with metric $\rho(\gamma, \xi) = \frac{1}{\min\{k : \gamma(k) \neq \xi(k)\}}$.
(b) $T : \Omega \to \Omega$ is continuous.
(c) $T^n\gamma(t) = \gamma(n+t)$ for $n, t \in \mathbf{N}$.
(d) $\rho(\gamma, \xi) < \frac{1}{n}$ if and only if $\gamma(j) = \xi(j)$, $1 \leq j \leq n$.

The partition $\mathbf{N} = \bigcup_{i=1}^{r} C_i$ may be used to define a point $\alpha \in \Omega$ by $\alpha(n) = i$ if and only if $n \in C_i$. Let $X = \overline{\{T^m\alpha : m \in \mathbf{N}\}}$.

Exercise 1.6. Show that the restriction of T to X is a continuous self-map of X.

The pair (X, T) is an example of a *dynamical system.* (Actually, (Ω, T) is a dynamical system as well, but, having nothing whatsoever to do with our given partition, isn't the one we are interested in right now.) We are ready to apply **vdW4** to the system (X, T).

By **vdW4** there exists $x \in X$ and $n \in \mathbf{N}$ such that $\rho(x, T^{in}x) < 1$, $1 \leq i < k$. In particular, $x(1) = T^n x(1) = \cdots = T^{(k-1)n}x(1)$, or

$$x(1) = x(n+1) = x(2n+1) = \cdots = x((k-1)n+1).$$

But $x \in X = \overline{\{T^m\alpha : m \in \mathbf{N}\}}$, meaning that for some $m \in \mathbf{N}$ we have $\rho(T^m\alpha, x) < \frac{1}{(k-1)n+1}$. In particular, if we let $i = x(1)$ this gives

$$i = T^m\alpha(1) = T^m\alpha(n+1) = T^m\alpha(2n+1) = \cdots = T^m\alpha((k-1)n+1),$$

or

$$i = \alpha(m+1) = \alpha(m+1+n) = \alpha(m+1+2n) = \cdots = \alpha(m+1+(k-1)n).$$

In other words, the k-term arithmetic progression

$$\{(m+1), (m+1)+n, (m+1)+2n, \cdots, (m+1)+(k-1)n\}$$

is contained in C_i.

□

Exercise 1.7. Show that **vdW1** implies **vdW4**.

The primary purpose of these notes is to show methods for proving Ramsey theory results via dynamics. However, we will at times give or at least sketch combinatorial proofs to many of the results, partly in order to compare the methods and partly out of independent interest. In the next few sections we will be giving dynamical proofs of various extensions of van der Waerden's theorem. Right now, let us examine a combinatorial proof.

We will work Exercise 1.4 in such a way as may be adapted to an inductive (and entirely combinatorial) proof of van der Waerden's theorem. (We'll leave the necessary adaptations as exercises.) All that we assume is that $N(2,r)$, $r \in \mathbf{N}$, has been previously determined. (Of course, one easily verifies that $N(2,r) = r+1$.) The following exercise is needed for the proof. For it, we introduce the notation $x \sim y$, which means that x and y lie in the same cell of a partition.

Exercise 1.8. Show that if $M = N(2,3^k) + k$ then for any 3-coloring of $\{1,\cdots,M\}$ there exists a recurrent k-block. That is, for some positive integers m and n we have

$$m \sim (m+n),\ \ (m+1) \sim (m+n+1),\ \ \cdots,(m+k-1) \sim (m+n+k-1).$$

We now commence with the solution to Exercise 1.4. Let

$$\begin{aligned} M_1 &= N(2,3), \\ M_2 &= N(2,3^{2M_1}) + 2M_1, \\ M_3 &= N(2,3^{2M_2}) + 2M_2. \end{aligned}$$

We claim that $N(3,3) \leq 2M_3$. To see this, let $\{1,\cdots,2M_3\} = C_1 \cup C_2 \cup C_3$ be an arbitrary 3-cell partition.

By the previous exercise and the definition of M_3 there exist a recurrent $2M_2$-block in $\{1,\cdots,M_3\}$, say

$$\{s,s+1,\cdots,s+2M_2-1\} \sim \{s+n_3,s+1+n_3,\cdots,s+2M_2-1+n_3\}.$$

Likewise, $\{s,s+1,\cdots,s+M_2-1\}$ contains a recurrent $2M_1$-block, say

$$\{t,t+1,\cdots,t+2M_1-1\} \sim \{t+n_2,t+n_2+1,\cdots,t+n_2+2M_1-1\}.$$

Finally $\{t,t+1,\cdots,t+M_1-1\}$ contains a recurrent element, say $x \sim (x+n_1)$. Since $x+n_1$ is contained in a block which recurs after a gap of n_2, and since $x+n_1+n_2$ is contained in a block which recurs after a gap of n_3, we may write, reindexing the cells if necessary,

$$\{x,x+n_1,x+n_1+n_2,x+n_1+n_2+n_3\} \subset C_1.$$

If $x + 2n_1 \in C_1$ then $\{x, x+n_1, x+2n_1\} \subset C_1$ and we are done. If not, then we may write (again reindexing if necessary)

$$\{x + 2n_1, x + 2n_1 + n_2, x + 2n_1 + n_2 + n_3\} \subset C_2.$$

If now $x + 2n_1 + 2n_2 \in C_1$ then $\{x, x+n_1+n_2, x+2n_1+2n_2\} \subset C_1$ and we are done, while if $x+2n_1+2n_2 \in C_2$ then $\{x+2n_1, x+2n_1+n_2, x+2n_1+2n_2\} \subset C_2$ and we are done. Suppose then that $x + 2n_1 + 2n_2 \in C_3$. Then

$$\{x + 2n_1 + 2n_2, x + 2n_1 + 2n_2 + n_3\} \subset C_3.$$

We leave it to the reader to verify that no matter which cell $x + 2n_1 + 2n_2 + 2n_3$ lies in, it forms part of a monochromatic 3-progression.

□

Exercise 1.9. Let $r \in \mathbf{N}$ be arbitrary. Modify the above argument to find an upper bound for $N(3, r)$ in terms of various values of $N(2, t)$.

Exercise 1.10. Let $k, r \in \mathbf{N}$ be arbitary. Modify the above argument to find an upper bound for $N(k, r)$ in terms of various values of $N(k-1, t)$, thus providing a proof of van der Waerden's theorem.

1.3 Multidimensional van der Waerden: Gallai's theorem.

Van der Waerden's theorem, which deals with colorings of $\mathbf{N}$ (or $\mathbf{Z}$), admits a natural $\mathbf{N}^k$ (or $\mathbf{Z}^k$) generalization. This multidimensional van der Waerden theorem (see for example [F2]), due to Gallai (also called Grünwald in the literature), has as van der Waerden's theorem does many equivalent formulations. First we give an affine form. Notice that the case $k = 1$ is just **vdW3**.

MvdW3. Let $k \in \mathbf{N}$. For any finite partition of $\mathbf{Z}^k$ one of the cells contains affine images of every finite set.

Notice that we choose to live in $\mathbf{Z}^k$ rather than in $\mathbf{N}^k$. We'll have more to say about this later.

Exercise 1.11. Formulate versions of Gallai's theorem **MvdW1** and **MvdW2** by analogy with **vdW1** and **vdW2** from the first section. Prove their equivalence with **MvdW3**.

Here now is a topological dynamics version of the theorem. It is this version whose proof we will sketch at the end of the section.

MvdW4. ([FW].) Let $k \in \mathbf{N}$ and $\epsilon > 0$. If X is a compact metric space and $T_1, \cdots, T_k$ are commuting homeomorphisms of X then there exists $x \in X$ and $n \in \mathbf{Z}$, $n \neq 0$, such that $\rho(x, T_i^n x) < \epsilon$, $1 \leq i \leq k$.

In the last section we showed how a topological recurrence result gave rise to a chromatic result and left the reverse implication as an exercise. Here we will do the opposite, showing that **MvdW3** gives **MvdW4** and leaving the converse to an exercise.

Suppose then that $k \in \mathbf{N}$, $\epsilon > 0$, X is a compact metric space and $T_1, \cdots, T_k$ are commuting homeomorphisms of X. Let $U_1, \cdots, U_r$ be a covering of X by pairwise disjoint sets of less than ϵ diameter. Let $y \in X$ and determine a partition $\mathbf{Z}^k = \bigcup_{i=1}^r C_i$ by the rule $(n_1, \cdots, n_k) \in C_i$ if and only if $T_1^{n_1} \cdots T_k^{n_k} y \in U_i$. According to **MvdW3**, one of the cells C_i contains an affine image of the set

$$F = \{(0, \cdots, 0), (1, 0, \cdots, 0), (0, 1, 0, \cdots, 0), \cdots, (0, \cdots, 0, 1)\}. \qquad (1.1)$$

That is, there exists $(n_1, \cdots, n_k) \in \mathbf{Z}^k$ and $n \in \mathbf{Z}$, $n \neq 0$, such that

$$\begin{aligned}&(n_1, \cdots, n_k) + F = \{(n_1, \cdots, n_k), (n_1 + n, n_2, \cdots, n_k),\\ &(n_1, n_2 + n, n_3, \cdots, n_k), \cdots, (n_1, \cdots, n_{k-1}, n_k + n)\} \subset C_i.\end{aligned}$$

Letting $x = T_1^{n_1} \cdots T_k^{n_k} y$, we then have $\{x, T_1^n x, \cdots, T_k^n x\} \subset U_i$. Since U_i is of diameter less than ϵ, we are done.

□

Exercise 1.12. Show that **MvdW4** implies **MvdW3**.

The proof just given uses only affine images of the very special configurations (1.1) (we call such configurations *simplices*), suggesting that the following formulation, which does not on the face of it seem as strong as **MvdW3**, is in fact equivalent to it.

MvdW5. Let $k \in \mathbf{N}$. For any finite partition of $\mathbf{Z}^k$, one of the cells contains an affine image of the set

$$\{(0, \cdots, 0), (1, 0, \cdots, 0), (0, 1, 0, \cdots, 0), \cdots (0, \cdots, 0, 1)\}.$$

Exercise 1.13. Tell why **MvdW5** is equivalent to the other formulations of the multidimensional van der Waerden theorem, but that the case $k = 1$ of **MvdW5** is weaker than the case $k = 1$ of **MvdW3**.

Exercise 1.14. Prove directly (that is, without recourse to a dynamical argument) from the case $k = 2$ of **MvdW5** that for any finite coloring of $\mathbf{Z}$ one has a monochromatic arithmetic progression of length 3.

We will now indicate the way to a proof (by induction on k) of the topological multidimensional van der Waerden theorem **MvdW4**. The proof follows [BPT].

If X is a compact metric space and $S_1, \cdots, S_t$ are commuting continuous homeomorphisms of X, then $(X, S_1, \cdots, S_t)$ is an example of a *dynamical system*. (We have used this terminology already in the case $t = 1$.) If $Y \subset X$ and $S_i^{-1} Y \subset Y$, $1 \leq i \leq t$, then Y is said to be an *invariant* set.

Exercise 1.15. Prove using Zorn's Lemma that if $(X, S_1, \cdots, S_t)$ is a dynamical system then there exists a non-empty, *minimal* closed invariant set Y, that is, a closed invariant set Y none of whose proper non-empty subsets are closed and invariant. Show that the restriction of S_i to Y is a homeomorphism of Y, $1 \le i \le k$.

Consider, for $k \in \mathbf{N}$, the following two statements:

$\mathcal{S}_k$: For every $\epsilon > 0$ and every compact metric space X, if $T_1, \cdots, T_k$ are commuting homeomorphisms of X then there exists $x \in X$ and $n \in \mathbf{N}$ such that $\rho(x, T_i^n x) < \epsilon$, $1 \le i \le k$.

$\mathcal{T}_k$: For every $\epsilon > 0$ and every compact metric space X, if $T_1, \cdots, T_k$ and R are commuting homeomorphisms of X such that $(X, T_1, \cdots, T_k, R)$ is a minimal system (that is, X is a minimal closed invariant set for this family of homeomorphisms), then for every non-empty open set U there exists $n \in \mathbf{N}$ such that

$$U \cap T_1^{-n} U \cap \cdots \cap T_k^{-n} U \neq \emptyset.$$

$\mathcal{T}_k$ states, in effect, that in a minimal system every non-empty open set is multiply recurrent. We remark that the minimality condition is necessary. Indeed, let $X = [0, 1]$ and put $Tx = x^2$. Then T is a (non-minimal) homeomorphism, but the set $(\frac{1}{4}, \frac{1}{2})$ never comes back to itself under T.

In order to prove **MvdW4** it is sufficient to show that $\mathcal{S}_k$ holds for all $k \in \mathbf{N}$.

Exercise 1.16. Show that $\mathcal{S}_1$ holds.

Our method of proof is induction on k and consists of two steps: $\mathcal{S}_k \Rightarrow \mathcal{T}_k$ and $\mathcal{T}_k \Rightarrow \mathcal{S}_{k+1}$.

$\mathcal{S}_k \Rightarrow \mathcal{T}_k$: Suppose that $(X, T_1, \cdots, T_k, R)$ is a minimal system and $U \subset X$ is open and non-empty. Choose $u \in U$ and $\epsilon > 0$ such that $B_\epsilon(u) \subset U$ and let $V = B_{\frac{\epsilon}{2}}(u) \subset U$. Then any member of X a distance of less than $\frac{\epsilon}{2}$ from V will be contained in U. Let G be the group of homeomorphisms generated by $T_1, \cdots, T_k, R$.

Exercise 1.17. Use minimality to show that $\bigcup_{S \in G} S^{-1} V = X$.

Hence by compactness of X there exist $S_1, \cdots, S_r \in G$ such that $X = \bigcup_{i=1}^r S_i^{-1} V$.

Exercise 1.18. There exists $\delta > 0$ so small that, for all $y, z \in X$, $\rho(y, z) < \delta$ implies $\rho(S_i y, S_i z) < \frac{\epsilon}{2}$, $1 \le i \le r$.

According to $\mathcal{S}_k$, there exists $y \in X$ and $n \in \mathbf{N}$ such that $\rho(y, T_i^n y) < \delta$, $1 \le i \le k$. Fix j with $y \in S_j^{-1} V$ and set $x = S_j y$. Then $x \in V$, and moreover by the properties of δ, $\rho(x, T_i^n x) < \frac{\epsilon}{2}$, $1 \le i \le k$. Hence $\{x, T_1^n x, \cdots, T_k^n x\} \subset U$, that is, $x \in (U \cap T_1^{-n} U \cap \cdots \cap T_k^{-n} U)$.

All that remains for the proof of **MvdW4** is that $\mathcal{T}_k$ implies $\mathcal{S}_{k+1}$. We do this for $k = 2$ (all the ideas needed are present here) and leave the general case as an exercise.

$\mathcal{T}_2 \Rightarrow \mathcal{S}_3$: Let X be a compact metric space and suppose that T_1, T_2, T_3 are commuting homeomorphisms of X. By Exercise 1.15 we may assume that the system (X, T_1, T_2, T_3) is minimal.

Let U_0 be a non-empty open set of diameter $< \frac{\epsilon}{2}$. By $\mathcal{T}_2$ there exists $n_1 \in \mathbf{N}$ such that $\left(U_0 \cap (T_1T_3^{-1})^{-n_1}U_0 \cap (T_2T_3^{-1})^{-n_1}U_0\right) \neq \emptyset$. Let U_1 be a non-empty open subset of diameter $< \frac{\epsilon}{2}$ such that

$$U_1 \subset T_3^{-n_1}\left(U_0 \cap (T_1T_3^{-1})^{-n_1}U_0 \cap (T_2T_3^{-1})^{-n_1}U_0\right) = \left(T_1^{-n_1}U_0 \cap T_2^{-n_1}U_0 \cap T_3^{-n_1}U_0\right).$$

Suppose now that non-empty open sets of diameter $< \frac{\epsilon}{2}$ $U_0, U_1, \cdots, U_l$ and $n_1, \cdots, n_l \in \mathbf{N}$ have been chosen such that:

$$\begin{aligned} U_j \subset &\left(T_1^{-(n_j+n_{j-1}+\cdots+n_{i+1})}U_i \cap T_2^{-(n_j+n_{j-1}+\cdots+n_{i+1})}U_i\right. \\ &\left.\cap\, T_3^{-(n_j+n_{j-1}+\cdots+n_{i+1})}U_i\right), \quad 1 \le i < j \le l. \end{aligned} \tag{1.2}$$

We want to choose U_{l+1} and n_{l+1} such that (1.2) is valid with l replaced by $l+1$. This will establish that sequences $(U_n)_{n=0}^{\infty}$ and $(n_i)_{i=1}^{\infty}$ may be chosen so that (1.2) holds for all $l \in \mathbf{N}$.

By $\mathcal{T}_2$ there exists n_{l+1} such that $\left(U_l \cap (T_1T_3^{-1})^{-n_{l+1}}U_l \cap (T_2T_3^{-1})^{-n_{l+1}}U_l\right) \neq \emptyset$. Let U_{l+1} be a non-empty open set of diameter $< \frac{\epsilon}{2}$ such that

$$\begin{aligned} U_{l+1} &\subset T_3^{-n_{l+1}}\left(U_l \cap (T_1T_3^{-1})^{-n_{l+1}}U_l \cap (T_2T_3^{-1})^{-n_{l+1}}U_l\right) \\ &= \left(T_1^{-n_{l+1}}U_l \cap T_2^{-n_{l+1}}U_l \cap T_3^{-n_{l+1}}U_l\right). \end{aligned}$$

Using this inclusion together with the case $j = l$ of (1.2) gives

$$\begin{aligned} U_{l+1} \subset &\left(T_1^{-(n_{l+1}+n_l+\cdots+n_{i+1})}U_i \cap T_2^{-(n_{l+1}+n_l+\cdots+n_{i+1})}U_i \cap\right. \\ &\left. T_3^{-(n_{l+1}+n_l+\cdots+n_{i+1})}U_i\right), \quad 0 \le i < l+1. \end{aligned}$$

Suppose now that $(U_n)_{n=0}^{\infty}$ and $(n_i)_{i=1}^{\infty}$ have been chosen such that (1.2) holds for all $l \in \mathbf{N}$. Let $x_n \in U_n$, $n = 0, 1, \cdots$. By compactness we have $\rho(x_i, x_j) < \frac{\epsilon}{2}$ for some $i < j$. Moreover, (1.2) tells us that if we let $n = n_j + n_{j-1} + \cdots + n_{i+1}$ then $\{T_1^n x_j, T_2^n x_j, T_3^n x_j\} \subset U_i$. But $x_i \in U_i$ as well, and the diameter of U_i is less than $\frac{\epsilon}{2}$. Therefore, $\rho(x_j, T_m^n x_j) < \epsilon$, $m = 1, 2, 3$. Let $x = x_j$.

□

Exercise 1.19. Modify the proof above to show that $\mathcal{T}_k \Rightarrow \mathcal{S}_{k+1}$ for $k > 2$, thus completing the proof of **MvdW4**.

Let us give one last, minimal formulation of Gallai's theorem, which is motivated by the $\mathcal{T}_k$ property of the proof.

MvdW6. Let $k \in \mathbf{N}$. If $(X, T_1, \cdots, T_k)$ is a minimal system and U is a non-empty open set then there exists $n \in \mathbf{N}$ such that

$$U \cap T_1^{-n}U \cap \cdots \cap T_k^{-n}U \neq \emptyset.$$

The special case of **MvdW6** which corresponds to the T_i's being powers of the same operator T is important for the next section, hence we formulate it separately.

vdW5. Let $k \in \mathbf{N}$ and $l_1, \cdots, l_k \in \mathbf{Z}$. If (X, T) is a minimal system and U is a non-empty open set then there exists $n \in \mathbf{N}$ such that

$$U \cap T^{-l_1 n} U \cap \cdots \cap T^{-l_k n} U \neq \emptyset.$$

Exercise 1.20. Show that **vdW5** is equivalent to **vdW1-4**.

Exercise 1.21. Show that if $(X, S_1, S_2, \cdots, S_t)$ is minimal then the set of $x \in X$ for which there exists a sequence $(n_k)_{k=1}^{\infty} \subset \mathbf{N}$ with $S_i^{n_k} x \to x$ for $i = 1, 2, \cdots, t$ simultaneously is residual.

The proof given in this section, while having a certain combinatorial flavor, is different than traditional combinatorial proofs (such as the one outlined in the last section) in that no iterative applications of previously determined finite bounds are required for the inductive step. The topological structure, in particular the existence of minimal subsystems, is what makes this possible. Indeed, consider that van der Waerden's theorem is equivalent to the statement that for every r and k, if $\gamma \in X = \{1, 2, \cdots, r\}^{\mathbf{Z}}$ then letter occurs in γ along an arithmetic progression of length k. If it should happen that the orbit closure of γ under the shift is minimal, things become simpler because of the following fact, for which we need a definition: if G is an abelian semigroup, then a set $E \subset G$ is said to be *syndetic* if there exists a finite set $F \subset G$ such that $G = \bigcup_{g \in F} g^{-1} E$, where $g^{-1} E = \{s \in G : gs \in E\}$. Once may check that in $\mathbf{Z}$ or $\mathbf{N}$ the syndetic sets are those with *bounded gaps*; namely, those E for which there exists $N \in \mathbf{N}$ having the property that for every a, one of $a, a+1, \cdots, a+N-1$ lies in E.

Exercise 1.22. If (X, T) is minimal then every $x \in X$ is *uniformly recurrent*; namely, for every neighborhood U of x the set $\{n : T^n x \in U\}$ is syndetic.

In the present context, this means that if γ is a member of a minimal set in X then every finite word occuring in γ occurs syndetically. In order to show that some letter occurs in γ along arithmetic progressions of length k, the combinatorial proof outlined in the previous section of course works, but may be simplifed considerably. Indeed, consider that Exercise 1.8, that any long enough interval contains some recurrent block of fixed length, becomes unnecessary; *every block* recurs. Thus the need to specify $M_1, M_2, M_3, \cdots$ in advance is eliminated.

Of course, γ need not be a member of minimal set. Nevertheless, observe that if ξ is in the orbit closure of γ then every finite word, and hence every monochromatic arithmetic progression, occuring in ξ occurs in γ as well (up to a shift). By choosing ξ in a minimal subset of the orbit closure of γ, therefore, we gain all of the aforementioned advantages without sacrificing anything.

1.4 A polynomial van der Waerden theorem.

In [BL1] V. Bergelson and A. Leibman proved a "polynomialized" extension of van der Waerden's theorem. We will state and outline the proof of a special, one-dimensional case of their result. The general, multidimensional version will be formulated at the end of the section, although the question of its proof will be deferred until Section 1.7.

PvdW1. Let $r \in \mathbf{N}$ and suppose $A \subset \mathbf{Z}[x]$ is a finite family of polynomials containing 0 with $p(0) = 0$, $p \in A$. For any r-coloring of $\mathbf{Z}$ there exists a monochromatic configuration of the form

$$\{m + p(n) : p \in A\}, \quad m, n \in \mathbf{Z}, \; n \neq 0.$$

Notice that the case of polynomial families A containing only linear polynomials corresponds to van der Waerden's theorem. Here now is a dynamical formulation of the polynomial van der Waerden theorem.

PvdW2. Suppose $A \subset \mathbf{Z}[x]$ is finite with $p(0) = 0$, $p \in A$. If (X, T) is a dynamical system (where T is a homeomorphism) and $\epsilon > 0$ then there exists $x \in X$ and $n \in \mathbf{Z}$, $n \neq 0$, such that $\rho(x, T^{p(n)}x) < \epsilon$, $p \in A$.

The full proof of **PvdW2** will be outlined in a series of exercises. Right now we will demonstrate the method by doing the simplest non-linear case.

Proof of **PvdW2** for $A = \{0, n^2\}$. As usual we may assume that the system (X, T) is minimal.

We claim it is possible to construct sequences $(U_i)_{i=0}^{\infty}$ of open sets of diameter $< \frac{\epsilon}{2}$ and natural numbers $(n_i)_{i=1}^{\infty}$ such that

$$T^{(n_j + n_{j-1} + \cdots + n_{i+1})^2} U_j \subset U_i, \; 0 \leq i < j. \tag{1.3}$$

Let U_0 be any non-empty open set of diameter $< \frac{\epsilon}{2}$. Pick $n_1 \in \mathbf{N}$ and let U_1 be a non-empty set of diameter $< \frac{\epsilon}{2}$ with $U_1 \subset T^{-n_1^2} U_0$. Suppose $U_0, U_1, \cdots, U_l$ and $n_1, \cdots, n_l$ have been chosen. By **vdW5** there exists $n_{l+1} \in \mathbf{N}$ such that

$$(U_l \cap T^{-2n_l n_{l+1}} U_l \cap T^{-2(n_l + n_{l-1})n_{l+1}} U_l \cap \cdots \cap T^{-2(n_l + \cdots + n_1)n_{l+1}} U_l) \neq \emptyset.$$

Let U_{l+1} be a non-empty open set of diameter $< \frac{\epsilon}{2}$ such that

$$U_{l+1} \subset T^{-n_{l+1}^2} (U_l \cap T^{-2n_l n_{l+1}} U_l \cap T^{-2(n_l + n_{l-1})n_{l+1}} U_l \cap \cdots \cap T^{-2(n_l + \cdots + n_1)n_{l+1}} U_l).$$

Exercise 1.23. The sequences $(U_i)_{i=0}^{\infty}$ and $(n_i)_{i=1}^{\infty}$ so constructed satisfy (1.3).

Let now $x_i \in U_i$, $i = 0, 1, \cdots$. By compactness there exists $i < j$ with $\rho(x_i, x_j) < \frac{\epsilon}{2}$. Letting $n = n_j + n_{j-1} + \cdots + n_{i+1}$, we have by (1.3) $\rho(T^{n^2} x_j, x_i) < \frac{\epsilon}{2}$. It follows that $\rho(x_j, T^{n^2} x_j) < \epsilon$. Let $x = x_j$.

□

Notice that the proof uses the minimal linear van der Waerden theorem **vdW5**. As a matter of fact, **vdW5** and a minor modification of the above argument are sufficient to handle any finite family of second degree polynomials whose leading coefficients are equal.

Exercise 1.24. Modify the proof above to supply a proof via **vdW5** of **PvdW2** for the case $A = \{0, n^2 + an, n^2 + bn\}$, where $a, b \in \mathbf{Z}$. Explain why the method will also work for any family of the form $\{0, cn^2 + a_1 n, cn^2 + a_2 n, \cdots, cn^2 + a_k n\}$.

Exercise 1.25. Provide a proof of **PvdW2** for the family $\{0, n, n^2\}$ (use Exercise 1.24). Generalize to families containing a single quadratic polynomial and finitely many linear polynomials.

The simplest family of polynomials for which we have yet to see a proof of **PvdW2** is something like $\{0, n, n^2, n^2 + n\}$. As it turns out, the minimal linear van der Waerden theorem **vdW5** is, by itself, an insufficient tool for the handling of this case. What we need is a minimal version of the case dealt with in Exercise 1.24. To this end let us first formulate a minimal version of the polynomial van der Waerden theorem in general.

PvdW3. Suppose $A \subset \mathbf{Z}[x]$ is a finite family of polynomials containing 0 and such that $p(0) = 0$, $p \in A$. If (X, T) is a minimal dynamical system (where T is a homeomorphism) and U is a non-empty open set then there exists $n \in \mathbf{Z}$, $n \neq 0$, such that

$$\Big(\bigcap_{p \in A} T^{-p(n)} U\Big) \neq \emptyset.$$

The following exercise satisfies our immediate need and will be of later use as well.

Exercise 1.26. Show that if **PvdW2** holds for a certain family of polynomials A then **PvdW3** holds for A as well.

Exercise 1.27. Using **PvdW3** show, under the conditions appearing there, that the set of $x \in X$ for which there exists a sequence $(n_k)_{k=1}^{\infty} \subset \mathbf{N}$ such that $T^{p(n_k)} x \to x$ for all $p \in A$ simultaneously is residual.

Proof of **PvdW2** for $A = \{0, n, n^2, n^2 + n\}$. In anticipation of the general case we will adopt a few notational conveniences. Let $p_1(n) = n$, $p_2(n) = n^2$ and $p_3(n) = n^2 + n$. More importantly, if $p(x) \in \mathbf{Z}[x]$ we will write $p^{(2)}(x, y) = p(x + y) - p(x) - p(y)$.

Exercise 1.28. If $0 \neq p(x) \in \mathbf{Z}[x]$ with $p(0) = 0$ and $n \in \mathbf{N}$ then $q(x) = p^{(2)}(n, x) \in \mathbf{Z}[x]$ satisfies $q(0) = 0$ and $\deg q = \deg p - 1$.

We assume that (X, T) is minimal. Let $\epsilon > 0$. We seek a sequence $(U_i)_{i=0}^{\infty}$ of non-empty open sets having diameter $< \frac{\epsilon}{2}$ and a sequence $(n_i)_{i=1}^{\infty} \subset \mathbf{N}$ such that

$$T^{p_k(n_j + n_{j-1} + \cdots + n_{i+1})} U_j \subset U_i, \quad 0 \leq i < j, \ k = 1, 2, 3. \tag{1.4}$$

Let U_0 be a non-empty open set of diameter $< \frac{\epsilon}{2}$. Notice that, writing $(p_2 - p_1)(x) = p_2(x) - p_1(x)$ and $(p_3 - p_1)(x) = p_3(x) - p_1(x)$, $(p_2 - p_1)$ and $(p_3 - p_1)$ are quadratic polynomials having the same leading coefficient, so that $\{0, (p_2 - p_1), (p_3 - p_1)\}$ is a family of the type considered in Exercise 1.24. Therefore by this exercise, together with Exercise 1.26, we may pick $n_1 \in \mathbf{N}$ such that

$$(U_0 \cap T^{-(p_2-p_1)(n_1)}U_0 \cap T^{-(p_3-p_1)(n_1)}U_0) \neq \emptyset.$$

Let U_1 be a non-empty set of diameter $< \frac{\epsilon}{2}$ with

$$\begin{aligned} U_1 &\subset T^{-p_1(n_1)}(U_0 \cap T^{-(p_2-p_1)(n_1)}U_0 \cap T^{-(p_3-p_1)(n_1)}U_0) \\ &= (T^{-p_1(n_1)}U_0 \cap T^{-p_2(n_1)}U_0 \cap T^{-p_3(n_1)}U_0) \end{aligned}.$$

Then (1.4) holds for $i = 0$, $j = 1$.

Suppose $U_0, U_1, \cdots, U_l$ and $n_1, \cdots, n_l$ have been chosen. The family

$$\begin{aligned} A' = \big\{&(p_k - p_1)(x), (p_k - p_1)(x) + p_k^{(2)}(n_l, x), (p_k - p_1)(x) + p_k^{(2)}(n_{l-1} + n_l, x), \\ &\cdots, (p_k - p_1)(x) + p_k^{(2)}(n_1 + \cdots + n_l, x) : \ k = 1, 2\big\} \end{aligned}$$

is again of the Exercise 1.24 type. Hence, there exists $n_{l+1} \in \mathbf{N}$ such that

$$\bigcap_{q \in A'} T^{-q(n_{l+1})}U_l \neq \emptyset.$$

Let U_{l+1} be a non-empty open set of diameter $< \frac{\epsilon}{2}$ such that

$$U_{l+1} \subset T^{-p_1(n_{l+1})}\Big(\bigcap_{q \in A} T^{-q(n_{l+1})}U_l\Big).$$

Exercise 1.29. Show that the sequences $(U_i)_{i=0}^{\infty}$ and $(n_i)_{i=1}^{\infty}$ satisfy (1.4). Complete the proof.

□

Exercise 1.30. Modify the above proof to show that **PvdW2** holds for families of the form $A = \{0, an, bn^2+c_1n, bn^2+c_2n, \cdots, bn^2+c_kn\}$, i.e., families with one linear polynomial and finitely many quadratic polynomials having equal leading coefficients.

Exercise 1.31. Using Exercises 1.30 and 1.26 to show that **PvdW2** holds for polynomial families having two linear polynomials and finitely many quadratic polynomials having equal leading coefficients. Generalize to families with k linear terms and finitely many quadratic terms with equal leading coefficient by inducting on k.

Exercise 1.32. Use Exercises 1.31 and 1.26 to show that **PvdW2** holds for the family $A = \{0, n^2, 2n^2\}$.

In the exercises a few increasingly complex special cases of **PvdW2** have been treated, culminating in the case $A = \{0, n^2, 2n^2\}$. Each of these cases in turn required (the minimal form of) the previous case for its proof. The reader may have guessed by now that what was being viewed were the initial steps in a general induction scheme which can be used to prove **PvdW2** in general. Bergelson coined the term *PET-induction* for this "polynomial exhaustion technique" (see [B2], [BL1], [BL2]).

At the heart of the technique lies an equivalence relation on polynomials. Namely, we write $p(x) \sim q(x)$ if p and q have the same degree and the same leading coefficient. Hence, for example,

$$(-3x^3 - 17x^2) \not\sim (3x^3 + 17x^2) \sim (3x^3 - 14x) \not\sim (3x^4 + 3x^3).$$

This equivalence relation allows us to associate a *weight vector* to any finite family of polynomials. Namely, we say that $(a_1, \cdots, a_d)$ is the weight vector of the family $A \subset \mathbf{Z}[n]$ if the polynomial in A of highest degree is of degree d and if a_i is equal to the number of ith degree equivalence classes under $\sim$ represented in A, $1 \le i \le d$.

Example. The family $\{n^3+2n^2, 2n^3+6n, 2n^3, n^2, n^2-n, 2n, 3n, 17n\}$ has weight vector $(3, 1, 2)$.

If $w = (w_1, \cdots, w_d)$ is a weight vector, let $\mathcal{S}_w$ stand for the assertion "**PvdW2** holds for all families with weight vector equal to w." We may classify the special cases posed earlier according to the weight vector of the class of polynomial families being considered. **vdW5**, for example, which served as an initial case, corresponds to $\mathcal{S}_{(k)}$, $k \in \mathbf{N}$. Recall that its proof went by induction on k. Exercise 1.24 is $\mathcal{S}_{(0,1)}$, while Exercise 1.30 is $\mathcal{S}_{(1,1)}$. Exercise 1.31 asks first for $\mathcal{S}_{(2,1)}$ then for $\mathcal{S}_{(k,1)}$, $k \in \mathbf{N}$, and suggests inducting on k to achieve it. Finally the family in Exercise 1.32 has weight vector $(0, 2)$.

The reader may have already guessed that the proof of **PvdW2** will be achieved by induction on $w(A)$ (the weight vector of A). He may even have guessed the well ordering on weight vectors which is used for this induction. Namely, we write $(a_1, a_2, \cdots, a_k) < (b_1, \cdots, b_n)$ if (i) $k < n$, or (ii) $k = n$ and there exists j, $1 \le j \le k$, with $a_k < b_k$ and $a_i = b_i$, $j < i \le k$.

We will now sketch the major elements in the induction step required for the proof of **PvdW2**, leaving some details as exercises. Most of the required ideas already appear (if in infancy) earlier in this section. Suppose $A \subset \mathbf{Z}[x]$ is a finite family of polynomials having zero constant term and suppose that **PvdW2** (and hence by Exercise 1.26 **PdvW3** as well) holds for any family of polynomials having lesser weight vector than $w(A)$.

Let (X, T) be a minimal system and let $\epsilon > 0$. Our strategy is to construct (as always) a sequence $(U_i)_{i=0}^{\infty}$ of non-empty open sets of diameter $< \frac{\epsilon}{2}$ and a sequence $(n_i)_{i=1}^{\infty} \subset \mathbf{N}$ such that

$$T^{p(n_j+n_{j-1}+\cdots+n_{i+1})} U_j \subset U_i, \quad 0 \le i \le j,\ p \in A,\ p \ne 0. \tag{1.5}$$

U_0 may be chosen arbitrarily.

Suppose that $U_0, \cdots, U_l$ and $n_1, \cdots, n_l$ have been chosen. Let $p_1 \in A$ be of the minimal non-zero degree appearing in A and put

$$A' = \{p(x) + p^{(2)}(n_{i+1} + n_{i+2} + \cdots + n_l, x) - p_1(x) : \ 0 \le i \le l, \ p \in A, \ p \ne 0\}.$$

In particular (taking the degenerate case $i = l$) we have $p(x) - p_1(x) \in A'$ for all non-zero $p(x) \in A$.

Exercise 1.33. Show that $w(A') < w(A)$.

By the induction hypothesis, there exists $n_{l+1} \in \mathbf{N}$ such that

$$\bigcap_{q \in A'} T^{-q(n_{l+1})} U_l \ne \emptyset.$$

Let U_{l+1} be an open set of diameter $< \frac{\epsilon}{2}$ with

$$U_{l+1} \subset T^{-p_1(n_{l+1})} \Big(\bigcap_{q \in A'} T^{-q(n_{l+1})} U_l \Big).$$

Exercise 1.34. Show that the sequences $(U_i)_{i=0}^{\infty}$ and $(n_i)_{i=1}^{\infty}$ satisfy (1.5). Complete the proof of **PvdW2**.

As promised, here now is the Bergelson-Leibman polynomial van der Waerden theorem in its full multidimensional generality.

MPvdW1. ([BL1].) Let $r, k \in \mathbf{N}$ and suppose A is a family of polynomial mappings $p : \mathbf{Z} \to \mathbf{Z}^k$ with $p(0) = (0, \cdots, 0)$, $p \in A$. For any r-coloring of $\mathbf{Z}^k$ there exists a monochromatic configuration of the form

$$\{m + p(n) : p \in A\}, \ \ m \in \mathbf{Z}^k, \ 0 \ne n \in \mathbf{Z}.$$

Exercise 1.35. Formulate a dynamical version **MPvdW2** of the above theorem and demonstrate its equivalence to **MPvdW1**.

In Section 1.7 we shall prove a theorem which will contain **MPvdW** as a special case.

1.5 $\mathbf{Z}_k$ and IP van der Waerden theorems.

In the next two sections we will give three progressively more general theorems having a van der Waerden flavor. The purpose here is to show the range of dynamical formulations for various Ramsey type combinatorial statements once we step outside of $\mathbf{Z}^k$. The dynamical formulation for the first theorem will involve a group action by homeomorphisms of a compact space. Unlike in the previous sections, however, the group is not finitely generated, and has torsion. The dynamical formulation for the second theorem will involve "almost-actions" of a badly non-cancellative semigroup. Finally the dynamical formulation for the third theorem (the Hales-Jewett theorem) will have very cumbersome combinatorial conditions attached to it. Moreover, we don't have a suitable dynamical proof of it!

Suppose that $k, n \in \mathbf{Z}$. For convenience, we will assume that k is prime. Put $\mathbf{Z}_k = \{0, 1, \cdots, k-1\}$ and let $+$ denote addition (mod k) on $\mathbf{Z}_k$. A k-element subset of $\mathbf{Z}_k^n$ of the form $\{a + iw : i \in \mathbf{Z}_k\}$, where a and $0 \neq w$ are elements of $\mathbf{Z}_k^n$, will be called an *affine line.* (Multiplication here of the "scalar" i by the "vector" w is coordinate-wise and modulo k.) Notice that an affine line is just a shifted copy of $\mathbf{Z}_k$.

Example. $\{(1,0,2,1,2,0,1), (1,0,0,1,0,2,1), (1,0,1,1,1,1,1)\}$ is an affine line in $\mathbf{Z}_3^7$. It has the form $\{a + iw : i \in \mathbf{Z}_3\}$, where $a = (1,0,2,1,2,0,1)$ and $w = (0,0,1,0,1,2,0)$.

ZkvdW1. Let $k, r \in \mathbf{N}$. There exists $n = n(k,r) \in \mathbf{N}$ having the property that for any r-coloring of $\mathbf{Z}_k^n$ there exists a monochromatic affine line.

We bring forth now another version of this theorem. For $k \in \mathbf{N}$ we denote by $\mathbf{Z}_k^\infty$ the direct sum of countably many copies of $\mathbf{Z}_k$. We may identify $\mathbf{Z}_k^\infty$ with the set of sequences $(a_1, a_2, \cdots, a_t, 0, 0, \cdots)$, where $a_i \in \mathbf{Z}_k$ and all but finitely many of the a_i's are zero. Affine lines are still of the form $\{a + iw : i \in \mathbf{Z}_k\}$, only now a and $0 \neq w$ are elements of $\mathbf{Z}_k^\infty$.

ZkvdW2. Let $k \in \mathbf{N}$ be prime. For any finite coloring of $\mathbf{Z}_k^\infty$, there exists an affine line.

We will show that **ZkvdW1** imples **ZkvdW2**. Let $k, r \in \mathbf{N}$ and let $n = n(k,r)$. Given an r-coloring $\mathbf{Z}_k^\infty = \bigcup_{i=1}^r C_i$, induce an r-coloring $\mathbf{Z}_k^n = \bigcup_{i=1}^r D_i$ by the rule $(w_1, \cdots, w_n) \in D_i$ if and only if $(w_1, \cdots, w_n, 0, 0, \cdots) \in C_i$. By **ZkvdW1**, there exists i such that D_i contains an affine line. This line clearly corresponds to an affine line in C_i.

Due to the fact that the correspondence $(w_1, \cdots, w_n) \to (w_1, \cdots, w_n, 0, \cdots)$ is not $1-1$, the converse is somewhat trickier. We leave it as an exercise.

Exercise 1.36. Show that **ZkvdW2** implies **ZkvdW1.**

We are now going to give a dynamical formulation for this theorem. In order to do so, we need to expand our notion of dynamical system. Let G be an arbitrary semigroup and let X be a compact metric space. If $\{T_g\}_{g\in G}$ is a family of continuous self-maps of X (homeomorphisms, if G is a group) satisfying $T_{gh} = T_g \circ T_h$ for all $g, h \in G$, then $\{T_g\}_{g\in G}$ will be called a G-action by continuous self maps (or homeomorphisms, if G is a group) of X and the pair $\left(X, \{T_g\}_{g\in G}\right)$ will be called a dynamical system. (One may check that this extends our previous notion of dynamical system.)

ZkvdW3. Suppose that X is a compact metric space, let $k \in \mathbf{N}$ be prime, and let $\{T_w\}_{w\in\mathbf{Z}_k^\infty}$ be a $\mathbf{Z}_k^\infty$-action by homeomorphisms of X. For every $\epsilon > 0$ there exists $x \in X$ and $w \in \mathbf{Z}_k^\infty$, $w \neq 0$, such that $\rho(x, T_{iw}x) < \epsilon$, $i \in \mathbf{Z}_k$.

Remark. k doesn't have to be prime in order for the conclusion to hold. As we said earlier, we are only assuming k to be prime for the sake of convenience in

ZkvdW1-2, and we assume it here in order to maintain the equivalence. Later in this section we will see a more general theorem with no restriction on k.

ZkvdW3 $\Rightarrow$ **ZkvdW2**: The ideas in the following construction should be familiar by now. Let $k \in \mathbf{N}$ be prime and suppose that $\mathbf{Z}_k^\infty = \bigcup_{i=1}^r C_i$. Put $\Omega = \{1, 2, \cdots, r\}^{\mathbf{Z}_k^\infty}$.

Exercise 1.37. Ω is a compact metric space with metric

$$\rho(\gamma, \eta) = \frac{1}{1 + \min\{n : \exists y = (y_1, \cdots, y_n, 0, \cdots) \in \mathbf{Z}_k^\infty,\ y_n \neq 0,\ \text{with } \gamma(y) \neq \eta(y)\}}.$$

For $w = (w_1, \cdots, w_n, 0, 0, \cdots) \in \mathbf{Z}_k^\infty$, define a transformation $T_w : \Omega \to \Omega$ by $T_w\gamma(y) = \gamma(y + w)$, $\gamma \in \Omega$, $y \in \mathbf{Z}_k^\infty$.

Exercise 1.38. $\{T_w\}_{w \in \mathbf{Z}_k^\infty}$ is a $\mathbf{Z}_k^\infty$-action by homeomorphisms of Ω.

Let $\xi(w) = i$ if and only if $w \in C_i$. Then $\xi \in \Omega$. Put $X = \overline{\{T_w\xi : w \in \mathbf{Z}_k^\infty\}}$.

Exercise 1.39. Restricted to X, $\{T_w\}_{w \in \mathbf{Z}_k^\infty}$ is a $\mathbf{Z}_k^\infty$-action by homeomorphisms of X.

According to **ZkvdW3**, there exists $x \in X$ and $0 \neq w \in \mathbf{Z}_k^\infty$ such that $\rho(x, T_{iw}x) < 1$, $i \in \mathbf{Z}_k$. But x is in the closure of the orbit of ξ, so for some $v \in \mathbf{Z}_k^\infty$, $\rho\big(T_v\xi, T_{iw}(T_v\xi)\big) < 1$ for all $i \in \mathbf{Z}_k$. In particular,

$$T_v\xi(0) = T_wT_v(0) = T_{2w}T_v(0) = \cdots = T_{(k-1)w}T_v(0),$$

which is say that $\xi(v) = \xi(v + w) = \xi(v + 2w) = \cdots = \xi\big(v + (k-1)w\big)$. In other words, letting $i = \xi(v)$, $\{v, v + w, v + 2w, \cdots, v + (k-1)w\} \subset C_i$.

$\square$

Rather than prove **ZkvdW3**, we will move to our next theorem, which is more general (and which we *will* prove). Let us denote by $\mathcal{F}$ the set of all finite subsets of $\mathbf{N}$. For $\alpha, \beta \in \mathcal{F}$, we write $\alpha < \beta$ in the event that for all $i \in \alpha$ and all $j \in \beta$ we have $i < j$. ($<$ is a partial order on $\mathcal{F}$.) Suppose that we have a subset of an abelian semigroup that is indexed by $\mathcal{F}$ (an *$\mathcal{F}$-sequence*), say $(n_\alpha)_{\alpha \in \mathcal{F}}$, with the property that $n_{\alpha \cup \beta} = n_\alpha + n_\beta$ whenever $\alpha \cap \beta = \emptyset$. Such a set is called an *IP-set*, or *IP-sequence*. If the elements of an IP-sequence are transformations (such as homeomorphisms of some compact metric space), we will generally use the term *IP-system* instead. Two IP-systems $(T_\alpha)_{\alpha \in \mathcal{F}}$ and $(S_\alpha)_{\alpha \in \mathcal{F}}$ are said to commute if $T_\alpha S_\beta = S_\beta T_\alpha$ for all $\alpha, \beta \in f$. (IP stands for "idempotent". The relevence of this designation will become clear in the next section, where a relationship between such sets and the existence of idempotents in some associated compact semigroups will be explored.)

An IP-system $\{T_\alpha\}_{\alpha \in \mathcal{F}}$ of homeomorphisms of a compact space is something like a semigroup action, but not quite. $\mathcal{F}$ is a commutative semigroup under the operation of union, albeit a highly non-cancellative one (obviously one can have

$\alpha \cup \beta = \gamma \cup \beta$ and yet have $\alpha \neq \gamma$). However, the relationship $T_{\alpha\cup\beta} = T_\alpha T_\beta$ need not hold when $\alpha \cap \beta \neq \emptyset$.

Nevertheless, IP-systems are close enough approximations to semigroup actions to have many of their recurrence properties. Moreover, IP-systems are, in some sense, "cheaper to come by" than semigroup actions. For example, if (X, T) is a dynamical system, where T is a homeomorphism, then $\{T^n : n \in \mathbf{Z}\}$ is a $\mathbf{Z}$-action. Oftentimes in applications it is helpful to be able to restrict to a subset of $\mathbf{Z}$ which both has strong recurrence properties and whose members satisfy some additional condition. Naturally any subgroup will have the desired recurrence properties, but the subgroups of $\mathbf{Z}$ are just $\{k\mathbf{Z} : k \in \mathbf{N} \cup \{0\}\}$. Most of these are big, and the whole collection is countable. IP-sets in $\mathbf{Z}$, however, can be much smaller, and they are much more plentiful: the family of IP-sets in $\mathbf{Z}$ has cardinality c. Therefore it is much more likely that one will be able to find an IP-set each of whose members satisfies the "additional condition" than it is that one will find a suitable subgroup. (More will be said about this in Section 2.2 below.)

IPvdW1. (See [FW] and [B5].) Suppose that X is a compact metric space, $k \in \mathbf{N}$, and let $\{T_\alpha^{(1)}\}_{\alpha\in\mathcal{F}}, \cdots, \{T_\alpha^{(k)}\}_{\alpha\in\mathcal{F}}$ be commuting IP-systems of homeomorphisms of X. For any $\epsilon > 0$ there exists $x \in X$ and $\alpha \in \mathcal{F}$ such that $\rho(x, T_\alpha^{(i)}x) < \epsilon$, $1 \leq i \leq k$.

Exercise 1.40. Show that **ZkvdW3** is a consequence of the special case of **IPvdW1** corresponding to k IP-systems which are the powers of a single IP-system in homeomorphisms which are cyclic of order k.

Exercise 1.41. Show that **IPvdW1** $\Rightarrow$ **MvdW4**.

The following is a combinatorial formulation of the above "IP van der Waerden theorem". For $k \in \mathbf{N}$, let $\mathbf{Z}^{k\times\infty}$ denote the set of functions from $\{1, 2, \cdots, k\} \times \mathbf{N}$ to $\mathbf{Z}$ which vanish at infinity. Alternatively, $\mathbf{Z}^{k\times\infty}$ is the set of $k \times \infty$ matrices with integer entries all but finitely many of which are non-zero. $\mathbf{Z}^{k\times\infty}$ is isomorphic to $\mathbf{Z}_k^\infty$, of course. For $\alpha \in \mathcal{F}$ and $1 \leq j \leq k$, let $\overline{\alpha}_j$ denote the element of $\mathbf{Z}^{k\times\infty}$ determined by the rule $\overline{\alpha}_j(i, n) = 1$ if $i = j$ and $n \in \alpha$ and $\overline{\alpha}_j = 0$ otherwise.

IPvdW2. Let $k \in \mathbf{N}$. For any finite coloring of $\mathbf{Z}^{k\times\infty}$ there exists a monochromatic configuration of the form

$$\{w, w + \overline{\alpha}_1, w + \overline{\alpha}_2, \cdots, w + \overline{\alpha}_k\},$$

where $w \in \mathbf{Z}^{k\times\infty}$ and $\alpha \in \mathcal{F}$.

An example of this type of configuration for $k = 3$ is

$$\left\{ \begin{pmatrix} -2 & 8 & 2 & 1 & 9 & 0 & 8 \\ 4 & 3 & 0 & -8 & 7 & 2 & 1 \\ -6 & 9 & 5 & -1 & 0 & 5 & -9 \end{pmatrix}, \begin{pmatrix} -2 & 9 & 2 & 2 & 9 & 0 & 9 \\ 4 & 3 & 0 & -8 & 7 & 2 & 1 \\ -6 & 9 & 5 & -1 & 0 & 5 & -9 \end{pmatrix}, \right.$$
$$\left. \begin{pmatrix} -2 & 8 & 2 & 1 & 9 & 0 & 8 \\ 4 & 4 & 0 & -7 & 7 & 2 & 2 \\ -6 & 9 & 5 & -1 & 0 & 5 & -9 \end{pmatrix}, \begin{pmatrix} -2 & 8 & 2 & 1 & 9 & 0 & 8 \\ 4 & 3 & 0 & -8 & 7 & 2 & 1 \\ -6 & 10 & 5 & 0 & 0 & 5 & -8 \end{pmatrix} \right\}.$$

Exercise 1.42. Use **IPvdW2** to establish the following extension of van der Waerden's theorem: for any $k \in \mathbf{N}$, any finite partition of $\mathbf{N}$, and any IP-set $\Gamma \subset \mathbf{N}$, there exists a monochromatic arithmetic progression of length k whose common difference comes from Γ; namely, there exists in one cell of the partition a configuration of the form $\{a, a+n, a+2n, \cdots, a+(k-1)n\}$, where $n \in \Gamma$.

IPvdW2⇒IPvdW1: Let $X = \bigcup_{i=1}^{r} U_i$ be a partition of X into sets of diameter less than ϵ. Put $T_{ij} = T^{(i)}_{\{j\}}$. For $w \in \mathbf{Z}^{k\times\infty}$, put

$$T(w) = \prod_{1\le i\le k, j\in\mathbf{N}} T_{ij}^{w(i,j)}.$$

(Recall that $w(i,j) = 0$ for all but finitely many pairs (i,j).) Let $y \in X$ be fixed. We create a partition $\mathbf{Z}^{k\times\infty} = \bigcup_{i=1}^{r} C_i$ according to the rule $w \in C_k$ if and only if $T(w)y \in U_k$. According to **IPvdW2**, there exists j, $1 \le j \le r$, and a configuration $\{w, w+\overline{\alpha}_1, \cdots, \overline{\alpha}_k\} \subset C_j$. Let $x = T(w)y$.

Exercise 1.43. Show that $x \in U_j$ and $T^{(i)}_\alpha x \in U_j$, $1 \le i \le k$.

□

Exercise 1.44. Show that **IPvdW1** implies **IPvdW2**.

Let us now prove the IP van der Waerden theorem. The form we shall use is that of **IPvdW1**. The reader may notice that the proof is quite similar to the proof of the multidimensional van der Waerden theorem given in Section 1.3 ([BPT]; see also [B5]). First we have some preliminary discussion.

Suppose that X is a compact metric space , G is an abelian group, and $\{T_g\}_{g\in G}$ is a G-action by homeomorphisms of X. The system $\left(X, \{T_g\}_{g\in G}\right)$ is said to be *minimal* if there are no closed, non-empty, proper G-invariant subsets of X. (Y is said to be G-invariant if $Y \subset T_g^{-1}Y$ for all $g \in G$.)

Exercise 1.45. Show that for any dynamical system $(X, \{T_g\}_{g\in G})$, where G is an abelian group, there exists a non-empty closed subset $Y \subset X$ such that the restriction of $\{T_g\}_{g\in G}$ to Y is a minimal G-action by homeomorphisms of Y (use Zorn's Lemma).

Recall that if $\alpha, \beta \in \mathcal{F}$ then $\alpha < \beta$ if for every $i \in \alpha$ and every $j \in \beta$ we have $i < j$. Consider now the following assertions, for $k \in \mathbf{N}$.

$\mathcal{S}_k$: For any $\epsilon > 0$ and any compact metric space X, if $(T^{(1)}_\alpha)_{\alpha\in\mathcal{F}}, \cdots, (T^{(k)}_\alpha)_{\alpha\in\mathcal{F}}$ are commuting IP-systems of homeomorphisms of X then there exists $x \in X$ and $\alpha \in \mathcal{F}$ such that $p(x, T^{(i)}_\alpha x) < \epsilon$, $1 \le i \le k$.

$\mathcal{T}_k$: Suppose that $\epsilon > 0$ and X is a compact metric space. If G is an abelian group of homeomorphisms acting minimally on X and $(T^{(1)}_\alpha)_{\alpha\in\mathcal{F}}, \cdots, (T^{(k)}_\alpha)_{\alpha\in\mathcal{F}}$ are IP-systems in G, then for every non-empty open $U \subset X$ there exists $\alpha \in \mathcal{F}$ such that

$$\left(U \cap (T^{(1)}_\alpha)^{-1}U \cap (T^{(2)}_\alpha)^{-1}U \cap \cdots \cap (T^{(k)}_\alpha)^{-1}U\right) \ne \emptyset.$$

In order to establish **IPvdW1** it is sufficient to show that (a) $\mathcal{S}_1$ holds, (b) $\mathcal{S}_k \Rightarrow \mathcal{T}_k$, and (c) $\mathcal{T}_k \Rightarrow \mathcal{S}_{k+1}$.

Exercise 1.46. Show that $\mathcal{S}_1$ holds and that $\mathcal{S}_k \Rightarrow \mathcal{T}_k$, $k \in \mathbf{N}$.

Exercise 1.47. Suppose that $\mathcal{T}_k$ holds. Then under the conditions appearing there, for every $\beta \in \mathcal{F}$ we may in fact choose $\alpha > \beta$ such that the conclusion holds for this α.

Finally, we show that $\mathcal{T}_k$ implies $\mathcal{S}_{k+1}$. Again, the proof should look familiar. We claim there exist non-empty open sets of diameter $< \frac{\epsilon}{2}$ $(U_i)_{i=0}^{\infty}$ and a sequence $(\alpha_i)_{i=1}^{\infty} \subset \mathcal{F}$ with $\alpha_1 < \alpha_2 < \cdots$ such that

$$U_j \subset \bigcap_{n=1}^{k+1} \left(T^{(n)}_{\alpha_j \cup \alpha_{j-1} \cup \cdots \cup \alpha_{i+1}}\right)^{-1} U_i, \ 1 \le i < j. \tag{1.6}$$

Let U_0 be arbitrary. Having chosen $U_0, \cdots, U_l$ and $\alpha_1, \cdots, \alpha_l$ such that (1.6) holds for $j \le l$, choose by $\mathcal{T}_k$ and Exercise 1.47 $\alpha_{l+1} \in \mathcal{F}$ with $\alpha_{l+1} > \alpha_l$ such that

$$\bigcap_{i=1}^{k} \left((T^{(k+1)}_{\alpha_{l+1}})^{-1} T^{(i)}_{\alpha_{l+1}}\right)^{-1} U_l \neq \emptyset.$$

Let U_{l+1} be a non-empty open subset of diameter $< \epsilon$ such that

$$\begin{aligned} U_{l+1} &\subset (T^{(k+1)}_{\alpha_{l+1}})^{-1} \left(\bigcap_{i=1}^{k} \left((T^{(k+1)}_{\alpha_{l+1}})^{-1} T^{(i)}_{\alpha_{l+1}}\right)^{-1} U_l \right) \\ &= \bigcap_{i=1}^{k+1} (T^{(i)}_{\alpha_{l+1}})^{-1} U_l. \end{aligned}$$

This coupled with the case $j = l$ of (1.6) gives

$$U_{l+1} \subset \bigcap_{n=1}^{k+1} (T^{(n)}_{\alpha_{l+1} \cup \alpha_l \cup \cdots \cup \alpha_{i+1}})^{-1} U_i, \ 0 \le i \le l,$$

so that (1.6) holds for $j \le l+1$, establishing the claim.

Let $x_n \in U_n$, $n = 0, 1, \cdots$. By compactness we have $\rho(x_i, x_j) < \frac{\epsilon}{2}$ for some $i < j$. One may now easily show that $\rho(x, T^{(i)}_\alpha) < \epsilon$, where $\alpha = (\alpha_j \cup \alpha_{j-1} \cup \cdots \cup \alpha_{i+1})$ and $x = x_j$.

□

Here is yet another version of the IP van der Waerden theorem.

IPvdW3. Let $k \in \mathbf{N}$ and let G be a countable abelian group. Suppose that $(n^{(i)}_\alpha)_{\alpha \in \mathcal{F}}$ are IP-sets in G, $1 \le i \le k$. Then for any finite coloring of G there exists a monochromatic configuration of the form

$$\{a, a + n^{(1)}_\alpha, \cdots, a + n^{(k)}_\alpha\}.$$

Exercise 1.48. Show that **IPvdW3** is equivalent to **IPvdW1** and **IPvdW2**.

1.6 The Hales-Jewett Theorem.

Let $k \in \mathbf{N}$ and put $\Lambda_k = \{1, \cdots, k\}$. For $n \in \mathbf{N}$, elements of Λ_k^n (i.e. n-tuples taken from Λ_k) will be called *words* and will be written as $x = x_1 x_2 \cdots x_n$ (where, of course, $x_i \in \Lambda_k$, $1 \le i \le n$). A k-element subset of Λ_k^n, $\{w^{(1)}, w^{(2)}, \cdots, w^{(k)}\}$, will be called a *combinatorial line* if there exists a non-empty subset $D \subset \{1, \cdots, n\}$ such that

(a) $w_i^{(j)} = j$ if $i \in D$, and

(b) $w_i^{(j)} = w_i^{(m)}$ if $i \in \{1, \cdots, n\} \setminus D$ and $1 \le j, m \le k$.

Example. $\{3211231, 3221232, 3231233\}$ is a combinatorial line in Λ_3^7.

Combinatorial lines may be conveniently expressed in matrix form, the words forming the rows. The combinatorial line of the previous example would then look like:

$$\begin{pmatrix} 3 & 2 & 1 & 1 & 2 & 3 & 1 \\ 3 & 2 & 2 & 1 & 2 & 3 & 2 \\ 3 & 2 & 3 & 1 & 2 & 3 & 3 \end{pmatrix}.$$

Another way to denote combinatorial lines is through the use of *variable words.* A variable word is a word on the alphabet $\{1, 2, \cdots, k, x\}$ in which the symbol x appears at least once. We denote a variable word by, for example $w(x)$, and x acts as a variable in the sense that $w(i)$, for $i = 1, 2, \cdots, k$, refers to the word which results by substituting the letter i for the letter x everywhere it appears in $w(x)$. There is thus a natural 1-1 correspondence between combinatorial lines and variable words. Given a variable word $w(x)$, the combinatorial line associated with $w(x)$ is $\{w(1), w(2), \cdots, w(k)\}$. If $k = 3$ and $w(x) = 32x123x$, this is exactly the combinatorial line above.

Notice that if we identify Λ_k with $\mathbf{Z}_k$ then any combinatorial line is also an affine line, as defined in the previous section (but not vice-versa). Therefore the Hales-Jewett theorem, which we now state, is a refinement of **ZkvdW1-3**.

HJ1. ([HJ].) Let $k, r \in \mathbf{N}$. There exists $n = n(k, r) \in \mathbf{N}$ such that for any r-coloring of Λ_k^n, there exists a monochromatic combinatorial line.

The Hales-Jewett theorem also extends the IP van der Waerden theorem. Indeed, let us show that **HJ1$\Rightarrow$IPvdW1**. Let X be a compact metric space , let $k \in \mathbf{N}$ and let $\{T_\alpha^{(i)}\}_{\alpha \in \mathcal{F}}$, $1 \le i \le k$ be commuting IP-systems of self-maps of X. We will assume that $T_\alpha^{(1)} = I$ for all $\alpha \in \mathcal{F}$ (so, really, we will be doing the $k-1$ case of **IPvdW1**). Let $\epsilon > 0$. We must find $x \in X$ such that $\rho(x, T_\alpha^{(i)} x) < \epsilon$, $1 \le i \le k$.

Let $X = \bigcup_{i=1}^r A_i$ be a partition of X into cells of diameter less than ϵ, and let $y \in X$ be arbitrary. Put $n = n(k, r)$ as in **HJ1** and define an r-cell partition $\Lambda_k^n = \bigcup_{i=1}^r C_i$ by the rule $w_1 w_2 \cdots w_n \in C_j$ if and only if $T_{\{1\}}^{(w_1)} T_{\{2\}}^{(w_2)} \cdots T_{\{n\}}^{(w_n)} y \in A_j$. According to **HJ1** there exists j, $1 \le j \le r$, and a variable word $w(x) =$

$v_1v_2\cdots v_n$ such that $\{w(i) : i = 1, 2, \cdots, k\} \subset C_j$. Let $\alpha = \{i \in \{1, \cdots, n\} : v_i = x\}$ and let $x = \left(\prod_{v_i \neq x} T_{\{i\}}^{(v_i)}\right) y$.

Exercise 1.49. Show that $\{x = T_\alpha^{(1)} x, T_\alpha^{(2)} x, \cdots, T_\alpha^{(k)} x\} \subset A_j$.

Hence $\rho(x, T_\alpha^{(i)} x) < \epsilon$, $1 \leq i \leq k$.

□

Here is another version of the Hales-Jewett theorem. For $m \in \mathbf{N}$ let $\mathcal{F}_m$ be the power set of $\{1, \cdots, m\}$, less the empty set.

HJ2. Let $k, r \in \mathbf{N}$. There exists $m = m(k, r) \in \mathbf{N}$ such that for any r-coloring of $\mathcal{F}_m^k$ there exists a monochromatic "simplex" of the form

$$\big\{(\alpha_1, \alpha_2, \cdots, \alpha_k), (\alpha_1 \cup \beta, \alpha_2, \cdots, \alpha_k), (\alpha_1, \alpha_2 \cup \beta, \alpha_3, \cdots, \alpha_k), \\ \cdots, (\alpha_1, \alpha_2, \cdots, \alpha_{k-1}, \alpha_k \cup \beta)\big\},$$

where $\beta \neq \emptyset$ and where $\beta \cap (\alpha_1 \cup \cdots \cup \alpha_k) = \emptyset$.

Exercise 1.50. Show that **HJ1** implies **HJ2**.

Let us now see that **HJ2** implies **HJ1**. Let $k, r \in \mathbf{N}$ and pick $l \in \mathbf{N}$ with $2^l \geq k$. Let $n = m(l, r)$ (as in **HJ2**) and let $\pi : \mathcal{F}_n^l \to \Lambda_k^n$ be the map which sends $(\alpha_1, \cdots, \alpha_l)$ first to the word $w_1 w_2 \cdots w_n \in \Lambda_{2^l}^n$, where $w_i - 1$ has as its binary expansion $1_{\alpha_1}(i) 1_{\alpha_2}(i) \cdots 1_{\alpha_l}(i)$, $1 \leq i \leq n$, then changes any letter greater than k to k. We now claim that any r-coloring of Λ_k^n lifts via π^{-1} to an r-coloring of $\mathcal{F}_n^l$, and that the image under π of any monochromatic simplex in $\mathcal{F}_n^l$ (and such exists by **HJ2**) is a monochromatic combinatorial line in Λ_k^n, thus completing the proof.

Exercise 1.51. Prove the claim.

□

Here now is a combinatorial proof of the Hales-Jewett theorem.

Proof (sketch) of HJ1. We will proceed by induction on k.

Exercise 1.52. Show that the conclusion of **HJ1** holds for $k = 2$.

Assume that $k > 2$ and that the conclusion of **HJ1** holds for k replaced by $k - 1$. Let $r \in \mathbf{N}$ and suppose we are given an r-coloring c of $\mathcal{W}_k$. Let:

$$\begin{aligned}
N_1 &= N(k-1, k^{(r-1)r}) \\
N_2 &= N(k-1, k^{(N_1+r-2)r}) \\
N_3 &= N(k-1, k^{(N_1+N_2+r-3)r}) \\
&\vdots \\
N_{r-1} &= N(k-1, k^{(N_1+\cdots+N_{r-2}+1)r}) \\
N_r &= N(k-1, k^{(N_1+\cdots+N_{r-1})r}).
\end{aligned}$$

We claim there exists an N_r-letter variable word $w_r(x)$ such that for every $(N_1 + \cdots + N_{r-1})$-letter word w the set $\{ww_r(1), ww_r(2), \cdots, ww_r(k-1)\}$ is monochromatic. To see this: for $u, v \in \mathcal{W}$, write $u \sim v$ if $c(wu) = c(wv)$ for every $(N_1 + \cdots + N_{r-1})$-letter word w.

Exercise 1.53. The equivalence classes of $\sim$ form a partition of $\mathcal{W}$ with at most $k^{(N_1+\cdots+N_{r-1})r}$ cells.

By the definition of N_r, there exists $w_r(x)$ with the required properties.

Exercise 1.54. Having chosen N_j-letter variable words $w_j(x)$, $j = r, r-1, \cdots, i+1$, where $1 \leq i < r$, show that it is possible to choose an N_i-letter variable word $w_i(x)$ such that for every $(N_1+\cdots+N_{i-1})$-letter word w and every $(N_{i+1} + N_{i+2} + \cdots + N_r)$-letter word v of the form $v = w_{i+1}(n_{i+1}) \cdots w_r(n_r)$, where $1 \leq n_{i+1}, \cdots, n_r \leq k$, the set $\big\{ww_i(1)v, ww_i(2)v, \cdots, ww_i(k-1)v\big\}$ is monochromatic.

Let now:

$$\begin{aligned}
v_0 &= w_1(k)w_2(k)w_3(k)\cdots w_{r-1}(k)w_r(k)\\
v_1 &= w_1(1)w_2(k)w_3(k)\cdots w_{r-1}(k)w_r(k)\\
v_2 &= w_1(1)w_2(1)w_3(k)\cdots w_{r-1}(k)w_r(k)\\
&\vdots\\
v_{r-1} &= w_1(1)w_2(1)w_3(1)\cdots w_{r-1}(1)w_r(k)\\
v_r &= w_1(1)w_2(1)w_3(1)\cdots w_{r-1}(1)w_r(1).
\end{aligned}$$

By the pigeonhole principle there exists i and j with $0 \leq i < j \leq r$ such that $c(v_i) = c(v_j)$ Define a variable word $w(x)$ by

$$w(x) = w_1(1)\cdots w_i(1)w_{i+1}(x)w_{i+2}(x)\cdots w_j(x)w_{j+1}(k)\cdots w_r(k).$$

Exercise 1.55. Show that $\{w(1), w(2), \cdots, w(k)\}$ is a monochromatic combinatorial line.

$\square$

A familiar motto of Ramsey theory is "lines implies spaces", and nowhere is this motto more vividly illustrated than in the ease with which the Hales-Jewett theorem implies a "multidimensional" version of itself which we now formulate. Let $k, n \in \mathbf{N}$. A word on the alphabet $\{1, 2, \cdots, k, x_1, \cdots, x_n\}$ for which (a) all of the letters $x_1, \cdots, x_n$ appear at least once, and (b) for $1 \leq i < j \leq n$, every occurence of x_i precedes every occurence of x_j, will be called an *n-variable word.* We denote an n-variable word by, for example, $w(x_1, \cdots, x_n)$. If $t_i \in \{1, \cdots, k\}$, $1 \leq i \leq n$, then by $w(t_1, \cdots, t_n)$ we mean the word obtained by substituting t_1 for x_1, t_2 for x_2, etc. The set of all words $w(t_1, \cdots, t_n)$ which may be obtained by substituting in this manner is called a *combinatorial n-space.*

HJ3. Let $k, r, n \in \mathbf{N}$. There exists $m = m(k, r, n) \in \mathbf{N}$ having the property that for any r-coloring of Λ_k^m, there exists a monochromatic combinatorial n-space.

Exercise 1.56. Prove that **HJ1**$\Rightarrow$**HJ3**. (*Hint: use induced colorings similar to those pertaining to the equivalence relation* $\sim$ *just above Exercise 1.53.*)

The proof of the Hales-Jewett theorem we have given is slightly cumbersome in that one has to specify $N_1, N_2, \cdots, N_r$ in advance. (Recall that such was the case in the combinatorial proof of van der Waerden's theorem as well.) In the proofs we have seen up to now proceeding via recurrence in topological dynamics, this is unnecessary. By utilizing the topological structure (minimality and invertibility in particular) and the shift invariance of the sought-after structures, one is able to take things one step at a time. This simplification of matters is one of the aesthetic hallmarks of the tradition.

In trying to generalize such methods to give a proof of the Hales-Jewett theorem, difficulties arise. One reason we can't proceed here as in Section 1.5 is that if we treat Λ_k^n as if it were $\mathbf{Z}_k^n$ then the class of combinatorial lines is not shift-invariant! A natural alternative is to treat the words of Λ_k^n as words in a free semigroup on the alphabet Λ_k. This is done in Section 2.3, where we will in fact see a more "dynamical" proof of the Hales-Jewett theorem (actually of a much stronger, *infinitary* version of the Hales-Jewett theorem). The methods of this chapter do not seem to mix all that well with either the non-commutativity or the non-invertible nature of a free semigroup action, however.

Question. Can the methods of this chapter be modified to give a topological proof of the Hales-Jewett theorem?

We do remark that in [BL2] a proof of the Hales-Jewett theorem is presented that appears on the surface to be topological, however (as noted by the authors) neither continuity of the maps nor completeness of the base space is used anywhere in the proof. Unsurprisingly, all the necessary constants N_i are specified in advance.

1.7 Recurrence for VIP-systems.

Let G be an additive abelian group and let $d \in \mathbf{N}$. Let $(f_\gamma)_{\gamma\in\mathcal{F}}$ be any $\mathcal{F}$-sequence in G having the property that $f_\gamma \neq e$ for some γ with $|\gamma| = d$, and $f_\gamma = e$ for every γ with $|\gamma| > d$. For $\alpha \in \mathcal{F}$, set

$$v_\alpha = \sum_{\gamma\subset\alpha} f_\gamma. \tag{1.7}$$

Then $(v_\alpha)_{\alpha\in\mathcal{F}}$ is said to be a *VIP-system* of *degree d*. (VIP-systems were introduced in [BFM]; see also [M2].) Notice that VIP-systems of degree 1 are just IP-sets. Sometimes it is convenient to restrict the domain of a VIP system to those elements of $\mathcal{F}$ that are disjoint from some $\beta \in \mathcal{F}$.

Exercise 1.57. Let $(n_\alpha^{(1)})_{\alpha\in\mathcal{F}}, (n_\alpha^{(2)})_{\alpha\in\mathcal{F}}, \cdots, (n_\alpha^{(k)})_{\alpha\in\mathcal{F}}$ be IP-sets in $\mathbf{Z}$ and let $p_i(x_1, \cdots, x_m)$ be polynomials in $\mathbf{Q}[x_1, \cdots, x_m]$, $1 \leq i \leq r$. Then

$$\Big((p_1(n_\alpha^{(1)}, n_\alpha^{(2)}, \cdots, n_\alpha^{(k)}), \cdots, p_r(n_\alpha^{(1)}, n_\alpha^{(2)}, \cdots, n_\alpha^{(k)}))\Big)_{\alpha\in\mathcal{F}}$$

is a VIP-system in $\mathbf{Z}^r$. Moreover, the degree of this VIP-system is less than or equal to the largest degree of the p_i's.

Notice that linear combinations of VIP-systems are again VIP-systems. If $\mathcal{A}$ is a collection of sets, we denote by $FU(\mathcal{A})$ the family of all non-trivial finite unions of the sets in $\mathcal{A}$. Also we let $FU_\emptyset(\mathcal{A}) = FU(\mathcal{A}) \cup \{\emptyset\}$. The following exercise gives an alternate characterization of VIP-systems.

Exercise 1.58. Show that $(v_\alpha)_{\alpha\in\mathcal{F}} \subset G$ is a VIP-system of degree $\leq d$ if and only if for all pairwise disjoint $\alpha_0, \alpha_1, \cdots, \alpha_d \in \mathcal{F}$,

$$\sum_{\gamma\in FU(\alpha_0,\cdots,\alpha_d)} (-1)^{|\gamma|} v_\gamma = e.$$

Suppose $(v_\alpha)_{\alpha\in\mathcal{F}}$ is a VIP-system. For disjoint $\alpha, \beta \in \mathcal{F}$, put $v^{(2)}_{\alpha,\beta} = v_{\alpha\cup\beta} v_\alpha^{-1} v_\beta^{-1}$.

Exercise 1.59. Show that

$$v^{(2)}_{\alpha,\beta} = \sum_{\gamma\subset\alpha\cup\beta, \gamma\not\subset\alpha, \gamma\not\subset\beta} f_\gamma.$$

Show also that if $(v_\alpha)_{\alpha\in\mathcal{F}}$ is regular then for fixed $\beta \in \mathcal{F}$, $\left(v^{(2)}_{\alpha,\beta}\right)_{\alpha\in\mathcal{F},\alpha\cap\beta=\emptyset}$ is a VIP-system of degree $d-1$.

Suppose $(v_\alpha)_{\alpha\in\mathcal{F}}$ is a VIP-system of degree d given by (1.7). We define the *leading term* of $(v_\alpha)_{\alpha\in\mathcal{F}}$ to be the VIP-system $(u_\alpha)_{\alpha\in\mathcal{F}}$, where

$$u_\alpha = \sum_{\gamma\subset\alpha, |\gamma|=d} f_\gamma.$$

Clearly the leading term has degree d as well.

VIPvdW1. Suppose that X is a compact metric space, G is a commutative group of homeomorphisms on X and $k \in \mathbf{N}$.

$$(V^{(1)}_\alpha)_{\alpha\in\mathcal{F}}, (V^{(2)}_\alpha)_{\alpha\in\mathcal{F}}, \cdots, (V^{(k)}_\alpha)_{\alpha\in\mathcal{F}}$$

are VIP-systems in G then for any $\epsilon > 0$ there exists $x \in X$ and $\alpha \in \mathcal{F}$ such that $\rho(x, V^{(i)}_\alpha x) < \epsilon$, $1 \leq i \leq k$.

Rather than prove this result in general, we will do a special case to illustrate the technique. The case of VIP-systems of degree 1 is just **IPvdW**, from Section 1.5. Therefore, the first non-trivial case yet to be considered is for $k = 1$ and a single VIP-system V_α of degree 2. Also we may assume without loss of generality that G acts minimally on X.

Let $\epsilon > 0$. Our plan is to construct a sequence of non-empty open sets $(U_i)_{i=0}^\infty$ of diameter $< \frac{\epsilon}{2}$ and a pairwise disjoint sequence $(\alpha_i)_{i=1}^\infty \subset \mathcal{F}$ such that

$$V_{\alpha_j\cup\alpha_{j-1}\cup\cdots\cup\alpha_{i+1}} U_j \subset U_i,\ 0 \leq i < j. \tag{1.8}$$

Supposing this has been done, we complete the proof in the standard way, letting $x_j \in U_j$ and finding $i \neq j$ with $\rho(x_i, x_j) < \frac{\epsilon}{2}$, etc.

Let U_0 be any small enough open set, let $\alpha_1 \in \mathcal{F}$ be arbitrary and let $U_1 \subset V_{\alpha_1}^{-1} U_0$ be small enough. Having chosen $U_0, U_1, \cdots, U_t$ and $\alpha_1, \alpha_2, \cdots, \alpha_t$, choose α_{t+1} disjoint from $\alpha_1 \cup \cdots \cup \alpha_t$ such that the set

$$E = \big(U_t \cap (V^{(2)}_{\alpha_{t+1},\alpha_t})^{-1} U_t \cap (V^{(2)}_{\alpha_{t+1},\alpha_t \cup \alpha_{t-1}})^{-1} U_t \cap \cdots$$
$$\cap (V^{(2)}_{\alpha_{t+1},\alpha_t \cup \alpha_{t-1} \cup \cdots \cup \alpha_1})^{-1} U_t\big) \neq \emptyset.$$

(Such α_{t+1} may be found by $\mathcal{T}_k$ in Section 1.5; the reader may take a moment to convince himself that in fact it can be chosen disjoint from $\alpha_1 \cup \cdots \cup \alpha_t$.) We now let $U_{t+1} \subset V_{\alpha_{t+1}}^{-1} E$ be any small enough non-empty open set. Then for $1 \leq i \leq t$, $U_{t+1} \subset (V_{\alpha_{t+1}})^{-1} (V^{(2)}_{\alpha_t \cup \cdots \cup \alpha_{i+1}})^{-1} U_t$, and by the induction hypothesis $V_{\alpha_t \cup \cdots \cup \alpha_{i+1}} U_t \subset U_i$. It follows that

$$V_{\alpha_{t+1} \cup \alpha_t \cup \cdots \cup \alpha_{i+1}} U_{t+1} = V_{\alpha_{t+1}} V^{(2)}_{\alpha_t \cup \cdots \cup \alpha_{i+1}} V_{\alpha_t \cup \cdots \cup \alpha_{i+1}} U_t \subset U_i,$$

as required.

□

Now we shall say a few words about the general proof of **VIPvdW1**. Given a set A of VIP-systems, if $w, v \in A$ then we write $w \sim v$ if the leading terms of v and w coincide; that is, if w and v have the same degree d and the degree of $v - w$ is less than d. $\sim$ is an equivalence relation on A. Similar to the situation in Section 1.4, this equivalence relation allows us to assign a *weight vector* to A. Namely, we say that $(a_1, \cdots, a_d)$ is the weight vector of A if the VIP-system in A of highest degree is of degree d and if a_i is equal to the number of ith degree equivalence classes under $\sim$ represented in A, $1 \leq i \leq d$. The set of weight vectors is ordered exactly as in Section 1.4, namely we write $(a_1, a_2, \cdots, a_k) < (b_1, \cdots, b_n)$ if (i) $k < n$, or (ii) $k = n$ and there exists j, $1 \leq j \leq k$, with $a_k < b_k$ and $a_i = b_i$, $j < i \leq k$.

The inductive scheme for the proof of **VIPvdW1** is the same PET-inductive procedure we have seen before. One shows that the conclusion holds for a set A of VIP-systems, provided it holds for every set of VIP-systems having weight vector preceding the weight vector of A. Or, more accurately, provided the following minimal version holds:

VIPvdW2. Suppose that X is a compact metric space, G is a commutative group of homeomorphisms acting minimally on X and $k \in \mathbf{N}$. If

$$(V_\alpha^{(1)})_{\alpha \in \mathcal{F}}, (V_\alpha^{(2)})_{\alpha \in \mathcal{F}}, \cdots, (V_\alpha^{(k)})_{\alpha \in \mathcal{F}}$$

are VIP-systems in G then for any open set U there exists $\alpha \in \mathcal{F}$ such that

$$\bigcap_{i=1}^{k} (V_\alpha^{(i)})^{-1} U = \emptyset.$$

Exercise 1.60. Prove **VIPvdW1** by filling in the details of the following argument:

(a) Let A be a finite set of VIP systems and let $T \subset \mathcal{F}$ be a finite set. If $W \in A$ is of minimal degree then $A' = \{\alpha \to V_{\alpha\cup\beta}B_\beta^{-1}W_\alpha^{-1} : \beta \in T, \alpha \cap \beta = \emptyset, V \in A\}$ is a set of VIP systems and precedes A.

(b) Let A be a finite set of VIP systems. If **VIPvdW2** holds for every set of VIP systems preceding A then **VIPvdW1** holds for A. (*Hint: construct, using part (a), a sequence of small non-empty open sets* (U_i) *and a pairwise disjoint sequence* $(\alpha_i) \subset \mathcal{F}$ *with* $V_{\alpha_j\cup\cdots\cup\alpha_{i+1}}U_j \subset U_i$, $0 \le i < j$.)

(c) If **VIPvdW1** holds for a set A of VIP systems then **VIPvdW2** holds for A as well.

Finally, we give a purely combinatorial formulation of this result.

VIPvdW3. Let G be an additive abelian group and let $k \in \mathbf{N}$. If

$$(v_\alpha^{(1)})_{\alpha\in\mathcal{F}}, (v_\alpha^{(2)})_{\alpha\in\mathcal{F}}, \cdots, (v_\alpha^{(k)})_{\alpha\in\mathcal{F}}$$

are VIP-systems in G then for any $r \in \mathbf{N}$ and any finite partition $G = \bigcup_{i=1}^r C_i$, there exists i with $1 \le i \le r$, $a \in G$, and $\alpha \in \mathcal{F}$ such that

$$\left\{a + v_\alpha^{(1)}, a + v_\alpha^{(2)}, \cdots, a + v_\alpha^{(k)}\right\} \subset C_i.$$

Exercise 1.61. Show the equivalence of **VIPvdW3** and **VIPvdW1**.

Exercise 1.62. Derive **MPvdW1** (Section 1.4) as a consequence of **VIPvdW3** and Exercise 1.57.

The previous exercise illustrates one usage of **VIPvdW3**. There are many others, however there are limitations as well. For example, it is limited to groups. But one can easily define VIP-systems in semigroups as well. Shouldn't there be a corresponding theorem for finite colorings of semigroups? How about badly non-cancellative semigroups? For example, take the set $S = \mathcal{F}(\mathbf{N}\times\mathbf{N})$ ($\mathcal{F}(A)$ denoting the family of finite non-empty subsets of A), which is a (non-cancellative) semigroup under union. Is it the case that for any finite coloring of S, there exists a monochromatic pair $\{\alpha, \alpha \cup (\beta \times \beta)\}$, where $\alpha \in S$ and $\beta \in \mathcal{F}$, with $\alpha \cap (\beta \times \beta) = \emptyset$? The answer is yes, but there are two obstacles to deriving that from **VIPvdW**. The first is that S is merely a semigroup. That in itself might not be so bad. Worse still, even if one could push the result for semigroups (and show that in this semigroup, the function $\beta \to \beta \times \beta$ is a VIP-system), we would not be getting the disjointness condition. And, without the disjointness condition, the result is meaningless. Of course one can get $\{\alpha, \alpha \cup (\beta \times \beta)\}$ in one color, if one lets $\beta \times \beta \subset \alpha$.

In the next section we shall prove a result, due to Bergelson and Leibman, that settles this matter and at the same time implies every other major result of this chapter. This theorem is the polynomial Hales-Jewett theorem.

1.8 The Bergelson-Leibman coloring theorem.

Let $l \in \mathbf{N}$. A *set-monomial* (over $\mathbf{N}^l$) in the variable X is an expression $m(X) = S_1 \times S_2 \times \cdots \times S_l$, where for each i, $1 \leq i \leq l$, S_i is either the symbol X or a non-empty singleton subset of $\mathbf{N}$ (these are called *coordinate coefficients*). The *degree* of the monomial is the number of times the symbol X appears in the list $S_1, \cdots, S_l$. For example, taking $l = 3$, $m(X) = \{5\} \times X \times X$ is a set-monomial of degree 2, while $m(X) = X \times \{17\} \times \{2\}$ is a set-monomial of degree 1. A *set-polynomial* is an expression of the form $p(X) = m_1(X) \cup m_2(X) \cup \cdots \cup m_k(X)$, where $k \in \mathbf{N}$ and $m_1(X), \cdots, m_k(X)$ are set-monomials. The degree of a set-polynomial is the largest degree of its set-monomial "summands", and its *constant term* consists of the "sum" of those m_i that are constant, i.e. of degree zero.

We will also consider set-polynomials $P(X,Y)$ of two variables, which are defined in the obvious way. In particular, if $P(X)$ is a set polynomial, define $P^{(2)}(X,Y) = P(X \cup Y) \setminus \big(P(X) \cup P(Y)\big)$. ($\setminus$ behaves formally in this expression exactly as it would if X and Y were disjoint sets.) Notice that if $P(X,Y)$ is a set polynomial of two variables and $A \subset \mathbf{N}$ is a finite set then $Q(X) = P(A, X)$ becomes, in a natural manner, a set polynomial of the single variable X.

Example. If $P(X) = X \times X$ then $P^{(2)}(X,Y) = (X \times Y) \cup (Y \times X)$.

Letting $\mathcal{F}(S)$ denote the family of non-empty finite subsets of a set S, any non-empty set polynomial $p(A)$ determines a function from $\mathcal{F}(\mathbf{N})$ to $\mathcal{F}(\mathbf{N}^l)$ in the obvious way (interpreting the symbol $\times$ as Cartesian product and the symbol $\cup$ as union). The following theorem, due to Bergelson and Leibman (see [BL2, Theorem 3.5]), may be viewed as a polynomial version of the Hales-Jewett theorem.

BL. Let $l \in \mathbf{N}$ and let $\mathcal{P}$ be a finite family of set-polynomials over $\mathbf{N}^l$ whose constant terms are empty. Let $I \subset \mathbf{N}$ be any finite set and let $r \in \mathbf{N}$. There exists a finite set $N \subset \mathbf{N}$, with $N \cap I = \emptyset$, such that if $\mathcal{F}\big(\bigcup_{P(X) \in \mathcal{P}} P(N)\big) = \bigcup_{i=1}^r C_i$ then there exists i, $1 \leq i \leq r$, some non-empty $B \subset N$, and some $A \subset \bigcup_{P(X) \in \mathcal{P}} P(N)$ with $A \cap P(B) = \emptyset$ for all $P \in \mathcal{P}$ and

$$\big\{A \cup P(B) : P(X) \in \mathcal{P}\big\} \subset C_i.$$

The proof of **BL** is the same sort of induction that we have already seen in Sections 1.4 and 1.7. For set polynomials P and Q, we write $P \sim Q$ if they have the same degree d and if their degree d terms (i.e. constituent monomials) coincide. If $\mathcal{P}$ is a finite set of set polynomials then $(w_1, \cdots, w_d)$ is the weight vector of $\mathcal{P}$ if d is the maximum degree of the members of $\mathcal{P}$ and if w_i is the number of equivalence classes under $\sim$ represented by set polynomials in $\mathcal{P}$ of degree i, $1 \leq i \leq d$. The set of weight vectors is ordered lexographically, as usual, and we shall induct upon this set.

Since a set polynomial is a union of set monomials, the family of set polynomials is partially ordered by containment. If $\mathcal{P}$ is a finite set of set polynomials, then we say that $Q \in \mathcal{P}$ is *minimal* if $Q \subset P$ for every $P \in \mathcal{P}$. Clearly minimal elements need not exist; however in the proof of **BL** we follow Bergelson and Leibman in assuming initially that there is a minimal element.

Proof of BL. We use PET induction. Namely, assume the validity of the conclusion for every family of set-polynomials $\mathcal{Q}$ with weight vector preceding that of $\mathcal{P}$. Initially we shall also assume that $\mathcal{P}$ contains a minimal element Q. Later we will eliminate this assumption.

Let $H \subset \mathbf{N}$ contain I as well as all numbers that appear as a coordinate coefficient in any of the members of $\mathcal{P}$. Let $\mathcal{P}_0$ consist of all set-polynomials of the form $P \setminus Q$, where $P \in \mathcal{P}$.

Exercise 1.63. The weight vector of $\mathcal{P}_0$ precedes that of $\mathcal{P}$.

Let $N_0 \subset \mathbf{N}$ be chosen with $H \cap N_0 = \emptyset$ and with the property that for any r-coloring of $\bigcup_{R(X)\in\mathcal{P}_0} R(N_0)$, there exists a set $A \subset \bigcup_{R(X)\in\mathcal{P}_0} R(N_0)$ and a non-empty set $B \subset N_0$ such that $A \cap \bigcup_{R(X)\in\mathcal{P}_0} R(B) = \emptyset$ and such that $\{A \cup R(B) : R \in \mathcal{P}_0\}$ is monochromatic.

Having chosen $N_0, N_1, \cdots, N_{t-1}$, let $\mathcal{P}_t$ consist of all set polynomials of the form $R(X) = P^{(2)}(S, X) \cup (P \setminus Q)(X)$, where $P(X) \in \mathcal{P}$ and $S \subset \bigcup_{i=0}^{t-1} N_i$.

Exercise 1.64. The weight vector of $\mathcal{P}_t$ precedes that of $\mathcal{P}$.

Let $L = \left|\bigcup_{i=0}^{t-1} \bigcup_{q\in\mathcal{P}_i\cup\{Q\}} q(N_i)\right|$. Choose N_t such that

(i) $N_t \cap \left(H \cup \bigcup_{i=0}^{t-1} N_i\right) = \emptyset$.

(ii) For any r^{2^L}-coloring of $\bigcup_{R(X)\in\mathcal{P}_t} R(N_t)$, there exists $A \subset \bigcup_{R(X)\in\mathcal{P}_t} R(N_t)$ and a non-empty set $B \subset N_t$ such that $A \cap \bigcup_{R(X)\in\mathcal{P}_t} R(B) = \emptyset$ and $\{A \cup R(B) : R(X) \in \mathcal{P}_t\}$ is monochromatic.

Continue until $N_0, \cdots, N_r$ have been chosen.

Let now $N = N_0 \cup \cdots \cup N_r$ (notice $N \cap I = \emptyset$) and fix an r-coloring of $\bigcup_{q\in\mathcal{P}} q(N)$. Henceforth, we use the notation $S_1 \sim S_2$ to indicate that S_1 and S_2 lie in the same cell for this r-coloring.

Let $A_r \subset \bigcup_{q\in\mathcal{P}_r} q(N_r)$ and $\emptyset \neq B_r \subset N_r$ be chosen such that $A_r \cap \bigcup_{q\in\mathcal{P}_r} q(B_r) = \emptyset$ and such that for any subset $S \subset \bigcup_{i=0}^{r-1} \bigcup_{q\in\mathcal{P}_i\cup\{Q\}} q(N_i)$, the family

$$\{S \cup Q(N_r) \cup A_r \cup q(B_r) : q \in \mathcal{P}_r\}$$

is monochromatic.

Exercise 1.65. This is possible. (*Hint: each subset S determines an r-coloring, and there are 2^L possible choices for S.*)

Having chosen $A_r, A_{r-1}, \cdots, A_{t+1}$ and $B_r, B_{r-1}, \cdots, B_{t+1}$, choose $A_t \subset \bigcup_{q\in\mathcal{P}_t} q(N_t)$ and non-empty $B_t \subset N_t$ such that $A_t \cap \bigcup_{q\in\mathcal{P}_t} q(B_t) = \emptyset$ and such that for any set $S \subset \bigcup_{i=0}^{t-1} \bigcup_{q\in\mathcal{P}_i\cup\{Q\}} q(N_i)$, the family

$$\{S \cup \bigcup_{i=t}^{r} (Q(N_i) \cup A_i) \cup q(B_t) : q \in \mathcal{P}_t\}$$

is monochromatic.

Continue until A_0 and B_0 have been chosen.

For $0 \le i \le r$, let $X_i = \bigcup_{j=0}^{r} \left(A_j \cup Q(N_j)\right) \setminus \bigcup_{j=0}^{t} Q(B_j)$. We claim that for all $P \in \mathcal{P}$, and $0 \le s \le t \le r$,

$$X_t \cup P\Big(\bigcup_{k=s+1}^{t} B_k\Big) \sim X_s. \tag{1.9}$$

We establish (1.9) by induction on $t - s$. For $s = t$ this is of course obvious. Suppose then that $s < t$ and the validity of (1.9) is known for t replaced by $t-1$, i.e.

$$X_{t-1} \cup P(\bigcup_{k=s+1}^{t-1} B_k) \sim X_s. \tag{1.10}$$

The left hand side of (1.10) may be rewritten

$$\Big(\bigcup_{i=0}^{t-1} (A_i \cup Q(N_i)) \setminus \bigcup_{i=0}^{t-1} Q(B_i)\Big) \cup (\bigcup_{i=t}^{r} A_i \cup Q(N_i)) \cup P(\bigcup_{k=s+1}^{t-1} B_k).$$

Set

$$S = \Big(\bigcup_{i=0}^{t-1} (A_i \cup Q(N_i)) \setminus \bigcup_{i=0}^{t-1} Q(B_i)\Big) \cup P(\bigcup_{k=s+1}^{t-1} B_k).$$

According to the requirement whereby A_r and B_r were chosen, the family

$$\{S \cup \bigcup_{i=t}^{r} (Q(N_i) \cup A_i) \cup q(B_t) : q \in \mathcal{P}_t\}$$

is monochromatic. In particular, since $\emptyset$ and $R(X) = P^{(2)}(\bigcup_{k=s+1}^{t-1} B_k, X) \cup (P \setminus Q)(X)$ are both in $\mathcal{P}_t$,

$$S \cup \bigcup_{i=t}^{r} (Q(N_i) \cup A_i) \sim S \cup \bigcup_{i=t}^{r} (Q(N_i) \cap A_i) \cup \Big(P^{(2)}(\bigcup_{k=s+1}^{t-1} B_k, B_t) \cup (P \setminus Q)(B_t)\Big).$$

In other words,

$$\begin{aligned}&\Big(\bigcup_{i=0}^{r} (A_i \cup Q(N_i)) \setminus \bigcup_{i=0}^{t-1} Q(B_i)\Big) \cup P(\bigcup_{k=s+1}^{t-1} B_k) \\ \sim &\Big(\bigcup_{i=0}^{r} (A_i \cup Q(N_i)) \setminus \bigcup_{i=0}^{t-1} Q(B_i)\Big) \cup \Big(P(\bigcup_{k=s+1}^{t} B_k) \setminus Q(B_t)\Big).\end{aligned}$$

That is,

$$X_{t-1} \cup P(\bigcup_{k=s+1}^{t-1} B_k) \sim X_{t-1} \cup \Big(P(\bigcup_{k=s+1}^{t} B_k) \setminus Q(B_t)\Big) = X_t \cup P(\bigcup_{k=s+1}^{t} B_k).$$

Hence by (1.10), $X_s \sim X_t \cup P\big(\bigcup_{k=s+1}^{t} B_k\big)$ and the claim is established.

By the pigeonhole principle, for some $0 \le s < t \le r$ we have $X_s \sim X_t$. For this t, we have

$$X_t \sim X_t \cup P(\bigcup_{k=s+1}^{t} B_k) \text{ for all } P \in \mathcal{P}.$$

Hence we need only show that $X_t \cap \bigcup_{P\in\mathcal{P}} P\big(\bigcup_{k=s+1}^{t} B_k\big) = \emptyset$. Since

$$X_t = \bigcup_{i=0}^{r} (A_i \cup Q(N_i)) \setminus \bigcup_{i=0}^{t} Q(B_i),$$

it suffices to show that

$$\bigcup_{i=0}^{r} (A_i \cup Q(N_i)) \cap P(\bigcup_{k=s+1}^{t} B_k) \subset \bigcup_{i=0}^{t} Q(B_i) \tag{1.11}$$

for all $P \in \mathcal{P}$.

Exercise 1.66. Show that

$$P(\bigcup_{k=s+1}^{t} B_k) = (\bigcup_{k=s+1}^{t} P(B_k)) \cup \bigcup_{k=s+2}^{t} P^{(2)}(\bigcup_{j=s+1}^{k-1} B_j, B_k).$$

Suppose that $v \in A_i$ for some i, $0 \le i \le r$. Then $v \in R(N_i)$ for some $R \in \mathcal{P}_i$, hence some coordinates of v lie in N_i and the rest lie in $H \cup N_0 \cup \cdots \cup N_{i-1}$. Moreover, $v \notin R(B_i)$ for any $R \in \mathcal{P}_i$, that is, for every $P \in \mathcal{P}$, and $S \subset N_0 \cup \cdots \cup N_{i-1}$,

$$v \notin P^{(2)}(S, B_i) \cup (P \setminus Q)(B_i). \tag{1.12}$$

If $v \in P(\bigcup_{k=s+1}^{t} B_k)$ for some $P \in \mathcal{P}$, then by Exercise 1.66 and what we know about the coordinates of v, $v \in P(B_i) \cup P^{(2)}(\bigcup_{j=s+1}^{i-1} B_j, B_i)$. Comparing this to (1.12), we get $v \in Q(B_i)$.

Suppose on the other hand that $v \in Q(N_i)$. As before, if $v \in P(\bigcup_{k=s+1}^{t} B_k)$ for some $P \in \mathcal{P}$ then $v \in P(B_i) \cup P^{(2)}(\bigcup_{j=s+1}^{i-1} B_j, B_i)$. But all the coordinates of v come from $H \cup N_i$. Therefore $v \in P(B_i) = (P \setminus Q)(B_i) \cup Q(B_i)$. Since $H \cap N_i = \emptyset$, there is a unique set monomial $R(X)$ whose coefficients come from H such that $v \in R(N_i)$. $R(X)$ is clearly contained in $Q(X)$ since $v \in Q(N_i)$. Hence $R(X)$ is not contained in $(P \setminus Q)(X)$, whence $v \notin (P \setminus Q)(B_i)$, againimplying that $v \in Q(B_i)$.

This establishes (1.11) and hence completes the induction step, modulo the assumption $\mathcal{P}$ contains a minimal element. In order to complete the proof, we shall reduce the general case to this special case. This reduction shall take place in two stages.

The first stage is to show that the conclusion is valid for a special class of families $\mathcal{P}$, provided it is valid for every family having a minimal element and the same weight matrix as $\mathcal{P}$. Accordingly, define $\mathcal{P}$ to be *special* if there exist *non-empty* set-monomials $\{P_i : P \in \mathcal{P}, 1 \le i \le \deg(P)\}$ such that (i) $\deg P_i = i$, (ii) $P = \bigcup_{i=1}^{\deg P} P_i$ for all $P \in \mathcal{P}$, and (iii) for $P, R \in \mathcal{P}$, $1 \le i \le \deg(P)$ and $1 \le j \le \deg(R)$, P_i and R_j are either equal (in which case, of course, $i = j$) or have coordinate coefficients derived from disjoint sets (so that, in particular, $\text{range}(P_i) \cap \text{range}(R_j) = \emptyset$).

Let $\mathcal{P}$ be a special finite family of set-polynomials and assume that the conclusion is valid for every system $\mathcal{Q}$ having a minimal element and the same weight matrix as $\mathcal{P}$. Let $Q \in \mathcal{P}$ be of minimal degree and put $\mathcal{Q} = \{Q \cup P : P \in \mathcal{P}\}$. Clearly Q is a minimal element for $\mathcal{Q}$.

Exercise 1.67. Show that $\mathcal{Q}$ has the same weight matrix as $\mathcal{P}$.

Again we let H be a set containing I and all possible coordinate coefficients of members of $\mathcal{P}$. Given a point $v = (v_1, v_2, \cdots, v_l) \in \mathbf{N}^l$, define $sp(v) = \{v_1, v_2, \cdots, v_l\} \setminus H$.

Exercise 1.68. If $v \in P(B)$ for any finite set $B \subset \mathbf{N}$ and $P \in \mathcal{P}$, then (a) $sp(v) \subset B$ and (b) $v \in P\big(sp(v)\big)$.

For any set $A \subset \mathbf{N}^l$, let $\varphi(A)$ be the subset of A such that $v \in A \setminus \varphi(A)$ if and only if there exist $i \in \mathbf{N}$, $R \in \mathcal{P}$, and $w \in A$ such that (i) $sp(v) = sp(w)$, (ii) $v \in \text{range}(Q_i)$, (iii) $w \in \text{range}(R_i)$, and (iv) $R_i \ne Q_i$.

Exercise 1.69. Show that for $A_1, A_2 \subset \mathbf{N}^l$ and $B \subset \mathbf{N}$,
(a) $\varphi(A_1 \cup A_2) \subset \varphi(A_1) \cup \varphi(A_2)$.
(b) $\varphi\big(P(B) \cup Q(B)\big) = P(B)$.

Choose N, with $N \cap H = \emptyset$, to satisfy the conclusion for the family $\mathcal{Q}$. Let now $\mathcal{F}\big(\bigcup_{P(X)\in\mathcal{P}} P(N)\big) = \bigcup_{i=1}^{r} C_i$ be an arbitrary partition. We shall now create a new partition $\mathcal{F}\big(\bigcup_{P(X)\in\mathcal{P}} P(N)\big) = \bigcup_{i=1}^{r} D_i$ by the rule $A \in D_i$ if and only if $\varphi(A) \in C_i$.

By hypothesis, there exists a set $A' \subset \bigcup_{P(X)\in\mathcal{Q}} P(N)$, a non-empty set $B \subset N$ and a number i, $1 \le i \le r$, such that

$$\{A' \cup P(N) : P \in \mathcal{Q}\} = \{A' \cup P(N) \cup Q(N) : P \in \mathcal{P}\} \subset D_i,$$

and such that

$$A' \cap \bigcup_{P(X)\in\mathcal{Q}} P(B) = A' \cap \bigcup_{P(X)\in\mathcal{P}} \big(P(B) \cup Q(B)\big) = \emptyset.$$

Let $A = \varphi(A')$. We claim that $\{A \cup P(B) : P \in \mathcal{P}\} \subset C_i$. In order to see this, it suffices to show that

$$\varphi\big(A' \cup P(B) \cup Q(B)\big) = A \cup P(B) \text{ for every } P \in \mathcal{P}.$$

Once this is done, the proof of this step will be complete, for clearly $A \cap P(B) = \emptyset$ for every $P \in \mathcal{P}$.

Exercise 1.70. $\varphi(A' \cup P(B) \cup Q(B)) \subset A \cup P(B)$. for every $P \in \mathcal{P}$.

In order to establish the reverse containment, assume there is some $v \in (A \cup P(B)) \setminus \varphi(A' \cup P(B) \cup Q(B))$. (We shall obtain a contradiction.) There exists $i \in \mathbf{N}$, $R \in \mathcal{P}$, and $w \in A' \cup P(B) \cup Q(B)$ such that (i) $sp(v) = sp(w)$, (ii) $v \in \text{range}(Q_i)$, (iii) $w \in \text{range}(R_i)$, and (iv) $R_i \neq Q_i$.

Case 1: $v \in A$. Then $w \notin A'$ (else v would not be in A). If $w \in P(B)$ then $w \in \text{range}(P_i)$ and $sp(v) = sp(w) \subset B$, so $v \in Q(B)$, a contradiction. So $w \in Q(B)$. But this is obviously false, for w is in the range of R_i and $R_i \neq Q_i$.

Case 2: $v \in P(B)$. Then $P_i = Q_i$ and $sp(v) \subset B$, so that $w \in R_i(B)$. But $R_i \cap (A' \cup P(B) \cup Q(B)) = \emptyset$.

The final step is to show that the conclusion holds for an arbitrary system $\mathcal{P}$, provided it holds for every special system having the same weight matrix as $\mathcal{P}$. Suppose then that $\mathcal{P}$ is such a system. For $P \in \mathcal{P}$, write $P(X) = \bigcup_{i=1}^{\deg P} P_i(X)$, where $\deg P_i = i$.

Exercise 1.71. There exists a system $\mathcal{P}'$ such that

(a) $\mathcal{P}'$ is in 1-1 correspondence with $\mathcal{P}$, $P \leftrightarrow P'$, with $\deg P' = \deg P$.

(b) for each $P \in \mathcal{P}$, $P'(X) = \bigcup_{i=1}^{\deg P} P_i'(X)$, where each $P_i'(X)$ is a *non-empty* set monomial of degree i.

(c) if $P, R \in \mathcal{P}$, and $P_i = R_i$, then $P_i' = R_i'$.

(d) $\mathcal{P}'$ is a special system.

Let $H \subset \mathbf{N}$ contain I and all coordinate coefficients of all members of both $\mathcal{P}$ and $\mathcal{P}'$. The notation $sp(v)$ will be used as before. For more detailed information about v, the following notation will be useful to us. Suppose $v = (v_1, v_2, \cdots, v_l) \in \mathbf{N}^l$. Let $v_{i_1}, \cdots, v_{i_t}$ be all of the v_i's that do not lie in H. (Here $1 \le i_1 < i_2 < \cdots < i_t \le l$. Then we write $\pi(v) = (v_{i_1}, \cdots, v_{i_t})$.

Exercise 1.72. Suppose $N \cap H = \emptyset$ and $v, w \in P_i'(N)$, where $P' \in \mathcal{P}'$ and $1 \le i \le \deg P'$. Then $v = w$ if and only if $\pi(v) = \pi(w)$.

We now create a map ψ taking $\mathcal{F}(\mathbf{N}^l)$ to $\mathcal{F}_\emptyset(\mathbf{N}^l)$ and having the properties

(a) $\psi(A_1 \cup A_2) = \psi(A_1) \cup \psi(A_2)$ for $A_1, A_2 \in \mathcal{F}(N^l)$.

(b) $\psi(P'(N)) = P(N)$ for all $N \in \mathcal{F}$, $P \in \mathcal{P}$.

Once ψ is defined on singletons, it will have a unique extension to $\mathcal{F}(\mathbf{N}^l)$ satisfying (a). Accordingly, let $v \in \mathbf{N}^l$. If $v \notin \bigcup_{P \in \mathcal{P}} \text{range}(P')$, then put $\psi(\{v\}) = \emptyset$. Otherwise choose i and $P \in \mathcal{P}$ such that $v \in \text{range}(P_i')$ and let

$$\psi(\{v\}) = \{w : w \in \text{range}(P_i), \pi(w) = \pi(v)\}.$$

Extend ψ to $\mathcal{F}(\mathbf{N}^l)$ in accordance with property (a).

Exercise 1.73. Property (b) is satisfied by ψ.

Choose $N \subset \mathbf{N}$ with $N \cap H = \emptyset$ to satisfy the conclusion for the family $\mathcal{P}'$. Suppose now that $\mathcal{F}(\mathbf{N}^l) = \bigcup_{i=1}^r C_i$. Create a partition $\mathcal{F}(\mathbf{N}^l) = \bigcup_{i=1}^r D_i$ by the rule $A \in D_i$ if and only if $\psi(A) \in C_i$.

By hypothesis, there exist sets $A' \subset \bigcup_{P\in\mathcal{P}} P'(N)$ and$B \in \mathcal{F}(\mathbf{N})$, and i such that $\{A' \cup P'(B) : P \in \mathcal{P}\} \subset D_i$ and such that $A' \cap P'(B) = \emptyset$ for all $P \in \mathcal{P}$. Let $A = \psi(A')$. Clearly we have

$$A \cup P(B) = \psi(A') \cup \psi\big(P'(B)\big) = \psi\big(A' \cup P'(B)\big) \in C_i$$

for all $P \in \mathcal{P}$. Moreover,

$$A = \psi(A') \subset \psi\big(\bigcup_{P\in\mathcal{P}} P'(N)\big) = \bigcup_{P\in\mathcal{P}} \psi(P'(N)) = \bigcup_{P\in\mathcal{P}} P(N).$$

Hence we need only show that $A \cap P(B) = \emptyset$, that is, $\psi(A') \cap \psi\big(p'(B)\big) = \emptyset$, for all $P \in \mathcal{P}$.

Suppose not. That is, suppose there exists some $v \in A \cap P(B)$, where $P \in \mathcal{P}$. Since $v \in P(B)$, $sp(w) \subset B$. On the other hand, since $w \in \psi(A')$, there exists $v \in A'$ with $\pi(v) = \pi(w)$. In particular, $sp(v) \subset B$. But $v \in P'(N)$ for some $P \in \mathcal{P}$, so by Exercise 1.68 (b) $v \in P'\big(sp(v)\big) \subset P'(B)$. This contradicts the fact that $A' \cap P'(B) = \emptyset$, completing the proof.

□

Chapter 2

Infinitary Ramsey Theory

2.1 The theorems of Ramsey and Schur.

Together with van der Waerden's theorem (1927), the primary classical results of Ramsey theory are Schur's theorem (1916) and Ramsey's theorem (1930). In this section we prove Ramsey's theorem and derive Schur's theorem as a corollary of it. The proofs we give are natural and not new. The reader is invited to peruse [GRS] for more details.

If S is a set and $k \in \mathbf{N}$, let us call a subset $B \subset S$ a *k-subset* if $|B| = k$.

R1. ([Ram].) Let S be a countable infinite set and let $k, r \in \mathbf{N}$. If the k-subsets of S are r-colored then there exists an infinite subset $T \subset S$ having the property that the k-subsets of T comprise a monochromatic family.

Before proving Ramsey's theorem, let us note that it differs from the results of the previous chapter in that the monochromatic family sought is infinite in cardinality (hence it is an *infinitary* result). Moreover, this family has the same structure as the original family to be r-colored; namely it is the collection of k-subsets of a countably infinite set. (It is hoped that this helps to elucidate the distinction we draw between monochromatic *configurations* and monochromatic *substructures.*)

Proof of R1. For $k = 1$, the result is of course obvious. Let us first handle the case $k = 2$, $r = 2$. Let us denote the 2-coloring by a function c from the 2-subsets of S to $\{1, 2\}$.

Let $A_0 = S$. If possible, choose $b_1 \in A_0$ such that $A_1 = \{n \in A_0 : c(\{b_1, n\}) = 1\}$ has infinite cardinality. Assuming that $b_1, \cdots, b_{k-1}$ and $A_0 \supset A_1 \supset \cdots \supset A_{k-1}$ have been chosen, choose (if possible) $b_k \in A_{k-1}$ such that $A_k = \{n \in A_{k-1} : c(\{b_k, n\}) = 1\}$ has infinite cardinality. Continue choosing b_k's and A_k's as long as possible.

If this process fails to terminate (in other words, if for all k it is possible to choose such b_k and A_k) then $T = \{b_1, b_2, \cdots\}$ clearly has the property that $c(B) = 1$ for every 2-subset B of T. If, on the other hand, for some k it is impossible to choose b_k, in other words if for every $x \in A_{k-1}$ the set $A_k = \{n \in A_{k-1} : c(\{x,n\}) = 1\}$ has finite cardinality, then one easily checks that for some infinite set $T \subset A_{k-1}$, $c(B) = 2$ for all 2-subsets of T, completing the proof of this case.

Exercise 2.1. Use induction on r to establish **R1** for $k = 2$, $r \in \mathbf{N}$.

We will now show that validity of **R1** for $k = 2$ implies validity of **R1** for $k = 3$, $r = 2$.

Denote the given 2-coloring of the 3-subsets of S by c. Let $A_0 = S$. If possible, choose $b_1 \in A_0$ such that for some infinite set $A_1 \subset A_0$ we have $c(\{b_1, x, y\}) = 1$ for every $x, y \in A_1$ with $x \neq y$.

Exercise 2.2. If the above is not possible then for any $t \in A_0$ we may (using the $k = 2$ case of **R1**) find an infinite set $X \subset A_0$ such that $c(\{t, x, y\}) = 2$ for every $x, y \in A_1$ with $x \neq y$.

Assuming that $b_1, \cdots, b_{k-1}$ and $A_0 \supset A_1 \supset \cdots \supset A_{k-1}$ have been chosen, choose (if possible) $b_k \in A_{k-1}$ and infinite $A_k \subset A_{k-1}$ such that $c(\{b_k, x, y\}) = 1$ for every $x, y \in A_k$ with $x \neq y$. Continue choosing in this manner as long as possible.

If this process fails to terminate (in other words, if for all k it is possible to choose such b_k and A_k) then $T = \{b_1, b_2, \cdots\}$ clearly has the property that $c(B) = 1$ for every 3-subset of T. If, on the other hand, for some k it is impossible to choose b_k, in other words if for every $t \in A_{k-1}$ and for every infinite subset $X \subset A_{k-1}$ there exists $x, y \in X$ with $x \neq y$ and $c(\{t, x, y\}) = 2$, then one may easily show (basically by iterating Exercise 2.2) that for some infinite set $T \subset A_{k-1}$, $c(B) = 2$ for all 3-subsets of T.

Exercise 2.3. Adapt the above methods to complete (by double induction on r and k) the proof of Ramsey's theorem in general.

Infinitary theorems general imply finitary versions due to an elementary "compactness" principle. For example, a consequence of Ramsey's theorem is:

Corollary 2.1.1. For every $k, r, t \in \mathbf{N}$, there exists $N = N(k, r)$ such that if the k-subsets of $\{1, \cdots, N\}$ are r-colored then there exists a set $T \subset \{1, \cdots, N\}$ with $|T| = t$ such that the k-subsets of T are monochromatic.

Proof. Suppose not. Then for some k, t, r and every N there exists an r-coloring γ_N of the k-subsets of $\{1, \cdots, N\}$ such that for no set $T \subset \{1, \cdots, N\}$ with $|T| = t$ is the family of k-subsets of T monochromatic. Extend each function γ_N arbitrarily to an r-coloring of the k-subsets of $\mathbf{N}$. Since the space of r-colorings of the k-subsets of $\mathbf{N}$ is a compact metric space under the topology of pointwise convergence, we can choose a sequence $(N_i)_{i=1}^{\infty} \subset \mathbf{N}$ such that $\lim_{i \to \infty} \gamma_{N_1}(E) = \gamma(E)$ exists for every k-subset of $\mathbf{N}$. According to **R1**, there exists an infinite set T' such that the k-subsets of T' comprise a monochromatic family under the coloring γ. Let $T \subset T'$ with $|T| = t$. Then the k-subsets of T comprise a monochromatic family. But we can choose i such that $\gamma_{N_i}(E) = \gamma(E)$ for every k-subset E of T. This is a contradiction.

□

Next we come to Schur's theorem.

S1. ([S].) For any finite coloring of $\mathbf{N}$, there exist $x, y \in \mathbf{N}$ such that $\{x, y, x+y\}$ is monochromatic.

Proof. We will prove something somewhat stronger. A Δ-set in $\mathbf{N}$ is a set of the form $\{a_j - a_i : i, j \in \mathbf{N},\ i < j\}$, where $a_1 < a_2 < \cdots$ is an increasing sequence of natural numbers. What we will show is that for every finite coloring of a Δ-set, a monochromatic configuration $\{x, y, x+y\}$ may be found.

Suppose then that $r \in \mathbf{N}$ and that c is an r-coloring of a Δ-set $\{a_j - a_i : i, j \in \mathbf{N},\ i < j\}$. c induces an r-coloring d of the 2-subsets of $\mathbf{N}$; namely $d(\{i, j\}) = c(a_j - a_1)$ (for $i < j$). According to Ramsey's theorem, for some infinite subset $T \subset \mathbf{N}$, the 2-subsets of T are monochromatic for d. Let $l < m < n$ be taken from T. Then $\{m - l, n - m, n - l\}$ is monochromatic for c. Let $x = m - l$ and $y = n - m$.

□

Notice that, although Schur's theorem involves finite configurations, these configurations are not shift-invariant. (For example, $\{5, 12, 17\}$ is a Schur configuration, while $\{6, 13, 18\}$ is not.) Indeed, we shall see in the next section that Schur's theorem marks the first small step on the way to a full-blown substructure theorem (this result, due to Hindman, implies that for any finite coloring of $\mathbf{N}$ there exists a monochromatic IP-set) in the spirit of Ramsey's theorem.

As easily as Schur's theorem is proved, it is somewhat surprising that its generalizations are as difficult as they are. Hindman's theorem, for example, seems to be much deeper than Ramsey's theorem. However, even small steps in this direction can be somewhat daunting. For example, our method of proving Schur doesn't seem to work for configurations of the form $\{x, y, z, x+y, x+z, y+z, x+y+z\}$, even if higher orders of Ramsey's theorem are used. As a matter of fact, for the proof of the following generalized Schur theorem, we need to invoke van der Waerden's theorem.

If $\langle x_i \rangle$ is a sequence in an abelian group, let us denote by $FS(\langle x_i \rangle)$ the set of non-trivial finite sums of members of the sequence without repetition. That is, $FS(\langle x_i \rangle) = \{x_{i_1} + \cdots + x_{i_t} : t \in \mathbf{N},\ i_1 < \cdots < i_t\}$. Also we write $FS_0(\langle x_i \rangle) = FS(\langle x_i \rangle) \cup \{0\}$.

GS1. (See for example [GRS].) Let $k \in \mathbf{N}$. For any finite coloring of $\mathbf{N}$ there exist natural numbers $x_1, \cdots, x_k$ such that $FS(\{x_1, \cdots, x_k\})$ is monochromatic.

Here is a finitistic version.

GS2. Let $k, r \in \mathbf{N}$. There exists $M = M(k, r) \in \mathbf{N}$ having the property that for any r-coloring of $\{1, 2, \cdots, M\}$ there exist $x_1, \cdots, x_k$ such that $FS(\{x_1, \cdots, x_k\})$ is contained in $\{1, 2, \cdots, M\}$ and is monochromatic.

Finally, here is a set-theoretic version.

GS3. Let $k \in \mathbf{N}$. For any finite coloring of $\mathcal{F}$ there exist pairwise disjoint $\alpha_1, \cdots, \alpha_k$ such that $FU(\{\alpha_1, \cdots, \alpha_k\})$ is monochromatic.

Exercise 2.4. Show that **GS1** implies **GS2** and that **GS3** implies **GS1**.

Let us show now that **GS2** implies **GS3**.

Let $k, r \in \mathbf{N}$ and suppose an r-coloring of $\mathcal{F}$ is given. Let $M = M(k,r)$. Let $A_1 \subset \mathbf{N}$ be infinite and monochromatic. By Ramsey's theorem (with $k = 2$), there exists an infinite set $A_2 \subset A_1$ such that the family of 2-subsets of A_2 is monochromatic. Having chosen $A_1 \supset A_2 \supset A_3 \supset \cdots \supset A_{i-1}$, let $A_i \subset A_{i-1}$ be an infinite set such that the family of i-subsets of A_i is monochromatic (this requires Ramsey's theorem with $k = i$). Continue until A_M has been chosen.

Notice that for $1 \leq i \leq M$, the i-subsets of A_M are monochromatic. We therefore have an induced coloring of $\{1, \cdots, M\}$: assign to i the color of the i-subsets. By choice of M, we have for this coloring a monochromatic configuration $FS(\{x_1, \cdots, x_k\})$. Simply choose now pairwise disjoint subsets $\alpha_1, \cdots, \alpha_k$ of A_M with $|\alpha_i| = x_i$, $1 \leq i \leq k$. One easily checks that $FU(\{\alpha_1, \cdots, \alpha_k\})$ is monochromatic.

□

We now proceed to a proof of the generalized Schur theorem (see also [GRS]). First we have a lemma.

Lemma 2.1.2. Let $r, k \in \mathbf{N}$. There exists $M = M(k,r) \in \mathbf{N}$ such that for any r-coloring of $\{1, \cdots, M\}$ there exists $x_1, \cdots, x_k$ such that $FS(\{x_1, \cdots, x_k\}) \subset \{1, \cdots, M\}$ and such that for $1 \leq n_1 < n_2 < \cdots < n_t \leq k$, the color of $x_{n_1} + \cdots + x_{n_t}$ depends only on n_t.

Proof. Fix r. We use induction on k. For $k = 1$ things are clear. Suppose the lemma's conclusion is valid for k. Let $M = M(k+1, r)$ be large enough that for any r-coloring of $\{1, \cdots, M\}$ there exists an arithmetic progression of length $M(k,r) + 1$. Given now an r-coloring of $\{1, \cdots, M\}$, let $\{x_{k+1}, x_{k+1} + d, \cdots, x_{k+1} + M(r,k)d\}$ be monochromatic (with $d > 0$). By our induction hypothesis there exist $a_1, \cdots, a_k$ such that $FS(\{a_1, \cdots, a_k\}) \subset \{1, \cdots, M(k,r)\}$ and such that for $1 \leq n_1 < n_2 < \cdots < n_r \leq k$ the color of $da_{n_1} + \cdots + da_{n_t}$ depends only on n_t. Letting $x_i = da_i$, $1 \leq i \leq k$, we are done, since the set $\{x_{k+1} + u : u \in FS_0(\{x_1, \cdots, x_k\})\}$ is monochromatic as well.

□

Proof of **GS1**. Suppose that we are given an r-coloring of $\mathbf{N}$. Choose by the above lemma $y_1, y_2, \cdots, y_{(k-1)r+1} \in \mathbf{N}$ such that for $1 \leq n_1 < n_2 < \cdots < n_t \leq (k-1)r+1$, the color of $y_{n_1} + y_{n_2} + \cdots + y_{n_t}$ depends only on t. This induces an r-coloring of the set $\{n_1, n_2, \cdots, n_{(k-1)r+1}\}$ for which, by the pigeonhole principle, there exists a k-element monochromatic set $\{m_1, m_2, \cdots, m_k\}$. Letting $x_i = y_{n_i}$, $1 \leq i \leq k$, one easily checks that $FS(\{x_1, \cdots, x_k\})$ is monochromatic.

□

2.2 Hindman's theorem.

Hindman's theorem, which may be viewed as an infinitary version of the generalized Schur theorem, is the archetypical Ramsey-theoretic result dealing with monochromatic substructures.

As usual, we denote by $\mathcal{F}$ the family of all finite subsets of $\mathbf{N}$, and we let $\mathcal{F}_\emptyset = \mathcal{F} \cup \{\emptyset\}$. $(\mathcal{F}, \cup)$ is a semigroup, while $(\mathcal{F}_\emptyset, \cup)$ is a semigroup with identity. For $\alpha, \beta \in \mathcal{F}$, we write $\alpha < \beta$ if $i < j$ for every $i \in \alpha$ and every $j \in \beta$. If $(\alpha_i)_{i=1}^\infty \subset \mathcal{F}$ with $\alpha_1 < \alpha_2 < \cdots$, then the sub-family

$$\mathcal{F}^{(1)} = \Big\{ \bigcup_{i\in\beta} \alpha_i : \beta \in \mathcal{F} \Big\} = FU\big((\alpha_i)_{i=1}^\infty\big) \tag{2.1}$$

is called an *IP-ring.*

Exercise 2.5. $(\mathcal{F}^{(1)}, \cup)$ is isomorphic as a semigroup to $(\mathcal{F}, \cup)$.

Here now is Hindman's theorem. (See [HS, Corollary 5.17].)

H1. ([H1].) Let $\mathcal{F}^{(1)}$ be an IP-ring. For any finite coloring of $\mathcal{F}^{(1)}$, there exists a monochromatic IP-ring $\mathcal{F}^{(2)} \subset \mathcal{F}^{(1)}$.

Here is another formulation. Recall that an IP-set in $\mathbf{N}$ is an $\mathcal{F}$-sequence $(n_\alpha)_{\alpha\in\mathcal{F}}$ having the property that $n_{\alpha\cup\beta} = n_\alpha + n_\beta$ when $\alpha \cap \beta = \emptyset$.

H2. (See [HS, Corollary 5.10].) For any finite coloring of $\mathbf{N}$, there exists a monochromatic IP-set.

H1⇒H2: Suppose that a partition $\mathbf{N} = \bigcup_{i=1}^r C_i$ is given. $\mathbf{N}$ is equal to the IP-set generated by the powers of 2, namely $(m_\alpha)_{\alpha\in\mathcal{F}}$, where $m_\alpha = \sum_{i\in\alpha} 2^{i-1}$. Construct a partition $\mathcal{F} = \bigcup_{i=1}^r D_i$ by the rule $\alpha \in D_i$ if and only if $m_\alpha \in C_i$. According to **H1** there exists j and an IP-ring $\mathcal{F}^{(1)}$ such that $\mathcal{F}^{(1)} \subset D_j$, which implies that $(m_\alpha)_{\alpha\in\mathcal{F}^{(1)}} \subset C_j$.

□

The converse is somewhat more daunting. Since we will actually be proving **H1**, however, we leave it as an exercise.

Exercise 2.6. Show that **H2⇒H1**.

Proofs of Hindman's theorem abound. Hindman's original proof (see [H1]), elementary though difficult, was greatly simplified by Baumgartner ([Ba]; see also [H2]). The proof we shall give is more in the spirit of proofs given by Glazer (see also [H2]) and by Furstenberg (see [F2]). Indeed, it is perhaps closest to a proof of Furstenberg and Katznelson (see [FK3]).

A *compact left topological semigroup* is a semigroup S endowed with a topology with respect to which it is a compact Hausdorff space with respect to which the map $t \to ts$ is continuous for all $s \in S$. (Warning: some authors say "right" instead of "left". Notice in any event the asymmetry of the condition.) Recall that an element $t \in S$ is called an *idempotent* if $t^2 = t$.

Example. If X is a compact Hausdorf space then according to Tychonoff's theorem X^X is compact in the product topology, and, as is easily verified, Hausdorf. We claim that, in fact, X^X is a compact left topological semigroup under the operation of composition. To see this, let $g, f \in X^X$ and suppose that U is an open neighborhood of fg, where $fg(x) = f\big(g(x)\big)$. We claim that the map $k \to kg$ is continuous as f; that is, there exists an open neighborhood V of f such that $Vg \subset U$. Indeed, U contains an open subset containing fg which has the form $\{h \in X^X : h(x_i) \in U_i,\ 1 \le i \le t\}$, where $x_1, \cdots, x_t \in X$ and $U_1, \cdots, U_t$ are open subsets of X with $f(g(x_i)) \in U_i$. Let V be the open set in X^X

$$V = \big\{h \in X^X : h\big(g(x_i)\big) \in U_i,\ 1 \le i \le t\big\}.$$

Clearly $f \in V$, and $Vg \subset U$, establishing that X^X is a compact left topological semigroup.

Exercise 2.7. Let X^X be a compact Hausdorf space and let $f \in X^X$. Show that the map $g \to fg$ is continuous if and only if f is continuous.

Lemma 2.2.1. (Ellis; see [E].) Any compact left topological semigroup S possesses an idempotent.

Proof. Let $\mathcal{M}$ denote the family of non-empty closed subsets $P \subset S$ for which $P^2 \subset P$.

Exercise 2.8. Show using Zorn's Lemma that $\mathcal{M}$ contains a minimal element P with respect to inclusion.

Let $p \in P$. Then $Pp \subset P$ is compact (being the continuous image of a compact set), non-empty, and moreover $(Pp)^2 \subset P$, hence $Pp = P$. In particular the set $Q = \{q \in P : qp = p\} \subset P$ is non-empty and, being the continuous inverse image of a singleton, closed. Furthermore $Q^2 \subset Q$, so that $Q = P$. That is, $qp = p$ for all $q \in P$. In particular, $p^2 = p$.

□

The context in which we shall use the lemma of Ellis is the following: Suppose $r \in \mathbf{N}$ and let $X = \{1, \cdots, r\}^{\mathcal{F}_\emptyset}$ be the space of all r-colorings of $\mathcal{F}_\emptyset$.

Exercise 2.9. X is a compact metric space with metric

$$\rho(\gamma, \xi) = \frac{1}{1 + \max\{n : \gamma(\alpha) = \xi(\alpha) \text{ for all } \alpha \subset \{1, \cdots, n\}\}}.$$

The set X^X of *all* self-maps of X (continuous or not) is a compact space with the product topology, which is generated by sets having the form $\{T \in X^X : Tx \in U\}$ where $x \in X$ and $U \subset X$ is open. We may embed $\mathcal{F}$ in X^X as follows: for $\alpha, \beta \in \mathcal{F}$ and $x \in X$, put $T_\alpha x(\beta) = x(\alpha \cup \beta)$.

Exercise 2.10. $\{T_\alpha\}_{\alpha \in \mathcal{F}}$ is an $\mathcal{F}$-action by continuous self-maps of X.

If $A \subset \mathcal{F}_\emptyset$, denote by $\overline{\{T_\alpha\}_A}$ the closure (in X^X) of the set $\{T_\alpha : \alpha \in A\}$.

Exercise 2.11. Let $(\alpha)_{i=1}^{\infty} \subset \mathcal{F}$ with $\alpha_1 < \alpha_2 < \cdots$. Then $\bigcap_{k=1}^{\infty} \overline{\{T_\alpha\}_{FU[(\alpha_i)_{i=k}^{\infty}]}}$ is a (non-empty) compact left topological semigroup.

Proof of H1. Suppose $r \in \mathbf{N}$, c is an r-coloring of $\mathcal{F}$, and $\mathcal{F}^{(1)}$ is an IP-ring. By Exercise 2.11 the set $S = \bigcap_{k=1}^{\infty} \overline{\{T_\alpha\}_{FU[(\alpha_i)_{i=k}^{\infty}]}}$ is a compact left topological semigroup and hence by **E** contains an idempotent θ.

Choose $\beta_1 \in \mathcal{F}^{(1)}$ which approximates θ to the extent that

$$\begin{aligned} c(\beta_1) &= T_{\beta_1} c(\emptyset) = \theta c(\emptyset) \\ \text{and } \theta c(\beta_1) &= T_{\beta_1} \theta c(\emptyset) = \theta^2 c(\emptyset) = \theta c(\emptyset). \end{aligned}$$

Having chosen β_1, choose $\beta_2 \in \mathcal{F}^{(1)}$ with $\beta_1 < \beta_2$ (recall that $\theta \in S$) and such that

$$\begin{aligned} c(\beta_2) &= T_{\beta_2} c(\emptyset) = \theta c(\emptyset), \\ c(\beta_1 \cup \beta_2) &= T_{\beta_2} c(\beta_1) = \theta c(\beta_1) = \theta c(\emptyset), \\ \theta c(\beta_2) &= T_{\beta_2} \theta c(\emptyset) = \theta^2 c(\emptyset) = \theta c(\emptyset), \\ \text{and } \theta c(\beta_1 \cup \beta_2) &= T_{\beta_2} \theta c(\beta_1) = \theta^2 c(\beta_1) = \theta c(\emptyset). \end{aligned}$$

Hence

$$\theta c(\gamma) = c(\gamma) = \theta c(\emptyset) \tag{2.2}$$

for all $\gamma \in \{\beta_1, \beta_2, \beta_1 \cup \beta_2\}$.

Having chosen $\beta_1 < \beta_2 < \cdots < \beta_{k-1}$ from $\mathcal{F}^{(1)}$ such that (2.2) holds for all $\gamma \in FU\big((\beta_i)_{i=1}^{k-1}\big)$, use the fact that $\theta \in S$ to select $\beta_k \in \mathcal{F}^{(1)}$ with $\beta_{k-1} < \beta_k$ and such that

$$\begin{aligned} c(\beta_k \cup \gamma) &= T_{\beta_k} c(\gamma) = \theta c(\gamma) = \theta c(\emptyset) \\ \text{and } \theta c(\beta_k \cup \gamma) &= T_{\beta_k} \theta c(\gamma) = \theta^2 c(\gamma) = \theta c(\gamma) = \theta c(\emptyset) \end{aligned}$$

for all $\gamma \in FU_\emptyset\big((\beta_i)_{i=1}^{k-1}\big)$. Then (2.2) holds for all $\gamma \in FU\big((\beta_i)_{i=1}^{k}\big)$. Continuing in this fashion, the resulting IP-ring $\mathcal{F}^{(2)} = FU\big((\beta_i)_{i=1}^{\infty}\big)$ is monochromatic.

□

Exercise 2.12. If Γ is an IP-set in $\mathbf{N}$ and $\Gamma = \bigcup_{i=1}^{r} C_i$, then one of the cells C_i contains an IP-set. Conclude that if $k \in \mathbf{N}$ then there exists an IP-set $\Gamma' \subset (k\mathbf{N} \cap \Gamma)$.

When combined with the IP or VIP van der Waerden theorem, Hindman's theorem yields some interesting applications.

Theorem 2.2.2. (cf. eg. [HS, p. 307].) Let $p_1(x), \cdots, p_k(x) \in \mathbf{Z}[x]$ each have zero constant term. For any finite coloring of $\mathbf{N}$ there exists a monochromatic configuration of the form $\{n, a, a + p_1(n), a + p_2(n), \cdots, a + p_k(n)\}$.

The proof of this theorem will be outlined in a series of exercises. We introduce some new ideas in preparation for this.

Let G be an abelian semigroup. A subset $E \subset G$ is said to be *thick* if it intersects every syndetic set non-trivially (the definition of syndetic appears right before Exercise 1.22). E is said to be *piecewise syndetic* if it is the intersection of a thick set and a syndetic set.

Exercise 2.13. A set $E \subset G$ is thick if and only if for every finite set F there exists $g \in G$ such that $gF = \{gf : f \in F\} \subset E$.

Exercise 2.14. A subset of $\mathbf{N}$ (or $\mathbf{Z}$) is thick if and only if it contains arbitrarily long intervals. Every thick set in $\mathbf{N}$ contains an IP-set.

Exercise 2.15. Show that if B is piecewise syndetic and $B = B_1 \cup B_2 \cup \cdots \cup B_k$ then one of the B_i's is piecewise syndetic by completing the steps in the following argument:

(a) Having established this for $k = 2$, it follows for general k by induction.
(b) Let $X = \{0,1\}^{\mathbf{N}}$ and let T be the shift. If B is piecewise syndetic then $\overline{\{T^n 1_B : n \in \mathbf{N}\}}$ contains some 1_E, where E is syndetic.
(c) Suppose B is piecewise syndetic and $B = B_1 \cup B_2$. Let $Y = \{0,1,2\}^{\mathbf{N}}$ and let S be the shift. Define $\gamma \in Y$ by $\gamma(n) = 0$ if $n \in B^c$, $\gamma(n) = 1$ if $n \in B_1$, $\gamma(n) = 2$ if $n \in (B_2 \setminus B_1)$. Use part (b) to show that there exists $\xi \in Y$ such that $\{n : \xi(n) > 0\}$ is syndetic.
(d) Let ψ be a uniformly recurrent point in $\overline{\{S^n \xi : n \in \mathbf{N}\}}$ (which exists by Exercise 1.22). Then $\psi(n) = j > 0$ for some $n \in \mathbf{N}$. Hence $\{n : \psi(n) = j\}$ is syndetic and B_j is piecewise syndetic.

Exercise 2.16. Let $p_i(x) \in \mathbf{Z}[x]$ with $p_i(0) = 0$, $1 \le i \le k$, and let $\Gamma \subset \mathbf{N}$ be an IP-set. If $B \subset \mathbf{N}$ is piecewise syndetic then there exists a configuration of the form $\{a, a+p_1(n), a+p_2(n), \cdots, a+p_k(n)\} \subset B$, where $n \in \Gamma$. (*Hint: show it first for syndetic sets by using Exercise 1.57 and* **VIPvdW3**, *then apply part (b) from the previous exercise.*)

Proof of Theorem 2.2.2. Let $k \in \mathbf{N}$ and suppose that $\mathbf{N} = \bigcup_{i=1}^r C_i$. Perhaps renumbering the cells, we may assume that for some $t \le r$, $C_1, \cdots, C_t$ are piecewise syndetic and $C_{t+1}, \cdots, C_r$ are not piecewise syndetic. Then by Exercise 2.15, $(C_{t+1} \cup \cdots \cup C_r)$ fails to be piecewise syndetic. In particular, $(C_1 \cup \cdots \cup C_t)$ is thick, so contains an IP-set Γ (by Exercise 2.14). Hence by Exercise 2.12, some cell C_i contains an IP-set Γ', where $1 \le i \le t$. But C_i is also piecewise syndetic, so by Exercise 2.16 there exists a configuration of the form $\{a, a+p_1(n), a+p_2(n), \cdots, a+p_k(n)\} \subset C_i$, where $n \in \Gamma' \subset C_i$.

□

The phenomenon in evidence in Exercise 2.16 above may be more generally formulated. Let us call a family $\mathcal{A}$ of subsets of $\mathbf{N}$ *partition regular* if for any finite partition of $\mathbf{N}$, some cell of the partition contains a member of $\mathcal{A}$.

Exercise 2.17. If $\mathcal{A}$ is a shift-invariant family of finite subsets of $\mathbf{N}$ then $\mathcal{A}$ is partition regular if and only if every piecewise syndetic set contains a member of $\mathcal{A}$.

This characterization does not extend to families whose members have infinite cardinality, as is demonstrated in the following exercise:

Exercise 2.18. (a) There exists a sequence of intervals $(I_n)_{n\in\mathbf{N}}$ in $\mathbf{N}$ with $|I_n| = n$ and such that $((I_n - I_m) \cap (I_t - I_s)) \neq \emptyset$ if and only if $n = t$ and $m = s$. (b) Put $E = \bigcup_{n\in\mathbf{N}} I_{2n}$ and $F = \bigcup_{n\in\mathbf{N}} I_{2n-1}$. Let $\mathcal{A}$ be the family of shifted subsets of E having infinite cardinality. Then $\mathcal{A}$ is a partition regular family none of whose members are contained in the thick (and hence piecewise syndetic) set F.

However, we do have this:

Exercise 2.19. Every piecewise syndetic set contains a shifted IP-set.

Hindman's theorem has important ramifications for a certain mode of convergence along $\mathcal{F}$ we shall define presently. Suppose that $\{x_\alpha\}_{\alpha\in\mathcal{F}}$ is an $\mathcal{F}$-sequence in a topological space and $\mathcal{F}^{(1)}$ is an IP-ring. We write

$$\text{IP-}\lim_{\alpha\in\mathcal{F}^{(1)}} x_\alpha = z$$

if for every neighborhood U of z there exists $\beta \in \mathcal{F}$ having the property that for every $\alpha \in \mathcal{F}^{(1)}$ with $\alpha > \beta$, $x_\alpha \in U$.

An $\mathcal{F}$-sequence in a compact metric space is also called a *compact coloring* of $\mathcal{F}$. (Notice that the set of r-colorings of $\mathcal{F}$ corresponds with the set of $\mathcal{F}$-sequences in $\{1, 2, \cdots, r\}$, which may be viewed as a compact metric space. Hence the notion of compact coloring is an extension of the notion of finite coloring.)

H3. Suppose that X is a compact metric space and $\{x_\alpha\}_{\alpha\in\mathcal{F}}$ is an $\mathcal{F}$-sequence in X. Then for any IP-ring $\mathcal{F}^{(1)}$, there exists an IP-ring $\mathcal{F}^{(2)} \subset \mathcal{F}^{(1)}$ such that

$$\text{IP-}\lim_{\alpha\in\mathcal{F}^{(2)}} x_\alpha = x$$

exists.

Exercise 2.20. Show that **H3** implies **H1**.

Let us show now that **H1** implies **H3**. Recall that any compact metric space is totally bounded. Hence for every $\epsilon > 0$ there exists an $\frac{\epsilon}{2}$-net, or, equivalently, a finite covering of X by $\frac{\epsilon}{2}$-balls. Using Hindman's theorem, therefore, for any IP-ring $\mathcal{F}^{(1)}$ there exists an IP-ring $\mathcal{G} \subset \mathcal{F}^{(1)}$ having the property that the the diameter of $\{x_\alpha : \alpha \in \mathcal{G}\}$ is at most ϵ. Therefore, given $\mathcal{F}^{(1)}$ we may let $\mathcal{F}^{(1)} \supset \mathcal{G}^{(1)} \supset \mathcal{G}^{(2)} \supset \mathcal{G}^{(3)} \supset \cdots$ be a descending sequence of IP-rings such that the diameter of $\{x_\alpha : \alpha \in \mathcal{G}^{(n)}\}$ is at most $\frac{1}{n}$ for all $n \in \mathbf{N}$. Let now $\alpha_1 < \alpha_2 < \cdots$ be an increasing sequence in $\mathcal{F}$ with $\alpha_i \in \mathcal{G}^{(i)}$, $i \in \mathbf{N}$.

Exercise 2.21. Show that the IP-ring $\mathcal{F}^{(2)} = FU\big((\alpha_i)_{i=1}^{\infty}\big)$ has the required properties. (*Hint: utilize completeness of* X.)

□

Suppose that $\mathcal{F}^{(1)}$ is an IP-ring and $m \in \mathbf{N}$. We denote by $(\mathcal{F}^{(1)})^m_<$ the set of all k-tuples $(\alpha_1, \cdots, \alpha_m) \in (\mathcal{F}^{(1)})^k$ having the property $\alpha_1 < \alpha_2 < \cdots < \alpha_m$.

The following theorem, attributed independently to both Milliken ([Mi]) and Taylor ([T]), is an extension of Hindman's theorem. (See also [HS, Corollary 18.8].)

MT1. Suppose that $\mathcal{F}^{(1)}$ is an IP-ring and $m \in \mathbf{N}$. For any finite coloring of $(\mathcal{F}^{(1)})^m_<$, there exists an IP-ring $\mathcal{F}^{(2)} \subset \mathcal{F}^{(1)}$ such that $(\mathcal{F}^{(2)})^m_<$ is monochromatic.

Proof. The case $m = 1$ is of course just Hindman's theorem. We shall show that the conclusion holds for $m = 2$ and leave the induction step as an exercise. Let $\mathcal{F}^{(1)}$ be an IP-ring. Suppose we are given an r-coloring of $(\mathcal{F}^{(1)})^2_<$. Let $\alpha_1 \in \mathcal{F}^{(1)}$ and r-color the IP-ring $\mathcal{G}' = \{\beta \in \mathcal{F}^{(1)} : \alpha_1 < \beta\}$ according to the color of (α_1, β). By Hindman's theorem there exists an IP-ring $\mathcal{G}^{(2)} \subset \mathcal{G}'$ which is monochromatic for this coloring; that is, the color of (α_1, β) does not depend on β, $\beta \in \mathcal{G}^{(2)}$. Now let $\alpha_2 \in \mathcal{G}^{(2)}$. Let $\mathcal{G}' = \{\beta \in \mathcal{G}^{(2)} : \alpha_2 < \beta\}$.

Exercise 2.22. There exists an IP-ring $\mathcal{G}^{(3)} \subset \mathcal{G}'$ such that for any $\alpha \in FU\{\alpha_1, \alpha_2\}$, the color of (α, β) does not depend on $\beta \in \mathcal{G}^{(3)}$.

Having chosen $\mathcal{G}^{(3)}$ as in the previous exercise, let $\alpha_3 \in \mathcal{G}^{(3)}$. Continue. Having chosen $\alpha_1, \cdots, \alpha_n$ with $\alpha_1 < \cdots < \alpha_n$, and $\mathcal{F}^{(1)} \supset \mathcal{G}^{(2)} \supset \cdots \supset \mathcal{G}^{(n)}$, let $\mathcal{G}' = \{\beta \in \mathcal{G}^{(n)} : \alpha_n < \beta\}$. Similarly to the previous exercise, there exists, according to Hindman's theorem, an IP-ring $\mathcal{G}^{(n+1)} \subset \mathcal{G}'$ such that for all $\alpha \in FU\{\alpha_1, \cdots, \alpha_n\}$ the color of (α, β) does not depend on $\beta \in \mathcal{G}^{(n+1)}$ (in other words, is a function of α). Let $\alpha_{n+1} \in \mathcal{G}^{(n+1)}$. Once $(\alpha_n)_{n=1}^{\infty}$ have been chosen in this way, let $\mathcal{G}' = FU\{\alpha_1, \alpha_2, \cdots\}$. By construction, the color of (α, β), for $(\alpha, \beta) \in (\mathcal{G}')^2_<$, is a function of α alone. Therefore by Hindman's theorem there exists an IP-ring $\mathcal{F}^{(2)} \subset \mathcal{G}' \subset \mathcal{F}^{(1)}$ such that $(\mathcal{F}^{(2)})^2_<$ is monochromatic.

Exercise 2.23. Finish the proof of **MT1** by induction on m.

□

The notion of IP-convergence has a natural multi-parameter generalization. Suppose that X is a topological space, $\mathcal{F}^{(1)}$ is an IP-ring, $m \in \mathbf{N}$, and

$$\{x_{(\alpha_1, \cdots, \alpha_m)} : (\alpha_1, \cdots, \alpha_m) \in (\mathcal{F}^{(1)})^m_<\}$$

is a sequence in X indexed by $(\mathcal{F}^{(1)})^m_<$. Write

$$\operatorname*{IP-lim}_{(\alpha_1, \cdots, \alpha_m) \in (\mathcal{F}^{(1)})^m_<} x_{(\alpha_1, \cdots, \alpha_m)} = z$$

if for any neighborhood U of z, there exists $\alpha_0 \in \mathcal{F}^{(1)}$ such that for every $(\alpha_1, \cdots, \alpha_m) \in (\mathcal{F}^{(1)})^m_<$, $\alpha_1 > \alpha_0$, $x_{(\alpha_1, \cdots, \alpha_m)} \in U$.

The following version of the Milliken-Taylor theorem is the counterpart to the formulation **H3** of Hindman's theorem.

MT2. Suppose that X is a compact metric space, $m \in \mathbf{N}$, $\mathcal{F}^{(1)}$ is an IP-ring and $\{x_{(\alpha_1, \cdots, \alpha_m)} : (\alpha_1, \cdots, \alpha_m) \in (\mathcal{F}^{(1)})^m_<\}$ is a sequence in X indexed by $(\mathcal{F}^{(1)})^m_<$.

Then there exists an IP-ring $\mathcal{F}^{(2)} \subset \mathcal{F}^{(1)}$ such that

$$\operatorname*{IP-lim}_{(\alpha_1,\cdots,\alpha_m)\in(\mathcal{F}^{(2)})^m_<} x_{(\alpha_1,\cdots,\alpha_m)} = z$$

exists.

The following finitary version of the Milliken-Taylor theorem is obtainable from it by a simple compactness argument. (Alternatively, it could be proved without recourse to infinitary results by using an argument similar to that used for the generalized Schur theorem.) For any finite set $B = \{\beta_1, \cdots, \beta_N) \subset \mathcal{F}$, where $\beta_1 < \cdots < \beta_N$, and any $m \in \mathbf{N}$, let $\mathcal{F}^{(m)}(B)$ denote the family of m-tuples $(\alpha_1, \cdots, \alpha_m)$ with $\alpha_1 < \cdots < \alpha_m$, where each α_i is a union of β_j's.

Corollary 2.2.3 Let $r, m, t \in \mathbf{N}$. There exists $N = N(r, m, t)$ such that for any set $B = \{\beta_1, \cdots, \beta_N\} \subset \mathcal{F}$, where $\beta_1 < \cdots < \beta_N$, and any r-coloring of $\mathcal{F}^{(m)}(B)$, there exists a set $C = \{\gamma_1, \cdots, \gamma_t\}$, $\gamma_1 < \cdots < \gamma_t$, where each γ_i is a union of β_j's, such that $\mathcal{F}^{(m)}(C)$ is monochromatic.

Exercise 2.24. (a) Show that **MT1** and **MT2** are equivalent. (b) Prove Corollary 2.2.3. *(Hint: use the compactness principle.)*

2.3 Infinitary Hales-Jewett: the Carlson-Simpson theorem.

The Carlson-Simpson theorem ([CS]) does for the Hales-Jewett theorem what Hindman's theorem does for Schur's theorem. For $k \in \mathbf{N}$, let $\mathcal{W}_k$ denote the free semigroup on the letters $\{1, \cdots, k\}$. (In Section 1.6 we denoted this alphabet by Λ_k and wrote Λ_k^n for the set of n-letter words on this alphabet. We are changing notation somewhat because, unlike in Section 1.6, we now will utilize the semigroup structure.) Variable words in the alphabet $\{1, \cdots, k, x\}$ are defined exactly as before (namely those words in which the letter x appears), and combinatorial lines as well (namely $\{w(1), \cdots, w(k)\}$, where $w(x)$ is a variable word and $w(i)$ corresponds to the word in $\mathcal{W}_k$ which results by substituting the letter i for each occurence of the letter x in $w(x)$). If $\{w_i(x)\}_{i=1}^{\infty}$ is a sequence of variable words then the set

$$\begin{aligned}&\mathcal{W}\{w_1(x_1)w_2(x_2)\cdots\}\\ =&\{w_1(i_1)w_2(i_2)\cdots w_n(i_n) : \ n \in \mathbf{N},\ i_t \in \{1, 2, \cdots, k\},\ 1 \le t \le n\}\end{aligned}$$

will be called a *$\mathcal{W}_k$-ring.* We will generally denote $\mathcal{W}_k$-rings by the symbols $\mathcal{W}^{(1)}$, $\mathcal{W}^{(2)}$, $\mathcal{V}^{(1)}$, etc.

The following theorem, sometimes referred to as the Carlson-Simpson theorem, is actually somewhat weaker than what Carlson and Simpson proved in [CS].

WCS. Suppose that $k \in \mathbf{N}$. For any finite coloring of a $\mathcal{W}_k$-ring $\mathcal{W}^{(1)}$, there exists a monochromatic $\mathcal{W}_k$-ring $\mathcal{W}^{(2)} \subset \mathcal{W}^{(1)}$.

The proof is from [FK3] and utilizes compact left topological semigroups. Let S be a compact left topological semigroup and let $J \subset S$ be non-empty and closed. If $SJ = \{sj : s \in S, j \in J\} \subset J$ then J is said to be a *left ideal.* If $JS \subset J$ then J is said to be a *right ideal.* If J is both a left and a right ideal then we call J a *two-sided ideal.* Any (left, right or two-sided) ideal, itself being a compact left topological semigroup, contains by Ellis' theorem an idempotent. If J is a left ideal of S which is minimal among left ideals with respect to inclusion, then we call J a *minimal left ideal.*

Exercise 2.25. Use Zorn's Lemma to show that every compact left topological semigroup contains a minimal left ideal.

Exercise 2.26. For any $x \in S$, Sx is a left ideal in S, hence if J is a minimal left ideal then $Sx = J$ for all $x \in J$.

Exercise 2.27. Let S be a compact left topological semigroup and suppose $I \subset S$ is a two-sided ideal. Then I contains every minimal left ideal of S.

Proposition 2.3.1. (eg. [HS, Theorem 2.9].) Let S be a compact left topological semigroup and let $\theta \in S$ be an idempotent. The following two conditions are equivalent:

(a) θ belongs to a minimal left ideal.
(b) The only idempotent $\phi \in S$ for which $\phi\theta = \theta\phi = \phi$ is $\phi = \theta$.

Proof. (a) $\Rightarrow$ (b). By Exercise 2.26, $S\theta$ is the minimal left ideal containing θ. Suppose that ϕ is an idempotent with $\phi\theta = \theta\phi = \phi$. Then $\phi = \phi\theta \in S\theta$, so that, again by Exercise 2.26, $S\phi = S\theta$. In particular, $\theta \in S\phi$, that is, for some ψ we have $\theta = \psi\phi$. Then $\phi = \theta\phi = \psi\phi^2 = \psi\phi = \theta$.

(b) $\Rightarrow$ (a). Let $H \subset S\theta$ be a minimal left ideal and let $\psi\theta \in H$ be idempotent. Let $\phi = \theta\psi\theta$. Then $\phi^2 = (\theta\psi\theta)(\theta\psi\theta) = \theta(\psi\theta)(\psi\theta) = \theta\psi\theta = \phi$ and $\phi\theta = \theta\phi = \phi$, hence $\phi = \theta$. But $\phi \in H$, hence $\theta \in H$.

□

An idempotent that possesses property (a), and hence property (b), of the proposition above is called a *minimal* idempotent. According to Exercise 2.27, therefore, any two-sided ideal contains every minimal idempotent.

Theorem 2.3.2. (eg. [HS, Theorem 2.23].) Let S be a compact left topological semigroup and let $\theta \in S$ be a minimal idempotent. If $k \in \mathbf{N}$ and $\mathcal{G} \subset S^k$ is a semigroup containing $(\theta, \theta, \cdots, \theta)$ then $(\theta, \theta, \cdots, \theta)$ is a minimal idempotent of $\mathcal{G}$ and therefore is contained in every two-sided ideal of $\mathcal{G}$.

Proof. Any idempotent in S^k is clearly of the form $(\phi_1, \cdots, \phi_k)$, where $\phi_i \in S$ is idempotent. Suppose $(\phi_1, \cdots, \phi_k) \in \mathcal{G}$ is idempotent with

$$(\phi_1, \cdots, \phi_k)(\theta, \cdots, \theta) = (\theta, \cdots, \theta)(\phi_1, \cdots, \phi_k) = (\phi_1, \cdots, \phi_k).$$

Then $\phi_i\theta = \theta\phi_i = \phi_i$, $1 \leq i \leq k$. But $\theta \in S$ is minimal, so $\phi_i = \theta$, $1 \leq i \leq k$. In other words, $(\phi_1, \cdots, \phi_k) = (\theta, \cdots, \theta)$. Hence $(\theta, \cdots, \theta)$ has property (b) of

Proposition 2.3.1, so that $(\theta, \cdots, \theta)$ is a minimal idempotent of G and hence according to Exercise 2.27 lies in every two-sided ideal of $\mathcal{G}$.

□

Let $k \in \mathbf{N}$ and denote by $\mathcal{W}_k$ the free semigroup on the alphabet $\{1, \cdots, k\}$. Let $r \in \mathbf{N}$ and put $X = \{1, \cdots, r\}^{\mathcal{W}_k \cup \{e\}}$, where e is an identity. Recall that X^X with the product topology forms a compact left topological semigroup under composition.

Lemma 2.3.3. Let $k \in \mathbf{N}$. If $A, B \subset (X^X)^k$ and A consists of k-tuples of continuous functions then $(\overline{A})(\overline{B}) \subset \overline{AB}$.

Proof. First we show that $A\overline{B} \subset \overline{AB}$. Let $a = (a_1, \cdots, a_k) \in A$ and let $\overline{b} = (\overline{b}_1, \cdots, \overline{b}_k) \in \overline{B}$. Let $\mathcal{U}$ be an open neighborhood of $a\overline{b}$. We will find $b \in B$ such that $ab \in \mathcal{U}$. There exist points $x_1, \cdots, x_l \in X$ and a collection of open sets $\{U_{i,j} : 1 \leq i \leq k,\ 1 \leq j \leq l\}$ such that $a_i(\overline{b}_i(x_j)) \in U_{i,j}$ and such that

$$\{f = (f_1, \cdots, f_k) \in (X^X)^k : f_i(x_j) \in U_{i,j} : 1 \leq i \leq k,\ 1 \leq j \leq l\} \subset \mathcal{U}.$$

Since for all i $\overline{a}_i$ is continuous there exists for all i and j a neighborhood $V_{i,j}$ of $\overline{b}_i(x_j)$ such that $\overline{a}_i(V_{i,j}) \subset U_{i,j}$. Let

$$\mathcal{V} = \{f = (f_1, \cdots, f_k) \in (X^X)^k : f_i(x_j) \in V_{i,j} : 1 \leq i \leq k,\ 1 \leq j \leq l\}.$$

Then $\mathcal{V}$ is a neighborhood of $\overline{b}$ and therefore contains a point $b = (b_1, \cdots, b_k) \in B$. Then $\overline{a}_i(b_i(x_j)) \subset U_{i,j}$ for all i and j. In particular, $ab \in \mathcal{U}$.

Next we show that $(\overline{A})(\overline{B}) \subset \overline{(A\overline{B})}$. This will complete the proof since a consequence of what we have just shown is that $\overline{(A\overline{B})} \subset \overline{AB}$. Let $\overline{a} = (\overline{a}_1, \cdots, \overline{a}_k) \in \overline{A}$ and let $\overline{b} = (\overline{b}_1, \cdots, \overline{b}_k) \in \overline{B}$. Let $\mathcal{U}$ be an open neighborhood of $\overline{a}\overline{b}$. We will find $a \in A$ such that $a\overline{b} \in \mathcal{U}$. There exist points $x_1, \cdots, x_l \in X$ and a collection of open sets $\{U_{i,j} : 1 \leq i \leq k,\ 1 \leq j \leq l\}$ such that $\overline{a}_i(\overline{b}_i(x_j)) \in U_{i,j}$ and such that

$$\{f = (f_1, \cdots, f_k) \in (X^X)^k : f_i(x_j) \in U_{i,j} : 1 \leq i \leq k,\ 1 \leq j \leq l\} \subset \mathcal{U}.$$

Since $\overline{a} \in \overline{A}$ there exists $a = (a_1, \cdots, a_k) \in A$ such that $a_i(\overline{b}_i(x_j)) \in U_{i,j}$ for all i, j. In particular, $a\overline{b} \in \mathcal{U}$.

□

We can embed $\mathcal{W}_k$ in X^X as follows: for $w \in \mathcal{W}_k$ let $T_w \in X^X$ be defined by $T_w\gamma(v) = \gamma(vw)$, where $\gamma \in X$ and $v \in \mathcal{W}_k$.

Exercise 2.28. Show that $\{T_w\}_{w \in \mathcal{W}_k}$ is a $\mathcal{W}_k$-action by continuous self-maps of X.

Let S be the closure in X^X of $\{T_w : w \in \mathcal{W}_k\}$. That is, $S = \overline{\{T_w\}_{\mathcal{W}_k}}$.

Exercise 2.29. Use Lemma 2.3.3 to show that S is a compact left topological subsemigroup of X^X.

Proof of **WCS**. Let $r \in \mathbf{N}$. Without loss of generality we may assume that $\mathcal{W}^{(1)} = \mathcal{W}_k$. Let $\mathcal{W}_k = \bigcup_{i=1}^r C_i$ be an r-coloring of $\mathcal{W}_k$ and let $\gamma \in X$ be determined by $\gamma(w) = i$ if and only if $w \in C_i$. Let $\mathcal{I}' \subset \mathcal{W}_k^k$ be the set of all k-tuples $\big(w(1), \cdots, w(k)\big)$, where $w(x)$ is a variable word. Let $\mathcal{G}' = \mathcal{I}' \cup \{(w, \cdots, w) : w \in \mathcal{W}_k\}$.

Exercise 2.30. $\mathcal{G}'$ is a subsemigroup of $\mathcal{W}_k^k$ and $\mathcal{I}'$ is subsemigroup of $\mathcal{G}'$ satisfying $\mathcal{G}'\mathcal{I}' \subset \mathcal{I}'$ and $\mathcal{I}'\mathcal{G}' \subset \mathcal{I}'$.

For $\mathbf{w} = (w_1, \cdots, w_k) \in \mathcal{W}_k^k$, let us write $T_{\mathbf{w}} = (T_{w_1}, \cdots, T_{w_k}) \in S^k$. Let now $\mathcal{G} = \overline{\{T_{\mathbf{w}}\}_{\mathcal{G}'}} \subset S^k$ and $\mathcal{I} = \overline{\{T_{\mathbf{w}}\}_{\mathcal{I}'}} \subset \mathcal{G}$.

Exercise 2.31. Use the above Lemma 2.3.3 to show that $\mathcal{G}$ is a compact left topological semigroup containing $\{(f, f, \cdots, f) : f \in S\}$ and $\mathcal{I}$ is a two-sided ideal in $\mathcal{G}$.

We are now ready to apply Theorem 2.3.2. Namely, let θ be any minimal idempotent in S. By Theorem 2.3.2, $(\theta, \cdots, \theta)$ is a minimal idempotent in $\mathcal{G}$, therefore $\mathcal{I}$, being a two-sided ideal in $\mathcal{G}$, contains $(\theta, \cdots, \theta)$. This implies that there exists a variable word $w_1(x)$ such that $\big((w_1(1), \cdots, w_1(k)\big) \in \mathcal{I}'$ approximates $(\theta, \cdots, \theta)$ so closely that

$$\begin{aligned}
&\gamma\big(w_1(i)\big) = T_{w_1(i)}\gamma(e) = \theta\gamma(e) \text{ and} \\
&\theta\gamma\big((w_1(i)\big) = T_{w_1(i)}\theta\gamma(e) = \theta^2\gamma(e) = \theta\gamma(e),\ 1 \le i \le k.
\end{aligned}$$

Having chosen variable words $w_1(x), \cdots, w_n(x)$ such that

$$\begin{aligned}
\gamma\big(w_1(i_1) \cdots w_n(i_n)\big) &= \theta\gamma(e) \text{ and} \\
\theta\gamma\big(w_1(i_1) \cdots w_n(i_n)\big) &= \theta\gamma(e)
\end{aligned}$$

for all choices $i_1, \cdots, i_n \in \{1, \cdots, k\}$, choose a variable word $w_{n+1}(x)$ having the property that $\big(w_{n+1}(1), \cdots, w_{n+1}(k)\big) \in \mathcal{I}'$ approximates $(\theta, \cdots, \theta)$ so closely that

$$\begin{aligned}
\gamma\big(w_1(i_1) \cdots w_{n+1}(i_{n+1})\big) &= T_{w_{n+1}(i_{n+1})}\gamma\big(w_1(i_1) \cdots w_n(i_n)\big) \\
&= \theta\gamma\big(w_1(i_1) \cdots w_n(i_n)\big) = \theta\gamma(e)
\end{aligned}$$

and

$$\begin{aligned}
\theta\gamma\big(w_1(i_1) \cdots w_{n+1}(i_{n+1})\big) &= T_{w_{n+1}(i_{n+1})}\theta\gamma\big(w_1(i_1) \cdots w_n(i_n)\big) \\
&= \theta^2\gamma\big(w_1(i_1) \cdots w_n(i_n)\big) \\
&= \theta\gamma\big(w_1(i_1) \cdots w_n(i_n)\big) = \theta\gamma(e)
\end{aligned}$$

for all choices of $i_1, \cdots, i_{n+1}$ taken from $\{1, \cdots, k\}$. Continue in this fashion. Once $w_i(x)$ has been chosen for all $i \in \mathbf{N}$, let $\mathcal{W}^{(2)} = \mathcal{W}\{w_1(x_1)w_2(x_2) \cdots\}$.

□

Exercise 2.32. (a) By altering slightly the proof of **WCS**, prove the following theorem of Furstenberg and Katznelson (see [FK3]): for any finite coloring of $\mathcal{W}_k$, there exists a sequence $(w_n(x))_{n=1}^{\infty}$ of variable words such that

$$\{w_{n_1}(t_1)w_{n_2}(t_2) \cdots w_{n_l}(t_l) : n_1 < \cdots < n_l,\ t_i \in \{1, 2, \cdots, k\},\ 1 \le i \le l\}$$

is monochromatic. Is it true that for any finite coloring of a set of this form, there is a monochromatic subset of the same form?

(b) Use part (a) to derive the following case of Furstenberg's "infinitary van der Waerden" theorem: Let $k \in \mathbf{N}$ and let $(n_\alpha)_{\alpha \in \mathcal{F}}$ be any IP-set in $\mathbf{N}$. There exists an IP-ring $\mathcal{F}^{(1)}$ and an IP-set $(a_\alpha)_{\alpha \in \mathcal{F}^{(1)}}$ such that the set

$$\bigcup_{\alpha \in \mathcal{F}^{(1)}} \{a_\alpha, a_\alpha + n_\alpha, a_\alpha + 2n_\alpha, \cdots, a_\alpha + (k-1)n_\alpha\}$$

is monochromatic.

As alluded to above, **WCS** is a consequence of, though weaker than, what is proved in [CS]. According to the *actual* Carlson-Simpson theorem, which we shall call **CS**, one can find a monochromatic $\mathcal{W}_k$-ring $\mathcal{W}\big(w_1(x_1)w_2(x_2)\cdots\big)$ for which each variable word $w_i(x)$ has x as its leftmost letter, $i \geq 2$ (this cannot be guaranteed for $i = 1$). In order to prove this, one considers the set $\mathcal{J}'$ of k-tuples $\big(w(1), \cdots, w(k)\big)$, where $w(x)$ is a variable word of this restricted type. Then by Lemma 2.3.3 $\mathcal{J} = \overline{\{T_w\}_{\mathcal{J}'}}$ is a right ideal and one uses the following fact in conjunction with Theorem 2.3.2.

Fact. If $\theta^2 = \theta \in L$, where L is a minimal left ideal of a compact left topological semigroup S, and $J \subset S$ is a right ideal, then there exists an idempotent $\phi \in J$ such that $\theta\phi = \theta$.

Exercise 2.33. Prove **CS** by completing the steps in the following argument.

(a) If J is a minimal right ideal and $x \in J$ then $xJ = J$.

(b) If L is a minimal left ideal, $\theta \in L$ is idempotent, and J is a right ideal then there exists $y \in J$ such that $\theta y = \theta$. (*Hint: without loss of generality, J is a minimal right ideal. LJ is a two-sided ideal. Use part (a).*)

(c) Prove the fact above. (*Hint: use part (b), taking $\phi = y\theta y$.*)

(d) Adapt the proof of **WCS** to give **CS**.

We will now outline a strictly combinatorial proof of **WCS** which is more in the flavor of the original proof given by Carlson and Simpson. This proof is based somewhat on a proof of Hindman's theorem due to Baumgartner ([Ba]). We need to introduce some terminology and notation. If $\mathcal{W}^{(1)}$ is a $\mathcal{W}_k$-ring then we will say that $w(x_1, \cdots, x_n)$ is an n-variable word *over* $\mathcal{W}^{(1)}$ if $\{w(t_1, \cdots, t_n) \in \mathcal{W}^{(1)} : 1 \leq t_i \leq k\} \subset \mathcal{W}^{(1)}$. In this case we will write

$$w(x_1, \cdots, x_n)^{-1}\mathcal{W}^{(1)} = \{v \in \mathcal{W}_k : w(t_1, \cdots, t_n)v \in \mathcal{W}^{(1)} \text{ for all } 1 \leq t_i \leq k\}$$

and remark that $w(x_1, \cdots, x_n)^{-1}\mathcal{W}^{(1)}$ is a $\mathcal{W}_k$-ring.

Exercise 2.34. Any $\mathcal{W}_k$-ring contained in $w(x_1, \cdots, x_n)^{-1}\mathcal{W}^{(1)}$ must have the form $w(x_1, \cdots, x_n)^{-1}\mathcal{V}$, where $\mathcal{V} \subset \mathcal{W}^{(1)}$ is a $\mathcal{W}_k$-ring and $w(x_1, \cdots, x_n)$ is an n-variable word over $\mathcal{V}$.

More generally, for any $E \subset \mathcal{W}_k$ we will write

$$w(x_1, \cdots, x_n)^{-1}E = \{v \in \mathcal{W}_k : w(t_1, \cdots, t_n)v \in E \text{ for all } 1 \leq t_i \leq k\}.$$

When $w(x_1, \cdots, x_n) = w_1(x_1) \cdots w_n(x_n)$, we may write

$$w_n(x_n)^{-1} \cdots w_1(x_1)^{-1} E = \big(w(x_1, \cdots, x_n)\big)^{-1} E.$$

Let $\mathcal{W}^{(1)}$ be a $\mathcal{W}_k$-ring. If $E \subset \mathcal{W}^{(1)}$, then we say that E is *large* for $\mathcal{W}^{(1)}$ if for every $\mathcal{W}_k$-ring $\mathcal{W}^{(2)} \subset \mathcal{W}^{(1)}$, $(E \cap \mathcal{W}^{(2)}) \neq \emptyset$. Large sets have a mild finite partition property:

Lemma 2.3.4. If E is large for $\mathcal{W}^{(1)}$ and $E = \bigcup_{i=1}^{r} E_i$ then for some i and some $\mathcal{W}_k$-ring $\mathcal{W}^{(2)} \subset \mathcal{W}^{(1)}$, E_i is large for $\mathcal{W}^{(2)}$.

Proof. If E_1 is large for $\mathcal{W}^{(1)}$ we are done, otherwise there exists a $\mathcal{W}_k$-ring $\mathcal{V}^{(1)} \subset \mathcal{W}^{(1)}$ with $(\mathcal{V}^{(1)} \cap E_1) = \emptyset$. If E_2 is large for $\mathcal{V}^{(1)}$ we are done, otherwise there exists a $\mathcal{W}_k$-ring $\mathcal{V}^{(2)} \subset \mathcal{V}^{(1)}$ such that $(\mathcal{V}^{(2)} \cap E_2) = \emptyset$. Continue choosing $\mathcal{W}_k$-rings $\mathcal{V}^{(1)} \supset \mathcal{V}^{(2)} \supset \cdots$. At each stage, if E_{i+1} is large for $\mathcal{V}^{(i)}$ we are done, otherwise ensure that $(\mathcal{V}^{(i+1)} \cap E_{i+1}) = \emptyset$. This process must terminate before we get to $\mathcal{V}^{(r)}$, for otherwise $(\mathcal{V}^{(r)} \cap E) = \emptyset$, a contradiction.

□

Here is the main tool we will use for proving **WCS**. Its proof utilizes the Hales-Jewett theorem.

Proposition 2.3.5. If E is large for $\mathcal{W}^{(1)}$ then there exists a variable word $c(x)$ over $\mathcal{W}^{(1)}$ and a $\mathcal{W}_k$-ring $\mathcal{V} \subset \mathcal{W}^{(1)}$ which contains $\{c(t) : 1 \leq t \leq k\}$ and such that $c(x)^{-1}E$ is large for $c(x)^{-1}\mathcal{V}$.

Proof. Choose, if possible, a variable word $w_1(x)$ over $\mathcal{W}^{(1)}$ such that $\{w_1(t_1) : 1 \leq t_1 \leq l\} \subset E^c$. Supposing this has been done, choose if possible a variable word $w_2(x)$ over $w_1(x)^{-1}\mathcal{W}^{(1)}$ such that $\{w_1(t_1)w_2(t_2) : 1 \leq t_1, t_2 \leq k\} \subset E^c$. Continue as long as this process remains possible, that is, having chosen $w_1(x), w_2(x), \cdots, w_r(x)$, choose if possible a variable word $w_{r+1}(x)$ over $\big(w_1(x_1) \cdots w_r(x_r)\big)^{-1}\mathcal{W}^{(1)}$ such that $\big\{w_1(t_1) \cdots w_{r+1}(t_r{+}1) : 1 \leq t_i \leq k\big\} \subset E^c$. Since E is large for $\mathcal{W}^{(1)}$, for some r it must become impossible to choose $w_{r+1}(x)$ with the aforementioned properties. (Otherwise, having chosen $w_i(x)$ for all $i \in \mathbf{N}$, E would not contain any member of $\mathcal{W}\big\{w_1(x_1)w_2(x_2)\cdots\big\}$, contradicting the fact that E is large for $\mathcal{W}^{(1)}$.) This means that for every variable word $w(x)$ over $\big(w_1(x_1) \cdots w_r(x_r)\big)^{-1}\mathcal{W}^{(1)}$, there exists a choice of $t_1, \cdots, t_{r+1}$ such that $w_1(t_1) \cdots w_r(t_r)w(t_{r+1}) \in E$.

By the Hales-Jewett theorem there exists N such that for any k^{r+1}-coloring of the length N words on k letters, there exists a monochromatic combinatorial line. Choose a word $u(x_1, x_2, \cdots, x_N)$ over $\big(w_1(x_1) \cdots w_r(x_r)\big)^{-1}\mathcal{W}^{(1)}$, and set

$$B = \{u(t_1, \cdots, t_N) : 1 \leq t_i \leq k\}.$$

Suppose now that $v(x)$ is an *arbitrary* variable word over

$$\big(w_1(x_1) \cdots w_r(x_r)u(x_{r+1}, \cdots, x_{r+N})\big)^{-1}\mathcal{W}^{(1)}.$$

Induce a k^{r+1}-coloring of B according to, for $b \in B$, which values of $t_1, \cdots, t_{r+1}$ make the statement $w_1(t_1) \cdots w_r(t_r) b v(t_{r+1}) \in E$ true. (That there is such a choice of $t_1, \cdots, t_{r+1}$ is a consequence of the fact that $bv(x)$ is a variable word over $\big(w_1(x_1) \cdots w_r(x_r)\big)^{-1} \mathcal{W}^{(1)}$. If there is more than one choice making the statement true, pick any of these for the purposes of creating the coloring.) For this k^{r+1}-coloring, there exists a monochromatic combinatorial line which we may identify as $\{b(t) : 1 \leq t \leq l\}$, where $b(x)$ is a variable word. In other words, there exists some *fixed* $t_1, \cdots, t_{n+1}$ for which $\{w_1(t_1) \cdots w_r(t_r) b(i) v(t_{r+1}) : 1 \leq i \leq k\} \subset E$. Since $w_1(t_1) \cdots w_r(t_r) b(x)$ is itself a variable word, we have that for every variable word $v(x)$ over $\big(w_1(x_1) \cdots w_r(x_r) u(x_{r+1}, \cdots, x_{r+N})\big)^{-1} \mathcal{W}^{(1)}$, there exists a variable word $c(x)$ and j such that $\{c(t)v(j) : 1 \leq t \leq k\} \subset E$. In particular, the set

$$F = \big\{ w \in (w_1 \cdots w_r u)^{-1} \mathcal{W}^{(1)} : \exists\, c(x) \text{ with } \{c(t) w : 1 \leq t \leq k\} \subset E \big\} \tag{2.3}$$

is large for $\big(w_1(x_1) \cdots w_r(x_r) u(x_{r+1}, \cdots, x_{r+N})\big)^{-1} \mathcal{W}^{(1)}$.

Notice, moreover, that the variable words $c(x)$ which are being used in (2.3) come from a finite set. Indeed, they are all of the form $w_1(t_1) \cdots w_r(t_r) b(x)$, where $b(x)$ corresponds to a combinatorial line in B. Hence we may form a finite partition of F by coloring $w \in F$ according to which of these variable words $c(x)$ makes the statement $\{c(t)w : 1 \leq t \leq k\} \subset E$ true (once again, if more than one $c(x)$ will do, pick one arbitrarily). We many naturally denote the cells of this partition by $\{F_{c(x)}\}$, $c(x)$ running over the aforementioned set. By Lemma 2.3.4, some $F_{c(x)}$ is large for a sub-system; that is, there exists a fixed variable word $c(x)$ and a $\mathcal{W}_k$-ring having the form

$$\begin{gathered} \big(w_1(x_1) \cdots w_r(x_r) u(x_{r+1}, \cdots, x_{r+N})\big)^{-1} \mathcal{V} \\ \subset \big(w_1(x_1) \cdots w_r(x_r) u(x_{r+1}, \cdots, x_{r+N})\big)^{-1} \mathcal{W}^{(1)} \end{gathered}$$

such that $F_{c(x)}$ is large for $c(x)^{-1}\mathcal{V} = \big(w_1(x_1) \cdots w_r(x_r) u(x_{r+1}, \cdots, x_{r+N})\big)^{-1} \mathcal{V}$. However, by construction $F_{c(x)} \subset c(x)^{-1} E$.

□

Proof of WCS. Let $k \in \mathbf{N}$, let $\mathcal{W}^{(1)}$ be a $\mathcal{W}_k$-ring and suppose that $\mathcal{W}^{(1)}$ is finitely colored. By Lemma 2.3.4, there exists a cell of this partition, call it E, and a $\mathcal{W}_k$-ring $\mathcal{V}^{(1)} \subset \mathcal{W}^{(1)}$ such that E is large for $\mathcal{V}^{(1)}$. Therefore, by Proposition 2.3.5, there exists a variable word $c_1(x)$ and a $\mathcal{W}_k$-ring $\mathcal{V}^{(2)} \subset \mathcal{V}^{(1)}$ such that $c_1(x)$ is a variable word over $\mathcal{V}^{(2)}$, and also such that $c_1(x)^{-1}E$ is large for $c_1(x)^{-1}\mathcal{V}^{(2)}$. By another application of the proposition, there exists a variable word $c_2(x)$ and a $\mathcal{W}_k$-ring $\mathcal{V}^{(3)} \subset \mathcal{V}^{(2)}$ such that $c_2(x)$ is a variable word over $c_1(x)^{-1}\mathcal{V}^{(3)}$, and furthermore such that $c_2(x)^{-1}c_1(x)^{-1}E$ is large for $c_2(x_2)^{-1}c_1(x_1)^{-1}\mathcal{V}^{(3)}$. Continue in this fashion. Namely, having chosen variable words $c_1(x), \cdots, c_{n-1}(x)$ and $\mathcal{W}_k$-rings $\mathcal{V}^{(1)} \supset \mathcal{V}^{(2)} \supset \cdots \supset \mathcal{V}^{(n)}$, with $c_{n-1}(x)^{-1} \cdots c_1(x)^{-1}E$ large for $c_n(x_n)^{-1} \cdots c_1(x_1)^{-1}\mathcal{V}^{(n)}$, choose a variable word $c_n(x)$ and a $\mathcal{W}_k$-ring $\mathcal{V}^{(n+1)} \subset \mathcal{V}^{(n)}$ such that $c_n(x)$ is a variable word over $c_{n-1}(x_{n-1})^{-1}\cdots c_1(x_1)^{-1}\mathcal{V}^{(n+1)}$, and such that $c_n(x)^{-1}\cdots c_1(x)^{-1}E$ is large for $c_n(x_n)^{-1}\cdots c_1(x_1)^{-1}\mathcal{V}^{(n+1)}$. Having chosen c_i, $i \in \mathbf{N}$, let $\mathcal{U} = \mathcal{W}\{c_1(x_1)c_2(x_2)\cdots\}$.

Exercise 2.35. Show that for any $n \in \mathbf{N}$ and any n-variable word $u(x_1, \cdots, x_n)$ over $\mathcal{U}$, $u(x_1, \cdots, x_n)^{-1}E$ is large for $u(x_1, \cdots, x_n)^{-1}\mathcal{U}$.

Since E is large for $\mathcal{V}^{(1)}$, and $\mathcal{U} \subset \mathcal{V}^{(1)}$, some combinatorial line from $\mathcal{U}$ is contained in E. That is, there exists a variable word $v_1(x)$ over $\mathcal{U}$ such that $\{v_1(t) : 1 \le t \le k\} \subset E$. By the exercise above, $v_1(x)^{-1}E$ is large for $v_1(x)^{-1}\mathcal{U}$. Therefore there exists a variable word $v_2(x)$ over $v_1(x)^{-1}\mathcal{U}$ such that $\{v_2(t) : 1 \le t \le k\} \subset v_1(x)^{-1}E$, that is, $\{v_1(t_1)v_2(t_2) : 1 \le t_1, t_2 \le k\} \subset E$. Continue in this fashion. Namely, having chosen variable words $v_1(x), \cdots, v_{n-1}(x)$, notice that $v_{n-1}(x_{n-1})^{-1}\cdots v_1(x_1)^{-1}E$ is large for $v_{n-1}(x_{n-1})^{-1}\cdots v_1(x_1)^{-1}\mathcal{U}$, therefore we may choose a variable word $v_n(x)$ over $v_{n-1}(x_{n-1})^{-1}\cdots v_1(x_1)^{-1}\mathcal{U}$ such that $\{v_n(t) : 1 \le t \le k\} \subset v_{n-1}(x_{n-1})^{-1}\cdots v_1(x_1)^{-1}E$, that is, $\{v_1(t_1)\cdots v_n(t_n) : 1 \le t_i \le k\} \subset E$. Once $v_i(x)$ has been chosen for all $i \in \mathbf{N}$, let $\mathcal{W}^{(2)} = \mathcal{W}\{v_1(x_1)v_2(x_2)\cdots\}$.

□

2.4 Carlson's Theorem.

Let A be a finite field and let $n \in \mathbf{N}$. Then A^n is a vector space over A. A translate of a t-dimensional vector subspace of A^n is called a *t-space*. The Graham-Leeb-Rothschild theorem follows.

GLR. ([GLR].) Let $r, n, t \in \mathbf{N}$. There exists $N = N(r, n, t)$ such that for any r-coloring of the n-spaces of A^N there exists a t-space V such that the family of n-spaces contained in V is monochromatic.

We won't prove **GLR** here. We mention it because it is a well known result of Ramsey theory that is related to a result that we will prove, namely Theorem 2.4.1 below. The reader is invited to compare the two.

For $k, M \in \mathbf{N}$, let us denote the set of words of length M on the alphabet $\{1, \cdots, k\}$ by $\mathcal{W}_k(M)$. Recall that a variable word over $\mathcal{W}_k$ is a word on the alphabet $\{1, 2, \cdots, k, x\}$ in which the symbol x appears at least once, and an n-variable word, $n \in \mathbf{N}$, is a word on the alphabet $\{1, \cdots, k, x_1, \cdots, x_n\}$ in which all the x_i's occur, and such that no occurence of x_{i+1} precedes an occurence of x_i, $1 \le i \le n-1$. The symbols $x_1, \cdots, x_n$ may be replaced, when convenient by other symbols such as $y_1, \cdots, y_n$ or $z_1, \cdots, z_n$. An n-variable word $w(x_1, \cdots, x_n)$ of length M will be called a variable word *over* $\mathcal{W}_k(M)$, and the set $\{w(t_1, t_2, \cdots, t_n) : 1 \le t_i \le k, i = 1, \cdots, n\}$ will be called the space associated with w. If w is a t-variable word and v is an n-variable word and the space associated with v is contained in the space associated with w, v will be called an *n-subword* of w. Another way of seeing this is, if $w(y_1, \cdots, y_t)$ is a t-variable word then the n-variable subwords of it (in the variables $x_1, \cdots, x_n$) are of the form $w(z_1, \cdots, z_t)$, where $z_1 \cdots z_t$ is an n-variable word over $\mathcal{W}_k(t)$.

Theorem 2.4.1. Let $k, r, n, t \in \mathbf{N}$ be given. There exists $M = M(k, r, n, t)$ such that for every r-coloring of the n-variable words over $\mathcal{W}_k(M)$ there exists a t-variable word all of whose n-subwords are the same color.

Exercise 2.36. Compare Theorem 2.4.1 with **GLR**. Does either easily imply the other?

The strategy of the proof of Theorem 2.4.1 is basically to first use the Hales-Jewett theorem to reduce to a subspace of words on which the color of variable words $w(x_1, \cdots, x_n)$ depends only on the locations of the variables $x_1, \cdots, x_n$ (that is, the color of two words which agree in this respect must be the same). The proof is then completed by invoking the finitistic Milliken-Taylor theorem.

Let $L = N(r, n, t)$, as in Corollary 2.2.3. Let M_1 be large enough that for any r-coloring of the n-variable words of length $M_1 + (L - 1)$ that have no variables occuring in the first M_1 places, there exists a variable word $w(x)$ of length M_1 such that for every n-variable word $v(x_1, \cdots, x_n)$ of length $L - 1$, the color of $w(t)v(x_1, \cdots, x_n)$ does not depend on t.

Exercise 2.37. Infer from the Hales-Jewett theorem that it is possible to choose such an M_1. (*Hint: for words w, the colors of $wv(x_1, \cdots, x_n)$, as v runs over all possible n-variable words of length $L - 1$, induces a coloring of such words w (albeit with a whole lot of colors).*)

Next let M_2 be large enough that for any r-coloring of the n-variable words of length $M_1 + M_2 + (L - 2)$ that have no variables occuring in places $\{M_1 + 1 \cdots, M_1 + M_2\}$, there exists a variable word $w(x)$ of length M_2 such that the color of any n-variable word having the form $v_1 w(t) v_2$, where v_1 is of length M_1 and v_2 is of length $(L - 2)$, does not depend on $t \in \{1, \cdots, k\}$. (In other words, the colors of $v_1 w(t) v_2$, $1 \leq t \leq k$, are all the same.)

Continue in this fashion. Namely, having chosen $M_1, \cdots, M_{i-1}$, choose M_i so large that for any r-coloring of the n-variable words of length $M_1 + M_2 + \cdots + M_i + (L - i)$ that have no variables occuring in places $\{M_1 + \cdots + M_{i-1} + 1, \cdots, M_1 + \cdots + M_i\}$, there exists a variable word $w(x)$ of length M_i such that the color of any n-variable word having the form $v_1 w(t) v_2$, where v_1 is of length $M_1 + \cdots + M_{i-1}$ and v_2 is of length $(L - i)$, does not depend on $t \in \{1, \cdots, k\}$. Continue until M_L has been chosen and set $M = M(k, r, n, t) = M_1 + M_2 + \cdots + M_L$.

Suppose now we are given an r-coloring of the n-variable words over $\mathcal{W}_k(M)$. By choice of M_L there exists a variable word $w_L(x)$ such that the color of $vw_L(t)$ does not depend on $t \in \{1, \cdots, k\}$ for any n-variable word v of length $M_1 + \cdots + M_{L-1}$. By choice of M_{L-1} there exists an M_{L-1} length variable word $w_{L-1}(x)$ such that the color of $vw_{L-1}(t)w_L(y)$ does not depend on $t \in \{1, \cdots, k\}$ for any v and any $y \in \{1, 2, \cdots, k, x_n\}$ for which the resulting products $vw_{L-1}(t)w_L(y)$ are n-variable words.

Continue in this fashion until $w_1(x)$, of length M_1, has been chosen such that the color of $w_1(t)w_2(y_2) \cdots w_L(y_L)$ does not depend on $t \in \{1, \cdots, k\}$ for any n-variable word $y_2 y_3 \cdots y_L$. The given r-coloring on the set of n-variable words $\{w_1(y_1)w_2(y_2) \cdots w_L(y_L)\}$ now lifts to a coloring of the set of n-variable words

$y_1 y_2 \cdots y_L$, that is, the set of n-variable words over $\mathcal{W}_k(L)$. This coloring has the property that the color of a word $w(x_1, \cdots, x_n)$ depends only on the places held by the variables $x_1, \cdots, x_n$. Hence this coloring further lifts to a coloring on the set of n-tuples $(\alpha_1, \cdots, \alpha_n)$ of subsets of $\{1, \cdots, L\}$ satisfying $\alpha_1 < \cdots < \alpha_n$ (here α_i corresponds to the set of places in which the symbol x_i occurs in a given word). Namely, the coloring lifts to $\mathcal{F}^{(n)}(B)$, where $B = \{\{1\}, \cdots, \{L\}\}$ (see the notation used for Corollary 2.2.3). By choice of L, there exists a family $C = \{\gamma_1, \cdots, \gamma_t\}$, where $\gamma_1 < \cdots < \gamma_t$, of subsets of $\{1, \cdots, L\}$ such that $\mathcal{F}^{(n)}(C)$ is monochromatic.

Hence, for the original coloring of $\mathcal{W}_k(M)$, if we let $v(z_1, \cdots, z_t)$ be any t-variable word of length L formed by taking $y_s = z_i$ for $s \in \gamma_i$, $1 \leq i \leq t$, in $w_1(y_1)w_2(y_2) \cdots w_L(y_L)$, and taking $y_s = 1$ elsewhere, then all of the n-subwords of $v(z_1, \cdots, z_t)$ have the same color.

Exercise 2.38. Verify the assertion made in the last paragraph.

□

In the remainder of this section we prove an infinitary version of Theorem 2.4.1. It is due to T. Carlson. Recall that a $\mathcal{W}_k$-ring is a set

$$\mathcal{W}^{(1)} = \{w_1(t_1)w_2(t_2) \cdots w_n(t_n) : n \in \mathbf{N}, t_i \in \{1, 2, \cdots, k\},\ 1 \leq i \leq n\},$$

where $(w_n(x))_{n=1}^{\infty}$ is a sequence of variable words over $\mathcal{W}_k$.

C1. ([Ca].) Suppose $k \in \mathbf{N}$ and let $\mathcal{W}^{(1)}$ be a $\mathcal{W}_k$-ring. For any finite coloring of the variable words over $\mathcal{W}^{(1)}$, there exists a $\mathcal{W}_k$-ring $\mathcal{W}^{(2)} \subset \mathcal{W}^{(1)}$ such that the set of variable words over $\mathcal{W}^{(2)}$ is monochromatic.

$\mathcal{W}_k$, recall, is the free semigroup on the alphabet $\{1, \cdots, k\}$. Notice that a copy of $\mathcal{W}_{k-1}$, consisting namely of those words not containing the letter k, sits inside $\mathcal{W}_k$. We will treat the letter k as a variable and introduce the following notation. Let $\mathcal{V}_k = \mathcal{W}_k \setminus \mathcal{W}_{k-1}$. Then $\mathcal{V}_k$ consists of all words in $\mathcal{W}_k$ which contain the letter k. Suppose we have a $\mathcal{W}_{k-1}$-ring $\mathcal{W}^{(1)} = \mathcal{W}\{w_1(x_1)w_2(x_2)\cdots\}$ (that is, each $w_i(x)$ is a variable word over $\mathcal{W}_{k-1}$, i.e. these variable words do not use the letter k). Let

$$\begin{aligned}\mathcal{V}^{(1)} &= \mathcal{V}\{w_1(x_1)w_2(x_2)\cdots\} \\ &= \{w_1(t_1)\cdots w_n(t_n) : n \in \mathbf{N},\ 1 \leq t_i \leq k \text{ and } t_i = k \text{ for some } i\} \subset \mathcal{V}_k.\end{aligned}$$

We call a set $\mathcal{V}^{(1)}$ formed in this way a *$\mathcal{V}_k$-system.* Notice that $\mathcal{V}^{(1)}$ may be identified with the family of variable words over $\mathcal{W}^{(1)}$ (by treating the letter k as the variable). Hence we have the following equivalent formulation of Carlson's theorem, which we shall prove.

C2. Suppose $k \in \mathbf{N}$. For any finite coloring of a $\mathcal{V}_k$-system $\mathcal{V}^{(1)}$, there exists a monochromatic $\mathcal{V}_k$-system $\mathcal{V}^{(2)} \subset \mathcal{V}^{(1)}$.

Proof. Our proof of this theorem goes by way of the methods of the last section and is again due to Furstenberg and Katznelson ([FK3]). Let $k, r \in \mathbf{N}$. Let

$X = \{1, \cdots, r\}^{\mathcal{W}_k \cup \{e\}}$, where e is an identity for $\mathcal{W}_k$, and embed $\mathcal{W}_k$ in X^X, putting $T_w\gamma(v) = \gamma(vw)$. Let S_k (in the last section it was S) be the closure of $\{T_w : w \in \mathcal{W}_k\}$ in X^X, and let S_{k-1} be the closure of $\{T_w : w \in \mathcal{W}_{k-1}\}$. Let $\mathcal{G}' \subset \mathcal{W}_k^k$ be the span of the set

$$\{(1, \cdots, 1), (2, \cdots, 2), \cdots, (k-1, \cdots, k-1), (1, 2, \cdots, k).$$

Let $\mathcal{I}' \subset \mathcal{G}'$ consist of those members of $\mathcal{G}$ which are not of the form $(w, \cdots, w)$. (Hence the coordinates of the k-tuples in $\mathcal{I}$ form combinatorial lines whose common part comes from $\{1, \cdots, k-1\}$.)

Exercise 2.39. $\mathcal{G}'$ is a subsemigroup of $\mathcal{W}_k^k$ and $\mathcal{I}'$ is a subsemigroup of $\mathcal{G}'$ satisfying $\mathcal{G}'\mathcal{I}' \subset \mathcal{I}'$ and $\mathcal{I}'\mathcal{G}' \subset \mathcal{I}'$.

For $\mathbf{w} = (w_1, \cdots, w_k) \in \mathcal{W}_k^k$, let us again write $T_{\mathbf{w}} = (T_{w_1}, \cdots, T_{w_k}) \in S_k^k$. Let now $\mathcal{G} = \overline{\{T_{\mathbf{w}}\}_{\mathcal{G}'}} \subset S_{k-1}^{k-1} \times S_k$ and $\mathcal{I} = \overline{\{T_{\mathbf{w}}\}_{\mathcal{I}'}} \subset \mathcal{G}$.

Exercise 2.40. Use Exercise 2.39 to show that $\mathcal{G}$ is a compact left topological semigroup containing $\{(f, f, \cdots, f) : f \in S_{k-1}\}$ and $\mathcal{I}$ is a two-sided ideal in $\mathcal{G}$.

Let θ be any minimal idempotent in S_{k-1}. By Exercise 2.40, $(\theta, \cdots, \theta) \in \mathcal{G}$. Consider $\mathcal{G}(\theta, \cdots, \theta)$. Being a left ideal in $\mathcal{G}$, it contains a minimal idempotent, say $(\phi_1, \cdots, \phi_k) = (\psi_1\theta, \cdots, \psi_k\theta)$.

Exercise 2.41. Show that $\gamma = (\theta\psi_1\theta, \cdots, \theta\psi_k\theta)$ is a minimal idempotent in $\mathcal{G}$, and that moreover, $(\theta, \cdots, \theta)\gamma = \gamma(\theta, \cdots, \theta) = \gamma$. Conclude by Proposition 2.3.1 that $\gamma = (\theta, \cdots, \theta, \phi)$ for some idempotent $\phi \in S_k$.

Recall that any two-sided ideal in a compact left topological semigroup contains every minimal idempotent. Therefore $(\theta, \cdots, \theta, \phi) \in \mathcal{I}$. We will inductively construct a sequence $\{w_n(x)\}$ of variable words with common part taken from $\{1, \cdots, k-1\}$ (in other words, such that $(w_n(1), \cdots, w_n(k)) \in \mathcal{I}'$) such that for all $n \in \mathbf{N}$,

$$\begin{aligned}
\gamma(w_1(t_1) \cdots w_n(t_n)) &= \phi\gamma(e) \text{ provided } t_i = k \text{ for some } i, \\
\theta\gamma(w_1(t_1) \cdots w_n(t_n)) &= \phi\gamma(e) \text{ provided } t_i = k \text{ for some } i, \text{ and} \\
\phi\gamma(w_1(t_1) \cdots w_n(t_n)) &= \phi\gamma(e) \text{ for all choice of } t_i.
\end{aligned} \tag{2.4}$$

We begin by choosing $w_1(x)$ such that $((w_1(1), \cdots, w_1(k)) \in \mathcal{I}'$ approximates $(\theta, \cdots, \theta, \phi)$ so closely that

$$\begin{aligned}
&\gamma(w_1(k)) = T_{w_1(k)}\gamma(e) = \phi\gamma(e), \\
&\theta\gamma((w_1(k)) = T_{w_1(k)}\theta\gamma(e) = \phi\theta\gamma(e) = \phi\gamma(e), \\
&\phi\gamma((w_1(i)) = T_{w_1(i)}\phi\gamma(e) = \theta\phi\gamma(e) = \phi\gamma(e), \ 1 \le i < k, \text{ and} \\
&\phi\gamma((w_1(k)) = T_{w_1(k)}\phi\gamma(e) = \phi^2\gamma(e) = \phi\gamma(e).
\end{aligned}$$

Then (2.4) holds for $n = 1$. Having chosen variable words $w_1(x), \cdots, w_n(x)$ such that (2.4) holds, choose $w_{n+1}(x)$ such that $\big(w_{n+1}(1), \cdots, w_{n+1}(k)\big) \in \mathcal{I}'$ approximates $(\theta, \cdots, \theta, \phi)$ so closely that

$$\begin{aligned}
&\phi\gamma\big(w_1(t_1)\cdots w_n(t_n)w_{n+1}(k)\big) = T_{w_{n+1}(k)}\phi\gamma\big(w_1(t_1)\cdots w_n(t_n)\big)\\
&\qquad = \phi^2\gamma\big(w_1(t_1)\cdots w_n(t_n)\big) = \phi\gamma\big(w_1(t_1)\cdots w_n(t_n)\big) = \phi\gamma(e),\\
&\phi\gamma\big(w_1(t_1)\cdots w_n(t_n)w_{n+1}(i)\big) = T_{w_{n+1}(i)}\phi\gamma\big(w_1(t_1)\cdots w_n(t_n)\big)\\
&\qquad = \theta\phi\gamma\big(w_1(t_1)\cdots w_n(t_n)\big) = \phi\gamma\big(w_1(t_1)\cdots w_n(t_n)\big) = \phi\gamma(e), 1 \le i < k,\\
&\theta\gamma\big(w_1(t_1)\cdots w_n(t_n)w_{n+1}(k)\big) = T_{w_{n+1}(k)}\theta\gamma\big(w_1(t_1)\cdots w_n(t_n)\big)\\
&\qquad = \phi\theta\gamma\big(w_1(t_1)\cdots w_n(t_n)\big) = \phi\gamma\big(w_1(t_1)\cdots w_n(t_n)\big) = \phi\gamma(e),\\
&\theta\gamma\big(w_1(t_1)\cdots w_n(t_n)w_{n+1}(i)\big) = T_{w_{n+1}(i)}\theta\gamma\big(w_1(t_1)\cdots w_n(t_n)\big)\\
&\qquad = \theta^2\gamma\big(w_1(t_1)\cdots w_n(t_n)\big) = \theta\gamma\big(w_1(t_1)\cdots w_n(t_n)\big) = \phi\gamma(e), 1 \le i < k,\\
&\qquad\qquad\qquad \text{provided some } t_m = k,\\
&\gamma\big(w_1(t_1)\cdots w_n(t_n)w_{n+1}(k)\big) = T_{w_{n+1}(k)}\gamma\big(w_1(t_1)\cdots w_n(t_n)\big)\\
&\qquad = \phi\gamma\big(w_1(t_1)\cdots w_n(t_n)\big) = \phi\gamma(e),\\
&\gamma\big(w_1(t_1)\cdots w_n(t_n)w_{n+1}(i)\big) = T_{w_{n+1}(i)}\gamma\big(w_1(t_1)\cdots w_n(t_n)\big)\\
&\qquad = \theta\gamma\big(w_1(t_1)\cdots w_n(t_n)\big) = \phi\gamma(e),\ 1 \le i < k,\ \text{provided some } t_m = k
\end{aligned}$$

for all choices (except where noted) of $t_1, \cdots, t_{n+1}$ taken from $\{1, \cdots, k\}$. Then (2.4) holds for n replaced by $n+1$. Continue in this fashion. Once $w_n(x)$ has been chosen for all $n \in \mathbf{N}$, let $\mathcal{V}^{(2)} = \mathcal{V}\{w_1(x_1)w_2(x_2)\cdots\}$.

□

Let us now discuss a theorem, due to Furstenberg and Katznelson, which stands in relation to Carlson's theorem as the Milliken-Taylor theorem stands in relation to Hindman's theorem.

FK. ([FK3].) Let $n \in \mathbf{N}$ and let $\mathcal{W}^{(1)}$ be a $\mathcal{W}_k$-ring. For any finite coloring of the n-variable words over $\mathcal{W}^{(1)}$, there exists a $\mathcal{W}_k$-ring $\mathcal{W}^{(2)} \subset \mathcal{W}^{(1)}$ such that the set of n-variable words over $\mathcal{W}^{(2)}$ is monochromatic.

Proof. We proceed by induction on n. The case $n = 1$ is just Carlson's theorem. Suppose the result has been proved for $n-1$. Let $\mathcal{W}^{(1)}$ be an $\mathcal{W}_k$-ring and suppose we are given a finite coloring of the n-variable words over $\mathcal{W}^{(1)}$. Let $\mathcal{V}^{(1)} = \mathcal{W}^{(1)}$. Choose a variable word $w_1(x)$ over $\mathcal{V}^{(1)}$ and color the $(n-1)$-variable words $u(x_1, \cdots, x_{n-1})$ over $w_1(x)^{-1}\mathcal{V}^{(1)}$ according to the color of the n-variable word $w_1(x_1)u(x_2, \cdots, x_n)$ for the given coloring. According to the induction hypothesis, there exists a $\mathcal{W}_k$-ring $w_1(x)^{-1}\mathcal{V}^{(2)} \subset w_1(x)^{-1}\mathcal{V}^{(1)}$ such that the set of $(n-1)$-variable words over $w_1(x)^{-1}\mathcal{V}^{(2)}$ is monochromatic for this coloring. Let $w_2(x)$ be any variable word over $w_1(x)^{-1}\mathcal{V}^{(2)}$. Color the $(n-1)$-variable words $u(x_1, \cdots, x_{n-1})$ over $\big(w_1(x)w_2(x)\big)^{-1}\mathcal{V}^{(2)}$ according to the $(2k+1)$-tuple consisting of the colors of the n-variable words in the set

$$\big\{w_1(a)w_2(b)u(x_2, \cdots, x_n) : a, b \in \{1, \cdots, k, x_1\},\ \text{at least one of } a, b = x_1\big\}$$

for the given coloring. By the induction hypothesis, there exists a $\mathcal{W}_k$-ring

$$\left(w_1(x)w_2(x)\right)^{-1}\mathcal{V}^{(3)} \subset \left(w_1(x)w_2(x)\right)^{-1}\mathcal{V}^{(2)}$$

such that, the set of $(n-1)$-variable words over $\left(w_1(x)w_2(x)\right)^{-1}\mathcal{V}^{(3)}$ is monochromatic (for this induced coloring).

Continue in this fashion. Namely, select a sequence of variable words $\left(w_i(x)\right)_{i=1}^{\infty}$ and a sequence of $\mathcal{W}_k$-rings $\mathcal{V}^{(1)} \supset \mathcal{V}^{(2)} \supset \cdots$ such that for all i $w_i(x)$ is a variable word over $\left(w_1(x)\cdots w_{i-1}(x)\right)^{-1}\mathcal{V}^{(i)}$, and such that the color of any n-variable word $v(x_1)u(x_2,\cdots.x_k)$ depends only on $v(x)$, and not on $u(x_1,\cdots,x_{n-1})$. Let $\mathcal{V} = \mathcal{W}\{w_1(x_1)w_2(x_2)\cdots\}$. Then the given coloring of n-variable words restricted to $\mathcal{V}$ reduces to a coloring of the single variable words of $\mathcal{V}$. By Carlson's theorem, there exists a $\mathcal{W}_k$ ring $\mathcal{W}^{(2)} \subset \mathcal{V}$ such that the set of single variable words for this coloring is monochromatic, which implies that the set of n-variable words of $\mathcal{W}^{(2)}$ for the given coloring is monochromatic.

□

Furstenberg and Katznelson actually proved something more general. Consider the function Γ which sends words on the alphabet $\{1,2,\cdots,k,x_1,\cdots,x_n\}$ to words on the alphabet $\{x_1,\cdots,x_n\}$ and which works by throwing out the letters $\{1,2,\cdots,k\}$, shortening any runs of 2 or more of the same x_i's to a singleton and pushing everything back together. (Words which originally contain none of the x_i's therefore get sent to an empty word.) For example, $\Gamma(32x_146x_38x_39x_2x_16x_1) = x_1x_3x_2x_1$. We denote the range of this function by $\mathcal{W}_n^*$. (This range is just the set of words over the alphabet $\{x_1,\cdots,x_n\}$ which contain no two consecutive occurences of any letter.) The following is the general form of their theorem (see [FK3]).

Exercise 2.42. For any finite coloring of the words over $\{1,2,\cdots,k,x_1,\cdots,x_n\}$ there exists a sequence of variable words $(w_n(x))_{n=1}^{\infty}$ over $\mathcal{W}_k$ such that for every $v \in \mathcal{W}_n^*$, the set

$$\big\{w = w_1(i_1)w_2(i_2)\cdots w_m(i_m): \\ m \in \mathbf{N}, i_t \in \{1,\cdots,k,x_1,\cdots,x_n\}, 1 \le t \le m,\ \Gamma(w) = v\big\}$$

is monochromatic.

FK corresponds to consideration of only the single instance $v = x_1x_2\cdots x_n$. The proof, however, is practically the same.

2.5 Central sets.

In this section, G will be any countable semigroup. Let $G_e = G \cup \{e\}$, where e is an identity for G (not necessarily an element of G). For $r \in \mathbf{N}$, put $X = \{0,1\}^{G_e}$. With the product topology, X is a compact, metrizable space. We consider the space $\Omega = X^X$ with the product topology, which is a compact left topological semigroup under composition, and embed G in Ω by putting $T_g\gamma(h) = \gamma(hg)$ for $\gamma \in X$ and $h \in G$. Let

$$S = \overline{\{T_g : g \in G\}}.$$

For every $A, B \subset G$ we have $(\overline{A})(\overline{B}) \subset \overline{AB}$. In particular, S is a (compact) semigroup.

A subset $C \subset G$ is called a *central set* if there exists a minimal idempotent $\theta \in S$ such that $\theta \in \overline{\{T_g : g \in C\}}$.

Exercise 2.43. If $G = \bigcup_{i=1}^r C_i$ then one of the cells C_i must be central.

In this section we establish a very strong combinatorial property of central sets due to Hindman, Malecki and Strauss. (In [HMS] it is shown that this property suffices to characterize central sets, however we shall not prove that here.)

A subset $E \subset G$ is said to be *left syndetic* if there exists a finite set $H \subset G$ such that $G = \bigcup_{h \in H} h^{-1}E$ (where $h^{-1}E = \{x : hx \in E\}$). E is said to be *left thick* if for every finite set $H \subset G$, there exists $x \in G$ such that $Hx \subset E$. Finally, E is said to be *left piecewise syndetic* if E is the intersection of left thick with a left syndetic set.

Exercise 2.44. Show that:

a. E is left thick if and only if E intersects every left syndetic set nontrivially.

b. E is left syndetic if and only if E intersects every left thick set nontrivially.

c. In the case of abelian G, the left and right notions above coincide and agree with the definition given in Chapter 1.

Suppose a family $\mathcal{J}$ of subsets of G has the property that the intersection of the members of any finite sub-family of $\mathcal{J}$ is left thick. Then we shall say that $\mathcal{J}$ is *collectionwise left thick.* A family $\mathcal{I}$ of subsets of G that is closed under finite intersections is said to be *collectionwise left piecewise syndetic* if for each $A \in \mathcal{I}$ there exists a finite set $K_A \subset G$ such that $\{K_A^{-1}A : A \in \mathcal{I}\}$ is collectionwise left thick. A family $\mathcal{I}$ of subsets that is not closed under finite intersections is collectionwise left piecewise syndetic if the family of finite intersections of members of $\mathcal{I}$ is collectionwise left piecewise syndetic.

The following proposition indicates a connection between minimal left ideals and the collectionwise piecewise syndetic notion.

Proposition 2.5.1. Let $J \subset S$ be a minimal left ideal, let $\theta \in J$, $t \in \mathbf{N}$ and $E \subset X$ with $|E| < \infty$. If $F \subset G$ is finite then:

(a) the set $B_{E,F} = \{g : \theta\gamma(hg) = \theta\gamma(h),\ \gamma \in E,\ h \in F\}$ is left syndetic.

(b) letting $P_{E,F} = \{g : \gamma(hg) = \theta\gamma(h),\ \gamma \in E,\ h \in F\}$ for all such pairs E, F, the family $\{P_{E,F}\}$ is collectionwise left piecewise syndetic.

Proof. (a) Suppose not. That is, assume that B_F^c is left thick. Let $(E_n)_{n=1}^\infty$ be an increasing sequence of finite sets whose union is G. For each $n \in \mathbf{N}$, there exists $l_n \in G$ such that $E_n l_n \subset B_F^c$. That is, for every $m \in E_n l_n$, there exists j, $1 \le j \le t$, and $h \in F$ such that $\theta\gamma_j(hm) \ne \theta\gamma_j(h)$. Equivalently, for every $b \in E_n$, $T_{l_n}\theta\gamma_j(hb) \ne \theta\gamma_j(h)$ for some j, $1 \le j \le t$, and some $h \in F$. Let ϕ be an accumulation point in S of $\{T_{l_n} : n \in \mathbf{N}\}$. Then for every $b \in G$, $\phi\theta\gamma_j(hb) \ne \theta\gamma_j(h)$ for some j, $1 \le j \le t$, and some $h \in F$. Since J is a minimal ideal, there exists $\psi \in S$ such that $\theta = \psi\phi\theta$. Hence we many choose $b \in G$ close enough to ψ that $T_b\phi\theta\gamma_j(h) = \psi\phi\theta\gamma_j(h) = \theta\gamma(h)$, $1 \le j \le t$, $h \in F$. That is, $\phi\theta\gamma_j(hb) = \theta\gamma_j(h)$ for all j, $1 \le j \le t$, and all $h \in F$. This is a contradiction.

(b) The family $\{P_{E,F}\}$ is clearly closed under finite intersections. By part (a), $B_{E,F}$ is left syndetic for all applicable pairs E, F, so we may choose finite sets $K_{E,F} \subset G$ such that $K_{E,F}^{-1}B_{E,F} = G$. Let $T_{E,F} = K_{E,F}^{-1}P_{E,F}$. Our goal is to show that the family of sets $\{T_{E,F}\}$ is collectionwise left thick. We claim that for fixed E and F, for every finite set $H \subset G$, the set W of $k \in G$ for which $Hk \subset T_{E,F}$ satisfies $\theta \in \overline{\{T_k : k \in W\}}$. This will suffice for the proof, for the family of such sets W clearly has the finite intersection property. Namely, given H and pairs E_i, F_i, $1 \le i \le r$, we can find k such that $Hk \subset \bigcap_{i=1}^r T_{E_i,F_i}$, showing that this intersection is left thick.

Pick a finite set $B \subset B_{E,F}$ such that $H \subset K_{E,F}^{-1}B$. Let W be the set of k for which

$$\gamma(hbk) = T_k\gamma(hb) = \theta\gamma(hb) = \theta\gamma(h),\ \gamma \in E, b \in B, h \in F.$$

Clearly $\theta \in \overline{\{T_k : k \in W\}}$. Also for any $k \in W$ we have $Bk \subset P_{E,F}$, which implies that $Hk \subset K_{E,F}^{-1}Bk \subset K_{E,F}^{-1}P_{E,F} = T_{E,F}$.

□

The *FP-tree* generated by a family of sequences $\{(x_i^{(F)})_{i=1}^\infty : F \in \mathcal{J}\}$ in G consists of all finite products $\{x_{i_1}^{(F)}x_{i_2}^{(F)}\cdots x_{i_k}^{(F)} : F \in \mathcal{J}, i_1 < i_2 < \cdots < i_k\}$. If $(x_1, x_2, \cdots, x_k)$ is a finite sequence in G which occurs at the beginning of one of the generating sequences then we define the set of successors of this sequence to be

$$B_{x_1,\cdots,x_k} = \{x_{k+1}^{(F)} : F \in \mathcal{J}, x_i^{(F)} = x_i, 1 \le i \le k\}.$$

Also put $B_\emptyset = \{x_1^{(F)} : F \in \mathcal{J}\}$.

HMS. ([HMS].) Suppose that G is a semigroup and $A \subset G$ is a central set. There exists an FP-tree in A whose successor sets comprise a collectionwise left piecewise syndetic family.

Proof. Let $\theta \in \overline{\{T_g : g \in A\}}$ be a minimal idempotent. We take as our family of generating sequences every sequence $(x_i)_{i=1}^\infty \subset G$ having the property that

$$1_A(x_{i_1}x_{i_2}\cdots x_{i_k}) = \theta 1_A(x_{i_1}x_{i_2}\cdots x_{i_k}) = \theta 1_A(e) = 1,\ k \in \mathbf{N},\ i_1 < i_2 < \cdots < i_k.$$

Clearly the FP-tree generated by these sequences lies in A. Moreover, one easily checks that each successor set is of the form $P_{E,F}$ for some finite set $F \subset G$ and some finite set $E \subset X$ (in fact for the sequence $x_1, \cdots, x_k$, one takes $E =$

$\{1_A, \theta 1_A\}$ and $F = \{x_{i_1} \cdots x_{i_t} : 1 \leq i_1 < \cdots < i_t \leq k\}$). Since the family $\{P_{E,F}\}$ is collectionwise left piecewise syndetic by Proposition 2.5.1 (b), we are done.

□

We close this section with a couple of applications of **HMS**. Recall that a family $\mathcal{A}$ of subsets of a countable semigroup G is called partition regular if for any finite partition of G, some cell must contain a member of $\mathcal{A}$. If for every $A \in \mathcal{A}$ and every $g \in G$, $gA \in \mathcal{A}$, $\mathcal{A}$ will be called a *left shift invariant* family. Right shift invariant and two-sided shift invariant families are defined similarly.

Lemma 2.5.2. Let G be a countable semigroup, let A be a left piecewise syndetic subset of G, and pick a finite set $H \subset G$ such that $\bigcup_{t \in H} t^{-1}A$ is left thick. Then there exists a left syndetic subset C of S (in fact $G = \bigcup_{t \in H} t^{-1}C$) such that whenever $F \subset C$ is a finite set, there exists $x \in G$ such that $Fx \subseteq A$.

Proof. Enumerate the members of G as $G = \{g_1, g_2, \cdots\}$ and set $G_N = \{g_1, \cdots, g_N\}$. For each $N \in \mathbf{N}$, choose $x_N \in G$ such that $G_N x_N \subset \bigcup_{t \in H} t^{-1}A$, and set $A_N = \{g \in G_N : gx_N \in A\}$. Since the topology of pointwise convergence (i.e. the product topology) on the space $\{0,1\}^G$ is compact and metrizable, for a sequence $(N_k)_{k=1}^{\infty}$, $\lim_{k \to \infty} 1_{A_{N_k}}(g)$ exists for all $g \in G$. Of course the limit function must be a characteristic function. Call it 1_C. To see that $G = \bigcup_{t \in H} t^{-1}C$, let $x \in G$ be arbitrary and choose k so large that $(\{x\} \cup Hx) \subset G_{N_k}$ and so that $1_{A_{N_k}}$ agrees with 1_C on Hx. Since $x \in G_{N_k}$ and $G_{N_k} x_{N_k} \subset \bigcup_{t \in H} t^{-1}A$, we have $txx_{N_k} \in A$ for some $t \in H$. Therefore since $tx \in G_{N_k}$, $tx \in A_{N_k}$. Hence $tx \in C$.

Finally, suppose $F \subset G$ is a finite set. Choose k large enough that $F \subset A_{N_k}$. Since $A_{N_k} x_{N_k} \subset A$, $Fx_{N_k} \subset A$.

□

Exercise 2.45. Use Lemma 2.5.2 to show that if $\mathcal{A}$ is a partition regular, two-sided shift invariant family of finite subsets of a countable semigroup G then any left piecewise syndetic subset of G contains a member of $\mathcal{A}$.

We now approach a result of Furstenberg known as the "central set theorem", a special case of which states that, for any $k \in \mathbf{N}$, one may find, in any central subset of $\mathbf{N}$, an IP-set of arithmetic progressions of length k. A consequence of this is that such an IP-set of arithmetic progressions may be found in one cell for any finite partition of $\mathbf{N}$. (We have already encountered this fact as Exercise 2.32 (b), in consequence of the Carlson-Simpson theorem.) This may be seen as a joint extension of van der Waerden's theorem and Hindman's theorem (which corresponds to the case $k = 1$).

To be precise, let $C \subset \mathbf{N}$ be a central set. Let us show that there exist IP-sets $(a_\alpha)_{\alpha \in \mathcal{F}}$ and $(n_\alpha)_{\alpha \in \mathcal{F}}$ in $\mathbf{N}$ such that for every $\alpha \in \mathcal{F}$,

$$\{a_\alpha, a_\alpha + n_\alpha, a_\alpha + 2n_\alpha, \cdots, a_\alpha + (k-1)n_\alpha\} \subset C. \tag{2.5}$$

By **HMS**, C contains an FP-tree, say generated by the family of sequences $\{(x_i^{(F)})_{i=1}^{\infty} : F \in \mathcal{J}\}$, whose successor sets comprise a collectionwise left piecewise syndetic family. Let $\mathcal{A} = \{\{a, a+n, \cdots, a+(k-1)n\} : a, n \in \mathbf{N}\}$. By van der Waerden's theorem $\mathcal{A}$ is a partition regular family, and of course $\mathcal{A}$ is shift invariant. Therefore by Exercise 2.45 every piecewise syndetic set, and hence every finite intersection of successor sets, contains an arithmetic progression of length k.

Let $A_1 = \{a_1, a_1 + n_1, \cdots, a_1 + (k-1)n_1\} \subset B_\emptyset$. (In particular, this progression lies in C.) Let

$$A_2 = \{a_2, a_2 + n_2, \cdots, a_2 + (k-1)n_2\} \subset \bigcap_{x_1 \in A_1} B_{x_1}.$$

Having chosen $a_1, \cdots, a_i$ and $n_1, \cdots, n_i$, with $A_i = \{a_i, a_i + n_i, \cdots, a_i + (k-1)n_i\}$, let

$$A_{i+1} = \{a_{i+1}, a_{i+1} + n_{i+1}, \cdots, a_{i+1} + (k-1)n_{i+1}\} \subset \bigcap_{x_j \in A_j, 1 \le j \le i} B_{x_1, \cdots, x_i}.$$

Continue choosing in this fashion, and set $a_\alpha = \sum_{i \in \alpha} a_i$, $n_\alpha = \sum_{i \in \alpha} n_i$, $\alpha \in \mathcal{F}$.

Exercise 2.46. Verify that for every $\alpha \in \mathcal{F}$, (2.5) holds.

We now formulate Furstenberg's central set theorem in general, leaving the proof as an exercise.

CST. Let $(n_\alpha^{(i)})_{\alpha \in \mathcal{F}}$ be IP-sets in $\mathbf{Z}$, $1 \le i \le m$. If $C \subset \mathbf{N}$ is a central set then there exists an IP-set $(a_\alpha)_{\alpha \in \mathcal{F}}$ in $\mathbf{N}$ and an IP-ring $\mathcal{F}^{(1)}$ such that

$$\left\{a_\alpha, a_\alpha + n_\alpha^{(1)}, \cdots, a_\alpha + n_\alpha^{(m)}\right\} \subset C \text{ for all } \alpha \in \mathcal{F}^{(1)}.$$

Exercise 2.47. Prove **CST.**

The above central set theorem admits the following polynomial generalization.

Theorem 2.5.3. ([M2], [HM].) Let G be an additive abelian semigroup and let

$$(v_\alpha)_{\alpha \in \mathcal{F}}, (w_\alpha)_{\alpha \in \mathcal{F}}, \cdots, (z_\alpha)_{\alpha \in \mathcal{F}}$$

be VIP-systems in G. For any $r \in \mathbf{N}$ and any r-coloring $G = \bigcup_{i=1}^r C_i$ there exists j, $1 \le j \le r$, an IP-ring $\mathcal{F}^{(1)}$, and an IP-set $(a_\alpha)_{\alpha \in \mathcal{F}^{(1)}}$ such that for all $\alpha \in \mathcal{F}^{(1)}$ we have

$$\{a_\alpha + v_\alpha, a_\alpha + w_\alpha, \cdots, a_\alpha + z_\alpha\} \subset C_j.$$

Rather than prove this result in general we will do a special case and leave the rest as an exercise. Let $(n_\alpha)_{\alpha \in \mathcal{F}}$ be an IP-set in $\mathbf{Z}$, and suppose $C \subset \mathbf{N}$ is a central set. We will find an IP-ring $\mathcal{F}^{(1)}$ and an IP-set $(a_\alpha)_{\alpha \in \mathcal{F}^{(1)}} \subset \mathbf{N}$ such that for all $\alpha \in \mathcal{F}^{(1)}$, $\{a_\alpha, a_\alpha + n_\alpha^2\} \subset C$.

By **HMS**, C contains an FP-tree, say generated by the family of sequences $\{(x_i^{(F)})_{i=1}^\infty : F \in \mathcal{J}\}$, whose successor sets comprise a collectionwise left piecewise syndetic family. Let $\mathcal{A}_1 = \{\{a, a+n_\alpha^2\} : a \in \mathbf{N}, \alpha \in \mathcal{F}\}$. $\mathcal{A}_1$ is a shift invariant partition regular family of subsets of $\mathbf{N}$ by VIPvdW3 and Exercise 1.57. Therefore by Exercise 2.45 every piecewise syndetic set, and hence every finite intersection of successor sets, contains a member of $\mathcal{A}_1$.

Choose a_1 and α_1 such that $A_1 = \{a_1, a_1 + n_{\alpha_1}^2\} \subset B_\emptyset$. Let

$$\mathcal{A}_2 = \{\{a, a+n_\alpha^2, a+n_\alpha^2 + 2n_{\alpha_1}n_\alpha\} : a \in \mathbf{N},\ \alpha > \alpha_1\}.$$

Then $\mathcal{A}_2$ is again partition regular. Choose a_2 and $\alpha_2 > \alpha_1$ such that

$$A_2 = \{a_2, a_2 + n_{\alpha_2}^2, a_2 + n_{\alpha_2}^2 + 2n_{\alpha_1}n_{\alpha_2}\} \subset \bigcap_{x_1 \in A_1} B_{x_1}.$$

One may now check that $\{a_\alpha, a_\alpha + n_\alpha^2\} \subset C$ for all $\emptyset \neq \alpha \subset \{1,2\}$ (we are taking, here and in what follows, $a_\alpha = \sum_{i\in\alpha} a_i$ and similarly for n_α).

Having chosen $a_1, \cdots, a_i$ and $\alpha_1 < \alpha_2 < \cdots < \alpha_i$, with

$$A_j = \{a_j, a_j + n_{\alpha_j}^2\} \cup \{a_j + n_{\alpha_j}^2 + 2n_\gamma n_{\alpha_j} :\ \gamma \in FU(\alpha_1, \cdots, \alpha_i)\},$$

let

$$\mathcal{A}_{i+1} = \{\{a, a+n_\alpha^2\} \cup \{a + n_\alpha^2 + 2n_\gamma n_\alpha : \gamma \in FU(\alpha_1, \cdots, \alpha_i)\} :\ a \in G,\ \alpha > \alpha_i\}.$$

Again, $\mathcal{A}_{i+1}$ is a partition regular family. Choose a_{i+1} and $\alpha_{i+1} > \alpha_i$ such that

$$\begin{aligned} &A_{i+1} \\ =&\{a_{i+1}, a_{i+1} + n_{\alpha_{i+1}}^2\} \cup \{a_{i+1} + n_{\alpha_{i+1}}^2 + 2n_\gamma n_{\alpha_{i+1}} :\ \gamma \in FU(\alpha_1, \cdots, \alpha_i)\} \\ \subset &\bigcap_{x_j \in A_j, 1\le j\le i} B_{x_1, \cdots, x_i}. \end{aligned}$$

Continue choosing in this fashion and let $\mathcal{F}^{(1)}$ be the IP-ring generated by the sequence $(\alpha_i)_{i=1}^\infty$. Let $\alpha \in \mathcal{F}^{(1)}$, say $\alpha = \alpha_{i_1} \cup \cdots \cup \alpha_{i_t}$, where $i_1 < i_2 < \cdots < i_t$. Then the sequence $(a_1, a_2, \cdots, a_{i_t})$ is a starter sequence, and a_α is a finite sum from this sequence, so $a_\alpha \in C$. Moreover, there are starter sequences containing

$$\begin{aligned} \{&a_{i_1} + n_{\alpha_{i_1}}^2, a_{i_2} + n_{\alpha_{i_2}}^2 + 2n_{\alpha_{i_1}}n_{\alpha_{i_2}}, a_{i_3} + n_{\alpha_{i_3}}^2 + 2n_{\alpha_{i_1}\cup\alpha_{i_2}}n_{\alpha_{i_3}}, \cdots \\ &\cdots, a_{i_t} + n_{\alpha_{i_t}}^2 + 2n_{\alpha_{i_1}\cup\alpha_{i_2}\cdots\cup\alpha_{i_{t-1}}}n_{\alpha_{i_t}}\}. \end{aligned}$$

The sum of the elements in this set is $a_\alpha + n_\alpha^2$.

□

Exercise 2.48. Prove Theorem 2.5.3 in general.

2.6 An infinitary polynomial Hales-Jewett theorem.

In this section we shall prove an infinitary version of **BL** (see Section 1.8), from [M3]. This result will also be a "set-polynomialization" of the (weak) Carlson-Simpson theorem **WCS** from Section 2.3. Our plan is as follows. First we prove such a theorem, but for ease of notation we restrict to two dimensions. Then we state a version for arbitrary dimension, leaving the analogous proof to the exercises.

Let $\mathcal{G}_1 = \mathcal{F}(\mathbf{N} \times \mathbf{N})$ be the set of all non-empty finite subsets of $\mathbf{N} \times \mathbf{N}$. $\mathcal{G}$ is an abelian semigroup under $\cup$ (that, is, union). For $N \geq 2$, let $\mathcal{G}_N = \mathcal{F}(\mathbf{N}^2 \setminus \{1, \cdots, N-1\}^2)$ consist of those members of $\mathcal{G}_1$ that are disjoint from $\{1, \cdots, N-1\}^2$. Let $\mathcal{G}_0 = \mathcal{G} \cup \{\emptyset\}$.

We put $X = \{0,1\}^{\mathcal{G}_0}$. Then X with the product topology may be seen as a compact metric space since $\mathcal{G}_0$ is countable. With respect to the product topology, X^X with composition of functions as the operation forms a compact left topological semigroup. We embed $\mathcal{G}_0$ in X^X by putting $T_E\gamma(A) = \gamma(A \cup E)$ for $\gamma \in X$ and $A, E \in \mathcal{G}_0$. Finally we let

$$S = \bigcap_{N=1}^{\infty} \overline{\{T_E : E \in \mathcal{G}_N\}}. \tag{2.6}$$

Exercise 2.49. S is a non-empty compact left topological semigroup.

Working in the semigroup S as defined in (2.6) (rather than in simply $\overline{\{T_E : E \in \mathcal{G}_0\}}$) is one way of dealing with the non-cancellativity of $\mathcal{G}_0$. The idea is, knowing what E and $E \cup A$ are, we can only recover what A is if we know something else; specifically, if we can assume that $(E \cap A) = \emptyset$, then we will know that $A = (E \cup A) \setminus A$. Thus A is "preserved" by (and only by) "disjoint shifts". Extending this to finite configurations, a configuration $\{A_1, \cdots, A_k\} \subset \mathcal{G}_0$ is in some sense "preserved" if one shifts by a set E which is disjoint from all of the A_i's. That is, if one knows E and $\{E \cup A_1, \cdots, E \cup A_k\}$, and that E is disjoint from the A_i's, one may recover $\{A_1, \cdots, A_k\}$. The classes of configurations we deal with in this section are, indeed, closed under these disjoint shifts, but badly non-closed under arbitrary shifts. By living in the semigroup S defined above, we can, given $\phi \in S$ and any configuration $\{A_1, \cdots, A_k\} \subset \mathcal{G}_0$, always approximate ϕ by some T_E, where E is disjoint from each A_i. This is important in the sequel.

Definition 2.6.1. Suppose $\mathcal{E} \subset \mathcal{G}$. $\mathcal{E}$ is said to be *strongly syndetic* if for every $M \in \mathbf{N}$, there exists $N \in \mathbf{N}$ such that for all $E \in \mathcal{G}_{N+1}$, there exists $C \subset \{1, \cdots N\}^2 \setminus \{1, \cdots, M\}^2$ such that $E \cup C \in \mathcal{E}$. $\mathcal{E}$ is said to be *strongly piecewise syndetic* if there exists a strongly syndetic set $\mathcal{B} \subset \mathcal{G}$ such that for every finite family $\mathcal{H} \subset \mathcal{B}$ and every $N \in \mathbf{N}$ there exists $E \in \mathcal{G}_{N+1}$ such that $(E \cup F) \in \mathcal{E}$ for every $F \in \mathcal{H}$.

Suppose now that $k \in \mathbf{N}$ and $p_1(B), \cdots, p_k(B)$ are set-polynomials over $\mathbf{N}^2$ having empty constant term. Let $\mathcal{A}$ be the family of configurations

$$\mathcal{A} = \big\{\{A \cup p_1(B), A \cup p_2(B), \cdots, A \cup p_k(B)\} : \\ A \in \mathcal{G}, B \in \mathcal{F}, \big(A \cap p_i(B)\big) = \emptyset,\ 1 \le i \le k\big\}.$$

Now, according to **BL**, for any $M \ge 0$ and for any finite coloring of $\mathcal{G}_{M+1}$ there exists a monochromatic member of $\mathcal{A}$. In particular, for every finite coloring of $\mathcal{G}_0$ there exists a monochromatic configuration $\{A\cup p_1(B), A\cup p_2(B), \cdots, A\cup p_k(B)\}$ with $\big(A\cup p_i(B)\big) \in \mathcal{G}_{M+1}$, $1 \le i \le k$. We will call such families (that are partition regular in $\mathcal{G}_{M+1}$ for all M) of configurations *strongly partition regular.*

We now partially justify our interest in strongly piecewise syndetic sets by the following proposition.

Proposition 2.6.2. Let $\mathcal{E} \subset \mathcal{G}$, let $M \in \mathbf{N}$ and let $\mathcal{A}$ be any strongly partition regular family of configurations that is closed under disjoint shifts.

(a) If $\mathcal{E}$ is strongly syndetic then $\mathcal{E}$ contains a member of $\mathcal{A}$ all of whose elements belong to $\mathcal{G}_{M+1}$.

(b) If $\mathcal{E}$ is strongly piecewise syndetic then $\mathcal{E}$ contains a member of $\mathcal{A}$ all of whose elements belong to $\mathcal{G}_{M+1}$.

Proof. (a) Let $N \in \mathbf{N}$ be large enough that for every $E \in \mathcal{G}_{N+1}$ there exists $A \subset \big(\{1,\cdots,N\}^2 \setminus \{1,\cdots,M\}^2\big)$ such that $(E \cup A) \in \mathcal{E}$. Indeed, finitely color $\mathcal{G}_{N+1}$ by assigning E a color according to which A accomplishes this (there are finitely many choices for A). For this coloring, there exists a monochromatic configuration $\mathcal{H} \in \mathcal{A}$, with $\mathcal{H} \subset \mathcal{G}_{N+1}$. Monochromaticity implies that for some fixed $A \subset \{1,\cdots,N\}^2 \setminus \{1,\cdots,M\}^2$, $A \cup \mathcal{H} = \{A \cup H : H \in \mathcal{H}\} \subset \mathcal{E}$. But $\mathcal{A}$ is closed under disjoint shifts, so $(A \cup \mathcal{H}) \in \mathcal{A}$. Moreover, $(A \cup \mathcal{H}) \subset \mathcal{G}_{M+1}$, so we are done.

(b) If $\mathcal{E}$ is strongly piecewise syndetic then there exists a strongly syndetic $\mathcal{B}$ such that every finite family contained in $\mathcal{B}$ can be moved (shifting by an element arbitrarily far out) into $\mathcal{E}$. Therefore this part follows from part (a).

□

Additional justification for our interest in strongly piecewise syndetic sets comes from the following proposition.

Proposition 2.6.3. Let J be a minimal left ideal in S and let $\theta \in J$. Let $t \in \mathbf{N}$, $\gamma_1, \cdots, \gamma_t \in X$ and $l \ge 0$. Then:

(a) the set $B_l = \{E : \theta\gamma_j(E \cup A) = \theta\gamma_j(A),\ 1 \le j \le t,\ A \subset \{1,2,\cdots,l\}^2\}$ is strongly syndetic.

(b) the set $P_l = \{E : \gamma_j(E \cup A) = \theta\gamma_j(A),\ 1 \le j \le t,\ A \subset \{1,2,\cdots,l\}^2\}$ is a strongly piecewise syndetic set.

Proof. (a) Suppose B_l is not strongly syndetic. Then there exists M such that for all $N > M$ there exists a set $E_N \in \mathcal{G}_{N+1}$ such that for every $C \subset$

$\{1,2,\cdots,N\}^2 \setminus \{1,2,\cdots,M\}^2$, $(E_N \cup C) \in B_l^c$; i.e., for some $A \subset \{1,2,\cdots,l\}^2$ and $1 \le j \le t$,

$$T_{E_N}\theta\gamma_j(C \cup A) = \theta\gamma_j(E_N \cup C \cup A) \neq \theta\gamma_j(A).$$

Let ϕ be an accumulation point in X^X of $\{T_{E_N} : N > M\}$. Then $\phi \in S$. Moreover, for every $C \in \mathcal{G}_{M+1}$, $\phi\theta\gamma_j(C \cup A) \neq \theta\gamma_j(A)$ for some $A \subset \{1,2,\cdots,l\}^2$ and some j with $1 \le j \le t$.

Using Exercise 2.26, pick $\psi \in S$ such that $\psi\phi\theta = \theta$. Finally, choose $C \in \mathcal{G}_{M+1}$ such that T_C is close enough to ψ to ensure that

$$\phi\theta\gamma_j(C \cup A) = T_C\phi\theta\gamma_j(A) = \psi\phi\theta\gamma_j(A) = \theta\gamma_j(A),\ 1 \le j \le t,\ A \subset \{1,2,\cdots,l\}^2.$$

This contradiction proves part (a).

(b) Let $\mathcal{H} \subset B_l$ be a finite family and let $N \in \mathbf{N}$. Pick $E \in \mathcal{G}_{N+1}$ such that T_E is close enough to θ to ensure that

$$\begin{aligned}\gamma_j(E \cup H \cup A) &= T_E\gamma_j(H \cup A) = \theta\gamma_j(H \cup A) = \theta\gamma_j(A),\\ &H \in \mathcal{H},\ 1 \le j \le t,\ A \subset \{1,2,\cdots,l\}^2.\end{aligned}$$

Then $(E \cup H) \in P_l$ for all $H \in \mathcal{H}$, so we are done.

□

Corollary 2.6.4. For any finite partition $\mathcal{G}_0 = \bigcup_{i=1}^r C_i$, one of the cells C_i is strongly piecewise syndetic.

Proof. Let $\theta \in S$ be a member of a minimal left ideal. We have

$$\theta \in \overline{\{T_E : E \in \mathcal{G}_0\}} = \bigcup_{i=1}^r \overline{\{T_E : E \in C_i\}},$$

so that for some j, $\theta \in \overline{\{T_E : E \in C_j\}}$. Note we can choose $E \in C_j$ such that $\theta 1_{C_j}(\emptyset) = T_E 1_{C_j}(\emptyset) = 1_{C_j}(E) = 1$. We now employ Proposition 2.6.3 with $l = 0$ and $t = 1$, taking γ_1 to be 1_{C_j}. Hence the set $\{E : 1_{C_j}(E) = \theta 1_{C_j}(\emptyset)\}$ is strongly piecewise syndetic. But this set is simply C_j.

□

Finally, we have the following.

Theorem 2.6.5. Let $J \subset S$ be a minimal left ideal and suppose $\theta \in J$. Suppose $k \in \mathbf{N}$ and let $\mathcal{A}$ be any strongly partition regular family of configurations of cardinality k in $\mathcal{G}_0$ that is closed under disjoint shifts. Let $N \in \mathbf{N}$ and let $V \subset (X^X)^k$ consist of all k-tuples $(T_{A_1}, T_{A_2}, \cdots, T_{A_k})$, where $\{A_1,\cdots,A_k\} \in \mathcal{A}$ with $A_i \in \mathcal{G}_{N+1}$, $1 \le i \le k$. Then $(\theta,\cdots,\theta) \in \overline{V}$.

Proof. We must show that for all $M, t \in \mathbf{N}$ and any choice of $\gamma_1,\cdots,\gamma_t \in X$, $E_1,\cdots,E_t \in \mathcal{G}_0$, there exists $\{A_1,\cdots,A_k\} \in \mathcal{A}$ such that $A_m \in \mathcal{G}_{M+1}$, $1 \le m \le k$, and

$$T_{A_m}\gamma_j(E_j) = \theta\gamma_j(E_j),\ 1 \le m \le k,\ 1 \le j \le t.$$

Let l be large enough that $E_i \subset \{1,2,\cdots,l\}^2$, $1 \le i \le t$. By Proposition 2.6.3 the set

$$\mathcal{B}_l = \{E : \gamma_j(E \cup A) = \theta\gamma_j(A),\ 1 \le j \le t,\ A \subset \{1,2,\cdots,l\}^2\}\}$$

is a strongly piecewise syndetic set. This implies that the (perhaps larger) set

$$\mathcal{B} = \{E : \gamma_j(E \cup E_j) = \theta\gamma_j(E_j),\ 1 \le j \le t\}$$

is a strongly piecewise syndetic set as well, so that by Proposition 2.6.2, $\mathcal{B}$ contains a configuration $\{A_1, \cdots, A_k\} \in \mathcal{A}$ with $A_m \in \mathcal{G}_{N+1}$, $1 \le m \le k$. We are done.

□

Taking θ to be a minimal *idempotent* in S, we get a notion of largeness for subsets of $\mathcal{G}_0$ that will be useful for us.

Definition 2.6.6. A family $\mathcal{C} \subset \mathcal{G}_0$ is said to be *strongly central* if there exists a minimal idempotent $\theta \in S$ such that $\theta 1_{\mathcal{C}}(\emptyset) = 1$.

Proposition 2.6.7. Strongly central sets are strongly piecewise syndetic. Moreover, if $r \in \mathbf{N}$ and $\mathcal{G}_0 = \bigcup_{i=1}^r C_i$ then some cell C_j is a strongly central set.

Proof. See the proof the Corollary 2.6.4, only take θ to be a minimal *idempotent*.

□

Finally, we will see the first fruits of our labors. Suppose that $(A_i)_{i=1}^\infty \subset \mathcal{G}_1$ is a sequence of pairwise disjoint sets, and that $(B_i)_{i=1}^\infty \subset \mathcal{F}$ is a sequence of pairwise disjoint sets such that, furthermore, $(B_i \times B_j) \cap A_k = \emptyset$ for $i,j,k \in \mathbf{N}$. For aesthetic reasons, we shall also require that there exists an increasing sequence $(M_i)_{i=1}^\infty \subset \mathbf{N}$ such that

$$B_i \subset \{M_{i-1}+1, \cdots, M_i\} \text{ and } A_i \subset \{1, \cdots, M_i\}^2 \setminus \{1, \cdots, M_{i-1}\}^2. \tag{2.7}$$

For $N \in \mathbf{N}$, let $\mathcal{M}_N^{(2)}$ be the set of $N \times N$ matrices with entries coming from $\{0,1\}$. Let $\mathcal{M} = \bigcup_{N=1}^\infty \mathcal{M}_N$. For $N \in \mathbf{N}$ and $M = (m_{ij}) \in \mathcal{M}_N^{(2)}$, let

$$K(M) = (A_1 \cup A_2 \cup \cdots \cup A_N) \cup \bigcup_{m_{ij}=1} (B_i \times B_j). \tag{2.8}$$

Letting N go over all of $\mathbf{N}$, we get a function $K : \mathcal{M}^{(2)} \to \mathcal{G}_0$. We shall refer to the range of any function K which arises in this manner, which may be represented as a sequence $(C_M)_{M \in \mathcal{M}^{(2)}}$, where $C_M = K(M)$, as an $\mathcal{M}^{(2)}$*-system.* We shall now prove an infinitary theorem concerning $\mathcal{M}^{(2)}$-systems.

Theorem 2.6.8. Let $\mathcal{C} \subset \mathcal{G}_0$ be strongly central. Then $\mathcal{C}$ contains an $\mathcal{M}^{(2)}$-system.

Proof. Let $\theta \in S$ be a minimal idempotent with $\theta 1_C(\emptyset) = 1$. Consider the family of configurations

$$\mathcal{A}_1 = \big\{\{A, A \cup B \times B\},\ A \in \mathcal{G},\ B \in \mathcal{F},\ A \cap (B \times B) = \emptyset\big\}$$

in $\mathcal{G}$. $\mathcal{A}_1$ is a strongly partition regular family, as we have noted previously. Furthermore, one easily sees that $\mathcal{A}_1$ is closed under disjoint shifts. Thus, if V is the set of ordered pairs $\big\{(T_A, T_{A\cup(B\times B)}) : \{A, A\cup(B\times B)\} \in \mathcal{A}_1\big\}$ in $(X^X)^2$, by Theorem 2.6.5 we have $(\theta, \theta) \in \overline{V}$. In particular, we may select $A_1 \in \mathcal{G}_1$ and $B_1 \in \mathcal{F}$ such that $\{A_1, A_1 \cup (B_1 \times B_1)\} \in \mathcal{A}_1$ and such that T_{A_1} and $T_{A_1\cup(B_1\times B_1)}$ are close enough to θ to ensure that

$$\begin{aligned}
&1_C(A_1) = T_{A_1} 1_C(\emptyset) = \theta 1_C(\emptyset) = 1,\\
&\theta 1_C(A_1) = T_{A_1}\theta 1_C(\emptyset) = \theta^2 1_C(\emptyset) = \theta 1_C(\emptyset),\\
&1_C(A_1 \cup (B_1 \times B_1)) = T_{A_1\cup(B_1\times B_1)} 1_C(\emptyset) = \theta 1_C(\emptyset) = 1, \text{ and}\\
&\theta 1_C(A_1 \cup (B_1 \times B_1)) = T_{A_1\cup(B_1\times B_1)}\theta 1_C(\emptyset) = \theta^2 1_C(\emptyset) = \theta 1_C(\emptyset) = 1.
\end{aligned}$$

Let M_1 be the smallest integer such that A_1 and $A_1 \cup (B_1 \times B_1)$ are each contained in $\{1, \cdots, M_1\}^2$.

Let now $\mathcal{A}_2$ be the family of configurations of the form

$$\big\{A \cup p_1(B), A \cup p_2(B), \cdots, A \cup p_8(B)\big\},$$

where $p_1(B), \cdots, p_8(B)$ are the 8 set-polynomials having empty constant term which consist of the union of some subset of the three set-polynomials $\{B \times B_1, B_1 \times B, B \times B\}$. $\mathcal{A}_2$ is a strongly partition regular family of configurations closed under disjoint shifts. Hence, by Theorem 2.6.3, if we let $V \subset (X^X)^8$ consist of all 8-tuples $(T_{D_1}, \cdots, T_{D_8})$, where $\{D_1, \cdots, D_8\} \in \mathcal{A}_2$ and each $D_i \in \mathcal{G}_{M_1+1}$, we have $(\theta, \cdots, \theta) \in \overline{V}$. Therefore, we may select $A_2 \in \mathcal{G}_{M_1+1}$ and $B_2 \in \mathcal{F}$ (containing no element less than $M_1 + 1$) such that

$$\begin{aligned}
&1_C(A_2 \cup p_i(B_2) \cup E) = T_{A_2\cup p_i(B_2)} 1_C(E) = \theta 1_C(E) = 1 \text{ and}\\
&\theta 1_C(A_2 \cup p_i(B_2) \cup E) = T_{A_2\cup p_i(B_2)}\theta 1_C(E) = \theta^2 1_C(E) = \theta 1_C(E) = 1,\\
&1 \le i \le 8,\ E \in \{A_1, A_1 \cup (B_1 \times B_1)\}.
\end{aligned}$$

Let M_2 be the smallest integer such that A_2 and $B_2 \times B_2$ lie in $\{1, \cdots, M_2\}^2$.

Let us take account of how the proof is progressing. We now have the sets A_1 and $A_1 \cup (B_1 \times B_1)$ in $\mathcal{C}$. These are exactly the images of the 1×1 matrices (0) and (1) respectively under the map K of (2.8). We also have

$$\big\{A_2 \cup A_1 \cup E : E \in FU\big(\{(B_1 \times B_1), (B_1 \times B_2), (B_2 \times B_1), (B_2 \times B_2)\}\big)\big\} \subset \mathcal{C}.$$

This family consists precisely of the images of the members of $\mathcal{M}_2^{(2)}$ under the map K as defined by (2.8). Moreover, we may continue in this fashion, utilizing the idempotence of θ. Namely, having chosen $A_1, \cdots, A_t \in \mathcal{G}_1$ and $B_1, \cdots, B_t \in$

$\mathcal{F}$ with $1_C(K(M)) = \theta 1_C(K(M)) = 1$ for all $M \in \mathcal{M}_t$, where K is given by (2.8), and $M_1 < M_2 < \cdots < M_t$ (such that (2.7) holds), we may find $A_{t+1} \in \mathcal{G}_{M_t+1}$ and $B_{t+1} \in \mathcal{F}$ (none of whose members are less than $M_t + 1$), such that

$$1_C(A_{t+1} \cup p(B_{t+1}) \cup E) = T_{A_{t+1}\cup p(B_{t+1})} 1_C(E) = \theta 1_C(E) = 1 \text{ and}$$
$$\theta 1_C(A_{t+1} \cup p(B_{t+1}) \cup E) = T_{A_{t+1}\cup p(B_{t+1})} \theta 1_C(E) = \theta^2 1_C(E) = \theta 1_C(E) = 1$$

for all $E \in K(\mathcal{M}_t^{(2)})$ and all set polynomials $p(B)$ which are a union of some (possibly none) of the monomials $(B \times B)$, $(B_j \times B)$ and $(B \times B_j)$, $1 \le j \le t$. We let M_{t+1} be the smallest integer such that $\{1, \cdots, M_{t+1}\}^2$ contains $K(\mathcal{M}_{t+1}^{(2)})$ (where, again, K is defined by (2.8); we keep mentioning it because we are building the map K as we go). Notice as well that now $1_C(K(M)) = \theta 1_C(K(M)) = 1$ for all $M \in \mathcal{M}_{t+1}$, so we can continue, thus completing the proof.

□

We now will change our focus slightly. Suppose we are given an increasing sequence $(N_i)_{i=1}^{\infty}$ of natural numbers, and a sequence of sets $B_i \subset \{N_{i-1} + 1, N_{i-1} + 2, \cdots, N_i\}$. For every $(l, m) \in \mathbf{N} \times \mathbf{N}$, let a_{lm} be the symbol x_{ij} if $(l, m) \in B_i \times B_j$. Otherwise, let $a_{lm} \in \{0, 1\}$. Then $V(x_{ij}) = (a_{lm})_{l,m\in\mathbf{N}}$ is an $\mathbf{N} \times \mathbf{N}$ matrix whose entries come from the set $\{0, 1\} \cup \{x_{ij} : i, j \in \mathbf{N}\}$. Moreover, for fixed $m \in \mathbf{N}$, the matrix $V_m(x_{ij}) = (a_{lm})_{l,m=1}^{N_m}$ is an $N_m \times N_m$ matrix whose entries come from the set $\{0, 1\} \cup \{x_{ij} : 1 \le i, j \le m\}$.

A matrix of this type induces a natural injection $(t_{ij})_{i,j=1}^{l} \to V_m(t_{ij})$ from $\mathcal{M}_l$ to $\mathcal{M}_{N_l}$. Namely, $V_m(t_{ij})$ is the $N_m \times N_m$ matrix which results by substituting t_{ij} for the symbol x_{ij} in the matrix $V_m(x_{ij}) = (a_{ij})_{i,j=1}^{N_m}$ constructed above. Hence, the $\mathbf{N} \times \mathbf{N}$ matrix $V(x_{ij}) = (a_{lm})_{l,m\in\mathbf{N}}$, together with the sequence $(N_m)_{m=1}^{\infty}$, induces such maps for all m; in other words, induces an injection of $\mathcal{M}$ into $\mathcal{M}$ (which takes $m \times m$ matrices to $N_m \times N_m$ matrices). We call the image of such a map an *$\mathcal{M}$-ring*. Specifically, the $\mathcal{M}$-ring generated by $(N_m)_{m=1}^{\infty}$ and the variable matrix $V(x_{ij}) = (a_{lm})$.

Hence for any $\mathcal{M}$-ring $\mathcal{N}$, there is an associated bijection $\varphi_{\mathcal{N}} : \mathcal{M} \to \mathcal{N}$, where φ arises as outlined above. We note that if $\mathcal{R}$ is another $\mathcal{M}$-ring and $\varphi_{\mathcal{R}} : \mathcal{M} \to \mathcal{R}$ the associated bijection, then $\varphi_{\mathcal{R}} \circ \varphi_{\mathcal{N}}$ is again a map arising in the fashion outlined above, so that $\varphi_{\mathcal{R}}(\mathcal{N})$ is again an $\mathcal{M}$-ring, called a *subring* of $\mathcal{R}$.

Theorem 2.6.9. ([M3].) Let $\mathcal{N}$ be an $\mathcal{M}$-ring. For any finite partition $\mathcal{N} = \bigcup_{i=1}^{r} C_i$, one of the cells C_i contains a subring of $\mathcal{N}$.

Proof. First of all, assume that the result is known for $\mathcal{N} = \mathcal{M}$. Now any finite coloring of a general $\mathcal{N}$ induces a coloring of $\mathcal{M}$ via the bijection $\varphi_{\mathcal{N}}$. Extracting a monochromatic $\mathcal{M}$-ring $\mathcal{R}$ for this induced coloring, $\varphi_{\mathcal{N}}(\mathcal{R})$ is a subring of $\mathcal{N}$ that is monochromatic for the original coloring. Hence we may assume without loss of generality that $\mathcal{N} = \mathcal{M}$.

Suppose, then, that $\mathcal{M} = \bigcup_{i=1}^{r} C_i$. We will induce an r-cell partition $\mathcal{G}_1 = \bigcup_{i=1}^{r} D_i$ as follows: for $E \in \mathcal{G}_1$, let N be the smallest integer such that $E \subset$

$\{1, \cdots, N\}^2$. Then let $E \in D_i$ if and only if the $N \times N$ matrix (a_{ij}), where $a_{ij} = 1$ if $(i,j) \in E$ and $a_{ij} = 0$ otherwise, is in C_i.

One of the cells D_b, where $1 \le b \le r$, must be strongly central and therefore by Theorem 2.6.8 contains an $\mathcal{M}^{(2)}$-system generated by sequences $(A_i)_{i=1}^{\infty} \subset \mathcal{G}_1$ and $(B_i)_{i=1}^{\infty} \subset \mathcal{F}$. Furthermore there is an associated increasing sequence $(M_l)_{l=1}^{\infty} \subset \mathbf{N}$ such that M_l is the least integer satisfying $A_l \cup (B_l \times B_l) \subset \{1, \cdots, M_l\}^2$. Put $B_i' = B_{2i-1}$, $A_i' = A_{2i} \cup A_{2i-1} \cup B_{2i} \times B_{2i}$, and $N_i = M_{2i}$, $i \in \mathbf{N}$. Let $V(x_{ij}) = (a_{ij})_{i,j \in \mathbf{N}}$ be the variable matrix obtained by letting $a_{kl} = x_{ij}$ if $(k,l) \in B_i' \times B_j'$, $a_{kl} = 1$ if $(k,l) \in \bigcup_{i=1}^{\infty} A_i'$, and $a_{kl} = 0$ otherwise.

We claim that the $\mathcal{M}$-ring $\mathcal{R}$ generated by $(N_i)_{i=1}^{\infty}$ and $V(x_{ij})$ is contained in C_b. To see this, let $l \in \mathbf{N}$ be arbitrary. We will show that $\varphi_{\mathcal{R}}(\mathcal{M}_l) \subset C_b$.

By hypothesis, every set having the form

$$E = (A_1' \cup A_2' \cup \cdots \cup A_l') \cup \Big(\bigcup_{r_{ij}=1} (B_i' \times B_j') \Big), \ (r_{ij}) \in \mathcal{M}_l \tag{2.9}$$

lies in D_b. Moreover, every set of this form has $N_l = M_{2l}$ as the least integer such that $\{1, \cdots, N_l\}^2$ contains it. (Recall that $A_l' = A_{2l-1} \cup A_{2l} \cup B_{2l} \times B_{2l}$.) That means that every $(a_{ij}) \in \mathcal{M}_{N_l}$ having the property that $a_{ij} = 1$ if and only if (i,j) lies in a given set of the form (2.9) lies in C_b. In other words, $\varphi_{\mathcal{R}}(\mathcal{M}_l) \subset C_b$.

□

Next we shall extend (leaving details of the proof to exercises) Theorem 2.6.9 in two senses. First note that the $\mathcal{M}$-rings defined previously could well be called $\mathcal{M}^{(2,2)}$-rings. The first 2 in this proposed superscript is owing to the fact that an $\mathcal{M}$-ring consists of 2-dimensional matrices, that is, indexed by $\{1, \cdots, N\}^2$ for some $N \in \mathbf{N}$. One might just as easily consider matrices (a_{ijk}) indexed by $\{1, \cdots, N\}^3$, or more general indexed by $\{1, \cdots, N\}^l$, $l \in \mathbf{N}$. The second 2 refers to the cardinality of the set from which the entries of the matrices are drawn. That set is $\{0, 1\}$. One might consider taking a set of cardinality k, such as $\{0, 1, \cdots, k-1\}$, as the set from which those entries are drawn.

As a matter of fact, neither of these considerations poses any obstacle to obtaining correspondingly more general versions of Theorems 2.6.8 and 2.6.9. Let us give a few of the details on the formulation of a more general version of Theorem 2.6.9, and its proof. For $N, l, k \in \mathbf{N}$, we will denote by $\mathcal{M}_N^{(l,k)}$ the set if all function (matrices) $A : \{1, \cdots, N\}^l \to \{0, 1, \cdots, k-1\}$. We now procede to define $\mathcal{M}^{(l,k)}$-rings. Suppose we are given an increasing sequence $(N_i)_{i=1}^{\infty}$ (let $N_0 = 0$) of natural numbers, and a sequence of sets $(B_1)_{i=1}^{\infty}$ with $B_i \subset \{N_{i-1}+1, N_{i-1}+2, \cdots, N_i\}$, $i \in \mathbf{N}$. For every $(i_1, \cdots, i_l), (j_1, \cdots, j_l) \in \mathbf{N}^l$, let $a_{i_1 i_2 \cdots i_l}$ be the symbol $x_{j_1 j_2 \cdots j_l}$ if $(i_1, i_2, \cdots, i_l) \in B_{j_1} \times B_{j_2} \times \cdots \times B_{j_l}$. Otherwise, let $a_{i_1 i_2 \cdots i_l} \in \{0, 1, \cdots, k-1\}$. Then $V(x_{j_1 j_2 \cdots j_l}) = (a_{i_1 i_2 \cdots i_l})_{i_1, i_2, \cdots, i_l \in \mathbf{N}}$ is a matrix indexed by $\mathbf{N}^l$ whose entries come from the set $\{0, 1, \cdots, k-1\} \cup \{x_{j_1 j_2 \cdots j_l} : j_1, \cdots, j_l \in \mathbf{N}\}$. Moreover, for fixed $m \in \mathbf{N}$, the matrix $V_m(x_{j_1 j_2 \cdots j_l}) = (a_{i_1 i_2 \cdots i_l})_{i_1, \cdots i_l = 1}^{N_m}$ is a matrix indexed by $\{1, \cdots, N_m\}^l$ whose entries come from the set $\{0, 1, \cdots, k-1\} \cup \{x_{j_1 j_2 \cdots j_l} : 1 \le j_1, \cdots, j_l \le m\}$. A matrix of this type induces an injection of $\mathcal{M}_m^{(l,k)}$ into $\mathcal{M}_{N_m}^{(l,k)}$. Letting m range over $\mathbf{N}$, the matrix $V(x_{j_1 j_2 \cdots j_l})$ induces a map from $\mathcal{M}^{(l,k)}$ to $\mathcal{M}^{(l,k)}$. We call the image of such a map an $\mathcal{M}^{(l,k)}$-*ring.* Specifically, the $\mathcal{M}^{(l,k)}$-ring generated by $(N_m)_{l=1}^{\infty}$ and the variable matrix $V(x_{j_1 j_2 \cdots j_l}) = (a_{i_1 i_2 \cdots i_l})_{i_1, i_2, \cdots, i_l \in \mathbf{N}}$. Subrings of $\mathcal{M}^{(l,k)}$-rings may be defined much as they were for $\mathcal{M}$-rings in Section 1.

We want to extend Theorem 2.6.9 to $\mathcal{M}^{(l,k)}$-rings. One way to accomplish this is to first extend Theorem 2.6.8 to more general type of systems; systems that are similar to $\mathcal{M}^{(2)}$-systems, except that whereas $\mathcal{M}^{(2)}$-systems consist of subsets of $\mathbf{N}^2$, these more general systems consist of k-tuples of subsets of $\mathbf{N}^l$. Of course, in order to accomplish this we need a correspondingly more general version of the Bergelson-Leibman coloring theorem, dealing with k-tuples of sets.

Theorem 2.6.10. Let $l, k, t \in \mathbf{N}$ and let $p_{i,j}(X)$, $1 \le i \le t$, $1 \le j \le k$ be set-polynomials over $\mathbf{N}^l$ whose constant terms are empty. Let $H \subset \mathbf{N}$ be any finite set and let $r \in \mathbf{N}$. There exists a finite set $M \subset \mathbf{N}$, with $M \cap H = \emptyset$, having the property that if $\mathcal{F}(\mathbf{N}^l)^k = \bigcup_{i=1}^r C_i$ then there exists some d with $1 \le d \le r$, some non-empty $N \subset M$, and some sets $A_1, A_2, \cdots, A_k \subset \bigcup_{i=1}^t \bigcup_{j=1}^k p_{i,j}(M)$, such that $A_s \cap p_{i,j}(N) = \emptyset$, $1 \le i \le t$, $1 \le j, s \le k$, and

$$\big\{ \big(A_1 \cup p_{i,1}(N), A_2 \cup p_{i,2}(N), \cdots, A_k \cup p_{i,k}(N)\big) : 1 \le i \le t \big\} \subset C_d.$$

Although the above formulation of the polynomial Hales-Jewett theorem is not given explicitly in [BL2], it is implicit in the exposition. Therefore, we shall omit the proof (which, at any rate, follows quite easily from **BL**; the key to seeing this is to identify the k-tuple of sets $(A_1, \cdots, A_{k-1})$ in $\mathbf{N}^l$ with the set $((\{1\} \times A_1) \cup (\{2\} \times A_2) \cup \cdots \cup (\{k-1\} \times A_{k-1}))$ in $\mathbf{N}^{l+1}$ and consider the family of set-polynomials $\big\{\{i\} \times p_j(X) : \ 1 \le i \le k-1, \ 1 \le j \le t)\big\}$.

Supposing one has the more general form of Theorem 2.6.8, one must still do something to get from there to a more general form of Theorem 2.6.9. Earlier this was accomplished quite easily, as there is a natural correspondence between subsets of $\{1, \cdots, N\}^2$ and $N \times N$ matrices whose entries are drawn from $\{0, 1\}$. The situation here is only slightly more complicated; there is a natural correspondence between k-tuples of subsets of $\{1, \cdots, N\}^l$ and $N \times N \times \cdots \times N$ (l times) "matrices" whose entries are drawn from $\{0, 1, \cdots, 2^k - 1\}$.

In order to better elucidate the argument (in getting to the more general form of Theorem 2.6.9), it may prove helpful to examine in some detail a *finitary* case of moving from k-tuples of sets to matrices. For convenience we again consider a case where $l = 2$. Let us denote, for $M > N$,

$$\begin{aligned} L(\{N+1, \cdots, M\}) =& (\{N+1, N+2, \cdots, M\} \times \{1, 2, \cdots, M\}) \\ & \cup (\{1, 2, \cdots, N\} \times \{N+1, N+2, \cdots, M\}). \end{aligned}$$

Notice that $L(\{N+1, \cdots, M\})$ is shaped like an L. The following corollary to Theorem 2.6.10 concerns itself with matrices indexed not by squares in the plane but by such L-shaped sets.

Corollary 2.6.11. Let $l \in \mathbf{N}$ and let $p_1(X), \cdots, p_t(X)$ be set-polynomials over $\mathbf{N}^l$ whose constant terms are empty. Let $N \in \mathbf{N}$ and let $r \in \mathbf{N}$. Suppose that $p_i(A) \cap p_j(B) = \emptyset$ for $i \neq j$ and every pair of sets A, B such that $A \cap \{1, \cdots, N\} = \emptyset$ and $B \cap \{1, \cdots, N\} = \emptyset$. Then for every $k, r \in \mathbf{N}$ there exists $M > N$ such that for any function $c : \{0, 1, \cdots, k-1\}^{L(\{N+1, \cdots, M\})} \to \{1, \cdots, r\}$, there exists some $v \in \{0, 1, \cdots, k-1\}^{L(\{N+1, \cdots, M\})}$ and some set $B \subset \{N+1, \cdots, M\}$ such that for every $u_1, u_2 \in \{0, 1, \cdots, k-1\}^{L(\{N+1, \cdots, M\})}$ that agree with v off of $\bigcup_{j=1}^t p_j(B)$, and with u_1 and u_2 each constant on every $p_j(B)$, $c(u_1) = c(u_2)$.

The content of Corollary 2.6.11 is basically that if eventually the set polynomials p_i have pairwise disjoint ranges then for any r-coloring of large enough (L-shaped) matrices whose coordinates are letters from the alphabet $\{0, 1, \cdots, k-1\}$, it is possible to choose a set B and a large enough matrix such that the color of the matrix remains constant over all possible values of the letters occuring on each $p_j(B)$ (provided that this letter is constant over each $p_j(B)$).

An example of this: say $A \subset \{1, \cdots, N\}$ and one has the three set polynomials $p_1(B) = A \times B$, $p_2(B) = B \times A$, and $p_3(B) = B \times B$. Then for $B \cap \{1, \cdots, N\} = \emptyset$, the sets $p_i(B)$ are pairwise disjoint. Hence, for any finite coloring matrices over the set $L(\{N+1, \cdots, M\})$, where M is large enough, on the alphabet $\{0, 1, \cdots, k-1\}$, there exists some matrix v and a set B such that for any replacement of the letters in v by a letter i_1 on $A \times B$, i_2 on $B \times B$, and i_3 on $B \times A$, the color of the resulting matrix does not depend on i_1, i_2, or i_3.

As for why Corollary 2.6.11 follows from Theorem 2.6.10, consider first of all that, given k, if we show Corollary 2.6.11 holds for k replaced by something bigger than k (say, 2^k), then it trivially holds for k as well (we can just consider colorings that identify certain letters). As mentioned earlier, k-tuples of finite subsets of $\mathbf{N}^2$ may be identified with $\mathbf{N} \times \mathbf{N}$ matrices with entries from the set $\{0, 1, \cdots, 2^k - 1\}$, all but finitely many of whose entries are zero. (Given such a k-tuple $(A_1, \cdots, A_k)$ and $x \in \mathbf{N}^2$, one can let a_x be the number whose binary representation is $1_{A_1}(x)1_{A_2}(x) \cdots 1_{A_k}(x)$.) Using this identification and considering the set of all polynomial k-tuples $(q_{i,1}(X), \cdots, q_{i,k}(X))$, where each $q_{i,j}$ is a union of some of the $p_i(X)$'s, Theorem 2.6.10 may be used to get Corollary 2.6.11 with k replaced by 2^k.

Exercise 2.50. Fill in the details of the above argument.

We now state a version of Theorem 2.6.9 for $\mathcal{M}^{(l,k)}$-rings.

Theorem 2.6.12. Let $l, k \in \mathbf{N}$ and let $\mathcal{N}$ be an $\mathcal{M}^{(l,k)}$-ring. For any finite partition $\mathcal{N} = \bigcup_{i=1}^r C_i$, one of the cells C_i contains a subring of $\mathcal{N}$.

Exercise 2.51. Prove Theorem 2.6.12.

An alternative approach to Theorem 2.6.12 is to derive it as a corollary of the following theorem, which can be obtained by defining, by analogy with the special case done in detail earlier, strongly partition regular, strongly syndetic, strongly central, etc. for subsets of $(\mathcal{G})^k$, and substituting Theorem 2.6.10 for **BL**.

Theorem 2.6.13. Let S be a set and let $k \geq 2$. For each $s \in S$, let T_s be a set and let $\mathcal{A}_s$ be a family of finite subsets of $(\mathcal{G}_1)^{k-1}$ that is strongly partition regular and closed under disjoint shifts. For every $s \in S$ and $t \in T_s$, let $A_{s,t} \in \mathcal{A}_s$, in such a way that $\mathcal{A}_s = \{A_{s,t} : t \in T_s\}$. Let $s_1 \in S$ and suppose that $\varphi : \{(s,t) : s \in S, t \in T_s\} \to S$ is a function. For any strongly central set $C \subset (\mathcal{G}_1)^k$, there exist sequences $(s_n)_{n=2}^{\infty} \subset S$ and $(t_n)_{n=1}^{\infty}$, with each $t_i \in T_{s_i}$, such that $\varphi(s_{n-1}, t_{n-1}) = s_n$ for $n \geq 2$ and such that the set of all unions $E_{n_1} \cup E_{n_2} \cup \cdots \cup E_{n_m}$, where $n_1 < n_2 < \cdots < n_m$ and $E_{n_i} \in A_{s_{n_i}, t_{n_i}}$, $1 \leq i \leq m$, is contained in C.

Exercise 2.52. Prove Theorem 2.6.13.

Chapter 3

Density Ramsey Theory

3.1 Measure Theoretic Preliminaries

In this section we review the basics of measure theory, in preparation for our treatment of density Ramsey theory via ergodic theory. Additional materials may be found in, for example, [Fol]. Let X be a set and suppose $\mathcal{A} \subset \mathcal{P}(X)$. $\mathcal{A}$ is called an *algebra* (or *field*) if $\mathcal{A}$ is closed under finite unions and complementation. An algebra which is closed under countable unions is called a *σ-algebra* (or *σ-field*).

Exercise 3.1. If $\mathcal{A}$ is an algebra then $\emptyset, X \in \mathcal{A}$ and $\mathcal{A}$ is closed under finite intersections. If $\mathcal{A}$ is a σ-algebra then $\mathcal{A}$ is closed under countable intersections.

Exercise 3.2. If $(\mathcal{A}_\gamma)_{\gamma\in\Gamma}$ is a collection of σ-algebras then $\bigcap_{\gamma\in\Gamma} \mathcal{A}_\gamma$ is a σ-algebra.

If $\mathcal{E} \subset \mathcal{P}(X)$ then we denote by $\mathcal{M}(\mathcal{E})$ the smallest σ-algebra containing $\mathcal{E}$ (which is the intersection of all σ-algebras containing $\mathcal{E}$). We call $\mathcal{M}(\mathcal{E})$ the σ-algebra *generated by $\mathcal{E}$*. For example, let X be a topological space. The σ-algebra generated by the open sets is called the *Borel σ-algebra.* Its members are called *Borel sets.*

If $\mathcal{A}$ is a σ-algebra of subsets of X, a *measure* on $\mathcal{A}$ is a function $\mu : \mathcal{A} \to [0,\infty]$ such that $\mu(\emptyset) = 0$ and $\mu\big(\bigcup_{i=1}^{\infty} A_i\big) = \sum_{i=1}^{\infty} \mu(A)$ for all pairwise disjoint sequences $(A_i)_{i=1}^{\infty} \subset \mathcal{A}$. The resulting triple $(X, \mathcal{A}, \mu)$ is called a *measure space.* If $\mu(X) = 1$ then $(X, \mathcal{A}, \mu)$ will be called a *probability space.* If for every $A \in \mathcal{A}$ with $\mu(A) = 0$ and each $B \subset A$ we have $B \in \mathcal{A}$ (whence $\mu(B) = 0$) then the measure μ is said to be *complete.* If $P(x)$ is a {true, false}-valued proposition depending on x and $\mu(\{x : P(x) \text{ is false}\}) = 0$ then we say that $P(x)$ holds *almost everywhere*, or *a.e.*

Exercise 3.3. Let $(X, \mathcal{A}, \mu)$ be a measure space and let $(A_i)_{i=1}^{\infty} \subset \mathcal{A}$. If $A_1 \subset A_2 \subset \cdots$ then $\mu\big(\bigcup_{i=1}^{\infty} A_i\big) = \lim_{i\to\infty} \mu(A_i)$. If $A_1 \supset A_2 \supset \cdots$ then $\mu\big(\bigcap_{i=1}^{\infty} A_i\big) = \lim_{i\to\infty} \mu(A_i)$.

We now will examine how to construct non-trivial examples of measures. If $\mathcal{A} \subset \mathcal{P}(X)$ is an algebra then a function $p : \mathcal{A} \to [0,\infty]$ will be called a *premeasure* if $\mu(\emptyset) = 0$, and if for every sequence $(A_i)_{i=1}^{\infty} \subset \mathcal{A}$ for which $\bigcup_{i=1}^{\infty} A_i \in \mathcal{A}$ we have $\mu\big(\bigcup_{i=1}^{\infty} A_i\big) = \sum_{i=1}^{\infty} \mu(A_i)$. A function $\mu^* : \mathcal{P}(X) \to [0,\infty]$ will be called an *outer measure* if $\mu^*(\emptyset) = 0$, $\mu^*(A) \leq \mu^*(B)$ whenever $A \subset B$, and if $\mu^*\big(\bigcup_{i=1}^{\infty} A_i\big) \leq \sum_{i=1}^{\infty} \mu^*(A_i)$.

Theorem 3.1.1. (Carathéodory; see [Fol].) Let X be a set, $\mathcal{A}$ be an algebra of subsets of X and p a premeasure on $\mathcal{A}$ for which $p(X) = 1$. For every $B \subset X$ let

$$\mu^*(B) = \inf\Big\{\sum_{i=1}^{\infty} p(A_i) : (A_i)_{i=1}^{\infty} \subset \mathcal{A},\ E \subset \bigcup_{i=1}^{\infty} A_i\Big\}.$$

Let $\mathcal{B} = \big\{B \subset X : \mu^*(B) + \mu^*(B^c) = 1\big\}$. Then μ^* is an outer measure on X which agrees with p on $\mathcal{A}$, $\mathcal{B}$ is a σ-algebra on X which contains $\mathcal{A}$, and the restriction of μ^* to $\mathcal{B}$ is a (probability) measure.

Proof. First we will show that μ^* is an outer measure. Clearly $\mu^*(\emptyset) = 0$ and $\mu^*(A) \leq \mu^*(B)$ for $A \subset B$. Suppose $(E_i)_{i=1}^{\infty}$ is a pairwise disjoint sequence of sets and let $\epsilon > 0$.

For every i we can find a sequence $(A_{i,j})_{j=1}^{\infty} \subset \mathcal{A}$ such that $E_i \subset \bigcup_{j=1}^{\infty} A_{i,j}$ and $\sum_{j=1}^{\infty} p(A_{i,j}) < \mu^*(E_i) + \frac{\epsilon}{2^i}$. Then

$$\sum_{i,j=1}^{\infty} p(A_{i,j}) < \Big(\sum_{i=1}^{\infty} \mu^*(E_i)\Big) + \epsilon.$$

But $\bigcup_{i=1}^{\infty} E_i \subset \bigcup_{i,j=1}^{\infty} A_{i,j}$. Hence, since ϵ is arbitary,

$$\mu^*\Big(\bigcup_{i=1}^{\infty} E_i\Big) \leq \Big(\sum_{i=1}^{\infty} \mu^*(E_i)\Big)$$

and μ^* is an outer measure.

Exercise 3.4. Show that $\mu^*(A) = p(A)$ for $A \in \mathcal{A}$, so that in particular $\mathcal{A} \subset \mathcal{B}$ and $\mu^*(X) = 1$. (*Hint: here is where one must use the fact that p is not merely finitely additive, but a premeasure.*)

Next, let us show simultaneously that $\mathcal{B}$ is an algebra and that μ^* is finitely additive on $\mathcal{B}$. Clearly $\mathcal{B}$ is closed under complementation. Let $B_1, B_2 \in \mathcal{B}$. Let $\epsilon > 0$ be arbitrary. Since $\mu^*(B_1) + \mu^*(B_1^c) = 1$, there exist sequence of sets (which without loss of generality we assume to be pairwise disjoint) $(A_i)_{i=1}^{\infty}$ and $(C_i)_{i=1}^{\infty}$ in $\mathcal{A}$ such that $B_1 \subset \bigcup_{i=1}^{\infty} A_i$ and $B_1^c \subset \bigcup_{i=1}^{\infty} C_i$, with

$$\sum_{i=1}^{\infty} \big(p(A_i) + p(C_i)\big) < 1 + \epsilon.$$

Let $\delta > 0$ be arbitrary. For all large enough N (we fix N) we have

$$\sum_{i=N+1}^{\infty} (p(A_i) + p(C_i)) \leq \delta, \text{ hence } \mu^*\Big(\bigcup_{i=1}^{N}(A_i \cup C_i)\Big) \geq 1 - \delta.$$

Let $D_j = C_j \setminus \big(\bigcup_{i=1}^{N} A_i\big)$, $1 \leq j \leq N$. Then $\bigcup_{i=1}^{N}(A_i \cup D_i) = \bigcup_{i=1}^{N}(A_i \cup C_i)$, so that

$$\sum_{i=1}^{N} (p(A_i) + p(D_i)) \geq 1 - \delta.$$

But

$$\sum_{i,j=1}^{N} (A_i \cap C_j) = \sum_{i=1}^{N} (p(A_i) + p(C_i)) - \sum_{i=1}^{N} (p(A_i) + p(D_i)) \leq 1 + \epsilon - (1 - \delta) = \epsilon + \delta.$$

Since δ is arbitrarily small and N is arbitrarily large we have $\sum_{i,j=1}^{\infty} p(A_i \cap C_j) \leq \epsilon$. Hence if we let $G = \bigcup_{i,j=1}^{\infty}(A_i \cap C_j)$, we have $\mu^*(G) \leq \epsilon$.

Let now $D_n = A_n \setminus \Big(\bigcup_{i=1}^{n} C_i\Big)$ and $F_n = C_n \setminus \Big(\bigcup_{i=1}^{n-1} A_i\Big)$, $n = 1, 2, \cdots$. Then the collection made up of the D_n's and F_n's is a pairwise disjoint collection from $\mathcal{A}$. Moreover, $B_1 \subset \big(\bigcup_{i=1}^{\infty} D_i\big) \cup G$ and $B_1^c \subset \big(\bigcup_{i=1}^{\infty} F_i\big) \cup G$.

By the same token, we may find sequences $(H_i)_{i=1}^{\infty}$ and $(I_i)_{i=1}^{\infty}$ in $\mathcal{A}$, and a set J with $\mu^*(J) \leq \epsilon$, such that the family made up of the H_i's and I_i's is pairwise disjoint and such that $B_2 \subset \big(\bigcup_{i=1}^{\infty} H_i\big) \cup J$ and $B_1^c \subset \big(\bigcup_{i=1}^{\infty} I_i\big) \cup J$.

We now have

$$B_1 \cap B_2 \subset \bigcup_{i,j=1}^{\infty} (D_i \cap H_j) \cup (G \cup J),$$

$$B_1 \cap B_2^c \subset \bigcup_{i,j=1}^{\infty} (D_i \cap I_j) \cup (G \cup J),$$

$$B_1^c \cap B_2 \subset \bigcup_{i,j=1}^{\infty} (F_i \cap H_j) \cup (G \cup J) \text{ and}$$

$$B_1^c \cap B_2^c \subset \bigcup_{i,j=1}^{\infty} (F_i \cap I_j) \cup (G \cup J).$$

The four families represented on the right are pairwise disjoint (from each other as well), and $\mu^*(G \cup J) \leq 2\epsilon$, so we have

$$\mu^*(B_1 \cap B_2) + \mu^*(B_1 \cap B_2^c) + \mu^*(B_1^c \cap B_2) + \mu^*(B_1^c \cap B_2^c) \leq 1 + 8\epsilon.$$

Since ϵ is arbitrary,

$$\mu^*(B_1 \cap B_2) + \mu^*(B_1 \cap B_2^c) + \mu^*(B_1^c \cap B_2) + \mu^*(B_1^c \cap B_2^c) = 1. \tag{3.1}$$

One now easily uses (3.1) to show that $\mu^*(B_1 \cup B_2) + \mu^*((B_1 \cup B_2)^c) = 1$, whence $(B_1 \cup B_2) \in \mathcal{B}$ (this shows that $\mathcal{B}$ is an algebra), and that if $(B_1 \cap B_2) = \emptyset$ then $\mu^*(B_1) + \mu^*(B_2) = \mu^*(B_1 \cup B_2)$, (this shows that μ^* is finitely additive on $\mathcal{B}$).

Any algebra which is closed under countable disjoint unions is a σ-algebra. Let $(B_i)_{i=1}^{\infty} \subset \mathcal{B}$ be a pairwise disjoint sequence. Put $B = \bigcup_{i=1}^{\infty} B_i$. Clearly $\mu^*(B) \geq \mu^*\big(\bigcup_{i=1}^{N} B_i\big)$ for each N, hence

$$\mu^*(B) \geq \lim_{N\to\infty} \mu^*\Big(\bigcup_{i=1}^{N} B_i\Big) = \lim_{N\to\infty} \sum_{i=1}^{N} \mu^*(B_i) = \sum_{i=1}^{\infty} \mu^*(B_i). \tag{3.2}$$

Recall that μ^* is subadditive, so in fact we have equality above. On the other hand $B^c \subset \bigcap_{i=1}^{N} B_i^c$ for each N, so that

$$\mu^*(B^c) \leq \lim_{N\to\infty} \mu^*\Big(\bigcap_{i=1}^{N} B_i^c\Big) = \lim_{N\to\infty} \Big(1 - \mu^*\Big(\bigcup_{i=1}^{N} B_i\Big)\Big) = 1 - \mu^*(B).$$

Together with subadditivity this gives $\mu^*(B) + \mu^*(B^c) = 1$, so that $B \in \mathcal{B}$ and $\mathcal{B}$ is a σ-algebra. On the other hand, countable additivity on μ^* restricted to $\mathcal{B}$ is a consequence of (3.2) and the sentence following it. Hence $\mu^*|_{\mathcal{B}}$ is a measure.

□

One can use the above theorem to carry out the construction of most common finite measure spaces. For example, taking X to be the unit interval $[0,1)$ in $\mathbf{R}$ and letting $\mathcal{A}$ be the algebra of finite disjoint unions of half-open intervals $[a,b)$ in X, define a premeasure on $\mathcal{A}$ by letting $p(I) = |I|$ for intervals I and extending additively to all of $\mathcal{A}$. The resulting measure on X is Lebesgue measure. One can do a similar construction on the unit cube in $\mathbf{R}^n$. Details are left to the reader.

Suppose X and Y are sets and $\mathcal{A}$ and $\mathcal{B}$ are σ-algebras on X and Y, respectively. A map $T : X \to Y$ is said to be $(\mathcal{A}, \mathcal{B})$-*measurable* if for every $B \in \mathcal{B}$ we have $T^{-1}B \in \mathcal{A}$. (If $Y = \mathbf{R}$ or $Y = [-\infty, \infty]$ is the extended reals and $\mathcal{B}$ is not mentioned explicitly then we shall always take $\mathcal{B}$ to be the Borel σ-algebra for the usual topology.) If μ is a measure on $\mathcal{A}$ and ν is a measure on $\mathcal{B}$ and $T : X \to Y$ is an $(\mathcal{A}, \mathcal{B})$-measurable map satisfying $\mu(T^{-1}B) = \nu(B)$ for every $B \in \mathcal{B}$ then T is said to be a *homomorphism* of measure spaces. If in addition T is a bijection and T^{-1} is measurable, then T is said to be an *isomorphism* of measure spaces, and the spaces $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ are said to be *measurably isomorphic.*

The most important example for us of measure preserving transformations will be *automorphisms*, specifically isomophisms $T : X \to X$ of a single probability space. If $(X, \mathcal{A}, \mu)$ is a probability space and $T : X \to X$ a measurable isomorphism we will say that the quadruple $(X, \mathcal{A}, \mu, T)$ is an *invertible measure preserving system.* These systems are the measurable analogs of topological dynamical systems consisting of a compact space and a homeomorphism of that space. Some examples of invertible measure preserving systems:

a. $X = [0, 1)$ with Lebesgue measure, $Tx = x + \alpha \pmod 1$, where $\alpha \in \mathbf{R}$ is fixed.

b. $X = [0, 1) \times [0, 1)$ with Lebesgue measure. $T(x, y) = (2x, \frac{1}{2}y)$ if $0 \leq x < \frac{1}{2}$, $T(x, y) = (2x - 1, \frac{1}{2}y + \frac{1}{2})$ if $\frac{1}{2} \leq x < 1$. (The *Baker's transformation.*)

c. $X = \{1, 2, \cdots, n\}$. $\mu(i) = \frac{1}{n}$, $1 \leq i \leq n$. $T =$ any permutation of X.

d. $X =$unit disk in $\mathbf{R}^2$ with normalized Lebesgue measure. $T =$ rotation by any fixed angle.

Exercise 3.5. If $\mathcal{E}$ generates $\mathcal{B}$ then $T : X \to Y$ is measurable if and only if $T^{-1}E \in \mathcal{A}$ for every $E \in \mathcal{E}$.

Exercise 3.6. If $(f_i)_{i=1}^{\infty}$ are measurable functions into the extended reals then $\sup_i f_i$, $\inf_i f_i$, $\limsup_i f_i$ and $\liminf_i f_i$ are measurable.

A *simple function* is a function of the form $\varphi(x) = \sum_{i=1}^{n} c_i 1_{A_i}(x)$, where $A_1, \cdots A_n$ are measurable sets and $c_1, \cdots, c_n \in \mathbf{R}$. We define the *integral* of φ to be $\int \varphi \, d\mu = \sum_{i=1}^{n} c_i \mu(A_i)$.

Exercise 3.7. If $f : X \to [0, \infty]$ is measurable, there exists a sequence $(\varphi_i)_{i=1}^{\infty}$ of simple functions with $0 \leq \varphi_1(x) < \varphi_2(x) < \cdots$ such that $\lim_{n\to\infty} \varphi_n(x) = f(x)$ for all $x \in X$.

We now extend the definition of the integral to non-negative measurable functions. Namely, if $f : X \to [0, \infty]$ is measurable, let

$$\int f \, d\mu = \sup \left\{ \int \varphi \, d\mu : 0 \leq \varphi \leq f, \ \varphi \text{ a simple function} \right\}.$$

Exercise 3.8. This definition of the integral agrees with the previous one if f is a simple function. *Hint: show first that for simple functions $\varphi_1 < \varphi_2$, we have $\int \varphi_1 \, d\mu < \int \varphi_2 \, d\mu$.*

Theorem 3.1.2. (Monotone convergence theorem.) Let $(X, \mathcal{A}, \mu)$ be a measure space. If $(f_i)_{i=1}^{\infty}$ is a non-decreasing sequence of non-negative measurable functions on X and we let $f(x) = \lim_{i\to\infty} f_i(x)$, then $\int f \, d\mu = \lim_{i\to\infty} \int f_i \, d\mu$.

Proof. Clearly $\int f_i \, d\mu \leq \int f \, d\mu$ for all i, so $\lim_{i\to\infty} \int f_i \, d\mu \leq \int f \, d\mu$. Let $\epsilon > 0$ be arbitrary and let $\varphi = \sum_{i=1}^{M} c_i 1_{B_i}$ be an arbitary non-negative simple function with $\varphi \leq f$. We will show that $\lim_{i\to\infty} \int f_i \, d\mu \geq (1 - \epsilon) \int \varphi \, d\mu$, which, since ϵ and φ are arbitrary, will get that $\lim_{i\to\infty} \int f_i \, d\mu \geq \int f \, d\mu$, completing the proof.

Let $A_i = \{x : f_i(x) \geq (1 - \epsilon)\varphi(x)\}$. Then for all j, $1 \leq j \leq M$, we have $(A_1 \cap B_j) \subset (A_2 \cap B_j) \subset \cdots$. Moreover, $\bigcup_{i=1}^{\infty} A_i = X$. Therefore, by Exercise

3.3, $\lim_{i\to\infty}\mu(A_i\cap B_j)=\mu(B_j)$. Hence

$$\begin{aligned}\lim_{i\to\infty}\int f_i\,d\mu &\geq \lim_{i\to\infty}\int f_i\cdot 1_{A_i}\,d\mu\\ &\geq \lim_{i\to\infty}\int(1-\epsilon)\varphi\cdot 1_{A_i}\,d\mu\\ &= \lim_{i\to\infty}(1-\epsilon)\sum_{j=1}^{M}c_j\mu(A_i\cap B_j)\\ &= (1-\epsilon)\sum_{j=1}^{M}c_j\mu(B_j)=(1-\epsilon)\int\varphi\,d\mu.\end{aligned}$$

□

Exercise 3.9. If $f\to[0,\infty]$ is measurable then $\int f\,d\mu=0$ if and only if $f(x)=0$ a.e.

Theorem 3.1.3. (Fatou's Lemma.) If $f_n:X\to[0,\infty]$ are measurable functions, $n\in\mathbf{N}$, then

a. $\int\liminf_{n\to\infty}f_n\,d\mu\leq\liminf_{n\to\infty}\int f_n\,d\mu$.

b. If $\mu(X)<\infty$ and there exists $K<\infty$ such that $f_n(x)\leq K$ for all x and all n then $\int\limsup_{n\to\infty}f_n\,d\mu\geq\limsup_{n\to\infty}\int f_n\,d\mu$.

Proof. (a) For all i we clearly have $\int\inf_{n\geq i}f_n\,d\mu\leq\inf_{n\geq i}\int f_n\,d\mu$. Notice that $inf_{n\geq i}f_n$ increases to $\liminf_{n\to\infty}f_n$ as $i\to\infty$. Hence by the monotone convergence theorem

$$\int\liminf_{n\to\infty}f_n\,d\mu=\lim_{i\to\infty}\int\inf_{n\geq i}f_n\,d\mu\leq\liminf_{n\to\infty}\int f_n\,d\mu.$$

To prove part (b), let $g_n=K-f_n$ and apply part (a).

□

Let us define $\int f\,d\mu$ for $f:X\to[-\infty,\infty]$ measurable. Let $f^+=\sup\{f,0\}$ and let $f^-=-\inf\{f,0\}$. If at least one of $\int f^+\,d\mu$ and $\int f^-\,d\mu$ is finite then we let $\int f\,d\mu=\int f^+\,d\mu-\int f^-\,d\mu$. If $\int f\,d\mu$ exists and is finite we say that f is *integrable.* One easily checks that f is integrable if and only if $\int|f|\,d\mu<\infty$.

Exercise 3.10. If $f(x)=g(x)$ a.e. then $\int f\,d\mu=\int g\,d\mu$.

Finally, we have the dominated convergence theorem.

Theorem 3.1.4. Suppose that $(f_i)_{i=1}^{\infty}$ is a sequence of measurable functions and $g\in L^1(X,\mathcal{A},\mu)$ with $|f_i|\leq g$ a.e. for all i. If $f(x)=\lim_{n\to\infty}f_n(x)$ exists a.e. then $f\in L^1(X,\mathcal{A},\mu)$ and $\int f\,d\mu=\lim_{n\to\infty}\int f_n\,d\mu$.

Proof. We have $g-f_n\geq 0$ a.e. Therefore by Theorem 3.1.3 (a),

$$\begin{aligned}\int g\,d\mu-\int f\,d\mu=\int\liminf_{n\to\infty}(g-f_n)\,d\mu &\leq\liminf_{n\to\infty}\int g-f_n\,d\mu\\ &=\int g\,d\mu-\limsup_{n\to\infty}\int f_n\,d\mu.\end{aligned}$$

However we also have $g + f_n \geq 0$ a.e., so that

$$\int g\,d\mu + \int f\,d\mu = \int \liminf_{n\to\infty}(g+f_n)\,d\mu \leq \liminf_{n\to\infty} \int g + f_n\,d\mu$$
$$= \int g\,d\mu + \liminf_{n\to\infty} \int f_n\,d\mu.$$

Hence

$$\limsup_{n\to\infty} \int f_n\,d\mu \leq \int f\,d\mu \leq \liminf_{n\to\infty} \int f_n\,d\mu.$$

□

3.2 Furstenberg correspondence.

In this section we will introduce the subject of density combinatorics. The following notion of "size" for subsets of $\mathbf{Z}$ is the key.

Definition 3.2.1. Let $E \subset \mathbf{Z}$. The *upper density* of E is the number

$$\overline{d}(E) = \limsup_{N\to\infty} \frac{|E \cap \{-N, -N+1, \cdots, N\}|}{2N+1}.$$

The *upper Banach density* of E is the number

$$d^*(E) = \limsup_{N-M\to\infty} \frac{|E \cap \{M, M+1, \cdots, N-1\}|}{N-M}.$$

Lower density $\underline{d}(E)$ and lower Banach density $d_*(E)$ are similarly defined (with lim sup replaced by lim inf). If $\overline{d}(E) = \underline{d}(E)$ then we may sometimes denote this common value by $d(E)$ and call it simply the *density* of E.

Exercise 3.11. Show that:

(a) $d_*(E) \leq \underline{d}(E) \leq \overline{d}(E) \leq d^*(E)$.

(b) $d^*(E \cup F) \leq d^*(E) + d^*(F)$ (and similarly for $\overline{d}$); $d_*(E \cup F) \geq d_*(E) + d_*(F)$ whenever $E \cap F = \emptyset$.

(c) $\overline{d}(E) = 1 - \underline{d}(E^c)$ (and similarly for Banach densities).

(d) If $d(E)$ and $d(F)$ exist and $E \cap F = \emptyset$ then $d(E \cup F) = d(E) + d(F)$.

(e) There exists a set E with $\overline{d}(E) = 1$ and $\underline{d}(E) = 0$.

(f) There exists an infinite pairwise disjoint family of sets each of whose members has upper density 1.

(g) If $d^*(E) > \frac{1}{k}$ and $n_1, \cdots, n_k \in \mathbf{Z}$ then for some $1 \leq i \neq j \leq k$ we have $d^*\big((E - n_i) \cap (E - n_j)\big) > 0$.

(h) Suppose $d^*(E) > \frac{1}{k}$. There exists t, with $2 \leq t \leq k$, and $x_1, \cdots, x_t \in \mathbf{Z}$ such that $\{x_1, x_2, \cdots, x_t, x_1 + x_2 + \cdots + x_t\} \subset E$.

According to (f) in the exercise above, $\overline{d}$ fails (quite badly), to be additive. (b) shows that $\overline{d}$ is at least finitely sub-additive, however be warned: $\overline{d}$ is not countably sub-additive since for example $\overline{d}(\{i\}) = 0$ for all i, while $\mathbf{Z} = \bigcup_{i\in\mathbf{Z}}\{i\}$.

Exercise 3.12. If $E \subset \mathbf{Z}$ is syndetic then $d_*(E) > 0$, and if E is piecewise syndetic then $d^*(E) > 0$.

Recall that a family $\mathcal{A}$ of subsets of $\mathbf{Z}$ is called partition regular if for every finite partition of $\mathbf{Z}$ we can find a member of $\mathcal{A}$ in one of the cells. Recall as well that a shift-invariant family $\mathcal{A}$ is partition regular if and only if every syndetic set contains a member of $\mathcal{A}$, and that a shift-invariant family $\mathcal{A}$ of finite sets is partition regular if and only if every piecewise syndetic set contains a member of $\mathcal{A}$.

We shall call a family $\mathcal{A}$ of subsets of $\mathbf{Z}$ *density regular* if every set of positive upper density contains a member of $\mathcal{A}$. According to the previous exercise, then, any density regular family is partition regular.

Exercise 3.13. Let $E \subset \mathbf{Z}$ have positive upper Banach density. Let $X = \{0,1\}^{\mathbf{Z}}$ and let T be the shift. Then $\overline{\{T^n 1_E : n \in \mathbf{Z}\}}$ contains some 1_B, where $\overline{d}(B) > 0$. Hence if $\mathcal{A}$ is a shift-invariant family of finite subsets then $\mathcal{A}$ is density regular if and only if every set having upper Banach density contains a member of $\mathcal{A}$, or, alternatively, if for every $\epsilon > 0$ there exists N such that for any set $E \subset \{1, \cdots, N\}$ with $|E| \geq N\epsilon$, E contains a member of $\mathcal{A}$.

This exercise, together with Exercise 2.17, shows that the role in density Ramsey theory of sets having positive upper Banach density is analogous to the role played by piecewise syndetic sets in partition Ramsey theory.

Exercise 3.14. Show that there exist sets in $\mathbf{Z}$ having positive upper Banach density and yet which fail to be piecewise syndetic.

Exercise 3.14 opens the door to the possibility that there might exist a shift-invariant family of finite sets which is partition regular but not density regular. There are obviously non-shift-invariant partition regular families of finite sets which are not density regular (a family consisting of a single one-element set, for example). We shall now see as well that there are shift-invariant partition regular families of infinite sets which are not density regular.

Theorem 3.2.2. The family of shifted IP-sets in $\mathbf{Z}$ is a shift-invariant family which is partition regular but not density regular.

Proof. The family in question is partition regular by Hindman's theorem. To see that it is not density regular, we will produce a set having positive upper density which contains no shift of an infinite IP-set.

For every $M \in \mathbf{N}_0$ we have

$$\overline{d}\Big(\bigcup_{k=0}^{M}(k + 3^{k+1}\mathbf{Z})\Big) \leq \sum_{k=0}^{M}\overline{d}(k + 3^{k+2}\mathbf{Z}) = \sum_{k=0}^{M}\frac{1}{3^{k+2}} < \frac{1}{6}.$$

It follows that there exists a sequence $0 < N_0 < N_1 < N_2 < \cdots$ such that for all $M \geq 0$,

$$\frac{\left|\bigcup_{k=0}^{M}(k + 3^{k+2}\mathbf{Z}) \cap \{-N_M, -N_M + 1, \cdots, N_M\}\right|}{2N_M + 1} < \frac{1}{6}.$$

Set $N_{-1} = 0$ and put $E = \big((k + 3^{k+2}\mathbf{Z}) \setminus \{-N_M, -N_M + 1, \cdots, N_M\}\big)$. Then

$$\frac{|E \cap \{-N_M, \cdots, N_M\}|}{2N_M + 1} < \frac{1}{6}$$

for every M, so that $\overline{d}(E) \leq \frac{1}{6}$. It follows that $\overline{d}(-E \cup E) \leq \frac{1}{3}$. Let $F = (-E \cup E)^c$. Plainly we have $\overline{d}(F) \geq \frac{2}{3}$. We claim F does not contain an infinite shifted IP-set. Indeed, let $k \in \mathbf{Z}$ and let Γ be an infinite shifted IP-set. Suppose $k \geq 0$. By Exercise 2.12 there exists $n \in \Gamma$ such that $n \in 3^{k+2}\mathbf{Z}$. Since $(k + 3^{k+2}\mathbf{Z}) \subset E$, we have $\big(E \cap (k + \Gamma)\big) \neq \emptyset$. Similarly, if $k \leq 0$ then $\big(-E \cap (k + \Gamma)\big) \neq \emptyset$. Either way, we cannot have $(k + \Gamma) \subset F$.

□

In 1984 V. Bergelson asked the more pertinent question of whether there exist shift-invariant families of *finite* sets that are partition regular but not density regular. The matter was rather more difficult to resolve than expected. The answer, which is *yes*, came 3 years later and is due to I. Kriz (see [K]). We will give a simpler construction of such a family (due to Rusza) in the next section.

Nevertheless, many of the shift-invariant partition regular families of configurations considered in the first section (arithmetic progressions, arithmetic progressions whose common difference comes from a fixed IP-set, polynomial progressions, etc.) have been shown to be density regular as well. Proving density regularity is, however, much more difficult than proving partition regularity. Hence, we will only be able to prove a sampling of the density versions of the partition theorems appearing in the first chapter.

Recall that we made our approach to partition Ramsey theory via topological dynamics, namely by an analysis of the recurrence properties of continuous self-maps of compact metric spaces. This approach was pioneered by Furstenberg and Weiss (see [FW]), and was actually modeled on the approach to density Ramsey theory Furstenberg had already established in [F1], which was via ergodic theory; specifically, by the analysis of the recurrence properties of measure-preserving transformations on probability spaces.

The key to understanding this link between density combinatorics and ergodic theory is a correspondence principle, called the *Furstenberg correspondence principle.* As a matter of fact, implicit in Chapter 1 is an analogous correspondence principle. Let us denote by $\mathcal{F}(\mathbf{Z})$ the family of finite subsets of $\mathbf{Z}$.

Theorem 3.2.3. (Topological correspondence principle.) Let $E \subset \mathbf{Z}$ be piecewise syndetic. Then there exists a minimal dynamical system (X, T), where T is a homeomorphism, and a non-empty open set $U \subset X$ such that

$$\Big\{\alpha \in \mathcal{F}(\mathbf{Z}) : \bigcap_{n \in \alpha} T^{-n}U \neq \emptyset\Big\} \subset \Big\{\alpha \in \mathcal{F}(\mathbf{Z}) : \bigcap_{n \in \alpha} (E - n) \neq \emptyset\Big\}.$$

Exercise 3.15. Prove Theorem 3.2.3. (Hint: look at parts (b) and (c) of Exercise 2.15.)

The following theorem represents the only application of Theorem 3.1.1 we shall need in the sequel. Let $X = \{0,1\}^{\mathbf{Z}}$. A *cylinder set* in X is a set of the form $C = \{\gamma \in X : \gamma(n_i) = \epsilon_i,\ 1 \leq i \leq t\}$, where $n_1, \cdots, n_t \in \mathbf{Z}$ and $\epsilon_1, \cdots, \epsilon_t \in \{0,1\}$. Note that the intersection of two cylinder sets is again a cylinder set, and the algebra $\mathcal{C}$ of finite unions of cylinder sets is a basis for the product topology on X.

Theorem 3.2.4. Let $X = \{0,1\}^{\mathbf{Z}}$ and let $T : X \to X$ be the shift $T\gamma(n) = \gamma(n+1)$. If $p : \mathcal{C} \to [0,1]$ is a finitely additive set function on the algebra $\mathcal{C}$ of finite unions of cylinder sets for which $p(X) = 1$ then p is a premeasure and extends to a measure μ on the σ-algebra $\mathcal{B}$ of Borel sets. If p is T-invariant on $\mathcal{C}$ then μ is T-invariant on $\mathcal{B}$.

Proof. Suppose $(C_i)_{i=1}^{\infty} \subset \mathcal{C}$ is a pairwise disjoint sequence of sets such that $\bigcup_{i=1}^{\infty} C_i \in \mathcal{C}$. Then in particular $\bigcup_{i=1}^{\infty} C_i$ is closed. Since the C_i's are open, we have $\bigcup_{i=1}^{N} C_i = \bigcup_{i=1}^{\infty} C_i$ for some N. In other words, since they are pairwise disjoint, all but finitely many of the C_i's must be empty. Thus $p\big(\bigcup_{i=1}^{\infty} C_i\big) = \sum_{i=1}^{\infty} p(C_i)$ follows by finite additivity of p. Hence p is a premeasure and by Theorem 3.1.1 p extends to a probability measure μ on a σ-algebra containing $\mathcal{C}$. Since $\mathcal{B}$ contains $\mathcal{C}$, we may take $\mathcal{B}$ to be the domain of μ.

Suppose p is T-invariant. Let $A \in \mathcal{B}$. Looking at the construction of μ in the proof of Theorem 3.1.1, for each $\epsilon > 0$ there exists a sequence $(C_i)_{i=1}^{\infty} \subset \mathcal{C}$ such that $A \subset \bigcup_{i=1}^{\infty} C_i$ and

$$\mu(A) \geq \sum_{i=1}^{\infty} p(C_i) - \epsilon = \sum_{i=1}^{\infty} p(T^{-1}C_i) - \epsilon \geq \mu(T^{-1}A) - \epsilon.$$

The last inequality comes from the fact that $T^{-1}A \subset \bigcup_{i=1}^{\infty} T^{-1}C_i$. Since ϵ is arbitrary this yields $\mu(A) \geq \mu(T^{-1}A)$. The reverse inequality is analogous, so $\mu(A) = \mu(T^{-1}A)$.

□

Theorem 3.2.5. (Furstenberg correspondence principle; see [B3].) Suppose $E \subset \mathbf{Z}$ with $d^*(E) > 0$. There exists an invertible measure preserving system $(X, \mathcal{A}, \mu, T)$ and a set $A \in \mathcal{A}$, with $\mu(A) = d^*(E)$, such that for all $\alpha \in \mathcal{F}(\mathbf{Z})$ we have

$$d^*\Big(\bigcap_{n\in\alpha}(E-n)\Big) \geq \mu\Big(\bigcap_{n\in\alpha} T^{-n}A\Big).$$

Proof. Let $X = \{0,1\}^{\mathbf{Z}}$ be the set of all functions $\gamma : \mathbf{Z} \to \{0,1\}$. As usual, let $T : X \to X$ be the shift: $T\gamma(n) = \gamma(n+1)$. Choose a sequence of intervals I_t in $\mathbf{Z}$ with $|I_t| \to \infty$ such that

$$\lim_{t\to\infty} \frac{|E \cap I_t|}{|I_t|} = d^*(E).$$

Let $\xi = 1_E \in X$ and let $A = \{\chi \in X : \chi(0) = 1\}$. Then

$$\lim_{t\to\infty} \frac{1}{|I_t|} \sum_{n\in I_t} 1_A(T^n\xi) = \lim_{t\to\infty} \frac{1}{|I_t|} \sum_{n\in I_t} 1_E(n) = d^*(E). \tag{3.3}$$

Exercise 3.16. Let $\mathcal{C}$ be the algebra of finite unions of cylinder sets. There exists a subsequence $(I_{t_s})_{s=1}^{\infty}$ such that

$$\lim_{s\to\infty} \frac{1}{|I_{t_s}|} \sum_{n\in I_{t_s}} 1_C(T^n\xi) = p(C) \tag{3.4}$$

exists for all $C \in \mathcal{C}$. (Hint: $\mathcal{C}$ is countable. Use a diagonal argument.)

The set function p defined on $\mathcal{C}$ by (3.4) is finitely additive and so by Theorem 3.2.4 is a premeasure which extends to a measure μ on the Borel σ-algebra, which we denote by $\mathcal{A}$. Moreover, p is T-invariant on $\mathcal{C}$, from which it follows that μ is T-invariant on $\mathcal{A}$. Also

$$\mu(A) = \lim_{t\to\infty} \frac{1}{|I_t|} \sum_{n\in I_t} 1_A(T^n\xi) = d^*(E).$$

Suppose now that $\alpha = \{n_1, \cdots, n_k\} \in \mathcal{F}(\mathbf{Z})$. We have

$$\begin{aligned}
&\mu\big(T^{-n_1}A \cap \cdots \cap T^{-n_k}A\big) \\
&= \lim_{s\to\infty} \frac{1}{|I_{t_s}|} \sum_{n\in I_{t_s}} 1_{T^{-n_1}A\cap\cdots\cap T^{-n_k}A}(T^n\xi) \\
&= \lim_{s\to\infty} \frac{1}{|I_{t_s}|} \sum_{n\in I_{t_s}} 1_{(E-n_1)\cap\cdots\cap(E-n_k)}(n) \\
&\le d^*\big((E-n_1)\cap\cdots\cap(E-n_k)\big).
\end{aligned} \tag{3.5}$$

□

The correspondence works in reverse, as well (this is mentioned in [F1] but not pursued). We will now establish this fact, which will prepare us for certain equivalences in the next section. Our exposition follows [BL1].

Lemma 3.2.6. Let $(E_i)_{i=1}^{\infty}$ be a sequence of measurable sets in a probability space $(X, \mathcal{B}, \mu)$. There exists $x \in X$ such that

$$\overline{d}(\{n : x \in E_n\}) \ge \limsup_{N\to\infty} \int \frac{1}{N} \sum_{n=1}^{N} 1_{E_n} \, d\mu.$$

Proof. Let $f_N = \frac{1}{N}\sum_{n=1}^{N} 1_{E_n}$. Then $f_N(x) \le 1$ for all x and all N, so by Fatou's lemma,

$$\int \limsup_{N\to\infty} f_N \, d\mu \ge \limsup_{N\to\infty} \int f_N \, d\mu.$$

Hence for some x,

$$\overline{d}\big(\{n : x \in E_n\}\big) = \limsup_{N\to\infty} f_N(x) \geq \limsup_{N\to\infty} \int \frac{1}{N}\sum_{n=1}^{N} 1_{E_n}\, d\mu.$$

$\square$

Proposition 3.2.7. (Reverse correspondence principle.) Let $(X, \mathcal{A}, \mu, T)$ be a measure preserving system and let $A \in \mathcal{A}$ with $\mu(A) > 0$. There exists a set $E \subset \mathbf{Z}$ with $\overline{d}(E) \geq \mu(A)$ such that

$$\Big\{\alpha \in \mathcal{F}(\mathbf{Z}) : \bigcap_{n\in\alpha}(E-n) \neq \emptyset\Big\} \subset \Big\{\alpha \in \mathcal{F}(\mathbf{Z}) : \mu\Big(\bigcap_{n\in\alpha} T^{-n}A\Big) > 0\Big\}.$$

Proof. For $\alpha \in \mathcal{F}(\mathbf{Z})$ let $E_\alpha = \bigcap_{n\in\alpha} T^{-n}A$. Let N be the union of all the E_α's which are of measure 0. Being a countable union, we have $\mu(N) = 0$. Let $B = A \setminus N$.

Exercise 3.17. For $\alpha \in \mathcal{F}$ we have $\mu\Big(\bigcap_{n\in\alpha} T^{-n}B\Big) = 0$ if and only if $\bigcap_{n\in\alpha} T^{-n}B = \emptyset$.

By Lemma 3.2.5, there exists $x \in X$ such that, letting $E = \{n : x \in T^{-n}B\}$, we have $\overline{d}(E) \geq \mu(B) = \mu(A)$. Observe now that if for some $\alpha \in \mathcal{F}(\mathbf{Z})$ we have $k \in \bigcap_{n\in\alpha}(E-n)$ then $T^k x \in \bigcap_{n\in\alpha} T^{-n}B$, so that by Exercise 3.17, $\mu\Big(\bigcap_{n\in\alpha} T^{-n}A\Big) > 0$.

$\square$

Exercise 3.18. Show that for any minimal system (X, T), there exists a syndetic set E such that

$$\Big\{\alpha \in \mathcal{F}(\mathbf{Z}) : \bigcap_{n\in\alpha}(E-n) \neq \emptyset\Big\} \subset \Big\{\alpha \in \mathcal{F}(\mathbf{Z}) : \bigcap_{n\in\alpha} T^{-n}U \neq \emptyset\Big\}.$$

3.3 Kriz' example.

In this section, we will be looking at difference sets of the form $E - E = \{x - y : x, y \in E\}$, where $E \subset \mathbf{Z}$ and either (a) $d^*(E) > 0$, or (b) E is piecewise syndetic. The reader is advised to attempt the following two exercises by combinatorial means.

Exercise 3.19. If $d^*(E) > 0$ then $E - E$ is syndetic. (*Hint: use Exercise 3.11 (g).*)

Exercise 3.20. If $k \in \mathbf{N}$ and $d^*(E_i) > 0$, $1 \leq i \leq k$, then $\bigcap_{i=1}^{k}(E_i - E_i)$ is syndetic. (*Hint: try using Ramsey's theorem.*)

Definition 3.3.1. A set $R \subset \mathbf{Z}$ is said to be:

(a) a *set of measure theoretic recurrence* if for every probability measure preserving system $(X, \mathcal{A}, \mu, T)$ and every $A \in \mathcal{A}$ with $\mu(A) > 0$ there exists $n \in R$ such that $\mu(A \cap T^{-n}A) > 0$.

(b) *density intersective* if for every $E \subset \mathbf{Z}$ with $d^*(E) > 0$ we have $\big((E - E) \cap R\big) \neq \emptyset$.

(c) a *set of topological recurrence* if for every minimal invertible system (X, T) and every non-empty open set $U \subset X$ there exists $n \in R$ such that $(U \cap T^{-n}U) \neq \emptyset$.

(d) *chromatically intersective* if for every piecewise syndetic set $E \subset \mathbf{Z}$ we have $\big((E - E) \cap R\big) \neq \emptyset$.

For more information about these notions, see [Rus], [For], [M] and [BM3]. Notice that the content of Exercise 3.19 is that thick sets are density intersective.

Exercise 3.21. If A is an infinite subset of $\mathbf{Z}$ then $R = A - A$ is a set of measure-theoretic recurrence.

The following exercise justifies the phrase "chromatic intersectivity".

Exercise 3.22. R is chromatically intersective if and only if for every finite partition $\mathbf{Z} = \bigcup_{i=1}^{r} C_i$ there exists a cell C_i of the partition for which $\big(R \cap (C_i - C_i)\big) \neq \emptyset$.

Exercise 3.23. R is density intersective if and only if the family $\mathcal{A} = \{(a, a+r) : a \in \mathbf{Z}, r \in R\}$ is density regular. R is chromatically intersective if and only if the family $\mathcal{A} = \{(a, a+r) : a \in \mathbf{Z}, r \in R\}$ is partition regular.

Exercise 3.24. If R is density intersective then R is chromatically intersective. (Hint: apply Exercise 3.12.)

Theorem 3.3.2. Let $R \subset \mathbf{Z}$.

(a) R is a set of measure theoretic recurrence if and only if R is density intersective.

(b) R is a set of topological recurrence if and only if R is chromatically intersective.

Proof. (a) Let R be a set of measure theoretic recurrence and suppose $d^*(E) > 0$. By Theorem 3.2.4 There exists a measure preserving system $(X, \mathcal{A}, \mu, T)$ and a measurable set A with $\mu(A) = d^*(A)$ such that $d^*\big(E \cap (E - n)\big) \geq \mu(A \cap T^{-n}A)$ for all $n \in \mathbf{Z}$. But for some $n \in R$ we have $\mu(A \cap T^{-n}A) > 0$, so that $d^*\big(E \cap (E - n)\big) > 0$, which in particular implies that $\big(R \cap (E - E)\big) \neq \emptyset$, so that R is density intersective. Conversely, if R is density intersective and $(X, \mathcal{A}, \mu, T)$ is a measure preserving system, then by Proposition 3.2.6 there exists E with $d^*(E) > 0$ having the property that for every n for which $\big(E \cap (E - n)\big) \neq \emptyset$ we have $\mu(A \cap T^{-n}A) > 0$. But for some $n \in R$ we have $n \in (E - E)$, which implies that $\big(E \cap (E - n)\big) \neq \emptyset$.

$\square$

Exercise 3.25. Mimic the proof of part (a) to prove part (b), substituting Theorem 3.2.3 and Exercise 3.18 for Theorem 3.2.4 and Proposition 3.2.6.

In light of Theorem 3.3.2, we can infer from Exercise 3.24 that every set of measure theoretic recurrence is a set of topological recurrence. It is instructive to see this in an alternative fashion; namely, as a consequence of the following theorem.

Theorem 3.3.3. For any minimal flow (X,T), there exists a T-invariant Borel measure μ on X assigning positive measure to non-empty open sets.

Proof. Let $x \in X$. Pick a sequence of intervals $(I_n)_{n=1}^{\infty}$ (again, using separability of $C(X)$ and a diagonal argument) such that $|I_n| \to \infty$ and such that

$$\lim_{n\to\infty} \frac{1}{|I_n|} \sum_{k\in I_n} f(T^k x) = L(f)$$

exists for every $f \in C(X)$. L is a positive linear function on $C(X)$ and $L(1) = 1$, so it comes by integration against a Borel measure μ. Moreover, one easily checks that $L(Tf) = L(f)$, so that μ is T-invariant. Finally, given a non-empty open set U, we can find a non-zero continuous function f supported inside of U and satisfying $0 \le f \le 1$. For some $\epsilon > 0$ the set $U' = f^{-1}\big((\epsilon, 1]\big)$ is non-empty (and open). Since (X,T) is minimal, the set $\{n : T^n x \in U'\}$ is syndetic. In particular this set has gaps bounded by some M. One now easily sees that $\mu(U) \ge \int f\, d\mu = L(f) \ge \frac{\epsilon}{M}$.

□

Recall that Bergelson had asked whether there are any finite, shift invariant, partition regular families of sets which are not density regular. Kriz showed in 1987 that there are by constructing a chromatically intersective set R which is not density intersective. (So that, in particular, the family $\mathcal{A} = \big\{\{a, a+r\} : a \in \mathbf{Z}, r \in R\}$ is partition regular but not density regular.) His construction ([K]) was in the context of graph theory and was somewhat cumbersome. We will present a simpler construction due to Rusza. Both constructions establish chromatic intersectivity with the help of the following theorem, which was a conjecture of Kneser proved by Lovász ([Lo]).

Theorem 3.3.4. Let $k, r \in \mathbf{N}$. Let E be the family of r-element subsets of $\{1, 2, \cdots 2r+k\}$. Given any k-coloring of E, there exists a disjoint pair of elements from E which are of the same color.

We need some terminology: if k is fixed and $R \subset \mathbf{Z}$ has the property that for any k-coloring of $\mathbf{Z}$, $\big(R \cap (C - C)\big) \ne \emptyset$, then R is said to be *k-intersective.* Thus chromatically intersective sets are those sets which are k-intersective for all k. It is easy to show that if a set R is k-intersective then for any $n \in \mathbf{N}$, the set $nR = \{nr : r \in R\}$ is also k-intersective.

Here now is the theorem of Kriz.

Theorem 3.3.5. There exists a set $R \subset \mathbf{N}$ such that R is chromatically intersective but not density intersective.

Proof. (Rusza; see [M1].) We will prove somewhat more, namely that for every $\epsilon > 0$ we can construct a chromatically intersective R and a set A with $\overline{d}(A) > \frac{1}{2} - \epsilon$ such that $(A - A) \cap r = \emptyset$. (This was shown by Kriz as well.)

We claim that for every $\epsilon > 0$ and every $k \in \mathbf{N}$ there exist $S = S(\epsilon, k) \in \mathbf{N}$ and pairwise disjoint sets $A' = A'(\epsilon, k)$, $B' = B'(\epsilon, k)$, and $R' = R'(\epsilon, k)$ in $\{0, 1, 2, \cdots, S-1\}$, such that

(a) R' is k-intersective,
(b) $|A'| > S\big(\frac{1}{2} - \epsilon\big)$,
(c) $|B'| > S\big(\frac{1}{2} - \epsilon\big)$,
(d) $\big((A' + R') \cap A'\big) = \big((B' + R') \cap B'\big) = \emptyset$,
(e) $\big((A' + R') \cup (B' + R')\big) \subset \{0, 1, \cdots, S-1\}$.

We will prove the claim. Let $r, N \in \mathbf{N}$ and let $p_1, \cdots, p_{2r+k}$ be odd primes. Write $M = p_1 \cdots p_{2r+k}$ and let $S = MN$. Put

$$\begin{aligned} A' = \big\{a : M \le a < (N-1)M,\ a \ne 0 (\text{mod } p_i),\ 1 \le i \le 2r + k, \\ \text{and } |\{i : a \ (\text{mod } p_i) \text{ is even }\}| < r\big\} \text{ and} \\ B' = \big\{b : M \le b < (N-1)M,\ b \ne 0 (\text{mod } p_i),\ 1 \le i \le 2r + k, \\ \text{and } |\{i : b \ (\text{mod } p_i) \text{ is odd }\}| < r\big\}. \end{aligned}$$

Exercise 3.26. If r, N and the primes are chosen large enough, conditions (b) and (c) above will be satisfied.

Let

$$R' = \Big\{r : 0 \le r < M \text{ and } |\{i : r \ (\text{mod } p_i) \in \{1, -1\}\}| \ge 2r\Big\}.$$

Let $a \in A'$ and $r \in R'$. a (mod p_i) is odd for more than $r + k$ i's, and r (mod p_i) is 1 or -1 for all but at most k i's. Hence $(a + r)$ (mod p_i) must be even for at least r i's, and hence is not in A'. In other words, $(A' + R') \cap A' = \emptyset$. One may just as easily show that $(B' + C') \cap B' = \emptyset$. This establishes (d).

Property (e) is obvious, so all that remains is (a). Namely, we must show that R' is k-intersective. Let

$$\begin{aligned} D = \Big\{d : 0 \le d < M - 1, \big|\{i : d = 2 \ (\text{mod } p_i)\}\big| = r, \\ \big|\{i : d = 2 \ (\text{mod } p_i)\}\big| = r + k\Big\}. \end{aligned}$$

Exercise 3.27. Suppose $D = \bigcup_{i=1}^{k} F_i$. Show there exist $d_1, d_2 \in D$ with $(d_1 - d_2) \in R'$. (Hint: identify $d \in D$ with the r-element subset $\{i : d = 2$ (mod $p_i)\}$ of $\{1, \cdots, 2r + k\}$ and apply Theorem 3.3.4.)

This gives (a), establishing the claim.

Suppose $\epsilon > 0$. Let $(\epsilon_k)_1^\infty$ be a sequence converging to zero sufficiently quickly that $\prod_{k=1}^{\infty} \epsilon_k \ge 1 - 2\epsilon$. For $k \in \mathbf{N}$ let $S_k = S(\epsilon_k, k)$, $A_k = A'(\epsilon_k, k)$, $B_k = B'(\epsilon_k, k)$, and $R_k = R'(\epsilon_k, k)$, as guaranteed by the claim above.

Exercise 3.28. Show that the set

$$A = \Big\{x_0 + x_1 S_1 + x_2 S_1 S_2 + \cdots + x_l S_1 S_2 \cdots S_l : l \in \mathbf{N},\ x_i \in \big(A_{i+1} \cup B_{i+1}\big),$$
$$0 \leq i \leq l, \big|\{i \in \{0, 1, \cdots, l\} : x_i \in A_{i+1}\}\big| \text{ is odd}\Big\}$$

satisfies $\overline{d}(A) \geq \frac{1}{2} - \epsilon$. *Hint: show first that*

$$\overline{d}\Big(\Big\{x_0 + x_1 S_1 + x_2 S_2 + \cdots + x_l S_l : l \in \mathbf{N},\ x_i \in \big(A_{i+1} \cup B_{i+1}\big), 0 \leq i \leq l\Big\}\Big) \geq 1 - 2\epsilon.$$

Let

$$R = R_1 \cup R_2 S_1 \cup R_3 S_1 S_2 \cup \cdots.$$

By the remark made before the statement of the theorem, R is $k-$intersective for all k and hence chromatically intersective. Suppose

$$a = \big(x_0 + x_1 S_1 + x_2 S_1 S_2 + \cdots + x_l S_1 S_2 \cdots S_l\big) \in A$$

and $r = r' S_1 S_2 \cdots S_t \in R$ (where $r' \in R_{t+1}$). If $t > l$ then $(a + r) \notin A$ due to the fact that $r' \notin (A_{t+1} \cup B_{t+1})$. If $t \leq l$, consider that if $x_t \in A_{t+1}$ then $(x_t + r') \notin A_{t+1}$ by construction. Hence the "evenness" requirement for inclusion in A, which is satisfied by a, is not satisfied by $a + r$. On the other hand, if $x_t \in B_{t+1}$ then $(x_t + r') \notin B_{t+1}$, so we meet with the same fact. Hence $(a + r) \notin A$, from which it follows that $\big(A \cap (A + R)\big) = \emptyset$. That is, $\big((A - A) \cap R\big) = \emptyset$.

□

3.4 Hilbert spaces.

In this section we discuss basic Hilbert space facts. More details can be found in, for example, [Fol].

Let V be a vector space over $\mathbf{R}$. A *norm* on V is a function from V to $[0, \infty)$, $x \to ||x||$, such that $||x + y|| \leq ||x|| + ||y||$ (the triangle inequality), $||kx|| = |k| \cdot ||x||$ for $k \in \mathbf{R}$ and $x \in V$, and $||x|| > 0$ whenever $x \neq 0$. If $|| \cdot ||$ is a norm on V then $\rho(x, y) = ||x - y||$ defines a metric on V. The topology associated with this metric is called the *norm topology* on V. A normed vector space which is complete with respect to the norm topology is called a *Banach space.*

Exercise 3.29. Suppose X is a compact metric space. Denote by $C(X)$ the set of all continuous real-valued functions on X. For $f \in C(X)$, put $||f||_u = \sup_{x \in X} |f(x)|$. Then $|| \cdot ||_u$ is a norm (called the *uniform norm*) with respect to which $C(X)$ is a Banach space.

Let V and U be normed vector spaces and let $T : V \to U$ be a linear map (i.e. $T(x+y) = T(x) + T(y)$ and $T(kx) = kT(x)$ for $x, y \in V$ and $k \in \mathbf{R}$). Put

$$||T|| = \sup\{||Tx|| : ||x|| = 1\}.$$

If $||T|| < \infty$ then T is said to be *bounded*. Let $\mathcal{L}(V,U)$ be the set of bounded linear maps from V to U.

Exercise 3.30. If T is bounded, T is continuous. If T is continuous at 0, then T is bounded.

Exercise 3.31. $\mathcal{L}(V,U)$ is a vector space and $||\cdot||$ is a norm (called the *operator norm*) on $\mathcal{L}(V,U)$. If U is a Banach space then $\mathcal{L}(V,U)$ is a Banach space.

An *inner product* on a vector space V over $\mathbf{R}$ is a function from $V \times V$ to $[0,\infty)$, $(x,y) \to \langle x,y\rangle$, such that

a. $\langle k_1x + k_2y, z\rangle = k_1\langle x,z\rangle + k_2\langle y,z\rangle$,
b. $\langle x,y\rangle = \langle y,x\rangle$, and
c. $\langle x,x\rangle > 0$ if $x \neq 0$.

Remark. We only deal here with vector spaces over $\mathbf{R}$. If one works with vector spaces over $\mathbf{C}$, condition *b.* must be replaced by $\langle x,y\rangle = \overline{\langle y,x\rangle}$.

Theorem 3.4.1. Suppose V is a vector space possessed of an inner product $\langle\cdot,\cdot\rangle$. Let $||x|| = \sqrt{\langle x,x\rangle}$, $x \in V$. Then

a. (The Schwarz inequality) $|\langle x,y\rangle| \leq ||x|| \cdot ||y||$, $x, y \in V$.
b. $||\cdot||$ is a norm on V.

Proof. a. If $||y|| = 0$ there is nothing to prove, so we assume this is not the case. For all $t \in \mathbf{R}$ we have

$$0 \leq \langle x-ty, x-ty\rangle = \langle x,x\rangle - 2t\langle x,y\rangle + t^2\langle y,y\rangle = ||x||^2 - 2t\langle x,y\rangle + t^2||y||^2. \quad (3.6)$$

The right-hand side of the above inequality is quadratic in t, and its minimum occurs at $t = \frac{\langle x,y\rangle}{||y||^2}$. Substituting this value of t into (3.6) gives us

$$0 \leq ||x||^2 - \frac{2(\langle x,y\rangle)^2}{||y||^2} + \frac{(\langle x,y\rangle)^2}{||y||^2} = ||x||^2 - \frac{(\langle x,y\rangle)^2}{||y||^2}.$$

This gives the result quite easily.

b. That $||x|| \neq 0$ if $x \neq 0$ and $||kx|| = |k| \cdot ||x||$ are clear. To get the triangle inequality, simply observe that by part a. we have

$$\begin{aligned} ||x+y||^2 = \langle x+y, x+y\rangle &= ||x||^2 + 2\langle x,y\rangle + ||y||^2 \\ &\leq ||x||^2 + 2||x|| \cdot ||y|| + ||y||^2 = (||x|| + ||y||)^2. \end{aligned}$$

$\square$

If V is a vector space over $\mathbf{R}$ equipped with an inner product, such that V is complete with respect to the norm $||x|| = \sqrt{\langle x, x\rangle}$, then V is said to be a *Hilbert space.*

Exercise 3.32. Let $n \in \mathbf{N}$. For $x = (x_1. \cdots, x_n), y = (y_1, \cdots, y_n) \in \mathbf{R}^n$, put $\langle x, y\rangle = x_1y_1 + x_2y_2 + \cdots + x_ny_n$. Then

a. $\langle \cdot, \cdot \rangle$ is an inner product with respect to which $\mathbf{R}^n$ is a Hilbert space.

b. For $y_1, \cdots, y_n \in \mathbf{R}$ we have $\left(\frac{1}{N}\sum_{n=1}^N y_n\right)^2 \leq \frac{1}{N}\sum_{n=1}^N y_n^2$. (*Hint: Let* $x = (1, 1, \cdots, 1)$ *and let* $y = (y_1, y_2, \cdots, y_n)$. *Apply the Schwarz inequality.*)

Let $(X, \mathcal{A}, \mu)$ be a measure space. As usual we identify two functions if they agree a.e. Put

$$L^2(X, \mathcal{A}, \mu) = \left\{f : X \to [-\infty, \infty]\,;\int f^2\, d\mu < \infty\right\}.$$

If $f, g \in L^2(X, \mathcal{A}, \mu)$ then $|\int fg\, d\mu| < \infty$. This is a consequence of the inequality $|f(x)g(x)| \leq \frac{1}{2}\left(f(x)^2 + g(x)^2\right)$. It follows that

$$\langle f, g\rangle = \int f(x)g(x)\, d\mu(x)$$

defines an inner product on $L^2(X, \mathcal{A}, \mu)$. Therefore

$$||f|| = \left(\int f^2\, d\mu\right)^{\frac{1}{2}}$$

defines a norm on $L^2(X, \mathcal{A}, \mu)$, called the L^2-*norm.*

Theorem 3.4.2. $L^2(X, \mathcal{A}, \mu)$ is complete in the norm topology, i.e. it is a Hilbert space.

Proof. Let $(f_n)_{n=1}^\infty$ be a Cauchy sequence in $L^2(X, \mathcal{A}, \mu)$. That is,

$$\lim_{n,m\to\infty} ||f_n - f_m|| = 0.$$

Choose a subsequence $(g_k)_{k=1}^\infty$, of the f_n's such that $||g_1|| + \sum_{k=1}^\infty ||g_{k+1} - g_k|| = T < \infty$. Let $G_N = |g_1| + \sum_{k=1}^{N-1} |g_{k+1} - g_k|$. Then $||G_N|| \leq T$ for all N, and $(G_N)_{N=1}^\infty$ is non-decreasing. Letting $G = \lim_{N\to\infty} G_N$, $||G|| \leq T$ by the monotone convergence theorem. In particular $G(x)$ is finite a.e. In other words, $g_1(x) + \sum_{k=1}^\infty \left(g_{k+1}(x) - g_k(x)\right)$ is absolutely convergent a.e. Hence $\lim_{k\to\infty} g_k(x) = f(x)$ exists and is finite a.e. We have $|f| \leq G$ and $|g_k| \leq G$ for all k, hence $f \in L^2(X, \mathcal{A}, \mu)$ and, since $|f - g_k| \leq 2G$ and $|f - g_k| \to 0$ a.e., by the dominated convergence theorem we have $||f - g_k|| \to 0$.

Let $\epsilon > 0$. There exists M such that for all $n > M$ we have $||f_n - g_k|| < \epsilon$ for all large enough k. Hence $||f_n - f|| \leq ||f_n - g_k|| + ||g_k - f||$. Letting $k \to \infty$

we get $||f_n - f|| \leq \epsilon$. Hence $||f_n - f|| \to \infty$. That is, $(f_n)_{n=1}^{\infty}$ converges to f in $L^2(X, \mathcal{A}, \mu)$.

$\square$

If $\mathcal{H}$ is a Hilbert space then $x, y \in \mathcal{H}$ are said to be *orthogonal* if $\langle x, y\rangle = 0$. If $M \subset \mathcal{H}$ then the *orthocomplement* of M is the set of vectors orthogonal to every member of M, namely $M^{\perp} = \{y \in \mathcal{H} : \langle x, y\rangle = 0 \text{ for all } x \in M\}$.

Exercise 3.33. $M^{\perp}$ is a closed linear subspace of $\mathcal{H}$.

Proposition 3.4.3. Let $\mathcal{H}$ be a Hilbert space.

a. (Pythagorean theorem.) If $x_1, x_2, \cdots, x_n \in \mathcal{H}$ are mutually orthogonal then $\left|\left|\sum_{i=1}^n x_i\right|\right|^2 = \sum_{i=1}^n ||x_i||^2$.

b. (Parallelogram law.) For all $a, b \in \mathcal{H}$, $||a+b||^2 + ||a-b||^2 = 2(||a||^2 + ||b||^2)$.

Proof. a. We have

$$\left|\left|\sum_{i=1}^n x_i\right|\right|^2 = \left\langle \sum_{i=1}^n x_i, \sum_{i=1}^n x_i \right\rangle = \sum_{i,j=1}^n \langle x_i, x_j\rangle = \sum_{i=1}^n ||x_i||^2.$$

Exercise 3.34. Prove part b.

$\square$

Theorem 3.4.4. Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{M}$ be a closed linear subspace. Then each $z \in \mathcal{H}$ may be uniquely expressed as a sum $z = x + y$, where $x \in \mathcal{M}$ and $y \in \mathcal{M}^{\perp}$. (We then say $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^{\perp}$.)

Proof. Let $z \in \mathcal{H}$ and put $\delta = \inf\{||z - x|| : x \in \mathcal{M}\}$. Let $\epsilon > 0$ be arbitrary. Suppose $x, y \in \mathcal{M}$ with $||z - x||, ||z - y|| \leq \delta + \epsilon$. Letting $a = z - x$, $b = y - z$ and applying Proposition 3.4.3 b., we have

$$2||z - x||^2 + 2||y - z||^2 = ||x - y||^2 + ||2z - x - y||^2.$$

But $(\frac{1}{2}x + \frac{1}{2}y) \in \mathcal{M}$, so $||z - (\frac{1}{2}x + \frac{1}{2}y)|| \geq \delta$. It follows that

$$||x - y||^2 = 2||z - x||^2 + 2||y - z||^2 - 4||z - (\frac{1}{2}x + \frac{1}{2}y)|| \leq 4\epsilon.$$

Hence if $(x_n)_{n=1}^{\infty} \subset \mathcal{M}$ is chosen with $\lim_{n\to\infty} ||z - x_n|| = \delta$ then $(x_n)_{n=1}^{\infty}$ is a Cauchy sequence which by completeness of $\mathcal{H}$ and the fact that $\mathcal{M}$ is closed converges to a point $x \in \mathcal{M}$. One easily sees that x is the unique point in $\mathcal{M}$ which is at a distance of δ from z.

We must now show that $(z - x) \in \mathcal{M}^{\perp}$. Let $w \in \mathcal{M}$ with $||w|| = 1$. $(x + tw) \in \mathcal{M}$ for all $\in \mathbf{R}$. Hence,

$$\begin{aligned}
\delta \leq ||z - (x + tw)||^2 &= \langle z - x - tw, z - x - tw\rangle \\
&= \langle z - x, z - x\rangle + 2\langle z - x, -tw\rangle + \langle -tw, -tw\rangle \\
&= ||z - x||^2 - 2t\langle z - x, w\rangle + t^2,
\end{aligned}$$

with equality if and only if $t = 0$. In other words, the minimum of the quadratic function of t on the right is at $t = 0$, hence the linear coefficient must be zero, that is, $\langle z - x, w\rangle = 0$.

Letting $y = z - x$ we therefore have $z = x + y$ where $x \in \mathcal{M}$ and $y \in \mathcal{M}^\perp$. If now we have another such decomposition $z = x' + y'$, where $x' \in \mathcal{M}$ and $y' \in \mathcal{M}^\perp$, then $x - x' = y - y'$ is in $\mathcal{M} \cap \mathcal{M}^\perp = \{0\}$. Hence $x = x'$ and $y = y'$.

□

If $\mathcal{M}$ is a closed linear subspace of a Hilbert space $\mathcal{H}$, define a linear map $P = P_{\mathcal{M}} : \mathcal{H} \to \mathcal{M}$ by $Pz = x$, where $z = x + y$, with $x \in \mathcal{M}$ and $y \in \mathcal{M}^\perp$. P is called the *orthogonal projection* onto $\mathcal{M}$.

Lemma 3.4.5. Let $\mathcal{H}$ be a Hilbert space and let P be the orthogonal projection onto a closed linear space $\mathcal{M} \subset \mathcal{H}$. Then

a. P is idempotent: $P^2 z = Pz$ for all $z \in \mathcal{H}$.
b. P is *self-adjoint*: $\langle Px, y\rangle = \langle x, Py\rangle$ for all $x, y \in \mathcal{H}$.
c. If $f, g, h \in \mathcal{H}$, with $f = g + h$, $\langle g, h\rangle = 0$, and $g \in \mathcal{M}$, then $||Pf|| \geq ||g||$.

Proof. a. For $x \in \mathcal{M}$, $x = x + 0$ is the unique decomposition of x into the sum of a vector in $\mathcal{M}$ and one in $\mathcal{M}^\perp$. Therefore $Px = x$. In particular, Since $Pz \in \mathcal{M}$ for all $z \in \mathcal{H}$, $P(Pz) = Pz$.

b. We have $y = Py + y'$, where $y' \in \mathcal{M}^\perp$. Hence

$$\langle Px, y\rangle = \langle Px, Py\rangle + \langle Px, y'\rangle = \langle Px, Py\rangle.$$

Similarly, $\langle x, Py\rangle = \langle Px, Py\rangle$.

c. We have $Pf = Pg + Ph = g + Ph$. Moreover, by part (b), $\langle g, Ph\rangle = \langle Pg, h\rangle = \langle g, h\rangle = 0$. Hence by the Pythagorean theorem, $||Pf||^2 = ||g||^2 + ||Ph||^2 \geq ||g||^2$.

□

The following serves as a sort of converse to Theorem 3.4.5 a.

Lemma 3.4.6. Let $\mathcal{H}$ be a Hilbert space. If $P : \mathcal{H} \to \mathcal{H}$ is a linear map with $||P|| \leq 1$ and $P^2 = P$ then P is the orthogonal projection onto its range.

Proof. That the range $P\mathcal{H}$ of P is a closed subspace is easily verified using the boundedness of P. We must show that for every $z \in \mathcal{H}$, $z = Pz + y$, where $y \in (P\mathcal{H})^\perp$. That is, we must show that $\langle Pz - z, Px\rangle = 0$ for all $x, z \in \mathcal{H}$ (which we now fix). Let $w = Pz - z$. Then $Pw = 0$, using idempotence of P. Hence by linearity and idempotence of P, for all $t \in \mathbf{R}$ we have $P(Px + tw) = Px$. Using the fact that $||P|| \leq 1$, we therefore have $||Px|| \leq ||Px + tw||$ for all t. But (cf. proof of Theorem 3.4.4) $||Px + tw||^2$ is a quadratic polynomial in t equal at $t = 0$ to $||Px||$. Hence its minimum occurs at $t = 0$ and its linear coefficient, namely, $2\langle Px, w\rangle$, is zero. That is, $\langle Px, Pz - z\rangle = 0$, as required.

□

Let $\mathcal{H}$ be a Hilbert space. An invertible isometry $U \in \mathcal{L}(\mathcal{H}, \mathcal{H})$ (that is, a bijection satisfying $||Ux|| = ||x||$ for all $x \in \mathcal{H}$) is called a *unitary operator*.

Exercise 3.35. If U is unitary then $\langle x, y\rangle = \langle U^n x, U^n y\rangle$ for all $x, y \in \mathcal{H}$ and $n \in \mathbf{Z}$.

Theorem 3.4.7. Let $\mathcal{H}$ be a Hilbert space and let U be a unitary operator on $\mathcal{H}$.

a. Putting $\mathcal{M}_1 = \{x \in \mathcal{H} : Ux = x\}$ and $\mathcal{M}_2 = \overline{\{y - Uy : y \in \mathcal{H}\}}$, we have $\mathcal{H} = \mathcal{M}_1 \oplus \mathcal{M}_2$.

b. (Unitary mean ergodic theorem) For every $x \in \mathcal{H}$,

$$\lim_{N\to\infty} \Big|\Big|\frac{1}{N}\sum_{n=1}^{N} U^n x - P_{\mathcal{M}_1} x\Big|\Big| = 0.$$

Proof. a. We must show that $\mathcal{M}_2 = \mathcal{M}_1^{\perp}$. If $x, y \in \mathcal{H}$ with $Ux = x$ then $\langle y - Uy, x\rangle = \langle y, x\rangle - \langle Uy, x\rangle = \langle y, x\rangle - \langle Uy, Ux\rangle = 0$. Since $\mathcal{M}_1^{\perp}$ is closed, this implies that $\mathcal{M}_2 \subset \mathcal{M}_1^{\perp}$. Let us now establish the reverse inclusion. Let $z \in \mathcal{M}_1^{\perp}$. Put $w = z - P_{\mathcal{M}_2} z$. Then $w \in \mathcal{M}_2^{\perp}$, so in particular for every $y \in \mathcal{H}$, $0 = \langle w, y - Uy\rangle$. That is, $\langle w, y\rangle = \langle w, Uy\rangle = \langle U^{-1}w, y\rangle$ for every $y \in \mathcal{H}$, so that $w = U^{-1}w$. Applying U to both sides, $w = Uw$, i.e. $w \in \mathcal{M}_1$. But we have already established that $\mathcal{M}_2 \subset \mathcal{M}_1^{\perp}$, so that $w = (z - P_{\mathcal{M}_2} z) \in \mathcal{M}_1^{\perp}$. Hence $w = 0$. That is, $z \in \mathcal{M}_2$.

b. We utilize the splitting given by part a. Let $x \in \mathcal{H}$. Then $x = x_1 + x_2$, where $x_1 = P_{\mathcal{M}_1} x \in \mathcal{M}_1$ and $x_2 \in \mathcal{M}_2$. We have

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} U^n x_1 = \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} x_1 = x_1 = P_{\mathcal{M}_1} x. \tag{3.7}$$

Exercise 3.36. $\{w \in \mathcal{H} : \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} U^n w = 0\}$ is closed and contains $\{y - Uy : y \in \mathcal{H}\}$. Hence it contains $\mathcal{M}_2$.

This exercise implies that, in particular,

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} U^n x_2 = 0. \tag{3.8}$$

Combining (3.7) and (3.8) gives the result.

□

Our interest in unitary operators owes itself to the fact that if $(X, \mathcal{A}, \mu)$ is a measure space and $T : X \to X$ is an invertible measure preserving transformation, then T induces a unitary operator (which we will normally denote by T as well) on $L^2(X, \mathcal{A}, \mu)$, given by $Tf(x) = f(Tx)$. (Invertibility of the unitary operator T follows from invertibility of the measure preserving transformation T. That T acts on $L^2(X, \mathcal{A}, \mu)$ as an isometry is a consequence of the fact that T is measure-perserving.)

Corollary 3.4.8. (The Mean Ergodic Theorem; see e.g. [P].) Suppose that $(X, \mathcal{A}, \mu, T)$ is a measure preserving system. Then

$$Pf = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} T^n f$$

exists in the norm topology for all $f \in L^2(X, \mathcal{A}, \mu)$. Moreover, P is the orthogonal projection onto the space of T-invariant functions.

A Hilbert space $\mathcal{H}$ is said to be *separable* if it contains a countable dense subset (in other words, if it is separable in the norm metric). The *weak topology* on $\mathcal{H}$ is the topology generated by all sets of the form $U_{\epsilon,g} = \{f \in \mathcal{H} : \langle f, g\rangle \leq \epsilon\}$, where $g \in \mathcal{H}$ and $\epsilon > 0$.

Exercise 3.37. If $\mathcal{H}$ is not finite dimensional (i.e. if there does not exist a finite spanning set for $\mathcal{H}$ as a vector space) then the weak topology is less fine than the norm topology on $\mathcal{H}$. (*Hint: show there exists a sequence of vectors all having norm 1 which converges weakly to 0 in the weak topology.*)

Exercise 3.38. Suppose $\mathcal{H}$ is separable and let $\mathcal{H}_1 = \{x \in \mathcal{H} : ||x|| \leq 1\}$ be the unit ball. Let $(x_n)_{n=1}^{\infty}$ be dense (for the norm topology) in $\mathcal{H}_1$. For $f, g \in \mathcal{H}_1$, put

$$\rho(f, g) = \sum_{i=1}^{\infty} \frac{|\langle f - g, \frac{x_i}{||x_i||}\rangle|}{2^i}. \tag{3.9}$$

Then

a. ρ is a metric on $\mathcal{H}_1$ which generates the weak topology.

b. $(\mathcal{H}_1, \rho)$ is compact. More generally, any closed bounded set in $\mathcal{H}$ is weakly compact.

c. If $\mathcal{H}$ is not finite dimensional then $\mathcal{H}$ with the weak topology is *not* metrizable.

Suppose $\mathcal{H}$ is a separable Hilbert space, U is a unitary operator on $\mathcal{H}$ and $(n_\alpha)_{\alpha\in\mathcal{F}} \subset \mathbf{N}$ is an IP-set. For any $f \in \mathcal{H}_1$, we may (by Exercise 3.38 part b. and **H3** from Section 2.2) choose an IP-ring $\mathcal{F}^{(1)}$ such that

$$\operatorname*{IP-lim}_{\alpha\in\mathcal{F}^{(1)}} U^{n_\alpha} f \tag{3.10}$$

exists in the weak topology.

Exercise 3.39. Use separability of $\mathcal{H}$ and the fact that U is unitary to show that there exists an IP-ring $\mathcal{F}^{(1)}$ such that the limit in (3.10) exists for all $f \in \mathcal{H}_1$. (*Hint: Take a countable dense set of f's and use a diagonal argument.*)

Theorem 3.4.9. (See [F2] or [FK2].) Let $\mathcal{H}$ be a Hilbert space, let U be a unitary operator on $\mathcal{H}$ and let $(n_\alpha)_{\alpha\in\mathcal{F}} \subset \mathbf{N}$ be an IP-set. Suppose that $\mathcal{F}^{(1)}$ is an IP-ring for which

$$Pf = \operatorname*{IP-lim}_{\alpha\in\mathcal{F}^{(1)}} U^{n_\alpha} f \tag{3.11}$$

exists weakly for all $f \in \mathcal{H}$. Then P is the orthogonal projection onto its range.

Proof. We shall need to use the following fact.

Exercise 3.40. Commutativity of Hilbert space operators is preserved by the passage to weak limits. In particular, P commutes with U.

Since it is obvious that $||P|| \leq 1$, by Lemma 3.4.6 all we must show is that $P = P^2$. Let $f \in \mathcal{H}$ with $||f|| \leq 1$. We will show that $P^2 f = Pf$. Restricting ourselves to the closed linear span of $\{U^n f : n \in \mathbf{Z}\}$ (which is separable), we may assume that $\mathcal{H}$ is separable. Let $\epsilon > 0$ be arbitrary, and let ρ be a metric for the weak topology on $\mathcal{H}_1$, as in (3.9). Fix $\alpha_0 \in \mathcal{F}$ having the property that

$$\rho(Pf, U^{n_\alpha} f) < \epsilon \text{ and } \rho(P^2 f, U^{n_\alpha} Pf) < \epsilon \text{ for all } \alpha \in \mathcal{F}^{(1)} \text{ with } \alpha > \alpha_0. \quad (3.12)$$

Fix some $\alpha > \alpha_0$ and choose $\beta \in \mathcal{F}^{(1)}$ with $\beta > \alpha$ such that

$$\rho(U^{n_\beta} U^{n_\alpha} f, PU^{n_\alpha} f) < \epsilon. \quad (3.13)$$

Considering now that $U^{n_\beta} U^{n_\alpha} = U^{n_{\alpha \cup \beta}}$ and $(\alpha \cup \beta) > \alpha_0$, (3.12) and (3.13) combine to give $\rho(P^2 f, Pf) < 3\epsilon$. Since ϵ is arbitrary, this completes the proof.

3.5 Sárközy's theorem.

In this section we will prove the following theorem ([Sá]; see also [F2], [B2], and [KM]).

Theorem 3.5.1 Let $p(x) \in \mathbf{Z}[x]$ with $p(0) = 0$. Then $\{p(n) : n \in \mathbf{N}\}$ is a set of measure-theoretic recurrence.

Exercise 3.41. Show that Theorem 3.5.1 is equivalent to *Sárközy's theorem*: if $p(x) \in \mathbf{Z}[x]$ with $p(0) = 0$ and $E \subset \mathbf{N}$ has positive upper Banach density then there exist $x, y \in E$ and $n \in \mathbf{N}$ with $x - y = p(n)$.

The proof of Theorem 3.5.1 we shall present is due V. Bergelson. It utilizes the following Lemma, which is motivated by van der Corput's fundamental inequality:

Lemma 3.5.2. ([B2].) Suppose that $\{x_n : n \in \mathbf{Z}\}$ is a bounded sequence of vectors in a Hilbert space $\mathcal{H}$. If for all $h \in \mathbf{Z}$, $h \neq 0$, we have

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \langle x_n, x_{n+h} \rangle = 0,$$

then

$$\lim_{N \to \infty} \left|\left| \frac{1}{N} \sum_{n=1}^{N} x_n \right|\right| = 0.$$

Proof. Let $\epsilon > 0$. Choose L with $L > ||x_n||$ for all n and fix $H \in \mathbf{N}$ with $\frac{L^2}{H} < \epsilon$. For $N \in \mathbf{N}$ let

$$\Psi_N = \frac{1}{N}\sum_{n=1}^{N}\Big(\frac{1}{H}\sum_{h=1}^{H} x_{n+h}\Big).$$

Exercise 3.42.

$$\lim_{N\to\infty}\Big|\Big|\frac{1}{N}\sum_{n=1}^{N} x_n - \Psi_N\Big|\Big| = 0.$$

All that remains is to show that $\lim_{N\to\infty}||\Psi_N|| < \epsilon$. Applying first the triangle inequality and then Exercise 3.32 b., we have

$$\begin{aligned}
||\Psi_N||^2 &= \Big|\Big|\frac{1}{N}\sum_{n=1}^{N}\Big(\frac{1}{H}\sum_{h=1}^{H} x_{n+h}\Big)\Big|\Big|^2 \\
&\leq \Big(\frac{1}{N}\sum_{n=1}^{N}\Big|\Big|\frac{1}{H}\sum_{h=1}^{H} x_{n+h}\Big|\Big|\Big)^2 \\
&\leq \frac{1}{N}\sum_{n=1}^{N}\Big|\Big|\frac{1}{H}\sum_{h=1}^{H} x_{n+h}\Big|\Big|^2 \\
&= \frac{1}{N}\sum_{n=1}^{N}\frac{1}{H^2}\sum_{h,k=1}^{H}\langle x_{n+h}, x_{n+k}\rangle \\
&= \sum_{r=-H}^{H}\frac{H-|r|}{H^2(N)}\sum_{u=1}^{N}\langle x_u, x_{u+r}\rangle + \Psi_N'',
\end{aligned}$$

where $\Psi_N'' \to 0$ as $N \to \infty$. The first summand, on the other hand, tends to at most $\frac{L^2}{H} < \epsilon$ for large N by hypothesis.

□

Proof of Theorem 3.5.1. Let $(X, \mathcal{A}, \mu, T)$ be a measure preserving system and put

$$\mathcal{H}_r = \overline{\{f \in L^2(X, \mathcal{A}, \mu) : T^k f = f \text{ for some } k \in \mathbf{N}\}}.$$

(The r in $\mathcal{H}_r$ is for *rational* spectrum.)

Exercise 3.43. Show that $\mathcal{H}_r$ is a subspace of $L^2(X, \mathcal{A}, \mu)$.

According to Theorem 3.4.4 and the previous exercise, $L^2(X, \mathcal{A}, \mu) = \mathcal{H}_r \oplus \mathcal{H}_r^{\perp}$. Denote by P_r the orthogonal projection onto $\mathcal{H}_r$. Let $A \in \mathcal{A}$ with $\mu(A) > 0$. Write $1_A = g + h$, where $g = P_r 1_A \in \mathcal{H}_r$ and $h = 1_A - g \in \mathcal{H}_r^{\perp}$.

Exercise 3.44. Show that $\langle g, 1_A\rangle \geq \mu(A)^2$. (*Hint: the constants are in $\mathcal{H}_r$. Use Lemma 3.4.5 c.*)

Let $\epsilon > 0$ be arbitrary. $g \in \mathcal{H}_r$, so we can find $f \in L^2(X, \mathcal{A}, \mu)$ and $k \in \mathbf{N}$ with $||f - g|| \leq \epsilon$ and $T^k f = f$. For all $n \in \mathbf{N}$, $k|p(kn)$, so that

$$||T^{p(kn)}g - g|| \leq ||T^{p(kn)}g - T^{p(kn)}f|| + ||T^{p(kn)}f - f|| + ||f - g|| \leq \epsilon + 0 + \epsilon = 2\epsilon.$$

It follows that for every $N \in \mathbf{N}$,

$$\left|\left|\frac{1}{N}\sum_{n=1}^{N} T^{p(kn)}g - g\right|\right| \leq 2\epsilon. \tag{3.14}$$

We claim that

$$\lim_{N\to\infty}\left|\left|\frac{1}{N}\sum_{n=1}^{N} T^{p(kn)}h\right|\right| = 0. \tag{3.15}$$

This will be established by induction on $\deg p$. For $\deg p = 1$ it follows from the mean ergodic theorem, because $\frac{1}{N}\sum_{n=1}^{N} T^{ln}h \to P_l h$ in norm, where P_l is the projection onto the space of T^l-invariant functions. Since the space of T^l-invariant functions is contained in $\mathcal{H}_r$, and $h \in \mathcal{H}_r^{\perp}$, we have $P_l h = 0$.

For $\deg p > 1$, we apply Lemma 3.5.2, with $x_n = T^{p(kn)}h$. For any $r \in \mathbf{N}$ we have

$$\begin{aligned}
&\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\langle x_n, x_{n+r}\rangle \\
&= \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\int T^{p(kn)}h\, T^{p(kn+kr)}h\, d\mu \\
&= \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\int h\, T^{p(kn+kr)-p(kn)}h\, d\mu \\
&= \lim_{N\to\infty}\int T^{-p(kr)}h\Big(\frac{1}{N}\sum_{n=1}^{N} T^{p(kn+kr)-p(kn)-p(kr)}h\Big)d\mu \\
&\leq \lim_{N\to\infty}||h||\cdot\left|\left|\frac{1}{N}\sum_{n=1}^{N} T^{p(kn+kr)-p(kn)-p(kr)}h\right|\right| = 0.
\end{aligned}$$

The last inequality is the Schwarz inequality. In the last line, we have used the induction hypothesis, as the degree of $q(kx) = p(kn + kx) - p(kn) - p(kr)$ is $(\deg p) - 1$. Moreover, $q(k \cdot 0) = 0$. This establishes the claim.

Combining (3.14) and (3.15), we get that for large enough N,

$$\left|\left|\frac{1}{N}\sum_{n=1}^{N} T^{p(kn)}1_A - g\right|\right| \leq 3\epsilon,$$

which implies that

$$\left|\frac{1}{N}\sum_{n=1}^{N}\langle T^{p(kn)}1_A, 1_A\rangle - \langle g, 1_A\rangle\right| = \left|\left\langle\left(\frac{1}{N}\sum_{n=1}^{N}T^{p(kn)}1_A\right) - g, 1_A\right\rangle\right|$$
$$\leq \left|\left|\frac{1}{N}\sum_{n=1}^{N}T^{p(kn)}1_A - g\right|\right| \cdot ||1_A|| \leq 3\epsilon.$$

In particular, for some n we have

$$\mu(A\cap T^{-p(kn)}A) = \langle T^{p(kn)}1_A, 1_A\rangle \geq \langle g, 1_A\rangle - 3\epsilon \geq \mu(A)^2 - 3\epsilon.$$

Since ϵ is arbitrary, we are done.

□

As a corollary (of the proof) we get that in fact for every $\epsilon > 0$ one may find n such that $\mu(A\cap T^{-p(n)}) \geq \mu(A)^2 - \epsilon$. In the next section, we show that n may be chosen from any prescribed IP-set.

3.6 Polynomial recurrence along IP-sets.

In this section, we offer a refinement of the result proved in the previous section. The approach follows [BFM], where a more general result is proved.

Theorem 3.6.1. Let $p(x) \in \mathbf{Z}[x]$ with $p(0) = 0$, let $(X, \mathcal{A}, \mu, T)$ be an invertible measure preserving system, and let $\mu(A) > 0$. Then for every IP-set $\Gamma \subset \mathbf{Z}$, there exists $n \in \Gamma$ such that $\mu(A\cap T^{-p(n)}A) > 0$.

Exercise 3.45. Show that Theorem 3.6.1 is equivalent to the fact that for any IP-set Γ, any polynomial $p(x) \in \mathbf{Z}[x]$, and any set $E \subset \mathbf{Z}$ with $d^*(E) > 0$ there exists $n \in \Gamma$ and $a \in \mathbf{Z}$ such that $\{a, a+p(n)\} \subset E$.

Theorem 3.6.1 tells us that IP-sets are "good" for recurrence along polynomials. This is significant in that difference sets (which according to Exercise 3.21 are themselves sets of recurrence) are not good along polynomials (see [F, p. 177]). The power of IP-sets for multiple recurrence was profoundly established in [FK2]. (In the next chapter we shall demonstrate this on the level of double recurrence.)

Our plan is to derive Theorem 3.6.1 as a corollary of the following Hilbert space fact.

Theorem 3.6.2. Let $(n_\alpha)_{\alpha\in\mathcal{F}} \subset \mathbf{Z}$ be an IP-set and let $p(x) \in \mathbf{Z}[x]$ be a polynomial having zero constant term. Let $\mathcal{H}$ be a Hilbert space and let U be a unitary operator on $\mathcal{H}$. If $\mathcal{F}^{(1)}$ is an IP-ring having the property that

$$Pf = \underset{\alpha\in\mathcal{F}^{(1)}}{\text{IP-lim}}\ U^{p(n_\alpha)}f$$

exist weakly for all $f \in \mathcal{H}$, then P is an orthogonal projection.

We show that Theorem 3.6.2 implies Theorem 3.6.1. Let $\Gamma = (n_\alpha)_{\alpha \in \mathcal{F}} \subset \mathbf{Z}$ be the IP-set in question. Recall the fact that T acts as a unitary operator on $L^2(X, \mathcal{A}, \mu)$. Let $A \in \mathcal{A}$ with $\mu(A) > 0$. Restricting attention to the subspace of $L^2(X, \mathcal{A}, \mu)$ spanned by the orbit under T of 1_A, we may assume that $L^2(X, \mathcal{A}, \mu)$ is separable.

Exercise 3.46. There exists an IP-ring $\mathcal{F}^{(1)}$ having the property that

$$\underset{\alpha \in \mathcal{F}^{(1)}}{\text{IP-lim}} \; T^{p(n_\alpha)} f = Pf$$

exists weakly for every $f \in L^2(X, \mathcal{A}, \mu)$. (*Hint: pick an IP-ring for which the limit in question exists for a countable dense set of f's.*)

By Theorem 3.6.2, P is an orthogonal projection.

Exercise 3.47. Show that $||P1_A|| \geq \mu(A)$. (*Hint: the constants are in the range of P. Use Lemma 3.4.5 c.*)

We now have

$$\begin{aligned} \underset{\alpha \in \mathcal{F}^{(1)}}{\text{IP-lim}} \; \mu(A \cap T^{-p(n_\alpha)} A) &= \underset{\alpha \in \mathcal{F}^{(1)}}{\text{IP-lim}} \; \langle 1_A, T^{p(n_\alpha)} 1_A \rangle \\ &= \langle 1_A, P1_A \rangle = ||P1_A||^2 \geq \mu(A)^2. \end{aligned}$$

□

Hence, once again, we may not only obtain an intersection of positive measure; we may guarantee that the size of this intersection is as close as desired to $\mu(A)^2$.

The proof of Theorem 3.6.2 we offer, from [BFM], has many similarities to the proof in the previous section. Namely, we prove the theorem via a Hilbert space splitting. We shall make use of the following fact.

Exercise 3.48. Let $\mathcal{H}$ be a Hilbert space. If $x_n \to x$ weakly and $||x_n|| \to ||x||$ then $x_n \to x$ in norm (i.e. $||x_n - x|| \to 0$). Similar statements hold for IP-limits.

The following lemma (a version of which appears in [FK2]) serves a purpose analogous to that served by Lemma 3.5.2 in the previous section.

Lemma 3.6.3. Suppose that $(x_\alpha)_{\alpha \in \mathcal{F}}$ is a bounded $\mathcal{F}$-sequence in a Hilbert space and $\mathcal{F}^{(1)}$ is an IP-ring. If

$$\underset{(\beta,\alpha) \in (\mathcal{F}^{(1)})^2_<}{\text{IP-lim}} \; \langle x_\alpha, x_{\alpha \cup \beta} \rangle = 0$$

then for some subring $\mathcal{F}^{(2)} \subset \mathcal{F}^{(1)}$,

$$\underset{\alpha \in \mathcal{F}^{(2)}}{\text{IP-lim}} \; x_\alpha = 0$$

in the weak topology.

Proof. Let $\mathcal{F}^{(2)} \subset \mathcal{F}^{(1)}$ be an IP-ring with the property that

$$\operatorname*{IP-lim}_{\alpha \in \mathcal{F}^{(2)}} x_\alpha = u$$

exists weakly. We have, for all $k \in \mathbf{N}$,

$$\operatorname*{IP-lim}_{(\alpha_1, \cdots, \alpha_k) \in (\mathcal{F}^{(2)})^k_<} \langle x_{\alpha_1 \cup \cdots \cup \alpha_k}, u \rangle = ||u||^2,$$

from which it follows that for all $m \in \mathbf{N}$,

$$\operatorname*{IP-lim}_{(\alpha_1, \cdots, \alpha_m) \in (\mathcal{F}^{(2)})^m_<} \left\langle \frac{1}{m} \sum_{k=1}^{m} x_{\alpha_k \cup \cdots \cup \alpha_m}, u \right\rangle = ||u||^2. \tag{3.16}$$

On the other hand,

$$\begin{aligned}
&\lim_{m \to \infty} \operatorname*{IP-lim}_{(\alpha_1, \cdots, \alpha_m) \in (\mathcal{F}^{(2)})^m_<} \left|\left| \frac{1}{m} \sum_{k=1}^{m} x_{\alpha_k \cup \cdots \cup \alpha_m} \right|\right|^2 \\
&= \lim_{m \to \infty} \operatorname*{IP-lim}_{(\alpha_1, \cdots, \alpha_m) \in (\mathcal{F}^{(2)})^m_<} \frac{1}{m^2} \sum_{k,j=1}^{m} \langle x_{\alpha_k \cup \cdots \cup \alpha_m}, x_{\alpha_j \cup \cdots \cup \alpha_m} \rangle \\
&= \lim_{m \to \infty} \frac{1}{m^2} \sum_{k,j=1}^{m} \operatorname*{IP-lim}_{(\alpha_1, \cdots, \alpha_m) \in (\mathcal{F}^{(2)})^m_<} \langle x_{\alpha_k \cup \cdots \cup \alpha_m}, x_{\alpha_j \cup \cdots \cup \alpha_m} \rangle \\
&= \lim_{m \to \infty} \frac{1}{m^2} \sum_{k=1}^{m} \left|\left| x_{\alpha_k \cup \cdots \cup \alpha_m} \right|\right|^2 = 0.
\end{aligned}$$

This together with (3.16) gives $u = 0$.

$\square$

Exercise 3.49. Let $(x_i)_{i=1}^\infty$ and $(y_i)_{i=1}^\infty$ be bounded sequences in a Hilbert space $\mathcal{H}$. Let B be a closed, bounded, separable subset of $\mathcal{H}$ containing both sequences and let ρ be any metric on B which yields the weak topology. If $||x_n - y_n|| \to 0$ then $\rho(x_n, y_n) \to 0$.

We first restrict to a special case.

Proof of Theorem 3.6.2 for the case $p(x) = x^2$. Let $x \in \mathcal{H}$. We will show that $Px = P^2x$. Since x is arbitrary, this will establish that P is idempotent. Since plainly $||P|| \leq 1$, By Lemma 3.4.6 this will imply that P is an orthogonal projection, completing the proof.

In showing that $Px = P^2x$ we may restrict attention to a separable subspace of $\mathcal{H}$ which contains the entire orbit $\{U^n x : n \in \mathbf{Z}\}$. Hence we may assume without loss of generality that $\mathcal{H}$ is separable. Therefore, by passing to a subring of $\mathcal{F}^{(1)}$ if necessary we may assume that

$$\underset{\alpha\in\mathcal{F}^{(1)}}{\text{IP-lim}}\ U^{kn_\alpha} f = P_k f$$

exists weakly for every $f \in \mathcal{H}$ and every $k \in \mathbf{N}$. By Theorem 3.4.9, each P_k is an orthogonal projection. Let

$$\mathcal{H}_r = \overline{\{f \in \mathcal{H} : P_k f = f \text{ for some } k \in \mathbf{N}\}}.$$

Exercise 3.50. $\mathcal{H}_r$ is a closed subspace. (*Hint: use Exercise 3.48.*)

(The r in $\mathcal{H}_r$ is for *rigid.*) We have $\mathcal{H} = \mathcal{H}_r \oplus \mathcal{H}_r^\perp$. Write $x = g + h$, where $g \in \mathcal{H}_r$ and $h \in \mathcal{H}_r^\perp$. The proof that $Px = P^2x$ consists of two steps: first we show that $Pg = P^2g$, then we show that $Ph = 0$ (so that $Px = Pg = P^2g = P(Pg) = P(Px) = P^2x$).

First step: $Pg = P^2g$. Notice that the set of vectors y for which $Py = P^2y$ is closed. Therefore it suffices to show that $P^2f = Pf$ for any $f \in \mathcal{H}$ for which there exists $k \in \mathbf{N}$ with $P_kf = f$. Fix such f and k. Let ρ be a metric for the weak topology on a closed ball which is big enough to contain the vectors we deal with below. By passing to a subring if necessary we may assume that for all $\alpha \in \mathcal{F}^{(1)}$, $k|n_\alpha$ (see Exercise 2.12). For all $\alpha, \beta \in \mathcal{F}^{(1)}$ with $\alpha < \beta$ we have

$$\begin{aligned}\rho(Pf, P^2f) \le \rho(Pf, U^{(n_\alpha+n_\beta)^2}f) &+ \rho(U^{(n_\alpha+n_\beta)^2}f, U^{n_\alpha^2+n_\beta^2}f)\\ &+ \rho(U^{n_\alpha^2+n_\beta^2}f, PU^{n_\alpha^2}f) + \rho(U^{n_\alpha^2}Pf, P^2f).\end{aligned}$$

(We have used here the fact that P commutes with U. See Exercise 3.40.) By choosing α far enough out (keeping in mind that $\beta > \alpha$ and $n_{\alpha\cup\beta} = n_\alpha + n_\beta$), the first and fourth quantities on the right may be made as small as desired. The third quantity may be made as small as desired by picking β far enough out after having fixed α. Finally, $||U^{(n_\alpha+n_\beta)^2}f - U^{n_\alpha^2+n_\beta^2}f|| = ||U^{2n_\beta n_\alpha}f - f||$ may be made as small as desired by picking β far enough out after fixing α (since $k|n_\alpha$ and $T^{kn_\beta}f \to f$ *in norm*, as may be determined from Exercise 3.48). It follows from Exercise 3.48 that the second quantity on the right may be chosen small. Hence $Pf = P^2f$ and by our earlier remark $Pg = P^2g$.

Second step: $Ph = 0$. We use Lemma 3.6.3. Namely, it is sufficient to demonstrate that

$$\underset{(\beta,\alpha)\in(\mathcal{F}^{(1)})^2_<}{\text{IP-lim}}\ \langle U^{n_\alpha^2}h, U^{(n_\alpha+n_\beta)^2}h\rangle = 0.$$

Rearranging the inner product in the limit, it looks like: $\langle U^{-n_\beta^2}h, U^{2n_\alpha n_\beta}h\rangle$. For any fixed β, this approaches $\langle U^{-n_\beta^2}h, P_{2n_\beta}h\rangle = 0$ as $\alpha \in \mathcal{F}^{(1)}$ goes to ∞, completing the proof.

□

We now proceed to the general proof. We require two lemmas.

Lemma 3.6.4. Suppose that $s \in \mathbf{N}$ and that $(v_\alpha)_{\alpha\in\mathcal{F}}$ is an $\mathcal{F}$-sequence in $\mathbf{Z}^s$. Then for any IP-ring $\mathcal{F}^{(1)}$ there exists $l \le s$, an $l-$dimensional subgroup $V \subset \mathbf{Z}^s$, and an IP-subring $\mathcal{F}^{(2)} \subset \mathcal{F}^{(1)}$, such that $\{v_\alpha : \alpha \in \mathcal{F}^{(2)}\} \subset V$ and such that (if $l > 0$) whenever $(\alpha_1,\cdots,\alpha_l) \in (\mathcal{F}^{(2)})^l_<$ the set $\{v_{\alpha_1},\cdots,v_{\alpha_l}\}$ is linearly independent.

Proof. Let $l \ge 0$ be minimal with respect to the property that there exists an $l-$dimensional subgroup $V \subset \mathbf{Z}^s$ and an IP-ring $\mathcal{G} \subset \mathcal{F}^{(1)}$ (both of which we now fix) such that $\{v_\alpha : \alpha \in \mathcal{G}\} \subset V$. If $l = 0$ we are done, so we now assume that $l > 0$. Then $\mathcal{G}^l_<$ is the union of the two sets

$$S_1 = \big\{(\alpha_1,\cdots,\alpha_l) \in \mathcal{G}^l_< : \{v_{\alpha_1},\cdots,v_{\alpha_l}\} \text{ is linearly independent}\big\},$$

$$S_2 = \big\{(\alpha_1,\cdots,\alpha_l) \in \mathcal{G}^l_< : \{v_{\alpha_1},\cdots,v_{\alpha_l}\} \text{ is linearly dependent}\big\}.$$

By the Milliken-Taylor theorem (see Section 2.2) there exists an IP-ring $\mathcal{F}^{(2)} \subset \mathcal{G}$ such that $(\mathcal{F}^{(2)})^l_< \subset S_i$, where either $i = 1$ or $i = 2$. Suppose $i = 2$. Let n be maximal with respect to the property that there exist $\alpha_1,\cdots,\alpha_n \in \mathcal{F}^{(2)}$ (which we now fix) with $\alpha_1 < \alpha_2 < \cdots < \alpha_n$ such that the set $\{v_{\alpha_1},\cdots,v_{\alpha_n}\}$ is linearly independent. If $n = 0$ then $v_\alpha = 0$ for all $\alpha \in \mathcal{F}^{(2)}$, contradicting the fact that $l > 0$. Therefore we may assume that $n > 0$. Let V' be the n dimensional subgroup consisting of all elements $v \in \mathbf{Z}^s$ for which, for some $k \in \mathbf{N}$, kv lies in the subgroup generated by $\{v_{\alpha_1},\cdots,v_{\alpha_n}\}$. V' contains v_α for every α in the IP-ring $\{\alpha \in \mathcal{F}^{(2)} : \alpha > \alpha_n\}$, contradicting the minimality of l. Hence $i = 1$, completing the proof.

□

Lemma 3.6.5. Suppose that $l \in \mathrm{N}$, $\mathcal{F}^{(1)}$ is an IP-ring, $\mathcal{H}$ is a Hilbert space and $\{P(\alpha)\}_{\alpha\in\mathcal{F}}$ is an $\mathcal{F}$-sequence of commuting orthogonal projections on $\mathcal{H}$ such that whenever $(\alpha_1,\cdots,\alpha_l) \in (\mathcal{F}^{(1)})^l_<$ and $f \in \mathcal{H}$ we have $\big(\prod_{i=1}^l P(\alpha_i)\big)f = 0$. Then

$$\operatorname*{IP-lim}_{\alpha\in\mathcal{F}^{(1)}} ||P(\alpha)f|| = 0.$$

Proof. It suffices to show that for an arbitrary sequence $\beta_1 < \beta_2 < \cdots$, where $\beta_i \in \mathcal{F}^{(1)}$, and $f \in \mathcal{H}$, we have $\lim_{i\to\infty} ||P(\beta_i)f|| = 0$. Fix N. For non-empty $A \subset \{1,\cdots,N\}$ put

$$\mathcal{H}_A = \Big(\bigcap_{i\in A} P(\beta_i)\mathcal{H}\Big) \cap \Big(\bigcap_{i\in\{1,\cdots,N\}\setminus A} \big(P(\beta_i)\mathcal{H}\big)^\perp\Big).$$

Also put $\mathcal{H}_\emptyset = \bigcap_{i=1}^N \big(P(\beta_i)\mathcal{H}\big)^\perp$. Then $\mathcal{H} = \oplus_{A\subset\{1,\cdots,N\},\,|A|<l}\mathcal{H}_A$. Let P_A denote the orthogonal projection onto $\mathcal{H}_A$. Then for each $g \in \mathcal{H}$,

$$||g||^2 = \sum_{A\subset\{1,\cdots,N\},\,|A|<l} ||P_A g||^2,$$

so that

$$\sum_{i=1}^{N} ||P(\beta_i)f||^2 = \sum_{i=1}^{N} \sum_{A\subset\{1,\cdots,N\}, i\in A} ||P_A f||^2$$
$$= \sum_{A\subset\{1,\cdots,N\},\ |A|<l} |A|\ ||P_A f||^2 \leq l||f||^2.$$

But this is true for any N, so $\sum_{i=1}^{\infty} ||P(\beta_i)f||^2 < \infty$ and $\lim_{i\to\infty} ||P(\beta_i)f|| = 0$, as desired.

□

Proof of Theorem 3.6.2. The proof is by induction on $d = \deg p(x)$. The case $d = 1$ is Theorem 3.4.9. Suppose now that $\deg p(x) = d$ and the theorem is valid for polynomials of degree less than d. As in the x^2 case, we may assume without loss of generality that $\mathcal{H}$ is separable, and we need only show that $P_p = P_p^2$.

We adopt the following notation:

$$P_q f = \text{IP-}\lim_{\alpha\in\mathcal{F}^{(1)}} U^{q(n_\alpha)} f, \tag{3.17}$$

where $q(x) \in \mathbf{Z}[x]$ with $q(0) = 0$. As usual, we shall assume, by passing to subrings if necessary, that all IP-limits encountered exist (including those in (3.17) for all $q(x)$ of degree less than d and all $f \in \mathcal{H}$–see Exercise 3.46).

Let $s = d - 1$. $W = \{q(x) \in \mathbf{Z}[x] : \deg q < d,\ q(0) = 0\}$ is isomorphic to $\mathbf{Z}^s$. For each $\alpha \in \mathcal{F}^{(1)}$ let

$$q^{(\alpha)}(x) = p(n_\alpha + x) - p(x) - p(n_\alpha).$$

Then $q^{(\alpha)}(x) \in W$. By Lemma 3.6.4, there exist $l \leq s$, an l–dimensional subgroup $V \subset W$, and an IP-ring $\mathcal{F}^{(2)} \subset \mathcal{F}^{(1)}$ such that $\{q^{(\alpha)} : \alpha \in \mathcal{F}^{(2)}\} \subset V$ and such that whenever $(\alpha_1, \cdots, \alpha_l) \in (\mathcal{F}^{(2)})^l_<$, the set $\{q^{(\alpha_i)} : 1 \leq i \leq l\}$ is linearly independent.

Exercise 3.51. For any $f \in \mathcal{H}$, the set $\{q \in V : P_q f = f\}$ is a subgroup of V.

It follows that if Y is an l–dimensional subgroup of V generated by the linearly independent set $\{q^{(\alpha_i)} : 1 \leq i \leq l\}$, where $\alpha_1 < \alpha_2 < \cdots < \alpha_l$, we have

$$\{f : P_q f = f \text{ for every } q \in Y\} = \Big(\prod_{i=1}^{l} P_{q^{(\alpha_i)}}\Big)\mathcal{H} = P_Y\mathcal{H}.$$

Exercise 3.52. Products of commuting orthogonal projections are orthogonal projections. In particular, P_Y is an orthogonal projection since the projections $P_{q^{(\alpha_i)}}$ commute.

For every $n \in \mathbf{N}$ let $V_n = n!W \cap V$. Then V_n is an l-dimensional subgroup of V for each n and P_{V_n} is an increasing sequence of orthogonal projections, so that $P = \lim_{n\to\infty} P_{V_n}$ is an orthogonal projection. Furthermore, for every l-dimensional subgroup $Y \subset V$, $V_n \subset Y$ for all large enough n. Hence

$$P\mathcal{H} = \overline{\{f \in \mathcal{H} : \; P_{V_n} f = f \text{ for some } n \in \mathbf{N}\}},$$

$$(P\mathcal{H})^{\perp} = \{f \in \mathcal{H} : \; P_Y f = 0 \text{ for every } l\text{-dimensional subgroup } Y \subset V\}.$$

According to our earlier remarks, all we must show is that for an arbitrarily chosen $f \in \mathcal{H}$, which we now fix, $P_p f = P_p^2 f$. We may assume that $||f|| < 1$. Write $f = g + h$, where $g \in P\mathcal{H}$ and $h \in (P\mathcal{H})^{\perp}$. Let $x_\alpha = U^{p(n_\alpha)}h$. We claim that $P_p h = \underset{\alpha\in\mathcal{F}^{(1)}}{\text{IP-lim}}\; x_\alpha = 0$. By Lemma 3.6.3 it suffices to show that

$$\underset{\alpha\in\mathcal{F}^{(2)}}{\text{IP-lim}}\;\underset{\beta\in\mathcal{F}^{(2)}}{\text{IP-lim}}\;\langle x_{\alpha\cup\beta}, x_\beta\rangle = 0.$$

Notice that by the properties ascribed to $\mathcal{F}^{(2)}$ earlier and the fact that $h \in (P\mathcal{H})^{\perp}$, we have that whenever $(\alpha_1, \cdots, \alpha_l) \in (\mathcal{F}^{(2)})^l_<$, $P_{q^{(\alpha_i)}} h = 0$. Therefore by Lemma 3.6.5 we have

$$\underset{\alpha\in\mathcal{F}^{(2)}}{\text{IP-lim}}\;||P_{q^{(\alpha)}} h|| = 0$$

and

$$\begin{aligned}
&\underset{\alpha\in\mathcal{F}^{(2)}}{\text{IP-lim}}\;\underset{\beta\in\mathcal{F}^{(2)}}{\text{IP-lim}}\;\langle U^{p(n_{\alpha\cup\beta})}h,\; U^{p(n_\beta)}h\rangle \\
=&\underset{\alpha\in\mathcal{F}^{(2)}}{\text{IP-lim}}\;\underset{\beta\in\mathcal{F}^{(2)}}{\text{IP-lim}}\;\langle U^{q^{(\alpha)}}h,\; U^{-p(n_\alpha)}h\rangle \\
=&\underset{\alpha\in\mathcal{F}^{(2)}}{\text{IP-lim}}\;\langle P_{q^{(\alpha)}}h,\; U^{-p(n_\alpha)}h\rangle \\
\le&\; ||h||\;\underset{\alpha\in\mathcal{F}^{(2)}}{\text{IP-lim}}\;||P_{q^{(\alpha)}}h|| = 0.
\end{aligned}$$

This establishes our claim. Next we show that $P_p g = P_p^2 g$. Let $\epsilon > 0$ be arbitrary. Choose g' with $||g'|| < 1$ and $n \in \mathbf{N}$ with $P_{V_n} g' = g'$ such that $||g - g'|| < \epsilon$. Let ρ be a metric on the unit ball of $\mathcal{H}$ for the weak topology satisfying $\rho(x, y) \le ||x - y||$. There exists $\alpha_0 \in \mathcal{F}^{(2)}$ such that for every $\alpha \in \mathcal{F}^{(2)}$ with $\alpha > \alpha_0$, $\rho\big(U^{p(n_\alpha)}g',\; P_p g'\big) < \epsilon$ and

$$\rho\big(U^{p(n_\alpha)}P_p g',\; P_p^2 g'\big) < \epsilon. \tag{3.18}$$

Let $\alpha \in \mathcal{F}^{(2)}$ be chosen with $\alpha > \alpha_0$ and such that $n!$ divides n_α. This will ensure that $q^{(\alpha)} \in V_n$. For every $\beta \in \mathcal{F}^{(2)}$, $\beta > \alpha$, we have $(\alpha\cup\beta) > \alpha_0$ as well, so that

$$\rho\big(U^{p(n_\alpha)+p(n_\beta)+q^{(\alpha)}(n_\beta)}g',\; P_p g'\big) = \rho\big(U^{p(n_{\alpha\cup\beta})}g',\; P_p g'\big) < \epsilon. \tag{3.19}$$

Since $q^{(\alpha)} \in V_n$ there exists $\beta_0 \in \mathcal{F}^{(2)}$, $\beta_0 > \alpha$, such that for every $\beta \in \mathcal{F}^{(2)}$ with $\beta > \beta_0$, $||U^{q^{(\alpha)}(n_\beta)}g' - g'|| < \epsilon$, which implies that

$$\rho\big(U^{p(n_\alpha)+p(n_\beta)+q^{(\alpha)}(n_\beta)}g', U^{p(n_\alpha)+p(n_\beta)}g'\big) < \epsilon. \tag{3.20}$$

We may now fix such a β with the further property that

$$\rho\big(U^{p(n_\alpha)+p(n_\beta)}g',\ U^{p(n_\alpha)}P_p g'\big) < \epsilon. \tag{3.21}$$

(We have used weak continuity of $U^{p(n_\alpha)}$.) Now (3.18)-(3.21) together with the triangle inequality give us $\rho\big(P_p g',\ P_p^2 g'\big) < 4\epsilon$. Recall that $||Px|| \leq ||x||$. Therefore

$$\rho\big(P_p g,\ P_p g'\big) < \epsilon \text{ and } \rho\big(P_p^2 g,\ P_p^2 g'\big) < \epsilon,$$

which gives us finally $\rho\big(P_p g,\ P_p^2 g\big) < 6\epsilon$. Since ϵ was arbitrary, we have

$$P_p f = P_p g = P_p^2 g = P_p^2 f.$$

This establishes that P_p is idempotent and completes the proof of Theorem 3.6.2.

$\square$

Chapter 4

Three Ergodic Roth Theorems

4.1 Ergodicity and weak mixing.

One of the goals of the presentation in the first three chapters was to be as self-contained as possible. Accordingly, with the exception of the chromatic result of Lovász used in Section 3.3, very little was assumed, beyond some basic analysis and point-set topology.

In proceeding to more difficult results in the fields of density combinatorics and recurrence in measure preserving systems, we shall abandon this policy. As a result, the proper audience for the next two chapters, in are presented some topical methods for dealing with multiple recurrence, is apt to consist of people who are already familiar with ergodic theory. We shall therefore take for granted whatever commonly used results from measure theory and ergodic theory we need, including the theory of Hilbert-Schmidt operators, disintegration of a measure over a sub-σ-algebra (including ergodic decomposition), conditional expectation, and Birkhoff's pointwise ergodic theorem.

Certain notions we will be using, such as "relative weak mixing", or "relative compactness", are derivatives of well-known "absolute" properties of measure preserving systems. familiarity with which is tacitly assumed; we will give only a brief synopsis of such basic matters in this section. The reader is invited to peruse for example [P] for more information. However, specialized material that is not likely to be familiar will be developed fully.

Ergodicity

An invertible measure preserving system $(X, \mathcal{A}, \mu, T)$ is said to be *ergodic* if the only measurable sets A for which $\mu(A \triangle T^{-1}A) = 0$ satisfy $\mu(A) \in \{0, 1\}$. Ergodicity is a "mixing" notion, as attested to by Theorem 4.1.1 below.

Note. In general, we will denote by $L^2(X, \mathcal{A}, \mu)$ the set of *real-valued* square integrable functions on X, and by $L^2_{\mathbb{C}}(X, \mathcal{A}, \mu)$ the set of *complex-valued* square integrable functions on X. $L^2(X, \mathcal{A}, \mu)$ is a real Hilbert space and $L^2_{\mathbb{C}}(X, \mathcal{A}, \mu)$ is a complex Hilbert space. We shall be mainly concerned with $L^2(X, \mathcal{A}, \mu)$.

Recall that T induces a unitary operator on either $L^2(X, \mathcal{A}, \mu)$ or $L^2_{\mathbf{C}}(X, \mathcal{A}, \mu)$ according to the rule $Tf(x) = f(Tx)$.

An *eigenfunction* for T is some $f \in L^2(X, \mathcal{A}, \mu)$ (respectively $L^2_c(X, \mathcal{A}, \mu)$) such that $f = \lambda f$ for some $\lambda \in \mathbf{R}$ (respectively $\mathbf{C}$). λ is called the *eigenvalue* corresponding to f. Since in our case T is always unitary, all eigenvalues λ must satisfy $|\lambda| = 1$. λ is called a *simple* eigenvalue if the dimension of the space $\{f : Tf = \lambda f\}$ is 1.

Theorem 4.1.1. Let $(X, \mathcal{A}, \mu, T)$ be a measure preserving system. The following are equivalent:

1. T is ergodic.
2. For all $A, B \in \mathcal{A}$, $\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mu(A \cap T^{-n}B) = \mu(A)\mu(B)$.
3. For all $f \in L^2(X, \mathcal{A}, \mu)$, $\lim_{N\to\to\infty} ||\frac{1}{N} \sum_{n=1}^{N} T^n f - \int f \, d\mu|| = 0$.
4. For all $f, g \in L^2$, $\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \int gT^n f \, d\mu = \left(\int f \, d\mu\right)\left(\int g \, d\mu\right)$.
5. 1 is a simple eigenvalue of the unitary operator induced by T.

Property 2 in the above theorem is sometimes called *mixing in the mean.* As for property 5, note that 1 is always an eigenvalue because $Tc = c$ for any constant function c.

Weak Mixing

Although ergodicity is a useful notion, for our purposes it isn't especially distinguishing. Indeed, we will see shortly that any system can be disintegrated into ergodic components, allowing us to assume ergodicity without loss of generality in the proofs of our recurrence theorems. There are many less general notions of mixing, the most useful of which for our purposes is that of *weak mixing.*

A system $(X, \mathcal{A}, \mu, T)$ is weak mixing if the product system $(X \times X, \mathcal{A} \otimes \mathcal{A}, \mu \times \mu, T \times T)$ is ergodic. (Here $\mathcal{A} \otimes \mathcal{A}$ is the σ-algebra on $X \times X$ generated by the measurable rectangles $\{A \times B : A, B \in \mathcal{A}\}$.) It is easily checked that any weak mixing system is therefore in particular ergodic, but there are plenty of ergodic systems which are not weak mixing. For example, let $X = \{\xi \in \mathbf{C} : |\lambda| = 1\}$ with Lebesgue measure and define T on X by $T\xi = e^{2\pi\alpha i}\xi$, where α is irrational. One may check that T is ergodic. However, letting $f(\xi) = \xi$, one checks that the function $f \otimes \overline{f}$, defined by $f \otimes \overline{f}(x, y) = f(x)\overline{f}(y)$, is a non-constant $T \times T$-invariant function.

In general, if f is non-constant, $Tf = \lambda f$, and $Tg = \overline{\lambda} g$ (there is always such a g if there is such an f–namely $g = \overline{f}$), then $f \otimes g$ will be a non-constant invariant function for the product system. Thus we see that the presence of non-constant eigenfunctions precludes weak mixing. Conversely, the absence of non-constant eigenfunctions implies weak mixing.

Definition 4.1.2. Let $(x_n)_{n\in\mathbf{Z}}$ be a sequence in a topological space. We write

$$D\text{-}\lim_{n\to\infty} x_n = x$$

if for every neighborhood U of x, $\overline{d}(\{n : x_n \notin U\}) = 0$.

Exercise 4.1. Let $\{x_n\}_{n=1}^{\infty}$ be a bounded sequence of real numbers. Then

(a) $D\text{-}\lim_{n\to\infty} x_n = 0$ if and only if $\frac{1}{N}\sum_{n=1}^{N} x_n^2 \to 0$.
(b) $D\text{-}\lim_{n\to\infty} x_n = x$ if and only if $\frac{1}{N}\sum_{n=1}^{N} x_n \to x$ and $\frac{1}{N}\sum_{n=1}^{N} x_n^2 \to x^2$.

Theorem 4.1.3. Let $(X, \mathcal{A}, \mu, T)$ be an invertible measure preserving system. The following are equivalent.

1. T is weak mixing.
2. For all $A, B \in \mathcal{A}$, $D\text{-}\lim_{n\to\infty} \mu(A \cap T^{-n}B) = \mu(A)\mu(B)$.
3. For all $f, g \in L^2$, $D\text{-}\lim_{n\to\infty} \int fT^n g\, d\mu = \left(\int f\, d\mu\right)\left(\int g\, d\mu\right)$.
4. $T \times T$ is weak mixing.
5. The constants are the only eigenfunctions for T in $L^2_{\mathbf{C}}(X, \mathcal{A}, \mu)$.

Property 2 in the above theorem is the source for the name "weak mixing". A seemingly more natural notion is that of "strong mixing"; T is strong mixing if $\lim_{n\to\infty} \mu(A \cap T^{-n}B) = \mu(A)\mu(B)$ for all $A, B \in \mathcal{A}$. However, this seeming naturalness is something of an illusion. As it turns out, the weak mixing property is far more useful for us, the characterization of the previous theorem being evidence for this. Further evidence is that weak mixing is easily seen to imply weak mixing of higher orders. This is the content of Theorem 4.1.4 below. (It is unknown whether strong mixing implies strong mixing of all orders.)

Exercise 4.2. Adapt the proof of Lemma 3.5.2 to establish the following fact from [B2]:

Suppose that $\{x_n : n \in \mathbf{Z}\}$ is a bounded sequence of vectors in a Hilbert space $\mathcal{H}$. If

$$D\text{-}\lim_{h} \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \langle x_n, x_{n+h}\rangle = 0,$$

then

$$\lim_{N\to\infty} \left|\left|\frac{1}{N}\sum_{n=1}^{N} x_n\right|\right| = 0.$$

Theorem 4.1.4. (See [FKO].) Let $(X, \mathcal{A}, \mu, T)$ be a weak mixing system and suppose that $f_0, \cdots, f_k \in L^{\infty}(X, \mathcal{A}, \mu)$. Then

$$\lim_{N\to\infty} \left|\left|\frac{1}{N}\sum_{n=1}^{N}\prod_{i=1}^{k} T^{in} f_i - \prod_{i=1}^{k}\int f_i\, d\mu\right|\right| = 0.$$

Proof. We proceed by induction on k. The case $k = 1$ holds by property 3 in Theorem 4.1.3. Suppose the result holds for $k-1$. We need to show that

$$\lim_{N\to\infty} \left|\left|\frac{1}{N}\sum_{n=1}^{N} T^n f_1 T^{2n} f_2 \cdots T^{kn} f_k - \prod_{i=1}^{k}\int f_i\, d\mu\right|\right| = 0. \tag{4.1}$$

Exercise 4.3. In establishing (4.1) it suffices to assume that $\int f_j \, d\mu = 0$ for some j. (*Hint: consider the identity*

$$\prod_{i=1}^{k} a_i - \prod_{i=1}^{k} b_i = (a_1 - b_1)b_2 \cdots b_k + a_1(a_2 - b_2)b_3 \cdots b_k + \cdots + a_1 \cdots a_{k-1}(a_k - b_k).)$$

Hence we must, under the assumption that $\int f_j \, d\mu = 0$ for some j, show that

$$\lim_{N\to\infty} \left\| \frac{1}{N} \sum_{n=1}^{N} T^n f_1 T^{2n} f_2 \cdots T^{kn} f_k \right\| = 0.$$

We apply Exercise 4.2. Namely set $x_n = \prod_{i=1}^{k} T^{in} f_i$. Then for all h,

$$\begin{aligned}
&D\text{-}\lim_{h} \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \langle x_n, x_{n+h} \rangle \\
=&D\text{-}\lim_{h} \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \int \Big(\prod_{i=1}^{k} T^{in} f_i \Big) \Big(\prod_{i=1}^{k} T^{i(n+h)} f_i \Big) \, d\mu \\
=&D\text{-}\lim_{h} \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \int (f_1 T^h f_1) \prod_{i=2}^{k} T^{(i-1)n} (f_i T^{ih} f_i) \, d\mu \\
=&D\text{-}\lim_{h} \prod_{i=1}^{k} \int f_i T^{ih} f_i \, d\mu = 0.
\end{aligned}$$

In moving to the last line, the induction hypothesis was used (in fact, only weak convergence). The last equality is a consequence of Theorem 4.1.3 property 3 and the fact that T^j is weak mixing. The conclusion now follows, at any rate, from Exercise 4.2.

□

As one can see from property 5 in Theorem 4.1.3, weak mixing is characterized by the absence of non-trivial eigenfunctions. Systems for which $L^2(X, \mathcal{A}, \mu)$ is spanned by eigenfunctions are therefore in some sense the counterpart to weakly mixing systems. Such systems are said to be *compact.* Equivalently, a system $(X, \mathcal{A}, \mu, T)$ is compact if for every $f \in L^2(X, \mathcal{A}, \mu)$ the orbit $\{T^n f : n \in \mathbf{Z}\}$ is a pre-compact subset of $L^2(X, \mathcal{A}, \mu)$ (hence the terminology *compact*).

In the next two chapters, our goal is to establish multiple recurrence for general single operator measure preserving systems. In the next section, we shall give a general flavor of the methodology to be employed by (a) establishing multiple recurrence for both weak mixing systems and for compact systems, and (b) presenting a Hilbert space splitting theorem (splitting $L^2(X, \mathcal{A}, \mu)$ into "compact" and "weak mixing" pieces) that will be sufficient for getting us to a proof of the first non-trivial case of multiple recurrence, namely double recurrence.

4.2 Roth's theorem.

The prototypical multiple recurrence result is Furstenberg's ergodic Szemerédi theorem, which states that if $\mu(A) > 0$ and $k \in \mathbf{N}$ then for some $n \neq 0$ one has $\mu(A \cap T^{-n}A \cap \cdots \cap T^{-kn}A) > 0$. In order to get a feel for the how the presence of compactness or weak mixing can be exploited in obtaining such results, let us now see why the theorem follows if $(X, \mathcal{A}, \mu, T)$ is either compact or weak mixing.

First suppose that $(X, \mathcal{A}, \mu, T)$ is compact, that is, that $\{T^n f : n \in \mathbf{Z}\}$ is precompact for every $f \in L^2(X, \mathcal{A}, \mu)$. In particular, $\{T^n 1_A : n \in \mathbf{Z}\}$ is totally bounded, so that there exist some integers $k \neq l$ such that, letting $n = k - l$,

$$\mu(A \setminus T^{-n}A) = \frac{1}{2}\mu(A \Delta T^{-n}A) = \frac{1}{2}||T^k 1_A - T^l 1_A|| < \frac{\mu(A)}{k(k-1)}.$$

Exercise 4.4. Show that $\mu(A \cap T^{-in}A) \leq \frac{i\mu(A)}{2k^2}$, $i \in \mathbf{N}$.

We now have

$$\begin{aligned}
&\mu(A \cap T^{-n}A \cap \cdots \cap T^{-kn}A)\\
=&\mu\Big(A \setminus \bigcup_{i=1}^{k}(A \setminus T^{-in}A)\Big)\\
\geq&\mu(A) - \sum_{i=1}^{k} \frac{i\mu(A)}{k(k-1)} \geq \frac{\mu(A)}{2}.
\end{aligned}$$

Next let us suppose that $(X, \mathcal{A}, \mu, T)$ is a weak mixing system. Letting $f_i = 1_A$ in Theorem 4.1.4, $1 \leq i \leq k$, and utilizing just weak convergence, we have

$$\begin{aligned}
&\mu(A \cap T^{-n}A \cap \cdots \cap T^{-kn}A)\\
=&\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \int fT^n f T^{2n} f \cdots T^{kn} f \, d\mu = \mu(A)^{k+1}.
\end{aligned}$$

In particular, $\mu(A \cap T^{-n}A \cap \cdots \cap T^{-kn}A) > 0$ for some $n \in \mathbf{N}$.

Exercise 4.5. Modify the argument given above to conclude that in the compact case

$$\liminf_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \mu(A \cap T^{-n}A \cap \cdots \cap T^{-kn}A) > 0.$$

More typically, the system $(X, \mathcal{A}, \mu, T)$ will be neither weak mixing nor compact. In other words, it will have non-constant eigenfunctions, but not all of $L^2(X, \mathcal{A}, \mu)$ will be spanned by these eigenfunctions. We will call the portion of $L^2(X, \mathcal{A}, \mu)$ spanned by the eigenfunctions the *compact portion* of $L^2(X, \mathcal{A}, \mu)$ and its orthocomplement the *weak mixing* portion. Our present goal is now to prove a theorem justifying these names. First, we remind the reader of the notion of a compact operator.

If $\mathcal{H}$ is a separable Hilbert space then a bounded linear operator T on $\mathcal{H}$ is said to be *compact* if the image of every bounded set is precompact. Equivalently, T is compact if for every bounded sequence $\{x_n\}_{n=1}^{\infty} \subset \mathcal{H}$, there exists a subsequence $\{n_k\} \subset \mathbf{N}$ such that $\lim_{k\to\infty} Tx_{n_k}$ exists in the norm topology.

A special class of compact operators on $L^2(X, \mathcal{A}, \mu)$ are those generated by *square summable kernals.* Let $H \in L^2(X \times X, \mathcal{A} \otimes \mathcal{A}, \mu \times \mu)$ and for $f \in L^2(X, \mathcal{A}, \mu)$ let $H * f$ be defined by

$$H * f(x) = \int_X H(x,y) f(y)\, d\mu(y).$$

That the operator $f \to H * f$ thus defined is linear and bounded is not hard to see; that it is compact is slightly less obvious (see for example [RS, Chapt. IV]).

Theorem 4.2.1. (See [KN].) Let $(X, \mathcal{A}, \mu, T)$ be an invertible measure preserving system. Put

$$\mathcal{H}_c = \left\{ f \in L^2(X, \mathcal{A}, \mu) : \{T^n f : n \in \mathbf{Z}\} \text{ is precompact} \right\}$$

and let

$$\mathcal{H}_{wm} = \left\{ g \in L^2(X, \mathcal{A}, \mu) : D\text{-}\lim \int f T^n g\, d\mu = 0 \text{ for all } f \in L^2(X, \mathcal{A}, \mu) \right\}.$$

Then $L^2(X, \mathcal{A}, \mu) = \mathcal{H}_c \oplus \mathcal{H}_{wm}$.

Proof. We begin with the following observation.

Exercise 4.6. $\mathcal{H}_c$ and $\mathcal{H}_{wm}$ are closed subspaces. Moreover, if $H \in L^2(X \times X, \mathcal{A} \otimes \mathcal{A}, \mu \times \mu)$ is $(T \times T)$-invariant then $H * f \in \mathcal{H}_c$ for every $f \in L^2(X, \mathcal{A}, \mu)$. (*Hint: show first that* $T(H * f) = H * Tf$.)

We must show that $\mathcal{H}_{wm} = \mathcal{H}_c^{\perp}$. Let $g \in \mathcal{H}_c^{\perp}$. By Exercise 4.6, g is orthogonal to $H * f$ for every $f \in L^2(X, \mathcal{A}, \mu)$ and every $(T \times T)$-invariant function $H \in L^2(X \times X, \mathcal{A} \otimes \mathcal{A}, \mu \times \mu)$. In particular,

$$\begin{aligned} 0 &= \int g(x) H * f(x)\, d\mu(x) \\ &= \int g(x) \int H(x,y) f(y)\, d\mu(y)\, d\mu(x) \\ &= \int (g \otimes f) H\, d\mu \times \mu. \end{aligned}$$

Thus we see that $g \otimes f$ is orthogonal to H for every $(T \times T)$-invariant H and every $f \in L^2(X, \mathcal{A}, \mu)$. In particular (taking $f = g$), $P(g \otimes g) = 0$, where P is the projection in $L^2(X \times X, \mathcal{A} \otimes \mathcal{A}, \mu \times \mu)$ onto the space of $(T \times T)$-invariant functions. Hence, making use of the mean ergodic theorem,

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N}\Big(\int fT^n g\, d\mu\Big)^2$$
$$= \lim_{N\to\infty} \int (f\otimes f)\frac{1}{N}\sum_{n=1}^{N}(T\times T)^n(g\otimes g)\, d\mu\times\mu$$
$$= \int (f\otimes f)P(g\otimes g)\, d\mu\times\mu = 0.$$

By Exercise 4.1 a. we have $D\text{-}\lim_{n\to\infty}\int fT^n g\, d\mu = 0$, so $g \in \mathcal{H}_{wm}$. This shows that $\mathcal{H}_c^{\perp} \subset \mathcal{H}_{wm}$.

Suppose now that $f \in \mathcal{H}_{wm}$. Write $f = f_1 + f_2$, where $f_1 \in \mathcal{H}_c$ and $f_2 \in \mathcal{H}_c^{\perp}$. By the result obtained in the previous paragraph, $f_2 \in \mathcal{H}_{wm}$. But $\mathcal{H}_{wm}$ is a subspace, so $f_1 = (f - f_2) \in \mathcal{H}_c \cap \mathcal{H}_{wm}$. Our goal is to show that $f_1 = 0$. Let $\epsilon < 0$ be so small that $\int f_1 h\, d\mu \geq \frac{||f_1||}{2}$ whenever $||f_1 - h|| \leq 2\epsilon$. Since $f_1 \in \mathcal{H}_c$, there exist functions $g_1, \cdots, g_k$ such that for every $n \in \mathbf{Z}$, $||T^n f_1 - g_i|| < \epsilon$ for some i, $1 \leq i \leq k$. It follows that for some i the set $E = \{n : ||T^n f_1 - g_i|| < \epsilon\}$ has positive lower density. According to Exercise 3.19, the set $E - E$ therefore is syndetic, having in particular lower density greater than some positive number δ.

Exercise 4.7. For every $n \in E - E$, $||T^n f_1 - f_1|| < 2\epsilon$.

It follows that

$$0 = \liminf_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N}\Big(\int f_1 T^n f_1\, d\mu\Big)^2 \geq \frac{\delta||f_1||^2}{4}.$$

Hence $||f_1|| = 0$ and $f = f_2 \in \mathcal{H}_c^{\perp}$. This shows that $\mathcal{H}_{wm} \subset \mathcal{H}_c^{\perp}$, completing the proof.

□

As an application of the Hilbert space splitting provided by Theorem 4.2.1, we will now prove the following *double recurrence* theorem for ergodic systems. The point for us in proving such a theorem is to eventually obtain Roth's theorem on arithmetic progressions ([Ro]), which states that in any set of positive density in $\mathbf{N}$ there exist arithmetic progressions of length three.

Theorem 4.2.2. (See [F1], [F2].) Let $(X, \mathcal{A}, \mu, T)$ be an ergodic invertible measure preserving system. If $A \in \mathcal{A}$ with $\mu(A) > 0$ then

$$\liminf_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N}\mu(A \cap T^{-n}A \cap T^{-2n}A) > 0.$$

Proof. Write $1_A = f + g$, where $f \in \mathcal{H}_c$ and $g \in \mathcal{H}_{wm}$. We have

$$\begin{aligned}
&\frac{1}{N}\sum_{n=1}^{N} \mu(A \cap T^{-n}A \cap T^{-2n}A) \\
=&\frac{1}{N}\sum_{n=1}^{N} \int 1_A T^n 1_A T^{2n} 1_A \, d\mu \qquad (4.2)\\
=&\frac{1}{N}\sum_{n=1}^{N} \int (f+g)T^n(f+g)T^{2n}(f+g)\, d\mu.
\end{aligned}$$

Expanding the product in the integral, we get eight terms. We would like to demonstrate that seven of these terms tend to zero. Our first task is to show that $\frac{1}{N}\sum_{n=1}^{N} T^n f T^{2n} g$, $\frac{1}{N}\sum_{n=1}^{N} T^n g T^{2n} f$, and $\frac{1}{N}\sum_{n=1}^{N} T^n g T^{2n} g$ all converge to zero in norm. Utilizing just weak convergence, this will eliminate 6 of the 8 terms in the expansion of (4.2). Since the proofs of these three facts are similar, we shall handle just the first. Namely, let $x_n = T^n f T^{2n} g$. We use Exercise 4.2.

$$\begin{aligned}
&D\text{-}\lim_{h} \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \langle x_n, x_{n+h}\rangle \\
=&D\text{-}\lim_{h} \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \int T^n f T^{2n} g T^{n+h} f T^{2n+2h} g \, d\mu \\
=&D\text{-}\lim_{h} \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \int (fT^h f) T^n (gT^{2h} g)\, d\mu \\
=&D\text{-}\lim_{h} \Big(\int fT^h f\, d\mu\Big)\Big(\int gT^{2h} g\, d\mu\Big) = 0.
\end{aligned}$$

Here we have used ergodicity in moving to the last line and the fact that $g \in \mathcal{H}_{wm}$ in the last equality. It follows now that $\|\frac{1}{N}\sum_{n=1}^{N} T^n f T^{2n} g\| \to 0$.

There are only two terms in (4.2) that remain to be dealt with. The following exercise narrows this list to one.

Exercise 4.8. Show that $\frac{1}{N}\sum_{n=1}^{N} \int g T^n f T^{2n} f \; d\mu \to 0$. (*Hint: multiply through by* T^{-2n} *in the integral and apply the argument above.*)

Hence we have

$$\liminf_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \mu(A \cap T^{-n}A \cap T^{-2n}A) = \liminf_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \int fT^n f T^{2n} f\, d\mu.$$

To see that this latter expression is positive, first note:

Exercise 4.9. Show that if $f_1, f_2 \in \mathcal{H}_c$ then $\sup\{f_1, f_2\} \in \mathcal{H}_c$.

A consequence of this exercise is that the positive portion of f is again contained in $\mathcal{H}_c$. However, the positive portion of f is at least as close to 1_A in

$L^2(X, \mathcal{A}, \mu)$ as f is. Since f is the projection of 1_A onto $\mathcal{H}_c$, f must coincide with its positive portion. That is, $f \geq 0$ a.e. The same argument shows that $f \leq 1$ a.e.

Exercise 4.10. $||f|| \geq \mu(A)$. In particular, f is strictly positive on a set of positive measure. (*Hint: use Lemma 3.4.5 c.*)

Exercise 4.11. There exists $\epsilon > 0$ such that for any $g, h \in L^2(X, \mathcal{A}, \mu)$ with $0 \leq g, h \leq 1$ with $||f - g|| < \epsilon$ and $||f - h|| < \epsilon$ we have $\int fgh\, d\mu \geq \frac{1}{2}\big(\int f^3\, d\mu\big)$. Moreover, for a syndetic set of n, $||T^n f - f|| < \epsilon$ and $||T^{2n} f - f|| < \epsilon$.

Hence for a syndetic set of n, $\int fT^n fT^{2n} f\, d\mu \geq \frac{1}{2}\big(\int f^3\, d\mu\big)$. It follows that

$$\liminf_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mu(A \cap T^{-n}A \cap T^{-2n}A) = \liminf \frac{1}{N} \sum_{n=1}^{N} \int fT^n fT^{2n} f\, d\mu > 0.$$

□

In order to obtain any combinatorial consequences from Theorem 4.2.2, we need first to eliminate the ergodicity assumption. We'll do this in the next section, simultaneously preparing ourselves for a more refined analysis of multiple recurrence.

4.3 Decomposition of measures on Lebesgue spaces.

Note the measure preserving system $(X, \mathcal{A}, \mu, T)$ constructed for the Furstenberg correspondence (see Theorem 3.2.5) is a *regular system*, namely X is compact metric, $\mathcal{A}$ is the Borel σ-algebra, μ is a regular Borel measure and T is a homeomorphism. It follows that in order to obtain combinatorial corollaries, it suffices to establish recurrence results for such systems. Indeed, this is a very useful thing to know, since many constructions on which the current proofs depend are possible only in specialized systems such as these. (Those constructions form the topic of the current section. For more details regarding them, see for example [Rud].) Note as well that the limitation to regular systems is temporary. Having obtained the combinatorial corollaries, we can invoke Theorem 3.2.7 (which is not limited to regular systems) to obtain the corresponding recurrence results for *general* systems.

Most of the material of this section deals with *factors* of measure preserving systems. A system $(Y, \mathcal{B}, \nu, S)$ is a factor of the system $(X, \mathcal{A}, \mu, T)$ if there exists a measure-preserving map $\pi : X \to Y$ for which $\pi(Tx) = S\pi(x)$ a.e. If π is a bimeasurable bijection (after discarding sets of measure 0 from X and Y) then the systems are said to be *isomorphic*.

For reasons we will disclose shortly, we shall actually consider a somewhat wider class of systems than regular systems. $(X, \mathcal{A}, \mu)$ is said to be a *Lebesgue space* if it is measurably isomorphic to a regular measure space. We shall call a system $(X, \mathcal{A}, \mu, T)$ a *Lebesgue system* if T is invertible and $(X, \mathcal{A}, \mu)$ is a Lebesgue space. As the following theorem shows, the distinction between regular systems and Lebesgue systems is subtle.

Theorem 4.3.1. Every regular system is Lebesgue, and every Lebesgue system is isomorphic to some regular system.

Our reason for working with Lebesgue systems, as well as for caring about factors, is the following theorem.

Theorem 4.3.2. Let $(X, \mathcal{A}, \mu, T)$ be a Lebesgue system and suppose that $\mathcal{C} \subset \mathcal{A}$ is a σ-algebra. There exists a Lebesgue system $(Y, \mathcal{B}, \nu, S)$ and a family of probability measures $\{\mu_y : y \in Y\}$ on X such that:

a. $(Y, \mathcal{B}, \nu, S)$ is a factor of $(X, \mathcal{A}, \mu, T)$ via a factor map $\pi : X \to Y$.

b. (X,μ_y) is a Lebesgue space and $\mu_y(\pi^{-1}(y)) = 1$ a.e.

c. For every integrable function f on X, $\int f\, d\mu = \int \left(\int f\, d\mu_y\right) d\nu(y)$.

d. Letting $Ef(x) = \int f\, d\mu_{\pi(x)}$ for $f \in L^1(X, \mathcal{A}, \mu)$, the restriction of E to $L^2(X, \mathcal{A}, \mu)$ is the projection onto $L^2(X, \mathcal{C}, \mu)$. Ef (sometimes we write $E(f|\mathcal{C})$) is the *conditional expectation* of f given $\mathcal{C}$.

e. For all $A \in \mathcal{A}$, $\mu_y(T^{-1}A) = \mu_{Sy}(A)$, a.e.

In particular, part d. shows us that for any $\mathcal{C}$-measurable function f, $f(x) = \int f\, d\mu_{\pi(x)}$ a.e. and is therefore identifiable with the $\mathcal{B}$-measurable function $g(y) = \int f\, d\mu_y$.

It is intuitively useful to think of X as being identified with the set $[0,1] \times [0,1]$, Y the set $\{0\} \times [0,1]$, and π the map that sends (x, y) to $(0, y)$. Then the "fibers" are the sets $\pi^{-1}(y) = [0,1] \times \{y\}$.

A special case of the measurable decomposition which is of considerable interest is when $\mathcal{C}$ is the σ-algebra of T-invariant sets. Then, since $\mathcal{B}$ may be identified with $\mathcal{C}$, every set in $\mathcal{B}$ is S-invariant; that is, S is the identity map on Y. A consequence of property e. in Theorem 4.3.2 is that for a.e. y, T preserves the measure μ_y. In fact, for a.e. y, T is ergodic with respect to μ_y. To see this, consider for example Birkhoff's pointwise ergodic theorem.

Theorem 4.3.3. Suppose that $(X, \mathcal{A}, \mu, T)$ is a measure preserving system and let $f \in L^1(X, \mathcal{A}, \mu)$. Then for a.e. $x \in X$,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} T^n f(x) = Ef(x),$$

where Ef is the conditional expectation of f given the σ-algebra of T-invariant functions.

In other words, for a.e. y and all $f \in L^1(X, \mathcal{A}, \mu)$, one has that for a.e. x (with respect to μ_y), $\frac{1}{N}\sum_{n=1}^{N} T^n f(x)$ tends to $\int f\, d\mu_y$. Choosing a countable dense set of f's we get that T is ergodic with respect to a.e. y.

We now demonstrate the process by which we obtain a recurrence result in general, having established it in the ergodic case:

Theorem 4.3.4. Let $(X, \mathcal{A}, \mu, T)$ be a Lebesgue invertible measure preserving system. If $A \in \mathcal{A}$ with $\mu(A) > 0$ then

$$\liminf_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mu(A \cap T^{-n}A \cap T^{-2n}A) > 0.$$

Proof. We have seen the ergodic case already. Let $\{\mu_y\}$ be the decomposition of μ over the σ-algebra of T-invariant sets. For a.e. y we have

$$\liminf_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mu_y(A \cap T^{-n}A \cap T^{-2n}A) > 0.$$

Choose a set $B \subset Y$ and some $\delta > 0$ such that $\nu(B) > 0$ and

$$\liminf_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mu_y(A \cap T^{-n}A \cap T^{-2n}A) > \delta$$

for $y \in B$. Then by Fatou's Lemma we have

$$\begin{aligned}
&\liminf_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mu(A \cap T^{-n}A \cap T^{-2n}A) \\
&= \liminf_{N\to\infty} \int \frac{1}{N} \sum_{n=1}^{N} \mu_y(A \cap T^{-n}A \cap T^{-2n}A) \\
&\geq \int \liminf_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mu_y(A \cap T^{-n}A \cap T^{-2n}A) \geq \nu(B)\delta > 0.
\end{aligned}$$

□

Theorem 4.3.5. Let $E \subset \mathbf{Z}$ with $d^*(E) > 0$. Then

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} d^*\big(E \cap (E-n) \cap (E-2n)\big) > 0.$$

Proof. According to Theorem 3.2.5, there exists an invertible measure preserving system $(X, \mathcal{A}, \mu, T)$ and a set of positive measure A for which $d^*\big(E \cap (E - n) \cap (E-2n)\big) \geq \mu(A \cap T^{-n}A \cap T^{-2n}A)$ for all $n \in \mathbf{Z}$. Moreover, a look at the proof of Theorem 3.2.5 shows that the system constructed there is regular and hence Lebesgue. Therefore the result follows from Theorem 4.3.4.

□

Notice that Theorem 4.3.5 implies Roth's theorem in particular; any set of positive upper Banach density contains three-term arithmetic progressions.

Now we see how to remove the dependence on the regularity of the system that encumbers Theorem 4.3.4.

Theorem 4.3.6. Let $(X, \mathcal{A}, \mu, T)$ be an invertible measure preserving system. If $A \in \mathcal{A}$ with $\mu(A) > 0$ then

$$\liminf_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mu(A \cap T^{-n}A \cap T^{-2n}A) > 0.$$

Proof. A consequence of the Birkhoff theorem is that for a.e. $x \in A$, $\overline{d}(R_x) > 0$, where $R_x = \{n \in \mathbf{Z} : T^n x \in A\}$. Hence by Theorem 4.3.5,

$$L_x = \liminf_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} 1_A(x) 1_A(T^n x) 1_A(T^{2n} x)$$
$$= \liminf \frac{1}{N} \sum_{n=1}^{N} \overline{d}\big(R_x \cap (R_x - n) \cap (R_x - 2n)\big) > 0$$

a.e. Choose a set B of positive measure and a $\delta > 0$ such that $L_x > \delta$ for $x \in B$. Then by Fatou's Lemma

$$\liminf_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mu(A \cap T^{-n} A \cap T^{-2n} A)$$
$$= \liminf_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \int 1_A(x) 1_A(T^n x) 1_A(T^{2n} x)\, d\mu$$
$$\geq \int \liminf_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} 1_A(x) 1_A(T^n x) 1_A(T^{2n} x)\, d\mu > \mu(B)\delta > 0.$$

□

In later sections, we shall take this process for granted, formulating the results for general systems but assuming in the proof that they are Lebesgue and ergodic.

Let us now go back to the situation outlined in Theorem 4.4.2. Namely, suppose we have a system $(X, \mathcal{A}, \mu, T)$ and a T-invariant σ-algebra $\mathcal{C}$. Form the system $(Y, \mathcal{B}, \nu, S)$ and the decomposition $\{\mu_y : y \in Y\}$. Define a measure $\mu \times_Y \mu$ on $(X \times X, \mathcal{A} \otimes \mathcal{A})$ by letting $\mu \times_Y \mu(A \times B) = \int \mu_y(A)\mu_y(B)\, d\nu(y)$. (It is of course sufficient to define $\mu \times_Y \mu$ on sets of this type. It extends to $\mathcal{A} \otimes \mathcal{A}$ uniquely.) We now have

$$\int f \otimes g \, d\mu \times_Y \mu = \int \Big(\int f \, d\mu_y\Big)\Big(\int g \, d\mu_y\Big)\, d\nu(y).$$

Exercise 4.12. Show that $\mu \times_Y \mu$ is preserved by $T \times T$.

The measure preserving system $(X \times X, \mathcal{A} \otimes \mathcal{A}, \mu \times_Y \mu, T \times T)$ is called the *conditional product system* relative to $\mathcal{C}$. We shall use this construction repeatedly.

So far in this section we have been considering factorization arising from a T-invariant σ-algebra. However, this would be of little use to us without a method for identifying T-invariant σ-algebras. The normal means by which we shall do this is to construct a subspace $\mathcal{L}$ of $L^2(X, \mathcal{A}, \mu)$ having properties we desire and to which the following well-known classical theorem applies.

Theorem 4.3.7. Let $\mathcal{L}$ be a closed subspace of $L^2(X, \mathcal{A}, \mu)$. Suppose there exists a spanning set $\mathcal{L}_0$ for $\mathcal{L}$ consisting of bounded functions having the property that for all $f, g \in \mathcal{L}_0$, $\min\{f, g\}$ and $\max\{f, g\}$ are in $\mathcal{L}$. Then there exists a σ-algebra $\mathcal{C} \subset \mathcal{A}$ such that $\mathcal{L} = L^2(X, \mathcal{C}, \mu)$. If $\mathcal{L}$ is T-invariant then $\mathcal{C}$ is as well.

The Roth theorem proved in this section serves as a kind of "maximal" result one can obtain using only "Hilbert space" methodology. In the next two sections, we shall prove two extensions of Theorem 4.3.6. In so doing, we shall need to employ Theorem 4.3.7, factorization over a T-invariant σ-algebra, decomposition of measures (not merely ergodic decomposition), and the conditional product construction. As a result, the proofs of these extensions more closely approximate the flavor of the proofs of the more general multiple recurrence theorems to be treated in the next chapter.

4.4 A Roth theorem for two commuting transformations

In this section we prove the following special case of [FK1].

Theorem 4.4.1. Let $(X, \mathcal{A}, \mu)$ be a measure space. If T and S are two invertible measure preserving transformations on X which commute with each other and $A \in \mathcal{A}$ with $\mu(A) > 0$ then

$$\liminf_{N-M\to\infty} \frac{1}{N-M} \sum_{n=M}^{N-1} \mu(A \cap T^{-n}A \cap S^{-n}A) > 0.$$

This theorem has combinatorial corollaries in $\mathbf{Z}^2$ which we shall not pursue explicitly, since for convenience we formulated our notions of density and Furstenberg corresponcend in $\mathbf{Z}$ only. However, taking T and S to the powers of the same transformation, one can obtain: if $E \subset \mathbf{Z}$ with $d^*(E) > 0$ and $k, l \in \mathbf{Z}$ then there exists $n \neq 0$ and $a \in \mathbf{Z}$ such that $\{a, a + kn, a + ln\} \subset E$.

The role of factors will be much stronger in this section than in the proof of the regular Roth theorem. Fix a system $(X, \mathcal{A}, \mu)$. We assume that it is Lebesgue.

Exercise 4.13. The set $\mathcal{B}$ of TS^{-1}-invariant sets is a σ-algebra that is both T-invariant and S-invariant.

Let $(Y, \mathcal{B}, \nu, R)$ be the induced factor. This system is simultaneously a factor of $(X, \mathcal{A}, \mu, T)$ and $(X, \mathcal{A}, \mu, S)$ under the same factor map $\pi : X \to Y$. In particular, $\pi(Tx) = R\pi(x) = \pi(Sx)$ a.e. Let $\{\mu_y : y \in Y\}$ be the decomposition of μ over Y.

Exercise 4.14. Show that $T \times S$ preserves the measure $\mu \times_Y \mu$.

We say that a function $f \in L^2(X, \mathcal{A}, \mu)$ is *T-compact over $\mathcal{B}$* if for every $\epsilon > 0$ there exist functions $g_1, \cdots, g_k \in L^2(X, \mathcal{A}, \mu)$ such that for every $n \in \mathbf{Z}$ and a.e. $y \in Y$ there exists some $s = s(n, y)$ with $1 \le s \le k$ such that $||T^n f - g_s||_{L^2(X,\mu_y)} < \epsilon$. S-compactness over $\mathcal{B}$ is defined similarly.

Suppose that $H \in L^2(X \times X, \mathcal{A} \otimes \mathcal{A}, \mu \times_Y \mu)$. We define two operators associated with H on $L^2(X, \mathcal{A}, \mu)$. These are given by

$$\mathbf{H} * \phi(x) = \int H(x,t)\phi(t)\, d\mu_{\pi(x)}$$

and

$$\phi * \mathbf{H}(x) = \int H(t,x)\phi(t)\, d\mu_{\pi(x)}.$$

These may be viewed either as operators on $L^2(X, \mathcal{A}, \mu)$, or as "bundles" of compact operators on $L^2(X, \mu_y)$.

Lemma 4.4.2. If $H \in L^2(X \times X, \mathcal{A} \otimes \mathcal{A}, \mu \times_Y \mu)$ is $(T \times S)$-invariant then for all $\phi \in L^\infty(X, \mathcal{A}, \mu)$, $\mathbf{H} * \phi$ may be approximated arbitrarily closely by a function that is T-compact over $\mathcal{B}$ and $\phi * \mathbf{H}$ may be approximated arbitrarily closely by a function that is S-compact over $\mathcal{B}$.

Proof. We will show only the first part, as the second is similar.

Exercise 4.15. Show that $T^n(\mathbf{H} * \phi) = \mathbf{H} * S^n\phi$.

Let $\epsilon > 0$. Choose a sequence $(\epsilon_i)_{i=1}^\infty$ of positive numbers with $\sum_{i=1}^\infty \epsilon_i < \epsilon$. For a.e. y, $\mathbf{H}$ acts as a compact operator on $L^2(X, \mu_y)$. It follows from this fact and Exercise 4.15 that for $i \in \mathbf{N}$ and a.e. y there exists $M = M(i,y)$ such that $\{T^n H * \phi : -M \leq n \leq M\}$ is ϵ_i-dense (for the $L^2(X, \mu_y)$ norm metric) in $\{T^n H * \phi : n \in \mathbf{Z}\}$. Let M_i be so large that $M_i > M(i,y)$ for all y outside of an exceptional set E_i with $\nu(E_i) < \epsilon_i$. Now put $f(x) = 0$ if $\pi(x) \in \bigcup_{i=1}^\infty E_i$ and $f(x) = H * \phi(x)$ otherwise.

Exercise 4.16. Show that $f(x)$ is compact over $\mathcal{B}$ and $||f - H * \phi|| < ||\phi||_\infty \sqrt{\epsilon}$.

As ϵ is arbitrary, this completes the proof.

□

Let $\mathcal{L}_1$ be the closure in $L^2(X, \mathcal{A}, \mu)$ of the set

$$\{\mathbf{H} * \phi : H \in L^2(X \times X, \mathcal{A} \otimes \mathcal{A}, \mu \times_Y \mu) \text{ is } (T \times S)\text{-invariant}, \phi \in L^\infty(X, \mathcal{A}, \mu)\}$$

and let $\mathcal{L}_2$ be the closure of the set

$$\{\phi * \mathbf{H} : H \in L^2(X \times X, \mathcal{A} \otimes \mathcal{A}, \mu \times_Y \mu) \text{ is } (T \times S)\text{-invariant}, \phi \in L^\infty(X, \mathcal{A}, \mu)\}.$$

Exercise 4.17. There exist σ-algebras $\mathcal{B}_1$ and $\mathcal{B}_2$ such that $\mathcal{L}_1 = L^2(X, \mathcal{B}_1, \mu)$ and $\mathcal{L}_2 = L^2(X, \mathcal{B}_2, \mu)$.

Exercise 4.18. Suppose $(b_n)_{n=0}^\infty \subset \mathbf{R}$ is bounded. If $\lim_{H\to\infty} \frac{1}{H} \sum_{h=1}^H b_h = 0$ then $\lim_{H\to\infty} \sum_{r=-H}^H \frac{H-|r|}{H^2} b_{|r|} = 0$.

Lemma 4.4.3. Suppose that $\{x_n : n \in \mathbf{Z}\}$ is a bounded sequence of vectors in a Hilbert space $\mathcal{H}$. If

$$\lim_{H\to\infty} \frac{1}{H}\sum_{h=1}^{H}\Big(\lim_{N-M\to\infty} \frac{1}{N-M}\sum_{n=M}^{N-1}\langle x_n, x_{n+h}\rangle\Big) = 0,$$

then

$$\lim_{N-M\to\infty}\left|\left|\frac{1}{N-M}\sum_{n=M}^{N-1} x_n\right|\right| = 0.$$

Proof. Let $\epsilon > 0$. Using Exercise 4.18, fix H large enough that

$$\sum_{r=-H}^{H}\frac{H-|r|}{H^2}\Big(\limsup_{N-M\to\infty}\frac{1}{N-M}\sum_{u=M}^{N-1}\langle x_u, x_{u+r}\rangle\Big) < \epsilon.$$

We have

$$\frac{1}{N-M}\sum_{n=M}^{N-1} x_n = \frac{1}{N-M}\sum_{n=M}^{N-1}\Big(\frac{1}{H}\sum_{h=1}^{H} x_{n+h}\Big) + \Psi'_{M,N} = \Psi_{M,N} + \Psi'_{M,N},$$

where $\limsup_{N-M\to\infty}\left|\left|\Psi'_{M,N}\right|\right| = 0$. Let's see that $\limsup_{N-M\to\infty}\left|\left|\Psi_{M,N}\right|\right| < \epsilon$. We have

$$\begin{aligned}
\left|\left|\Psi_{M,N}\right|\right|^2 &\le \frac{1}{N-M}\sum_{n=M}^{N-1}\left|\left|\frac{1}{H}\sum_{h=1}^{H} x_{n+h}\right|\right|^2 \\
&= \frac{1}{N-M}\sum_{n=M}^{N-1}\frac{1}{H^2}\sum_{h,k=1}^{H}\langle x_{n+h}, x_{n+k}\rangle \\
&= \sum_{r=-H}^{H}\frac{H-|r|}{H^2(N-M)}\sum_{u=M}^{N-1}\langle x_u, x_{u+r}\rangle + \Psi''_{M,N},
\end{aligned}$$

where $\Psi''_{M,N} \to 0$ as $N-M\to\infty$. By choice of H the last expression is less than ϵ when $N-M$ is sufficiently large.

□

Theorem 4.4.4. Suppose that f and g are in $L^2(X, \mathcal{A}, \mu)$ with either $f \in \mathcal{L}_1^\perp$ or $g \in \mathcal{L}_2^\perp$. Then $f \otimes g$ is orthogonal to H for every $(T\times S)$-invariant function $H \in L^2(X\times X, \mathcal{A}\otimes\mathcal{A}, \mu\times_Y \mu)$.

Proof. Suppose $f \in \mathcal{L}_1^\perp$ (the case of $g \in \mathcal{L}_2^\perp$ is similar). Then

$$\begin{aligned}
0 &= \int f(x)\mathbf{H} * g(x)\, d\mu(x) \\
&= \int_Y\int_{\pi^{-1}(y)} f(x)\mathbf{H} * g(x)\, d\mu_y(x)d\nu(y) \\
&= \int_Y\int_{\pi^{-1}(y)} f(x)\int_{\pi^{-1}(y)} H(x,t)g(t)\, d\mu_y(t)d\mu_y(x)d\nu(y) \\
&= \int (f\otimes g)H\, d\mu\times_Y \mu.
\end{aligned}$$

□

Recall that a set $E \subset \mathbf{Z}$ is thick if it contains arbitrarily long intervals.

Exercise 4.19. If E is thick, $0 < \delta < 1$, $(Z, \mathcal{C}, \xi, W)$ is an invertible measure preserving system, and $C \in \mathcal{C}$ with $\xi(C) > 0$ then there exists $n \in E$ with $\xi(C \cap T^{-n}C) > \delta\xi(C)^2$.

Lemma 4.4.5. Let $(Z, \mathcal{C}, \xi, W)$ be an invertible measure preserving system. Let $C \in \mathcal{C}$ with $\xi(C) > 0$ and let $0 < \delta < 1$. For any thick set E there exists an IP-set $(n_\alpha)_{\alpha\in\mathcal{F}} \subset E$ such that for every $\alpha \in \mathcal{F}_\emptyset$

$$\mu\Big(\bigcap_{\beta\subset\alpha} W^{-n_\beta}C\Big) > \big(\delta\xi(C)\big)^{2^{|\alpha|}}.$$

Proof. By Exercise 4.19 we may choose $n_1 \in E$ such that $\xi(C \cap T^{-n_1}C) > \delta\xi(C)^2$. Let $C_1 = (C \cap T^{-n_1}C)$. Again by Exercise 4.19 we may choose $n_2 \in \big(E \cap (E - n_1)\big)$ (which is again thick) such that $\xi(C_1 \cap T^{-n_2}C_1) > \delta\xi(C_1)^2 > \delta^3\mu(C)^4$. Notice now that $\xi(C \cap T^{-n_1}C \cap T^{-n_2}C \cap T^{-n_1-n_2}C) > \delta^3\xi(C)^4$.

Continuing in this fashion and setting $n_\alpha = \sum_{i\in\alpha} n_i$ completes the proof. □

Proof of Theorem 4.4.1. Let E be an arbitrary thick set, and let $A \in \mathcal{A}$ with $\mu(A) > 0$. Write $1_A = f = f_1 + g_1 = f_2 + g_2$, where $f_1 \in \mathcal{L}_1$, $g_1 \in \mathcal{L}_1^\perp$, $f_2 \in \mathcal{L}_2$, and $g_2 \in \mathcal{L}_2^\perp$.

Exercise 4.20. Use Theorem 4.4.4 to show that

$$\begin{aligned}
&\liminf_{N-M\to\infty} \frac{1}{N-M} \sum_{n=M}^{N-1} \mu(A \cap T^{-n}A \cap S^{-n}A) \\
&= \liminf_{N-M\to\infty} \frac{1}{N-M} \sum_{n=M}^{N-1} \int fT^n f_1 S^n f_2 \, d\mu.
\end{aligned}$$

The rest of the proof consists in showing that this latter limit is positive.

Exercise 4.21. Using the decomposition of μ over $\mathcal{B}_1$, show that $f_1(x) > 0$ for a.e. $x \in A$. Show similarly that $f_2(x) > 0$ for a.e. $x \in A$.

Therefore, there exists some $\eta > 0$, and a set $A' \subset A$ with $\mu(A') > 0$ such that $f_1(x)f_2(x) > \eta$ for all $x \in A'$. Furthermore, there exists $\gamma > 0$ and a set $B \subset Y$ with $\nu(B) = 2\xi > 0$ such that for all $y \in B$, $\mu_y(A') > \gamma$. It follows that $\int f f_1 f_2 \, d\mu_y > \eta\xi$ for all $y \in B$. Notice that ξ, η, and γ do not depend on the chosen thick set E.

Let $\epsilon = \frac{\eta\gamma}{16}$. We may approximate f_1 as closely as desired by a function ϕ_1 which is T-compact over $\mathcal{B}$. Likewise, we may approximate f_2 by a function ϕ_2 which is S-compact over $\mathcal{B}$. Choose such ϕ_1 and ϕ_2 so that there exists a set $B' \subset B$ with $\nu(B') > \xi$ such that for all $y \in B'$, $||f_1 - \phi_1||_y < \epsilon$ and $||f_2 - \phi_2||_y < \epsilon$.

There exists a finite family of functions $h_1, \cdots, h_l \in L^2(X, \mathcal{A}, \mu)$ having the property that for a.e. $y \in Y$ and all $n \in \mathbf{Z}$ there exist $k_1 = k_1(n, y)$ and $k_2 = k_2(n, y)$ such that $||T^n\phi_1 - h_{k_1}||_y < \epsilon$ and $||S^n\phi_2 - h_{k_2}||_y < \epsilon$.

Let $M = l^2 + 1$. By Lemma 4.4.5 (the full strength of the lemma is not needed), there exists some $s > 0$, depending only on $\mu(B')$ and M, and some integers $h_1, \cdots, h_M$, with $\mu(B' \cap T^{-h_1}B' \cap \cdots \cap T^{-h_M}B') > s$, and such that $(h_j - h_i) \in E$ whenever $1 \le i < j \le M$. (One simply lets $h_i = n_{\{1,\cdots,i\}}$, where the n_α's are as in that lemma.)

Fix a "typical" $y \in (B' \cap T^{-h_1}B' \cap \cdots \cap T^{-h_m}B')$. Since $M > l^2$, there exist numbers $i = i(y)$ and $j = j(y)$, with $1 \le i < j \le M$, such that $k_1(h_i, y) = k_1(h_j, y)$ and $k_2(h_i, y) = k_2(h_j, y)$ simultaneously. It follows by the triangle inequality that

$$||T^{h_i}\phi_1 - T^{h_j}\phi_1||_y < 2\epsilon \text{ and } ||S^{h_i}\phi_2 - S^{h_j}\phi_2||_y < 2\epsilon.$$

We now have $||\phi_1 - T^{h_j - h_i}\phi_1||_{T^{h_i}y} < 2\epsilon$. Also, since $T^{h_i}y \in B'$, we have $||\phi_1 - f_1||_{T^{h_i}y} < \epsilon$. On the other hand, since $T^{h_j}y \in B'$, we have

$$||T^{h_j - h_i}f_1 - T^{h_j - h_i}\phi_1||_{T^{h_i}y} = ||T^{h_j}f_1 - T^{h_j}\phi_1||_y = ||f_1 - \phi_1||_{T^{h_j}y} < \epsilon.$$

This, finally, gives $||f_1 - T^{h_j - h_i}f_1||_{T^{h_i}y} < 4\epsilon$. Similarly, $||f_2 - S^{h_j - h_i}f_2||_{T^{h_i}y} < 4\epsilon$.

Let $h = h(y) = h_{j(y)} - h_{i(y)}$. Then for $y \in (B' \cap T^{-h_1} \cap \cdots \cap T^{-h_M}B')$,

$$\int fT^h f_1 \; S^h f_2 \, d\mu_y > \int f f_1 f_2 \, d\mu_y - 8\epsilon > \frac{\eta\gamma}{2}.$$

Let $C \subset (B' \cap T^{-h_1}B' \cap \cdots \cap T^{-h_M}B')$ be a set satisfying $\nu(C) > \frac{s}{M^2}$ and having the property that $h_0 = h(y)$ is constant on C. Then

$$\int f \; T^{h_0} f_1 \; S^{h_0} f_2 \, d\mu > \frac{\eta\gamma s}{2M^2} > 0.$$

Recall that $h_0 \in E$. Let

$$\Gamma = \left\{ n : \int f \; T^n f_1 \; S^n f_2 \, d\mu > \frac{\eta\gamma s}{2M^2} \right\}.$$

Note that f_1 and f_2 did not depend on the arbitary thick set E, and we managed to find some $h_0 \in (E \cap \Gamma)$. It follows that Γ is syndetic, i.e. for some L, any interval of length L contains a member of Γ. It now follows that

$$\liminf_{N-M\to\infty} \frac{1}{N-M} \sum_{n=M}^{N-1} \int fT^n f_1 S^n f_2 \, d\mu \ge \frac{\eta\gamma s}{2M^2 L} > 0.$$

□

4.5 An IP Roth Theorem.

Our goal in this section is to prove the following theorem.

Theorem 4.5.1. Suppose that $\{n_\alpha\}_{\alpha\in\mathcal{F}} \subset \mathbf{Z}$ is an IP-set and $\mathcal{F}^{(0)}$ is an IP-ring. If $(X,\mathcal{A},\mu,T)$ is an invertible measure preserving system, and $A\in\mathcal{A}$ with $\mu(A)>0$, then for some IP-ring $\mathcal{F}^{(1)}\subset\mathcal{F}^{(0)}$,

$$\operatorname*{IP-lim}_{\alpha\in\mathcal{F}^{(1)}} \mu(A\cap T^{-n_\alpha}A\cap T^{-2n_\alpha}A)>0.$$

Theorem 4.5.1 is a very special case of the IP-Szemerédi theorem of Furstenberg and Katznelson ([FK2]). The proof follows [M4].

We fix an invertible Lebesgue measure preserving system $(X,\mathcal{A},\mu,T)$, a set $A\in\mathcal{A}$ with $\mu(A)>0$, an IP-set $\{n_\alpha\}_{\alpha\in\mathcal{F}}$ and an IP-ring $\mathcal{F}^{(0)}$. Let $\mathcal{F}^{(1)}$ be an IP-ring with the property that

$$\operatorname*{IP-lim}_{\alpha\in\mathcal{F}^{(1)}} \mu(A\cap T^{-n_\alpha}A\cap T^{-2n_\alpha}A)$$

exists, with the additional requirement that for all $f\in L^2(X,\mathcal{A},\mu)$,

$$\operatorname*{IP-lim}_{\alpha\in\mathcal{F}^{(1)}} T^{n_\alpha}f=Pf$$

exists in the weak topology. By Proposition 3.4.9, P is an orthogonal projection. Furthermore,

$$P\big(L^2(X,\mathcal{A},\mu)\big)=\big\{f\in L^2(X,\mathcal{A},\mu):\operatorname*{IP-lim}_{\alpha\in\mathcal{F}^{(1)}} \big|\big|T^{n_\alpha}f-f\big|\big|=0\big\}. \tag{4.3}$$

Exercise 4.22. Use the characterization (4.3) together with Theorem 4.3.7 to show that there exists a T-invariant σ-algebra $\mathcal{B}$ such that $P\big(L^2(X,\mathcal{A},\mu)\big)=L^2(X,\mathcal{B},\mu)$.

It follows that $Pf=E(f|\mathcal{B})$, that is, P is the projection onto $L^2(X,\mathcal{B},\mu)$. The factor determined by $\mathcal{B}$ will be denoted $(Y,\mathcal{B},\nu,S)$, and $(\tilde X,\tilde{\mathcal{A}},\tilde\mu,\tilde T)$ will denote the conditional product system $(X\times X,\mathcal{A}\otimes\mathcal{A},\mu\times_Y\mu,T\times T)$. Let $\mathcal{F}^{(2)}\subset\mathcal{F}^{(1)}$ be an IP-ring with the property that for all $H\in L^2(X\times X,\mathcal{A}\otimes\mathcal{A},\mu\times_Y\mu)$,

$$\operatorname*{IP-lim}_{\alpha\in\mathcal{F}^{(2)}} \tilde T^{n_\alpha}H=Q_1H$$

and

$$\operatorname*{IP-lim}_{\alpha\in\mathcal{F}^{(2)}} \tilde T^{2n_\alpha}H=Q_2H$$

exist in the weak topology. Again by Proposition 3.4.9, Q_1 and Q_2 are orthogonal projections.

Definition 4.5.2. Let $\{k_\alpha\}_{\alpha\in\mathcal{F}}$ be an IP-set. A function $f \in L^\infty(X,\mathcal{A},\mu)$ is *$\{k_\alpha\}$-almost periodic over $\mathcal{B}$* if for every $\epsilon > 0$, there exists a set $D \in \mathcal{B}$ with $\nu(D) < \epsilon$, and functions $g_1, \cdots, g_N \in L^2(X,\mathcal{A},\mu)$ having the property that for every $\delta > 0$, there exists $\alpha_0 \in \mathcal{F}^{(2)}$ such that for every $\alpha \in \mathcal{F}^{(2)}$ with $\alpha > \alpha_0$, there is a set $E(\alpha) \in \mathcal{B}$ with $\nu\big(E(\alpha)\big) < \delta$ having the property that for all $y \notin D \cup E(\alpha)$, there exists a number $i(y,\alpha)$, $1 \le i(y,\alpha) \le N$, with $\big|\big|T^{n_\alpha} f - g_{i(y,\alpha)}\big|\big|_y < \epsilon$.

We will denote by $\mathcal{L}_1$ the closure of the $\{n_\alpha\}$-almost periodic functions and by $\mathcal{L}_2$ the closure of the $\{2n_\alpha\}$-almost periodic functions.

Exercise 4.23. Use Theorem 4.3.7 to show that there exists σ-algebras $\mathcal{B}_1$ and $\mathcal{B}_2$ such that $\mathcal{L}_1 = L^2(X,\mathcal{B}_1,\mu)$ and $\mathcal{L}_2 = L^2(X,\mathcal{B}_2,\mu)$.

As usual, any $H \in L^2(X \times X, \mathcal{A}\otimes\mathcal{A}, \mu \times_Y \mu)$ defines an operator $\mathbf{H} : \phi \to \mathbf{H} * \phi$, by

$$\mathbf{H} * \phi(x) = \int H(x,t)\phi(t)\, d\mu_{\pi(x)}(t).$$

For a.e. $y \in Y$, $\mathbf{H}$ is compact on $L^2(X,\mu_y)$.

Lemma 4.5.3. If $H \in L^2(X \times X, \mathcal{A}\otimes\mathcal{A}, \mu \times_Y \mu)$ satisfies $Q_1 H = H$ (respectively $Q_2 H = H$) and $\phi \in L^\infty(X,\mathcal{A},\mu)$, then $\mathbf{H} * \phi$ is $\{n_\alpha\}$-almost periodic (respectively $\{2n_\alpha\}$-almost periodic) over $\mathcal{B}$.

Proof. Suppose that $H \in L^2(X \times X, \mathcal{A}\otimes\mathcal{A}, \mu \times_Y \mu)$ satisfies $Q_1 H = H$. (The other case is virtually identical.) Let $\epsilon > 0$ be arbitrary. Since for a.e. $y \in Y$ the operator $\mathbf{H}$ is a compact operator on $L^2(X,\mathcal{A},\mu_y)$, there exists a number $M(y) \in \mathbf{N}$ such that

$$\big\{\mathbf{H} * (T^j f) : -M(y) \le j \le M(y)\big\}$$

is $\frac{\epsilon}{2}$-dense in $\{\mathbf{H} * (T^j f) : j \in \mathbf{Z}\}$ (in $L^2(X,\mu_y)$). Let M be so large that $M > M(y)$ for all y outside of a set $D \in \mathcal{B}$ with $\nu(D) < \epsilon$, and let

$$\{g_1, \cdots, g_N\} = \big\{\mathbf{H} * (T^{-M}\phi), \mathbf{H} * (T^{-M+1}\phi), \cdots, \mathbf{H} * (T^M\phi)\big\}.$$

Then for any $y \in D^c$, and any $n \in \mathbf{Z}$, there exists some $j(y,n) \in \mathbf{N}$ with $1 \le j(y,n) \le M$ such that

$$\big|\big|\mathbf{H} * (T^n\phi) - g_{j(y,n)}\big|\big|_y < \frac{\epsilon}{2}.$$

For $y \in D^c$ and $\alpha \in \mathcal{F}$ let $i(y,\alpha) = j(y,n_\alpha)$. Suppose now that $\delta > 0$ is arbitrary. As

$$\begin{aligned}
&\operatorname*{IP-lim}_{\alpha\in\mathcal{F}^{(2)}} \big|\big|T^{n_\alpha}(\mathbf{H} * \phi) - \mathbf{H} * (T^{n_\alpha}\phi)\big|\big|^2 \\
&= \operatorname*{IP-lim}_{\alpha\in\mathcal{F}^{(2)}} \int \Big| \int \big(H(T^{n_\alpha}x, T^{n_\alpha}t) - H(x,t)\big)\phi(T^{n_\alpha}t)\, d\mu_{\pi(x)}(t)\Big|^2 d\mu(x) \\
&\le \operatorname*{IP-lim}_{\alpha\in\mathcal{F}^{(2)}} \int\int \big|H(T^{n_\alpha}x, T^{n_\alpha}t) - H(x,t)\big|^2 \big|\phi(T^{n_\alpha}t)\big|^2 d\mu_{\pi(x)}(t)\, d\mu(x) \\
&\le \operatorname*{IP-lim}_{\alpha\in\mathcal{F}^{(2)}} \big|\big|\tilde{T}^{n_\alpha} H - H\big|\big|^2_{L^2(X\times X,\mathcal{A}\otimes\mathcal{A},\mu\times_Y\mu)} \big|\big|\phi\big|\big|^2_\infty = 0,
\end{aligned}$$

there exists $\alpha_0 \in \mathcal{F}^{(2)}$ having the property that for every $\alpha \in \mathcal{F}^{(2)}$ with $\alpha > \alpha_0$,

$$\left|\left|T^{n_\alpha}(\mathbf{H} * \phi) - \mathbf{H} * (T^{n_\alpha}\phi)\right|\right|$$

is so small that

$$\left|\left|T^{n_\alpha}(\mathbf{H} * \phi) - \mathbf{H} * (T^{n_\alpha}\phi)\right|\right|_y < \frac{\epsilon}{2} \tag{4.4}$$

for all y outside of a set $E(\alpha)$, where $\nu\big(E(\alpha)\big) < \delta$. If then $y \notin D \cup E(\alpha)$, we have

$$\left|\left|T^{n_\alpha}(\mathbf{H} * \phi) - g_{i(y,n_\alpha)}\right|\right|_y < \epsilon.$$

□

Lemma 4.5.4. If $f \in L^\infty(X, \mathcal{A}, \mu)$ satisfies $E(f|\mathcal{B}_1) = 0$, then

$$\operatorname*{IP-lim}_{\alpha\in\mathcal{F}^{(2)}} \left|\left|P(fT^{n_\alpha}f)\right|\right| = 0.$$

If $f \in L^\infty(X, \mathcal{A}, \mu)$ satisfies $E(f|\mathcal{B}_2) = 0$, then

$$\operatorname*{IP-lim}_{\alpha\in\mathcal{F}^{(2)}} \left|\left|P(fT^{2n_\alpha}f)\right|\right| = 0.$$

Proof. Again, we prove only the first claim, as the second is similar. By Lemma 4.5.3 and the fact that $E(f|\mathcal{B}_1) = 0$, f is orthogonal to $\mathbf{H} * f$ for every $H \in L^2(X \times X, \mathcal{A} \otimes \mathcal{A}, \mu \times_Y \mu)$ satisfying $Q_1 H = H$. It follows that $f \otimes f$ is orthogonal to all $H \in L^2(X \times X, \mathcal{A} \otimes \mathcal{A}, \mu \times_Y \mu)$ which satisfy $Q_1 H = H$. To see this, note that

$$\begin{aligned}
&\int f \otimes f(x,t) H(x,t)\, d\tilde{\mu}(x,t) \\
&= \int f(x) \int H(x,t) f(t)\, d\mu_{\pi(x)}(t)\, d\mu(x) \\
&= \int f(x)\big(\mathbf{H} * f(x)\big)\, d\mu(x) = \langle \mathbf{H} * f, f\rangle = 0.
\end{aligned}$$

$f \otimes f$ is therefore orthogonal to $Q_1 H$ for all $H \in L^2(X \times X, \mathcal{A} \otimes \mathcal{A}, \mu \times_Y \mu)$, hence

$$\begin{aligned}
\operatorname*{IP-lim}_{\alpha\in\mathcal{F}^{(2)}} \left|\left|P(fT^{n_\alpha}f)\right|\right|^2 &= \operatorname*{IP-lim}_{\alpha\in\mathcal{F}^{(2)}} \int \left|\int f(x) T^{n_\alpha} f(x)\, d\mu_y(x)\right|^2 d\nu(y) \\
&= \operatorname*{IP-lim}_{\alpha\in\mathcal{F}^{(2)}} \int f(x) f(t) T^{n_\alpha} f(x) T^{n_\alpha} f(t)\, d\tilde{\mu}(x,t) \\
&= \int (f \otimes f) Q_1 (f \otimes f)\, d\tilde{\mu} = 0.
\end{aligned}$$

□

Lemma 4.5.5. If $f, g \in L^\infty(X, \mathcal{A}, \mu)$ with either $E(f|\mathcal{B}_1) = 0$ or $E(g|\mathcal{B}_2) = 0$, then there exists an IP-ring $\mathcal{F}^{(3)} \subset \mathcal{F}^{(2)}$ such that

$$\underset{\alpha\in\mathcal{F}^{(3)}}{\text{IP-lim}}\ T^{n_\alpha} f T^{2n_\alpha} g = 0$$

in the weak topology.

Proof. We will use Lemma 3.6.3. Let $x_\alpha = T^{n_\alpha} f T^{2n_\alpha} g$. Then

$$\begin{aligned}
&\underset{\beta\in\mathcal{F}^{(2)}}{\text{IP-lim}}\ \underset{\alpha\in\mathcal{F}^{(2)}}{\text{IP-lim}}\ \langle x_\alpha, x_{\alpha\cup\beta}\rangle \\
&= \underset{\beta\in\mathcal{F}^{(2)}}{\text{IP-lim}}\ \underset{\alpha\in\mathcal{F}^{(2)}}{\text{IP-lim}}\ \int T^{n_\alpha} f T^{2n_\alpha} g T^{n_\alpha+n_\beta} f T^{2n_\alpha+2n_\beta} g\, d\mu \\
&= \underset{\beta\in\mathcal{F}^{(2)}}{\text{IP-lim}}\ \underset{\alpha\in\mathcal{F}^{(2)}}{\text{IP-lim}}\ \int (f T^{n_\beta} f) T^{n_\alpha}(g T^{2n_\beta} g)\, d\mu \\
&= \underset{\beta\in\mathcal{F}^{(2)}}{\text{IP-lim}}\ \int P(f T^{n_\beta} f) P(g T^{2n_\beta} g)\, d\mu \\
&\leq \underset{\beta\in\mathcal{F}^{(2)}}{\text{IP-lim}}\ \left(\left\|P(f T^{n_\alpha} f)\right\|\right)\left(\left\|P(g T^{2n_\alpha} g)\right\|\right) = 0
\end{aligned}$$

by Lemma 4.5.4.

□

Lemma 4.5.6. Suppose that $\mathcal{G}^{(1)}$ is an IP-ring and that $\left\{x_{\alpha,\beta} : (\alpha,\beta) \in \left(\mathcal{G}^{(1)}\right)^2_<\right\} \subset \mathbf{R}$ satisfies

$$\underset{\beta\in\mathcal{G}^{(1)}}{\text{IP-lim}}\ x_{\alpha,\beta} = 0$$

for all $\alpha \in \mathcal{G}^{(1)}$. Then for any $\delta > 0$ there exists an IP-ring $\mathcal{G}^{(2)} \subset \mathcal{G}^{(1)}$ with the property that for all $(\alpha,\beta) \in \left(\mathcal{G}^{(2)}\right)^2_<$ we have $\left|x_{\alpha,\beta}\right| < \delta$.

Proof. Since we clearly cannot have $|x_{\alpha,\beta}| \geq \delta$ for all $(\alpha,\beta) \in (\mathcal{G}^{(2)})^2_<$ for any IP-ring $\mathcal{G}^{(2)} \subset \mathcal{G}^{(1)}$, the result follows from the Milliken-Taylor theorem (see **MT1** in Section 2.2).

□

Completion of the Proof of Theorem 4.5.1. We now proceed to show that

$$\underset{\alpha\in\mathcal{F}^{(1)}}{\text{IP-lim}}\ \mu(A \cap T^{-n_\alpha} A \cap T^{-2n_\alpha} A) > 0.$$

Let $f = 1_A$, and put $f_1 = E(f|\mathcal{B}_1)$, $f_2 = E(f|\mathcal{B}_2)$. Also put $h_1 = f - f_1$ and $h_2 = f - f_2$. According to Lemma 4.5.5 (more or less) we may choose an IP-ring $\mathcal{F}^{(3)} \subset \mathcal{F}^{(2)}$ such that

$$\begin{aligned}
&\underset{\alpha\in\mathcal{F}^{(3)}}{\text{IP-lim}}\ T^{n_\alpha} f_1 T^{2n_\alpha} h_2 \\
=&\underset{\alpha\in\mathcal{F}^{(3)}}{\text{IP-lim}}\ T^{n_\alpha} h_1 T^{2n_\alpha} f_2 \\
=&\underset{\alpha\in\mathcal{F}^{(3)}}{\text{IP-lim}}\ T^{n_\alpha} h_1 T^{2n_\alpha} h_2 = 0
\end{aligned}$$

in the weak topology. Then

$$\begin{aligned}
&\operatorname*{IP-lim}_{\alpha\in\mathcal{F}^{(1)}} \mu(A\cap T^{-n_\alpha}A\cap T^{-2n_\alpha}A)\\
=&\operatorname*{IP-lim}_{\alpha\in\mathcal{F}^{(3)}} \int fT^{n_\alpha}fT^{2n_\alpha}f\,d\mu\\
=&\operatorname*{IP-lim}_{\alpha\in\mathcal{F}^{(3)}} \int fT^{n_\alpha}(f_1+h_1)T^{2n_\alpha}(f_2+h_2)\,d\mu\\
=&\operatorname*{IP-lim}_{\alpha\in\mathcal{F}^{(3)}} \int fT^{n_\alpha}f_1T^{2n_\alpha}f_2\,d\mu.
\end{aligned}$$

We therefore need only show that

$$\operatorname*{IP-lim}_{\alpha\in\mathcal{F}^{(3)}} \int fT^{n_\alpha}f_1T^{2n_\alpha}f_2\,d\mu>0. \tag{4.5}$$

Exercise 4.24. Using measurable decomposition, show that $f_1(x)f_2(x)>0$ for a.e. $x\in A$.

Therefore, there exists some $a>0$ and a set $A'\subset A$ with $\mu(A')>0$ such that $f_1(x)f_2(x)>a$ for all $x\in A'$. Furthermore, there exist numbers $b,\xi>0$, and a set $B_1\in\mathcal{B}$ with $\nu(B_1)=5\xi>0$, such that for all $y\in B_1$, $\mu_y(A')>b$. It follows that

$$\int ff_1f_2\,d\mu_y>ab \tag{4.6}$$

for all $y\in B_1$.

Let $\epsilon=\frac{ab}{18}$. We may approximate f_1 by a function ϕ_1 which is $\{n_\alpha\}$-almost periodic over $\mathcal{B}$. Likewise, we may approximate f_2 by a function ϕ_2 which is $\{2n_\alpha\}$-almost periodic over $\mathcal{B}$. We make these approximations so close that there exists a set $B_2\subset B_1$ with $\nu(B_2)>4\xi$ such that for all $y\in B_2$ we have

$$\left|\left|f_1-\phi_1\right|\right|_y<\epsilon \tag{4.7}$$

and

$$\left|\left|f_2-\phi_2\right|\right|_y<\epsilon. \tag{4.8}$$

By definition, there exists a finite set $\{g_1,\cdots,g_M\}\subset L^2(X,\mathcal{A},\mu)$ and a set $D\in\mathcal{B}$ with $\nu(D)<\xi$ such that for every $\delta>0$ there exists $\alpha_0\in\mathcal{F}^{(3)}$ having the property that for every $\alpha\in\mathcal{F}^{(3)}$ with $\alpha>\alpha_0$, there exists $E(\alpha)\in\mathcal{B}$ with $\nu(E(\alpha))<\delta$ such that for every $y\notin D\cup E(\alpha)$, there exist numbers $i(y,\alpha)$ and $j(y,\alpha)$, $1\le i(y,\alpha),j(y,\alpha)\le M$, with $\left|\left|T^{n_\alpha}\phi_1-g_{i(y,\alpha)}\right|\right|_y<\epsilon$ and $\left|\left|T^{2n_\alpha}\phi_2-g_{j(y,\alpha)}\right|\right|_y<\epsilon$.

We claim that

$$\operatorname*{IP-lim}_{\alpha\in\mathcal{F}^{(3)}} \int fT^{n_\alpha}f_1T^{2n_\alpha}f_2\,d\mu\ge\frac{ab\xi}{4M^2}.$$

If this were not the case, we could pass to an IP-subring of $\mathcal{F}^{(3)}$ (continue to call it $\mathcal{F}^{(3)}$) having the property that for all $\alpha \in \mathcal{F}^{(3)}$,

$$\int fT^{n_\alpha} f_1 T^{2n_\alpha} f_2 \, d\mu < \frac{ab\xi}{4M^2}.$$

We will show that this is impossible by producing an $\alpha \in \mathcal{F}^{(3)}$ for which

$$\int fT^{n_\alpha} f_1 T^{2n_\alpha} f_2 \, d\mu > \frac{ab\xi}{4M^2}.$$

Let $N = M^2 + 1$. There exists $\alpha_0 \in \mathcal{F}^{(3)}$ such that for every $\alpha \in \mathcal{F}^{(3)}$ with $\alpha > \alpha_0$, there exists a set $E(\alpha) \in \mathcal{B}$ with $\nu\big(E(\alpha)\big) < 2^{-2N-2}\xi$ having the property that for every $y \notin D \cup E(\alpha)$, there exist $i(y,\alpha), j(y,\alpha) \in \mathbf{N}$ with $1 \le i(y,\alpha), j(y,\alpha) \le M$ such that

$$\big|\big|T^{n_\alpha}\phi_1 - g_{i(y,\alpha)}\big|\big|_y < \epsilon \tag{4.9}$$

and

$$\big|\big|T^{2n_\alpha}\phi_2 - g_{j(y,\alpha)}\big|\big|_y < \epsilon. \tag{4.10}$$

Let $B_3 = (B_2 \setminus D)$. Then $\nu(B_3) > 3\xi$. Since $B_3 \in \mathcal{B}$, there exists $\beta_0 \in \mathcal{F}^{(3)}$ with $\beta_0 > \alpha_0$ having the property that for any $\alpha \in \mathcal{F}^{(3)}$ with $\alpha > \beta_0$ we have

$$\nu(B_3 \triangle T^{-n_\alpha} B_3) < \frac{2^{-N}\xi}{3}, \tag{4.11}$$

which implies that

$$\nu(B_3 \triangle T^{-2n_\alpha} B_3) < \frac{2^{-N+1}\xi}{3}. \tag{4.12}$$

Recall that

$$\operatorname*{IP-lim}_{\beta \in \mathcal{F}^{(3)}} \big|\big|T^{n_\beta}h - T^{2n_\beta}h\big|\big| = \operatorname*{IP-lim}_{\beta \in \mathcal{F}^{(3)}} \big|\big|h - T^{n_\beta}h\big|\big| = 0$$

for all $h \in L^2(X, \mathcal{B}, \mu)$. It follows that for any $\alpha \in \mathcal{F}^{(3)}$,

$$\operatorname*{IP-lim}_{\beta \in \mathcal{F}^{(3)}} \int \Big| \big|\big|f_2 - T^{2n_\alpha} f_2\big|\big|_{T^{n_\beta}y} - \big|\big|f_2 - T^{2n_\alpha} f_2\big|\big|_{T^{2n_\beta}y} \Big|^2 d\nu(y) = 0.$$

By Lemma 4.5.6, for any $\delta > 0$ there exists an IP-ring $\mathcal{F}^{(4)} \subset \mathcal{F}^{(3)}$ having the property that for any $(\alpha, \beta) \in \big(\mathcal{F}^{(4)}\big)^2_<$,

$$\int \Big| \big|\big|f_2 - T^{2n_\alpha} f_2\big|\big|_{T^{n_\beta}y} - \big|\big|f_2 - T^{2n_\alpha} f_2\big|\big|_{T^{2n_\beta}y} \Big|^2 d\nu(y) < \delta.$$

By choosing δ small enough we may ensure that for all $(\alpha,\beta)\in\left(\mathcal{F}^{(4)}\right)^2_<$ we have a set $C(\alpha,\beta)\in\mathcal{B}$ with $\nu\big(C(\alpha,\beta)\big)<\frac{\xi}{2M^2}$ such that for all $y\notin C(\alpha,\beta)$ we have

$$\left|\big|\big|f_2-T^{2n_\alpha}f_2\big|\big|_{T^{n_\beta}y}-\big|\big|f_2-T^{2n_\alpha}f_2\big|\big|_{T^{2n_\beta}y}\right|<\epsilon.$$

Fix the IP-ring $\mathcal{F}^{(4)}$ for the remainder of the proof.

Fix some N-tuple $(\alpha_1,\cdots,\alpha_N)\in\left(\mathcal{F}^{(4)}\right)^N_<$ with $\alpha_1>\beta_0$. Let

$$B_3'=B_3\setminus\Big(\bigcup_{\alpha\in F\{\alpha_1,\cdots,\alpha_N\}}E(\alpha)\Big).$$

Recall that $FU_\emptyset(A)$ denotes the family of all finite unions of members of A. Now put

$$\begin{aligned}B_4=\bigg(B_3\cap\Big(\bigcap_{\beta\in FU_\emptyset\{\alpha_1,\cdots,\alpha_N\}}(T^{-n_\beta}B_3\cap T^{-2n_\beta}B_3)\Big)\bigg)\\ \setminus\Big(\bigcup_{\alpha,\beta\in FU_\emptyset\{\alpha_1,\cdots,\alpha_N\}}(E(\alpha)\cup T^{-n_\beta}E(\alpha)\cup T^{-2n_\beta}E(\alpha)\Big).\end{aligned}$$

Then

(i) $\nu(B_4)>\xi$.

(ii) For any $y\in B_3'$, (4.6), (4.7), (4.8), (4.9) and (4.10) hold.

(iii) For any $y\in B_4$ and $\beta\in F\{\alpha_1,\cdots,\alpha_N\}$, $T^{n_\beta}y\in B_3'$ and $T^{2n_\beta}y\in B_3'$.

Since $N=M^2+1$, for any $y\in B_4$ there exist numbers $l(y)$ and $m(y)$ with $1\le l(y)<m(y)\le N$ such that

$$i(y,\alpha_{l(y)}\cup\cdots\cup\alpha_N)=i(y,\alpha_{m(y)}\cup\cdots\cup\alpha_N)$$

and

$$j(y,\alpha_{l(y)}\cup\cdots\cup\alpha_N)=j(y,\alpha_{m(y)}\cup\cdots\cup\alpha_N).$$

Pick a set $B_5\subset B_4$ with $\nu(B_5)>\frac{\xi}{M^2}$ on which $l(y)=l$ and $m(y)=m$ are constant.

Recall that $n_{\alpha_m\cup\cdots\cup\alpha_N}=n_{\alpha_m}+\cdots+n_{\alpha_N}$, etc. Hence, for $y\in B_5$ we have by (4.9), (4.10) and the triangle inequality

$$\big|\big|T^{n_{\alpha_m}+\cdots+n_{\alpha_N}}\phi_1-T^{n_{\alpha_l}+\cdots+n_{\alpha_N}}\phi_1\big|\big|_y<2\epsilon \tag{4.13}$$

and

$$\big|\big|T^{2n_{\alpha_m}+\cdots+2n_{\alpha_N}}\phi_2-T^{2n_{\alpha_l}+\cdots+2n_{\alpha_N}}\phi_2\big|\big|_y<2\epsilon. \tag{4.14}$$

We may rewrite (4.13) as

$$\big|\big|\phi_1-T^{n_{\alpha_l}+\cdots+n_{\alpha_{m-1}}}\phi_1\big|\big|_{T^{n_{\alpha_m}+\cdots+n_{\alpha_N}}y}<2\epsilon. \tag{4.15}$$

Also, since $T^{n_{\alpha_m}+\cdots+n_{\alpha_N}} y \in B_3$, we have

$$||\phi_1 - f_1||_{T^{n_{\alpha_m}+\cdots+n_{\alpha_N}} y} < \epsilon. \tag{4.16}$$

On the other hand, since $T^{n_{\alpha_l}+\cdots+n_{\alpha_N}} y \in B_3$, we have

$$\begin{aligned} &||T^{n_{\alpha_l}+\cdots+n_{\alpha_{m-1}}} f_1 - T^{n_{\alpha_l}+\cdots+n_{\alpha_{m-1}}} \phi_1||_{T^{n_{\alpha_m}+\cdots+n_{\alpha_N}} y} \\ =&||f_1 - \phi_1||_{T^{n_{\alpha_l}+\cdots+n_{\alpha_N}} y} < \epsilon. \end{aligned} \tag{4.17}$$

(4.15), (4.16) and (4.17) give

$$||f_1 - T^{n_{\alpha_l}+\cdots+n_{\alpha_{m-1}}} f_1||_{T^{n_{\alpha_m}+\cdots+n_{\alpha_N}} y} < 4\epsilon. \tag{4.18}$$

Similarly, we may rewrite (4.14) as

$$||\phi_1 - T^{2n_{\alpha_l}+\cdots+2n_{\alpha_{m-1}}} \phi_1||_{T^{2n_{\alpha_m}+\cdots+2n_{\alpha_N}} y} < 2\epsilon. \tag{4.19}$$

Since $T^{2n_{\alpha_m}+\cdots+2n_{\alpha_N}} y \in B_3$, we have

$$||\phi_1 - f_1||_{T^{2n_{\alpha_m}+\cdots+2n_{\alpha_N}} y} < \epsilon, \tag{4.20}$$

and since $T^{2n_{\alpha_l}+\cdots+2n_{\alpha_N}} y \in B_3$ we have

$$\begin{aligned} &||T^{2n_{\alpha_l}+\cdots+2n_{\alpha_{m-1}}} f_1 - T^{2n_{\alpha_l}+\cdots+2n_{\alpha_{m-1}}} \phi_1||_{T^{2n_{\alpha_m}+\cdots+2n_{\alpha_N}} y} \\ =&||f_1 - \phi_1||_{T^{2n_{\alpha_l}+\cdots+2n_{\alpha_N}} y} < \epsilon. \end{aligned} \tag{4.21}$$

(4.19), (4.20), and (4.21) now give

$$||f_2 - T^{2n_{\alpha_l}+\cdots+2n_{\alpha_{m-1}}} f_2||_{T^{2n_{\alpha_m}+\cdots+2n_{\alpha_N}} y} < 4\epsilon. \tag{4.22}$$

Let $\alpha = \alpha_l \cup \cdots \cup \alpha_{m-1}$ and $\beta = \alpha_m \cup \cdots \cup \alpha_N$. Then $\beta > \alpha$, so according to the properties attributed to $\mathcal{F}^{(4)}$,

$$\left| ||f_2 - T^{2n_\alpha} f_2||_{T^{n_\beta} z} - ||f_2 - T^{2n_\alpha} f_2||_{T^{2n_\beta} z} \right| < \epsilon \tag{4.23}$$

For all $z \in Y$ outside of a set $C(\beta)$ with $\nu(C(\beta)) < \frac{\xi}{2M^2}$. Let $B_6 = B_5 \setminus C(\beta)$. Then $\nu(B_6) > \frac{\xi}{2M^2}$ and for all $y \in B_6$, (4.22) and (4.23) give

$$||f_2 - T^{2n_\alpha} f_2||_{T^{n_\beta} y} < 5\epsilon. \tag{4.24}$$

Since $T^{n_\beta} y \in B_3 \subset B_1$, we now have, by (4.6), (4.22) and (4.24),

$$\int f T^{n_\alpha} f_1 T^{2n_\alpha} f_2 \, d\mu_{T^{n_\beta} y} > \frac{ab}{2}$$

for all $y \in B_6$. Since $\nu(B_6) > \frac{\xi}{2M^2}$, we have

$$\int f T^{n_\alpha} f_1 T^{2n_\alpha} f_2 \, d\mu > \frac{ab\xi}{4M^2}.$$

In particular (4.5) holds, completing the proof of Theorem 4.5.1.

□

Exercise 4.25. Infer from Theorem 4.5.1 that if $\mathbf{Z} = \bigcup_{i=1}^r C_i$ then for some i, there exists $n \in C_i$ such that $d^*\big(C_i \cap (c_1 - n) \cap (C_1 - 2n)\big) > 0$.

Chapter 5

Two Szemerédi Theorems

5.1 Furstenberg's structure theorem.

In this chapter we prove our first "full-fledged" multiple recurrence theorems. The first one is due to Furstenberg, and is sufficient to give Szemerédi's theorem on arithmetic progressions (see the introduction) as a corollary. Furstenberg's method of proof has undergone several simplifying modifications, however the basic idea remains the same: transfinite induction through a chain of σ-algebras. One of the most important aspects of his work was the structure theorem itself, a brief exposition of a version of which is the main content of this section (see also [F1], [F2], and [FKO]). We also mention that Zimmer has developed a similar structure theory (see [Z]).

Theorem 5.1.1. (See [F1].) Let $(X, \mathcal{A}, \mu, T)$ be an invertible measure preserving system and let $\mu(A) > 0$. For any $k \in \mathbf{N}$,

$$\liminf_{N-M\to\infty} \frac{1}{N-M} \sum_{n=M}^{N-1} \mu(A \cap T^{-n}A \cap T^{-2n}A \cap \cdots \cap T^{-kn}A) > 0.$$

The proof we give here is based on one in [FKO] (note, however, the proof in [FKO] is not sufficient to give a uniform limit).

As usual, we will assume without loss of generality that $(X, \mathcal{A}, \mu, T)$ is Lebesgue and ergodic. (This goes for all the stated lemmas and theorems throughout the chapter.)

The typical situation we encounter in this chapter is as follows. Suppose $(Z, \mathcal{D}, \xi, R)$ is an ergodic, invertible Lebesgue system. Let $\mathcal{B} \subset \mathcal{A}$ be a pair of R-invariant σ-algebras contained in $\mathcal{D}$. Form the associated factors (which are Lebesgue and ergodic) $(X, \mathcal{A}, \mu, T)$ and $(Y, \mathcal{B}, \nu, S)$. Let $\pi_1 : Z \to X$ and $\pi_2 : Z \to Y$ be the associated factor maps. One might now ask whether $(Y, \mathcal{B}, \nu, S)$ is a factor of $(X, \mathcal{A}, \mu, T)$. The answer is yes. There exists a factor map $\pi_3 : X \to Y$

such that $\pi_3(\pi_1(z)) = \pi_2(z)$ a.e. For this reason, we will often say that $\mathcal{A}$ is an extension of $\mathcal{B}$.

Let $\{\mu_y : y \in Y\}$ be the decomposition of μ over Y. Recall that a function $f \in L^2(X, \mathcal{A}, \mu)$ is called T-compact (or just compact, since there is only one operator T to worry about now) over $\mathcal{B}$ if for every $\epsilon > 0$ there exist $g_1, \cdots, g_k \in L^2(X, \mathcal{A}, \mu)$ such that for all $n \in \mathbf{Z}$ and a.e. $y \in Y$ there exists some $s = s(n, y)$ with $1 \le s \le k$ such that $||T^n f - g_s||_{L^2(X,\mu_y)} < \epsilon$. We shall also write $f \in AP$ if f is compact. (AP stands for "almost periodic".) If $\{f \in L^2(X, \mathcal{A}, \mu) : f \text{ is compact over } \mathcal{B}\}$ is dense in $L^2(X, \mathcal{A}, \mu)$, we say that $(X, \mathcal{A}, \mu, T)$ is an *compact* extension of $(Y, \mathcal{B}, \nu, S)$, or simply that $\mathcal{A}$ is a compact extension of $\mathcal{B}$. If, on the other hand, the conditional product system $(X \times X, \mathcal{A} \otimes \mathcal{A}, \mu \times_Y \mu, T \times T)$ is ergodic, then we say that $\mathcal{A}$ is a *weak mixing* extension of $\mathcal{B}$.

The reader is invited to compare the notions of "relative" compactness and weak mixing with the well known "absolute" notions of compactness and weak mixing we have dealt with previously.

Exercise 5.1. Show that if $H \in L^2(X \times X, \mathcal{A} \otimes \mathcal{A}, \mu \times_Y \mu)$ is $(T \times T)$-invariant and $\phi \in L^\infty(X, \mathcal{A}, \mu)$ then $\mathbf{H} * \phi$ may be approximated arbitrarily closely by a function that is T-compact over $\mathcal{B}$ (*Hint: see the proof of Lemma 4.4.2.*)

Lemma 5.1.2. Suppose that $\mathcal{A}$ fails to be a weak mixing extension of $\mathcal{B}$. Then there exists a T-invariant σ-algebra $\mathcal{D} \subset \mathcal{A}$ such that $\mathcal{D}$ is a compact extension of $\mathcal{B}$.

Proof. Let

$$\mathcal{L} = \overline{\{\mathbf{H} * \phi : H \in L^2(\mu \times_Y \mu) \text{ is } (T \times T)\text{-invariant and } \phi \in L^\infty(X, \mathcal{A}, \mu)\}}.$$

Exercise 5.2. Show that $\mathcal{L}$ is a T-invariant subspace and that for all $f, g \in \mathcal{L}$ both $\min\{f, g\}$ and $\max\{f, g\}$ are in $\mathcal{L}$.

It follows from Theorem 4.3.7 that there exists a T-invariant σ-algebra $\mathcal{D} \subset \mathcal{A}$ such that $\mathcal{L} = L^2(X, \mathcal{D}, \mu)$. By Exercise 5.1, the T-compact over $\mathcal{B}$ functions are dense in $\mathcal{L}$, hence $\mathcal{D}$ is a compact extension of $\mathcal{B}$.

□

A a corollary, we get Furstenberg's structure theorem for ergodic systems.

Theorem 5.1.3. (Cf. [F2, Theorem 6.17].) Suppose that $(X, \mathcal{A}, \mu, T)$ is an invertible ergodic Lebesgue measure preserving system. There is an ordinal η and a system of T-invariant sub-σ algebras $\{\mathcal{A}_\xi \subset \mathcal{A} : \xi \le \eta\}$ such that:

(i) $\mathcal{A}_0 = \{A \in \mathcal{A} : \mu(A) \in \{0, 1\}\}$
(ii) For every $\xi < \eta$, $\mathcal{A}_{\xi+1}$ is a proper compact extension of $\mathcal{A}_\xi$.
(iii) If $\xi \le \eta$ is a limit ordinal then $\bigcup_{\xi' < \xi} \mathcal{A}_{\xi'}$ generates $\mathcal{A}_\xi$.
(iv) Either $\mathcal{A}_\eta = \mathcal{A}$ or else $\mathcal{A}$ is a weakly mixing extension of $\mathcal{A}_\eta$.

Proof. The set $\{\mathcal{A}_\xi\}$ is defined inductively. Suppose $\{\mathcal{A}_\gamma : \gamma < \xi\}$ have been defined. If ξ is not a successor ordinal, Let $\mathcal{A}_\xi$ be the σ-algebra generated by

$\{\mathcal{A}_\gamma : \gamma < \xi\}$. If ξ is a successor ordinal and $\mathcal{A}$ is a weakly mixing extension of $\mathcal{A}_{\xi-1}$, put $\eta = \xi - 1$ and stop. Otherwise, if $\mathcal{A}$ is a compact extension of $\mathcal{A}_{\xi-1}$ put $\mathcal{A}_\xi = \mathcal{A}$. If neither of these is true, let $\mathcal{A}_\xi$ be some non-trivial compact extension of $\mathcal{A}_{\xi-1}$, which exists by Lemma 5.1.2.

Exercise 5.3. This process must terminate. Indeed, by separability, η must be a countable ordinal.

□

The factor $\mathcal{A}_\eta$ appearing in the structure theorem is called the *maximal distal factor* of $\mathcal{A}$. In the remainder of this section, we show that in order to prove Theorem 5.1.1 for a general system $(X, \mathcal{A}, \mu, T)$, it suffices to establish that the conclusion holds when A is taken from its maximal distal factor. In other words, the validity of Theorem 5.1.1 passes through weakly mixing extensions. Let us start with a lemma.

Lemma 5.1.4. Suppose that $(X, \mathcal{A}, \mu, T)$ is a weakly mixing extension of $(Y, \mathcal{B}, \nu, S)$. Let Eh denote the conditional expectation of h given $\mathcal{B}$. Let $f, g \in L^2(X, \mathcal{A}, \mu)$. If $Ef = 0$ or $Eg = 0$ then

$$D\text{-}\lim_h \left|\left|E(fT^h g)\right|\right| = 0.$$

Proof. In light of Exercise 4.1 (a) it suffices to show that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \left|\left|E(fT^n g)\right|\right|^2 = 0.$$

We have

$$\begin{aligned}
&\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \left|\left|E(fT^n g)\right|\right|^2 \\
&= \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \int (f \otimes f)\tilde{T}^n (g \otimes g) d\tilde{\mu} \\
&= \Big(\int (f \otimes f) d\tilde{\mu}\Big)\Big(\int (g \otimes g) d\tilde{\mu}\Big) \\
&= \Big(\int (Ef)^2\, d\nu\Big)\Big(\int (Eg)^2\, d\nu\Big) = 0.
\end{aligned}$$

□

The reader may like to compare Lemma 5.1.4 with Theorem 4.1.3. The following uniform version of the mean ergodic theorem is proved in the same manner as Corollary 3.4.8.

Theorem 5.1.5. (Uniform mean ergodic theorem.) Suppose that $(X, \mathcal{A}, \mu, T)$ is a measure preserving system. Then

$$Pf = \lim_{N-M\to\infty} \frac{1}{N-M} \sum_{n=M}^{N-1} T^n f$$

exists in the norm topology for all $f \in L^2(X, \mathcal{A}, \mu)$. Moreover, P is the orthogonal projection onto the space of T-invariant functions.

Exercise 5.4. Prove Theorem 5.1.5.

We shall need yet another of Bergelson's van der Corput type lemmas.

Lemma 5.1.6. ([B2].) Suppose that $\{x_n : n \in \mathbf{Z}\}$ is a bounded sequence of vectors in a Hilbert space $\mathcal{H}$. If

$$D\text{-}\lim_h \lim_{N-M\to\infty} \frac{1}{N-M} \sum_{n=M}^{N-1} \langle x_n, x_{n+h}\rangle = 0,$$

then

$$\lim_{N-M\to\infty} \left|\left| \frac{1}{N-M} \sum_{n=M}^{N-1} x_n \right|\right| = 0.$$

Proof. Let $\epsilon > 0$. Using Exercise 4.17, we may fix H large enough that

$$\sum_{r=-H}^{H} \frac{H-|r|}{H^2} \limsup_{N-M\to\infty} \frac{1}{N-M} \sum_{u=M}^{N-1} \langle x_u, x_{u+r}\rangle < \epsilon.$$

We have

$$\frac{1}{N-M}\sum_{n=M}^{N-1} x_n = \frac{1}{N-M}\sum_{n=M}^{N-1}\left(\frac{1}{H}\sum_{h=1}^{H} x_{n+h}\right) + \Psi'_{M,N} = \Psi_{M,N} + \Psi'_{M,N},$$

where $\limsup_{N-M\to\infty} ||\Psi'_{M,N}|| = 0$. We show $\limsup_{N-M\to\infty} ||\Psi_{M,N}|| < \epsilon$.

$$\begin{aligned}
||\Psi_{M,N}||^2 &\le \frac{1}{N-M}\sum_{n=M}^{N-1} \left|\left| \frac{1}{H}\sum_{h=1}^{H} x_{n+h}\right|\right|^2 \\
&= \frac{1}{N-M}\sum_{n=M}^{N-1} \frac{1}{H^2} \sum_{h,k=1}^{H} \langle x_{n+h}, x_{n+k}\rangle \\
&= \sum_{r=-H}^{H} \frac{H-|r|}{H^2(N-M)} \sum_{u=M}^{N-1} \langle x_u, x_{u+r}\rangle + \Psi''_{M,N},
\end{aligned}$$

where $\Psi''_{M,N} \to 0$ as $N - M \to \infty$. By choice of H the last expression is less than ϵ when $N - M$ is sufficiently large.

$\square$

The following theorem may be compared to Theorem 4.1.4.

Theorem 5.1.7. Let $(X, \mathcal{A}, \mu, T)$ be a weakly mixing extension of $(Y, \mathcal{B}, \nu, S)$. Then for any $f_1, \cdots, f_k \in L^\infty(X, \mathcal{A}, \mu)$,

$$\lim_{N-M\to\infty} \left|\left| \frac{1}{N-M} \sum_{n=M}^{N-1} \left(\prod_{i=1}^{k} T^{in} f_i - \prod_{i=1}^{k} S^{in} E f_i \right) \right|\right| = 0.$$

Proof. The proof is by induction on k.

Exercise 5.5. Show that the set of T-invariant functions in $L^2(X, \mathcal{A}, \mu)$ is a subspace of $L^2(X, \mathcal{B}, \mu)$. Hence by Theorem 5.1.5 the conclusion of Theorem 5.1.7 holds for $k = 1$.

Suppose now that Theorem 5.1.7 holds for $k - 1$.

Exercise 5.6. Show that it is sufficent to show that the conclusion holds when $Ef_a = 0$ for some a, $1 \leq a \leq k$. (*Hint: see Exercise 4.3.*)

We must show in this case that

$$\lim_{N-M\to\infty} \left|\left| \frac{1}{N-M} \sum_{n=M}^{N-1} \left(\prod_{i=1}^{k} T^{in} f_i \right) \right|\right| = 0.$$

We use Lemma 5.1.6. Let $x_n = \prod_{i=1}^{k} T^{in} f_i$. Then

$$\begin{aligned}
&\lim_{N-M\to\infty} \frac{1}{N-M} \sum_{n=M}^{N-1} \langle x_n, x_{n+h} \rangle \\
&= \lim_{N-M\to\infty} \frac{1}{N-M} \sum_{n=M}^{N-1} \int \Big(\prod_{i=1}^{k} T^{in} f_i \Big) \Big(\prod_{i=1}^{k} T^{i(n+h)} f_i \Big)\, d\mu \\
&= \lim_{N-M\to\infty} \frac{1}{N-M} \sum_{n=M}^{N-1} \int (f_1 T^h f_1) T^n (f_2 T^{2h} f_2) \cdots T^{(k-1)n} (f_k T^{kh} f_k)\, d\mu \\
&= \lim_{N-M\to\infty} \frac{1}{N-M} \sum_{n=M}^{N-1} \int E(f_1 T^h f_1) S^n E(f_2 T^{2h} f_2) \cdots S^{(k-1)n} E(f_k T^{kh} f_k)\, d\mu \\
&\leq ||E(f_a T^h f_a)||_{L^2} \prod_{i\neq a} ||f_i||_\infty^2 .
\end{aligned} \tag{5.1}$$

In the second to last line we have used the induction hypothesis (utilizing weak convergence only). Since $D\text{-}\lim_h$ of the last quantity is zero by Lemma 5.1.4, this finishes the proof.

□

Suppose now that we know the conclusion of Theorem 5.1.1 holds for all $A \in \mathcal{A}_\eta$, the maximal distal factor. Let $A \in \mathcal{A}$ with $\mu(A) > 0$. Putting $f = 1_A$

in Theorem 5.1.5, we get

$$\liminf_{N-M\to\infty} \frac{1}{N-M} \sum_{n=M}^{N-1} \mu(A \cap T^{-n}A \cap T^{-2n}A \cap \cdots \cap T^{-kn}A)$$
$$= \liminf_{N-M\to\infty} \frac{1}{N-M} \sum_{n=M}^{N-1} \int EfT^{n}Ef \cdots T^{kn}Ef \, d\mu.$$

Exercise 5.7. Show that under the given assumptions the last line above is positive.

Hence in order to establish Theorem 5.1.1 in general it is sufficient to establish it for all A in the maximal distal factor. This is the content of the next section.

5.2 Szemerédi's theorem.

In the last section we saw that in order to establish Theorem 5.1.1, it suffices to establish that the conclusion holds for all A coming from its maximal distal factor $\mathcal{A}_\eta$. That is what we shall do in this section, using transfinite induction on the set of ordinals $\{\xi : \xi \leq \eta\}$ appearing in Theorem 5.1.3. Notice first that the conclusion trivially holds for all $A \in \mathcal{A}_0$. There are two cases to consider in pushing the induction, namely passage to successor ordinals and passage to limit ordinals.

In order to show that the conclusion passes to compact extensions (the successor ordinal case), we will employ the Hales-Jewett coloring theorem, in particular the version **HJ3** from Section 1.6. Also, our method of proof requires that we push a property somewhat stronger than the conclusion to Theorem 5.1.1. Recall that

$$FS(n_1, \cdots, n_t) = \{n_{i_1} + \cdots + n_{i_m} : 1 \leq m \leq t,\ 1 \leq i_1 < \cdots < i_m \leq t\}.$$

Recall as well that a subset of $\mathbf{Z}$ is thick if it contains arbitrarily large intervals, and a subset of $\mathbf{Z}$ is syndetic if it intersects every thick set non-trivially.

Definition 5.2.1. A T-invariant σ-algebra $\mathcal{B}$ is said to have the *SZ property* if for every $A \in \mathcal{B}$ with $\mu(A) > 0$ and every $k, t \in \mathbf{N}$, there exists $\delta > 0$ such that for every thick set $E \subset \mathbf{Z}$, there exist $n_1, \cdots, n_t \in \mathbf{Z}$ such that $FS(n_1, \cdots, n_t) \subset E$ and

$$\mu\Big(\bigcap_{\substack{0 \leq i_s \leq k \\ 1 \leq s \leq t}} T^{-(i_1 n_1 + \cdots + i_t n_t)}A \Big) > \delta.$$

If $\mathcal{B}$ has the SZ property, then for any $A \in \mathcal{B}$ with $\mu(A) > 0$, taking $t = 1$ in the above definition gives some $\delta > 0$ for which the set

$$\{n \in \mathbf{Z} :\ \mu(A \cap T^{-n}A \cap \cdots \cap T^{-kn}A) > \delta\}$$

is syndetic. In particular, this ensures that

$$\liminf_{N-M\to\infty} \frac{1}{N-M} \sum_{n=M}^{N-1} \mu(A \cap T^{-n}A \cap \cdots \cap T^{-kn}A) > 0.$$

Exercise 5.8. Suppose that $\mathcal{A}$ is a compact extension of $\mathcal{B}$ and $A \in \mathcal{A}$. For every $\epsilon > 0$ there exists a set $A' \in \mathcal{A}$ with $A' \subset A$, $1_{A'} \in AP$ and $\mu(A') > \mu(A) - \epsilon$.

Theorem 5.2.2. If $(X, \mathcal{A}, \mu, T)$ is a compact extension of $(Y, \mathcal{B}, \nu, S)$, and $\mathcal{B}$ has the SZ property, then $\mathcal{A}$ has the SZ property.

Proof. Let $A \in \mathcal{A}$ with $\mu(A) > 0$. By Exercise 5.8 we may assume without loss of generality that $f = 1_A \in AP$. Suppose that $t, k \in \mathbf{N}$. There exists some $c > 0$ and a set $B \in \mathcal{B}$ with $\nu(B) > 0$ such that for all $y \in B$, $\mu_y(A) > c$. Let $\epsilon = \frac{1}{2}\sqrt{\frac{c}{(k+1)^t}}$. Since $1_A \in AP$, there exist functions $g_1, \cdots, g_r \in L^2(X, \mathcal{A}, \mu)$ having the property that for any $n \in \mathbf{N}$, and a.e. $y \in Y$, there exists $s = s(n, y)$, $1 \le s \le r$, such that $\left|\left|T^n f - g_s\right|\right|_y < \epsilon$. We now let $N = m(k+1, r, l)$ as in **HJ3** in Section 1.6. That is, for any r-coloring of Λ_{k+1}^N (which in this context will be taken to be the set of length N words on the alphabet $\{0, 1, 2, \cdots, k\}$), there exists a monochromatic combinatorial l-space (see Section 1.6 for the definition of a combinatorial l-space).

Since $\mathcal{B}$ has the SZ property, there exists $\eta > 0$ such that for every thick set $E \subset \mathbf{Z}$, there exists $u_1, \cdots, u_N \in \mathbf{Z}$ such that $FS(u_1, \cdots, u_N) \subset E$ and

$$\mu\Big(\bigcap_{\substack{0 \le i_s \le k \\ 1 \le s \le t}} T^{-(i_1 n_1 + \cdots + i_N n_N)} B\Big) > \eta. \tag{5.2}$$

Let D be the number of t-variable words of length N on a $(k+1)$-letter alphabet and set $\delta = \frac{c\eta}{2D}$. We will show that in any thick set $E \subset \mathbf{Z}$ there exist $n_1, \cdots, n_t \in \mathbf{Z}$ such that $FS(n_1, \cdots, n_t) \subset E$ and

$$\mu\Big(\bigcap_{\substack{0 \le i_s \le k \\ 1 \le s \le t}} T^{-(i_1 n_1 + \cdots + i_t n_t)} A\Big) > \delta.$$

Let E be thick and choose $u_1, \cdots, u_N \in \mathbf{Z}$ with $FS(u_1, \cdots, u_N) \subset E$ such that (5.2) holds. For

$$y \in \bigcap_{\substack{0 \le i_s \le k \\ 1 \le s \le t}} T^{-(i_1 n_1 + \cdots + i_N n_N)} B, \tag{5.3}$$

form an r-cell partition of Λ_k^N by the rule $i_1 i_2 \cdots i_N \in C_j$ if $s\big(i_1 u_1 + \cdots + i_N u_N), y\big) = j$. In particular, if $i_1 i_2 \cdots i_N \in C_j$ then $\left|\left|T^{i_1 u_1 + \cdots + i_N u_N} f - g_j\right|\right|_y <$

ϵ. For this partition, there exists some j with $1 \le j \le r$ and a t-variable word $w(x_1, \cdots, x_t)$ over Λ_{k+1}^N such that

$$\{w(i_1, i_2, \cdots, i_t) : 0 \le i_s \le k,\ 1 \le s \le t\} \subset C_j.$$

Let L be the map which sends a word $i_1 i_2 \cdots i_N$ to the integer $i_1 u_1 + \cdots + i_N u_N$, and put $M = L\big(w(0, 0, \cdots, 0)\big)$. Put $n_m = L\big(w(0, 0, \cdots, 0, 1, 0, \cdots, 0)\big) - M$, $1 \le m \le t$. (Where the 1 in the arguments of w occurs in the mth place). □

Exercise 5.9. $L\big(w(i_1, \cdots, i_t)\big) = M + i_1 n_1 + i_2 n_2 + \cdots + i_t n_t$.

It follows from the previous exercise that

$$\left|\left|T^{M+i_1 n_1+\cdots+i_t n_t} f - g_j\right|\right|_y < \epsilon \ \ 0 \le i_s \le k,\ 1 \le s \le t.$$

Setting $\tilde{y} = S^M y$, we have

$$\left|\left|T^{i_1 n_1+\cdots+i_t n_t} f - g_j\right|\right|_{\tilde{y}} < \epsilon \ \ 0 \le i_s \le k,\ 1 \le s \le t.$$

Taking i_s to be zero, $1 \le s \le t$, and applying the triangle inequality,

$$\left|\left|T^{i_1 n_1+\cdots+i_t n_t} f - f\right|\right|_{\tilde{y}} < 2\epsilon \ \ 0 \le i_s \le k,\ 1 \le s \le t.$$

It follows that

$$\begin{aligned}&\mu_{\tilde{y}}\big(A \setminus T^{-(i_1 n_1+\cdots+i_t n_t)} A\big)\\ =&\frac{1}{2}\left|\left|T^{i_1 n_1+\cdots+i_t n_t} f - f\right|\right|_{\tilde{y}}^2 \le 2\epsilon^2, \ \ 0 \le i_s \le k,\ 1 \le s \le t.\end{aligned}$$

Recall that for the y we have chosen (see (5.3) above), $\tilde{y} = S^M y \in B$, so that in particular $\mu_{\tilde{y}}(A) \ge c$. Therefore, since $\epsilon = \frac{1}{2}\sqrt{\frac{c}{(k+1)^t}}$,

$$\mu_{\tilde{y}}\Big(\bigcap_{\substack{0 \le i_s \le k\\ 1 \le s \le t}} T^{-(i_1 n_1+\cdots+i_t n_t)} A\Big) \ge c - 2\epsilon^2 (k+1)^t = \frac{c}{2}.$$

The variable word $w(x_1, \cdots, x_t)$, and hence the numbers $n_1, \cdots, n_t$, depend measurably on y and are defined for all

$$y \in \bigcap_{\substack{0 \le i_s \le k\\ 1 \le s \le t}} T^{-(i_1 n_1+\cdots+i_N n_N)} B.$$

Recall that this set has measure greater than η. As there are D choices for $w(x_1, \cdots, x_t)$, and hence at most D choices for $n_1, \cdots, n_t$, there exists a set $H \in \mathcal{B}$ with $\nu(H) > \frac{\eta}{D}$ on which $n_1, \cdots, n_t$ are constant. Thus

$$\mu\Big(\bigcap_{\substack{0 \le i_s \le k\\ 1 \le s \le t}} T^{-(i_1 n_1+\cdots+i_t n_t)} A\Big) \ge \frac{c}{2}\nu(H) > \frac{c\eta}{2D} = \delta.$$

Only one piece of the puzzle remains. We must show that the SZ property passes to limit ordinals.

Exercise 5.10. If ξ is a limit ordinal (as in Theorem 5.1.3) then $\bigcup_{\xi'<\xi} \mathcal{A}_{\xi'}$ is dense in $\mathcal{A}_\xi$. (*Hint: let $\mathcal{L}$ be the closure in $L^2(X, \mathcal{A}, \mu)$ of the set of finite linear combinations of characteristic functions of sets in $\bigcup_{\xi'<\xi} \mathcal{A}_{\xi'}$ and apply Theorem 4.3.7.*)

Proposition 5.2.3. Suppose that $(X, \mathcal{A}, \mu, T)$ is a measure preserving system and that $\{\mathcal{A}_\xi\}$ is a totally ordered chain of T-invariant sub-σ-algebras of $\mathcal{A}$ having the SZ property. If $\bigcup_\xi \mathcal{A}_\xi$ is dense in $\mathcal{A}$ (in particular if $\mathcal{A}$ is the completion of the σ-algebra generated by $\bigcup_\xi \mathcal{A}_\xi$), then $\mathcal{A}$ has the SZ property.

Proof. Let $A \in \mathcal{A}$ with $\mu(A) > 0$ and let $k, t \in \mathbf{N}$. By Exercise 5.10 there exists some ξ and a set $B \in \mathcal{A}_\xi$ such that

$$\mu(A \triangle B) \leq \frac{\mu(A)}{4(k+1)^t}.$$

Let $(Y, \mathcal{A}_\xi, \nu, S)$ be the factor determined by $\mathcal{A}_\xi$. Let $\{\mu_y : y \in Y\}$ be the decomposition of μ over $\mathcal{A}_\xi$. Let $C = \{y \in B : \mu_y(A) \geq 1 - \frac{1}{2(k+1)^t}\}$.

$\square$

This completes the proof of Theorem 5.1.1.

Exercise 5.11. $\nu(C) > 0$.

Since $\mathcal{A}_\xi$ has the SZ property, there exists some $\alpha > 0$ having the property that in any thick set E we may find $n_1, \cdots, n_t \in \mathbf{Z}$ such that $FS(n_1, \cdots, n_t) \subset E$ and

$$\nu\Big(\bigcap_{\substack{0 \leq i_s \leq k \\ 1 \leq s \leq t}} T^{-(i_1 n_N + \cdots i_N n_N)} C\Big) > \alpha. \tag{5.4}$$

Let $\delta = \frac{\alpha}{2}$ and let E be any thick set. Find $n_1, \cdots, n_t \in \mathbf{Z}$ satisfying (5.4) and with $FS(n_1, \cdots, n_t) \subset E$. For any

$$y \in \bigcap_{\substack{0 \leq i_s \leq k \\ 1 \leq s \leq t}} T^{-(i_1 n_1 + \cdots + i_N n_N)} C,$$

$\mu_y(T^{-(i_1 n_1 + \cdots + i_N n_N)} A) \geq 1 - \frac{1}{2(k+1)^t}$ for all $0 \leq i_s \leq k$, $1 \leq s \leq t$, whence

$$\mu_y\Big(\bigcap_{\substack{0 \leq i_s \leq k \\ 1 \leq s \leq t}} T^{-(i_1 n_1 + \cdots + i_N n_N)} A\Big) \geq \frac{1}{2}.$$

Combining this with (5.4) we get

$$\mu\Big(\bigcap_{\substack{0 \leq i_s \leq k \\ 1 \leq s \leq t}} T^{-(i_1 n_1 + \cdots + i_N n_N)} A\Big) \geq \frac{1}{2}\alpha = \delta.$$

5.3 A polynomial Szemerédi theorem.

In this section we will prove the following theorem.

Theorem 5.3.1. ([BM1], [M4].) Assume that $(X, \mathcal{A}, \mu, T)$ is an invertible probability measure preserving system, $k \in \mathbf{N}$, $A \in \mathcal{A}$ with $\mu(A) > 0$, and $p_i(x) \in \mathbf{Q}[x]$ are polynomials satisfying $p_i(\mathbf{Z}) \subset \mathbf{Z}$ and $p_i(0) = 0$, $1 \leq i \leq k$. Then

$$\liminf_{N-M\to\infty} \frac{1}{N-M} \sum_{n=M}^{N-1} \mu(A \cap T^{p_1(n)}A \cap \cdots \cap T^{p_k(n)}A) > 0.$$

This is a uniform version of a special case of the main theorem from [BL1]. We obtain it by modifying the proof of Theorem 5.1.1 in several places. First off, the following supercedes Theorem 5.1.7.

Theorem 5.3.2. Suppose an ergodic system $(X, \mathcal{A}, \mu, T)$ is a weakly mixing extension of $(Y, \mathcal{B}, \nu, S)$, and that $p_1(x), \cdots, p_k(x) \in \mathbf{Q}[x]$ are non-zero, pairwise distinct polynomials with $p_i(\mathbf{Z}) \subset \mathbf{Z}$ and $p_i(0) = 0$, $1 \leq i \leq k$. Then for any $f_1, \cdots, f_k \in L^\infty(X, \mathcal{A}, \mu)$,

$$\lim_{N-M\to\infty} \left\| \frac{1}{N-M} \sum_{n=M}^{N-1} \Big(\prod_{i=1}^{k} T^{p_i(n)} f_i - \prod_{i=1}^{k} S^{p_i(n)} E(f_i|\mathcal{B}) \Big) \right\| = 0.$$

Proof. The reader is encouraged to review the notion of weight vectors and the PET-induction scheme introduced in Section 1.4. First we show that the conclusion holds if the weight vector of $P = \{p_1(x), \cdots, p_k(x)\}$ is $(1, 0, 0, \cdots)$. In this case $k = 1$ and $p_1(x) = jx$ for some non-zero integer j. We may write f_1 as the sum of two functions, one of which has zero conditional expectation over $\mathcal{B}$ and the other of which is $\mathcal{B}$-measurable, namely $f_1 = (f_1 - E(f_1|\mathcal{B})) + E(f_1|\mathcal{B})$. Since the conclusion obviously holds when f_1 is replaced by $E(f_1|\mathcal{B})$ (recall that E is idempotent), we need only show that the conclusion holds when f_1 is replaced by $\big(f_1 - E(f_1|\mathcal{B})\big)$, i.e. we may assume without loss of generality that $E(f_1|\mathcal{B}) = 0$. What we must show, then, is that

$$\lim_{N-M\to\infty} \left\| \frac{1}{N-M} \sum_{n=M}^{N-1} T^{jn} f_1 \right\| = 0.$$

However, by the uniform mean ergodic theorem (Theorem 5.1.5),

$$\lim_{N-M\to\infty} \frac{1}{N-M} \sum_{n=M}^{N-1} T^{jn} f_1 = P f_1,$$

in norm, where P is the projection onto the set of T^j-invariant functions. Since $(X, \mathcal{A}, \mu, T)$ is a weakly mixing extension of $(Y, \mathcal{B}, \nu, S)$, and $E(f_1|\mathcal{B}) = 0$, we have $Pf_1 = 0$. This completes the minimal weight vector case.

Suppose now that $Q = \{p_1(x), \cdots, p_k(x)\}$ is a family of non-zero, pairwise distinct polynomials having zero constant term, and that the conclusion holds for all P with $P < Q$. Reindexing if necessary, we may assume that $1 \le \deg p_1 \le \deg p_2 \le \cdots \le \deg p_k$. Let $f_1, \cdots, f_k \in L^\infty(X, \mathcal{A}, \mu)$. Suppose that $E(f_a|\mathcal{B}) = 0$ for some a, $1 \le a \le k$. We then must show that

$$\lim_{N-M\to\infty} \left\| \frac{1}{N-M} \sum_{n=M}^{N-1} \left(\prod_{i=1}^{k} T^{p_i(n)} f_i \right) \right\| = 0.$$

To see that the supposition is made without loss of generality, consider the identity

$$\begin{aligned} &\prod_{i=1}^{k} a_i - \prod_{i=1}^{k} b_i \\ =&(a_1 - b_1) b_2 b_3 \cdots b_k + a_1(a_2 - b_2) b_3 b_4 \cdots b_k + \cdots + a_1 a_2 \cdots a_{k-1}(a_k - b_k) \end{aligned}$$

with $a_i = T^{p_i(n)} f_i$ and $b_i = S^{p_i(n)} E(f_i|\mathcal{B})$, noting that on the right hand side we have a sum of terms each of which has at least one factor with zero expectation relative to $\mathcal{B}$.

We use Lemma 5.1.6. Let $x_n = \prod_{i=1}^{k} T^{p_i(n)} f_i$. Then

$$\begin{aligned} &\limsup_{N-M\to\infty} \frac{1}{N-M} \sum_{n=M}^{N-1} \langle x_n, x_{n+h} \rangle \\ =& \limsup_{N-M\to\infty} \frac{1}{N-M} \sum_{n=M}^{N-1} \int \left(\prod_{i=1}^{k} T^{p_i(n)} f_i \right) \left(\prod_{i=1}^{k} T^{p_i(n+h)} f_i \right) d\mu \\ =& \limsup_{N-M\to\infty} \frac{1}{N-M} \sum_{n=M}^{N-1} \int f_1 \left(\prod_{i=2}^{k} T^{p_i(n)-p_1(n)} f_i \right) \\ & \qquad \left(\prod_{i=1}^{k} T^{p_i(n+h)-p_1(n)-p_i(h)} (T^{p_i(h)} f_i) \right) d\mu. \end{aligned} \tag{5.5}$$

For any $h \in \mathbf{Z}$ let

$$\begin{aligned} P_h = &\{p_i(n) - p_1(n) : 2 \le i \le k\} \\ &\cup \{p_i(n+h) - p_1(n) - p_i(h) : \deg p_i \ge 2, 1 \le i \le k\}. \end{aligned}$$

P_h consist of polynomials with zero constant term. Furthermore, the equivalence class of polynomials in Q with degree and leading coefficient the same as $p_1(n)$ has been annihilated in P_h. All other equivalence classes consisting of polynomials in Q of the same degree as $p_1(n)$ have been preserved (although the leading coefficients of these classes have changed). Equivalence classes of higher degree are completely intact. New equivalence classes may exist, but if so they will be of lesser degree than $p_1(n)$. It follows that $P_h < Q$. We now consider two cases:

Case 1. $\deg p_1 \geq 2$. Then $\deg p_i \geq 2$, $1 \leq i \leq k$, and one may check that for all h outside of some finite set, P_h consists of $2k-1$ distinct polynomials. For these h, we use our induction hypothesis for the validity of the theorem conclusion for the family P_h (utilizing weak convergence only) and continue from (5.5):

$$= \limsup_{N-M\to\infty} \frac{1}{N-M} \sum_{n=M}^{N-1} \int E(f_1|\mathcal{B}) \left(\prod_{i=2}^{k} S^{p_i(n)-p_1(n)} E(f_i|\mathcal{B}) \right)$$
$$\left(\prod_{i=1}^{k} S^{p_i(n+h)-p_1(n)-p_i(h)} E(T^{p_i(h)} f_i|\mathcal{B}) \right) d\nu = 0.$$

This since $E(f_a|\mathcal{B}) = 0$.

Case 2. $\deg p_1 = \deg p_2 = \cdots = \deg p_t = 1 < \deg p_{t+1}$. (Of course, if all the p_i are of degree 1 then $t = k$ and there is no p_{t+1}.) In this case $p_1(n+h) - p_1(n) - p_1(h) = 0$, and $p_i(n+h) - p_1(n) - p_i(h) = p_i(n) - p_1(n)$, $2 \leq i \leq t$, so that P_h will consist of $2k-t-1$ elements (again, excepting a finite set of h's for which other relations might hold). In this case we write $p_i(n) = c_i n$, $1 \leq i \leq t$, and proceed from (5.5):

$$= \limsup_{N-M\to\infty} \frac{1}{N-M} \sum_{n=M}^{N-1} \int E(f_1 T^{c_1 h} f_1|\mathcal{B}) \left(\prod_{i=2}^{t} S^{p_i(n)-p_1(n)} E(f_i T^{c_i h} f_i|\mathcal{B}) \right)$$
$$\left(\prod_{i=t+1}^{k} S^{p_i(n)-p_1(n)} E(f_i|\mathcal{B}) S^{p_i(n+h)-p_1(n)-p_i(h)} E(T^{p_i(h)} f_i|\mathcal{B}) \right) d\nu.$$

If $t+1 \leq a \leq k$, this is zero. If $1 \leq a \leq t$, however, it will still be at most

$$\left(\left|\left| E(f_a T^{c_a h} f_a|\mathcal{B}) \right|\right|_{L^2(Y,\mathcal{B},\nu)} \right) \prod_{l\neq a} ||f_l||_\infty^2,$$

so that, by Lemma 5.1.4,

$$D\text{-}\lim_{h\to\infty} \limsup_{N-M\to\infty} \frac{1}{N-M} \sum_{n=M}^{N-1} \langle x_n, x_{n+h} \rangle$$
$$\leq D\text{-}\lim_{h\to\infty} ||E(f_a T^{c_a h} f_a|\mathcal{B})||_{L^2(Y,\mathcal{B},\nu)} \cdot \prod_{l\neq a} ||f_l||_\infty^2 = 0.$$

In either case, the conclusion to Lemma 5.1.5 says

$$\lim_{N-M\to\infty} \left|\left| \frac{1}{N-M} \sum_{n=M}^{N-1} \left(\prod_{i=1}^{k} T^{p_i(n)} f_i \right) \right|\right| = 0.$$

□

The following corollary is what we need.

Corollary 5.3.3. Suppose that $(X, \mathcal{A}, \mu, T)$ is an ergodic measure preserving system and denote by $(Y, \mathcal{A}_\eta, \nu, S)$ its maximal distal factor. If for all $A \in \mathcal{A}_\eta$ with $\nu(A) > 0$, we have

$$\limsup_{N-M\to\infty} \frac{1}{N-M} \sum_{n=M}^{N-1} \nu(A \cap S^{p_1(n)} A \cap \cdots \cap S^{p_k(n)} A)$$
$$= \limsup_{N-M\to\infty} \frac{1}{N-M} \sum_{n=M}^{N-1} \int 1_A S^{-p_1(n)} 1_A \cdots S^{-p_k(n)} 1_A \, d\nu > 0,$$

then for all $A \in \mathcal{A}$ with $\mu(A) > 0$,

$$\limsup_{N-M\to\infty} \frac{1}{N-M} \sum_{n=M}^{N-1} \mu(A \cap T^{p_1(n)} A \cap \cdots \cap T^{p_k(n)} A)$$
$$= \limsup_{N-M\to\infty} \frac{1}{N-M} \sum_{n=M}^{N-1} \int 1_A T^{-p_1(n)} 1_A \cdots T^{-p_k(n)} 1_A \, d\mu > 0.$$

Proof. By Theorem 5.1.3, $(X, \mathcal{A}, \mu, T)$ is either isomorphic to, or is a non-trivial weakly mixing extension of, $(Y, \mathcal{A}_\eta, \nu, S)$. In the former case there is nothing to prove, so we assume the latter. If $A \in \mathcal{A}$, then for some $\delta > 0$ we have

$$\nu(A_\delta) = \nu(\{y \in Y : \ \mu_y(A) \geq \delta\}) > 0.$$

We have $E(1_A|\mathcal{A}_\eta) > \delta 1_{A_\delta}$, so that by Theorem 5.3.1 (utilizing only weak convergence),

$$\limsup_{N-M\to\infty} \frac{1}{N-M} \sum_{n=M}^{N-1} \int 1_A T^{-p_1(n)} 1_A \cdots T^{-p_k(n)} 1_A \, d\mu$$
$$= \limsup_{N-M\to\infty} \frac{1}{N-M} \sum_{n=M}^{N-1} \int 1_A S^{-p_1(n)} E(1_A|\mathcal{A}_\eta) \cdots S^{-p_k(n)} E(1_A|\mathcal{A}_\eta) \, d\nu$$
$$= \limsup_{N-M\to\infty} \frac{1}{N-M} \sum_{n=M}^{N-1} \int E(1_A|\mathcal{A}_\eta) S^{-p_1(n)} E(1_A|\mathcal{A}_\eta) \cdots S^{-p_k(n)} E(1_A|\mathcal{A}_\eta) \, d\nu$$
$$\geq \delta^{k+1} \limsup_{N-M\to\infty} \frac{1}{N-M} \sum_{n=M}^{N-1} \int 1_{A_\delta} S^{-p_1(n)} 1_{A_\delta} \cdots S^{-p_k(n)} 1_{A_\delta} \, d\nu > 0.$$

□

According to Corollary 2.5, in order to establish Theorem 0.1 for an arbitrary system $(X, \mathcal{A}, \mu, T)$, it suffices to establish that the conclusion holds for its maximal distal factor $(X, \mathcal{A}_\eta, \mu, T)$. That is what we do now, by "polynomializing" Section 5.2.

Recall that for natural numbers $n_1, \cdots, n_t$,

$$FS(n_1, \cdots, n_t) = \{n_{i_1} + \cdots + n_{i_m} : 1 \le m \le t, 1 \le i_1 < \cdots < i_m \le t\}.$$

Definition 5.3.4. Suppose $(X, \mathcal{A}, \mu, T)$ is an invertible measure preserving system and that $\mathcal{B} \subset \mathcal{A}$ is a complete T-invariant sub-σ-algebra. $\mathcal{B}$ is said to have the *PSZ property* if for every $A \in \mathcal{B}$ with $\mu(A) > 0$, $t \in \mathbf{Z}$, and polynomials $p_1(x_1, \cdots x_t), \cdots, p_k(x_1, \cdots, x_t) \in \mathbf{Z}[x_1, \cdots, x_t]$ having zero constant term, there exists $\delta > 0$ such that in every thick set $E \subset \mathbf{Z}$, there exist $n_1, \cdots, n_t \in \mathbf{Z}$ such that $FS(n_1, \cdots, n_t) \subset E$ and

$$\mu(A \cap T^{p_1(n_1,\cdots,n_t)}A \cap \cdots \cap T^{p_k(n_1,\cdots,n_t)}A) > \delta.$$

The case $t = 1$, in particular, gives some $\delta > 0$ for which the set

$$\{n \in \mathbf{Z} : \ \mu(A \cap T^{p_1(n)}A \cap \cdots \cap T^{p_k(n)}A) > \delta\}$$

is syndetic, which insures that

$$\liminf_{N-M\to\infty} \frac{1}{N-M} \sum_{n=M}^{N-1} \mu(A \cap T^{p_1(n)}A \cap \cdots \cap T^{p_k(n)}A) > 0.$$

We must of course show that the maximal distal factor of any system has the PSZ property. As in Section 5.2, there are two parts to this task. We must show that the property passes to both limit and successor ordinals. First we handle successor ordinals.

Theorem 5.3.5. Suppose that $(X, \mathcal{A}, \mu, T)$ is an ergodic measure preserving system and that $\mathcal{B} \subset \mathcal{A}$ is a complete, T-invariant sub-σ-algebra having the PSZ property. If $(X, \mathcal{A}, \mu, T)$ is a compact extension of the factor $(Y, \mathcal{B}, \nu, S)$ determined by $\mathcal{B}$, then $\mathcal{A}$ has the PSZ property as well.

Proof. Suppose that $A \in \mathcal{A}$, $\mu(A) > 0$. By Exercise 5.8 we may assume without loss of generality that $f = 1_A \in AP$. Suppose that $t, k \in \mathbf{N}$ and that

$$p_1(x_1, \cdots, x_t), \cdots, p_k(x_1, \cdots, x_t) \in \mathbf{Z}[x_1, \cdots, x_t]$$

have zero constant term. There exists some $c > 0$ and a set $B \in \mathcal{B}$, $\nu(B) > 0$, such that for all $y \in B$, $\mu_y(A) > c$. Let $\epsilon = \sqrt{\frac{c}{8k}}$. Since $1_A \in AP$, there exist functions $g_1, \cdots, g_r \in L^2(X, \mathcal{A}, \mu)$ having the property that for any $n \in \mathbf{N}$, and a.e. $y \in Y$, there exists $s = s(n, y)$, $1 \le s \le r$, such that $\left|\left|T^n f - g_s\right|\right|_y < \epsilon$. For these numbers r, k, t and polynomials p_i, let $w, N \in \mathbf{N}$ and

$$\begin{gathered} Q = \{q_1(y_1, \cdots, y_N), \cdots, q_w(y_1, \cdots, y_N)\} \\ \subset \mathbf{Z}[y_1, \cdots, y_N], \ q_i(0, \cdots, 0) = 0, \ 1 \le i \le w \end{gathered}$$

have the property that for any r-cell partition $Q = \bigcup_{i=1}^r C_i$, there exists i, $1 \leq i \leq r$, $q \in Q$, and pairwise disjoint subsets $S_1, \cdots, S_t \subset \{1, \cdots, N\}$ such that substituting $x_m = \sum_{n \in S_m} y_n$, $1 \leq m \leq t$, we have

$$\{q(y_1, \cdots, y_N), q(y_1, \cdots, y_N) - p_1(x_1, \cdots, x_t), \cdots \\ , q(y_1, \cdots, y_N) - p_k(x_1, \cdots, x_t)\} \subset C_i.$$

(This is possible by Corollary 1.8.1.)

Since $\mathcal{B}$ has the PSZ property, there exists $\eta > 0$ such that for every thick set $E \subset \mathbf{Z}$, there exists $u_1, \cdots, u_N \in \mathbf{Z}$ such that $FS(u_1, \cdots, u_N) \subset E$ and

$$\nu\big(B \cap S^{q_1(u_1, \cdots, u_N)} B \cap \cdots \cap S^{q_w(u_1, \cdots, u_N)} B\big) > \eta.$$

Let D be the number of ways of choosing t non-empty, pairwise disjoint sets $S_1, \cdots, S_t \subset \{1, \cdots, N\}$, and set $\delta = \frac{c\eta}{2D}$. We want to show that in any thick set $E \subset \mathbf{Z}$ there exist $n_1, \cdots, n_t \in \mathbf{Z}$ such that $FS(n_1, \cdots, n_t) \subset E$ and

$$\mu\big(A \cap T^{p_1(n_1, \cdots, n_t)} A \cap \cdots \cap T^{p_k(n_1, \cdots, n_t)} A\big) > \delta.$$

Let E be thick. There exist $u_1, \cdots, u_N \in \mathbf{Z}$ such that $FS(u_1, \cdots, u_N) \subset E$ and

$$\nu\big(B \cap S^{q_1(u_1, \cdots, u_N)} B \cap \cdots \cap S^{q_w(u_1, \cdots, u_N)} B\big) > \eta.$$

Pick any $y \in \big(B \cap S^{q_1(u_1, \cdots, u_N)} B \cap \cdots \cap S^{q_w(u_1, \cdots, u_N)} B\big)$. Form an r-cell partition of Q, $Q = \bigcup_{i=1}^r C_i$, by $q_a(y_1, \cdots, y_N) \in C_i$ if and only if $s\big(q_a(u_1, \cdots, u_N), y\big) = i$, $1 \leq a \leq w$. In particular, if $q_a \in C_i$ then $\left|\left|T^{q_a(u_1, \cdots, u_N)} f - g_i\right|\right|_y < \epsilon$. For this partition, there exists some i, $1 \leq i \leq r$, some $q \in Q$, and pairwise disjoint subsets $S_1, \cdots, S_t \subset \{1, \cdots, N\}$ such that, under the substitution $x_m = \sum_{n \in S_m} y_n$, $1 \leq m \leq t$, we have

$$\{q(y_1, \cdots, y_N), q(y_1, \cdots, y_N) - p_1(x_1, \cdots, x_t), \cdots \\ , q(y_1, \cdots, y_N) - p_k(x_1, \cdots, x_t)\} \subset C_i.$$

In particular, making the analogous substitutions $n_m = \sum_{n \in S_m} u_n$, $1 \leq m \leq t$, we have $FS(n_1, \cdots, n_t) \subset E$, and furthermore, we have, setting $p_0(x_1, \cdots, x_t) = 0$,

$$\left|\left|T^{q(u_1, \cdots, u_N) - p_b(n_1, \cdots, n_t)} f - g_i\right|\right|_y < \epsilon; \quad 0 \leq b \leq k.$$

Setting $\tilde{y} = S^{-q(u_1, \cdots, u_N)} y$, we have

$$\left|\left|T^{-p_b(n_1, \cdots, n_t)} f - T^{-q(u_1, \cdots, u_N)} g_i\right|\right|_{\tilde{y}} < \epsilon; \quad 0 \leq b \leq k.$$

In particular, since this holds for $b = 0$, we have by the triangle inequality

$$\left|\left|T^{-p_b(n_1, \cdots, n_t)} f - f\right|\right|_{\tilde{y}} < 2\epsilon, \quad 1 \leq b \leq k.$$

It follows that

$$\mu_{\tilde{y}}\big(A \setminus T^{p_b(n_1,\cdots,n_t)}A\big) \le \left|\left|T^{-p_b(n_1,\cdots,n_t)}f - f\right|\right|_{\tilde{y}}^2 \le 4\epsilon^2; \quad 1 \le b \le k.$$

Moreover, $\tilde{y} \in B$, so that $\mu_{\tilde{y}}(A) \ge c$, therefore, since $\epsilon = \sqrt{\frac{c}{8k}}$,

$$\mu_{\tilde{y}}\big(A \cap T^{p_1(n_1,\cdots,n_t)}A \cap \cdots \cap T^{p_k(n_1,\cdots,n_t)}A\big) \ge c - 4k\epsilon^2 = \frac{c}{2}.$$

$S_1, \cdots, S_t$ depend measurably on y, therefore $n_1, \cdots, n_t$ are measurable functions of y defined on the set $\big(B \cap S^{q_1(u_1,\cdots,u_N)}B \cap \cdots \cap S^{q_w(u_1,\cdots,u_N)}B\big)$, which, recall, is of measure greater than η. Hence, as there are only D choices possible for $S_1, \cdots, S_t$, we may assume that for all $y \in H$, where $H \in \mathcal{B}$ satisfies $\nu(H) > \frac{\eta}{D}$, $n_1, \cdots, n_t$ are constant. For this choice of $n_1, \cdots, n_t$ we have

$$\mu\big(A \cap T^{p_1(n_1,\cdots,n_t)}A \cap \cdots \cap T^{p_t(n_1,\cdots,n_t)}A\big) \ge \frac{c}{2}\nu(H) > \frac{c\eta}{2D} = \delta.$$

□

Next we handle passage to limit ordinals.

Proposition 5.3.6. Suppose that $(X, \mathcal{A}, \mu, T)$ is a measure preserving system and that $\mathcal{A}_\xi$ is a totally ordered chain of sub-σ-algebras of $\mathcal{A}$ having the PSZ property. If $\bigcup_\xi \mathcal{A}_\xi$ is dense in $\mathcal{A}$, that is, if $\mathcal{A}$ is the completion of the σ-algebra generated by $\bigcup_\xi \mathcal{A}_\xi$, then $\mathcal{A}$ has the PSZ property.

Proof. Suppose $A \in \mathcal{A}$, $\mu(A) > 0$, $t, k \in \mathbf{N}$, and that

$$p_1(x_1, \cdots, x_t), \cdots, p_k(x_1, \cdots, x_t) \in \mathbf{Z}[x_1, \cdots, x_t]$$

are polynomials with zero constant term. There exists ξ and $B \in \mathcal{A}_\xi$ such that

$$\mu\big((A \setminus B) \cup (B \setminus A)\big) \le \frac{\mu(A)}{4(k+1)}.$$

Let $\int d\mu = \int_Y \int_X d\mu_y d\nu(y)$ be the decomposition of the measure μ over the factor $\mathcal{A}_\xi$. Let $C = \{y \in B : \ \mu_y(A) \ge 1 - \frac{1}{2(k+1)}\}$. It is easy to see that $\nu(C) > 0$. Since $\mathcal{A}_\xi$ has the PSZ property, there exists some $\alpha > 0$ having the property that in any thick set E we may find $n_1, \cdots, n_t \in \mathbf{Z}$ such that $FS(n_1, \cdots, n_t) \subset E$ and

$$\nu\big(C \cap S^{p_1(n_1,\cdots,n_t)}C \cap \cdots \cap S^{p_k(n_1,\cdots,n_t)}C\big) > \alpha. \tag{5.6}$$

Set $\delta = \frac{\alpha}{2}$ and let E be any thick set. Find $n_1, \cdots, n_t \in \mathbf{Z}$ satisfying (5.6) and with $FS(n_1, \cdots, n_t) \subset E$. For any $y \in \big(C \cap S^{p_1(n_1,\cdots,n_t)}C \cap \cdots \cap S^{p_k(n_1,\cdots,n_t)}C\big)$ we have $\mu_y(A), \mu_y(T^{p_1(n_1,\cdots,n_t)}A), \cdots, \mu_y(T^{p_k(n_1,\cdots,n_t)}A)$ all not less than $1-\frac{1}{2(k+1)}$, from which it follows that

$$\mu_y\big(A \cap T^{p_1(n_1,\cdots,n_t)}A) \cap \cdots \cap T^{p_k(n_1,\cdots,n_t)}A\big) \geq \frac{1}{2}.$$

Therefore,

$$\mu\big(A \cap T^{p_1(n_1,\cdots,n_t)}A) \cap \cdots \cap T^{p_k(n_1,\cdots,n_t)}A\big) > \frac{\alpha}{2} = \delta.$$

□

Exercise 5.12. By pasting together the pieces, complete the argument for the proof of Theorem 5.3.1.

References

[Ba] J.E. Baumgartner, A short proof of Hindman's theorem, *J. Combinatorial Theory (A)* **17** (1974), 384-386.

[B1] V. Bergelson, Sets of recurrence of $\mathbf{Z}^m$-actions and properties of sets of differences in $\mathbf{Z}^m$, *J. London Math. Soc.* (2) **31** (1985), 295-304.

[B2] V. Bergelson, Weakly mixing PET, *Ergodic Theory and Dynamical Systems*, **7** (1987), 337-349.

[B3] V. Bergelson, Ergodic Ramsey Theory, *Logic and Combinatorics* (edited by S. Simpson), *Contemporary Mathematics* **65** (1987), 63-87.

[B4] V. Bergelson, Ergodic Ramsey theory–an update, *Ergodic Theory of* $\mathbf{Z}^d$*-actions*, M. Pollicott and K. Schmidt, editors, London Mathematical Society Lecture Notes Series **228** (1996) 1-61.

[B5] V. Bergelson, Ergodic theory and Diophantine problems, *Symbolic Dynamics and its Applications*, F. Blanchard, R. Nogueira and A. Maass, eds. Cambridge University Press. 1999.

[BFM] V. Bergelson, H. Furstenberg and R. McCutcheon, IP-sets and polynomial recurrence, *Ergodic Theory and Dynamical Systems* **16** (1996), 963-974.

[BH] V. Bergelson and N. Hindman, Nonmetrizable topological dynamics and Ramsey theory, *Trans. Amer. Math. Soc.* **320** (1990) 293-320.

[BL1] V. Bergelson and A. Leibman, Polynomial extensions of van der Waerden's and Szemerédi's theorems, *Journal of AMS* **9** (1996) 725-753.

[BL2] V. Bergelson and A. Leibman, Set polynomials and a polynomial extension of the Hales-Jewett theorem, *Ann. Math.* **150** (1999).

[BM1] V. Bergelson and R. McCutcheon, Uniformity in the polynomial Szemerédi theorem, *Ergodic Theory of* $\mathbf{Z}^d$*-actions*, M. Pollicott and K. Schmidt, editors, London Mathematical Society Lecture Notes Series **228** (1996) 273-296.

[BM2] V. Bergelson and R. McCutcheon, A Polynomial IP Szemerédi Theorem, *Memoirs of the AMS.* To appear.

[BM3] V. Bergelson and R. McCutcheon, Recurrence for semigroup actions and a non-commutative Schur theorem, *Contemporary Mathematics.* **215** (1998), 205-222.

[BPT] A. Blaszczyk, S. Plewik, and S. Turek, Topological multidimensional van der Waerden theorem, *Comment. Math. Univ. Carolin.* **30** (1989) 783-787.

[C] C. Carathéodory, Die Homomorphieen von Somen und die Multiplikation von Inhalts funktionen, *Annalidella R. Scuola Normale Superiore di Pisa* (Ser. 2) **8** (1939) 105-130.

[Ca] T. Carlson, Some unifying principles in Ramsey theory, *Discrete Math.* **68** (1988), 117-169.

[CS] T. Carlson and S. Simpson, A dual form of Ramsey's theorem, *Adv. in Math.* **53** (1984) 265-290.

[E] R. Ellis, *Lectures on topological dynamics*, Benjamin, New York, 1969.

[ET] P. Erdös and P. Turán, On some sequences of integers, *J. London Math. Soc.* **11** (1936), 261-264.

[F1] H. Furstenberg, Ergodic behavior of diagonal measures and a theorem of Szemerédi on arithmetic progressions, *J. d'Analyse Math.* **31** (1977), 204-256.

[F2] H. Furstenberg, *Recurrence in Ergodic Theory and Combinatorial Number Theory*, Princeton University Press, 1981.

[FK1] H. Furstenberg and Y. Katznelson, An ergodic Szemerédi theorem for commuting transformations, *J. d'Analyse Math.* **34** (1978), 275-291.

[FK2] H. Furstenberg and Y. Katznelson, An ergodic Szemerédi theorem for IP-systems and combinatorial theory, *J. d'Analyse Math.* **45** (1985), 117-168.

[FK3] H. Furstenberg and Y. Katznelson, Idempotents in compact semigroups and Ramsey theory, *Israel J. Math.* **68** (1990), 257-270.

[FK4] H. Furstenberg and Y. Katznelson, A density version of the Hales-Jewett theorem, *J. d'Analyse Math.* **57** (1991), 64-119.

[FKO] H. Furstenberg, Y. Katznelson and D. Ornstein, The ergodic theoretical proof of Szemerédi's theorem, *Bull. Amer. Math. Soc.* **7** (1982), 527-552.

[Fol] G. Folland, *Real Analysis, Modern Techniques and Their Applications*, John Wiley and Sons, 1984.

[For] A. Forrest, Phd. Dissertation. The Ohio State University. 1990.

[FW] H. Furstenberg and B. Weiss, Topological dynamics and combinatorial number theory, *J. d'Analyse Math.* **34** (1978), 61-85.

[G1] W. T. Gowers, A new proof of Szemerédi's theorem for arithmetic progressions of length four, *Geom. Funct. Anal.* **8** (1998), 529-551.

[G2] W. T. Gowers, A new proof of Szemerédi's theorem. *Geom. Funct. Anal.* To appear.

[GLR] R. Graham, K. Leeb, B. Rothschild, Ramsey's theorem for a class of categories, *Adv. Math.* **8** (1972), 417-433.

[GRS] R. Graham, B. Rothschild and J. Spencer, *Ramsey Theory*, Wiley, New York, 1980.

[HJ] A.W. Hales and R.I. Jewett, Regularity and positional games, *Trans. AMS* **106** (1963), 222-229.

[H1] N. Hindman, Finite sums from sequences within cells of a partition of **N**, *J. Combinatorial Theory* (Series A) **17** (1974) 1-11.

[H2] N. Hindman, Ultrafilters and combinatorial number theory, *Number Theory Carbondale*, M. Nathanson, ed., *Lecture Notes in Math.* **751** (1979) 119-184.

[HMS] N. Hindman, A. Maleki and D. Strauss, Central sets and their combinatorial characterization, *J. Combin. Theory Ser. A*, **74** (1996) no. 2, 188-208.

[HM] N Hindman and R. McCutcheon, VIP systems in partial semigroups. Under review.

[HS] N. Hindman and D. Strauss, *Algebra in the Stone-Čech compactification: theory and applications*, de Gruyter, Berlin, 1998.

[KM] T. Kamae and M. Mendès France, Van der Corput's difference theorem, *Israel J. Math.* **31** (1978), 335-342.

[KN] B. O. Koopman and J. von Neumann, Dynamical systems of continuous spectra, *Proc. Nat. Acad. Sci. U.S.A.* **18** (1932), 255-263.

[K] I. Kriz, Large independent sets in shift-invariant graphs. Solution of Bergelson's problem, *Graphs and Combinatorics* **3** (1987), 145-158.

[L] A. Leibman, Multiple recurrence theorem for measure preserving actions of a nilpotent group, *Geom. Funct. Anal.* **8** (1998), 853-931.

[Lo] L. Lovàsz, Kneser's conjecture, chromatic number and homotopy, *J. Comb. Theory (A)* **25** (1978), 319-324.

[M1] R. McCutcheon, Three results in recurrence, *Ergodic Theory and its Connections with Harmonic Analysis* (edited by K. Petersen and I. Salama), *London Math. Soc. Lecture Notes Series* **205**, Cambridge University Press, 1995, 349-358.

[M2] R. McCutcheon, An infinitary version of the polynomial van der Waerden theorem. Jour. Comb. Th. (Series A) **86** (1999), 214-231.

[M3] R. McCutcheon, An infinitary version of the polynomial Hales-Jewett theorem. Under review.

[M4] R. McCutcheon, IP-sets, polynomials and multiple recurrence. Ph.D. dissertation, The Ohio State University, 1995.

[M] K. Milliken, Ramsey's Theorem with sums or unions, *J. Combinatorial Theory* (Series A) **18** (1975), 276-290.

[P] K. Petersen, *Ergodic Theory*, Cambridge Univ. Press, Cambridge, 1981.

[RS] F. Riesz and B. Szökefalvi-Nagy, *Functional Analysis*, Dover Publications, 1990.

[R] W. A. Rokhlin, Selected topics from the metric theory of dynamical systems, *Amer. Math. Soc. Transl. 2*, **49** (1966) 171-209.

[Ra] S. P. Radziszowski, Small Ramsey numbers, *The Electronic Journal of Combinatorics*, **1** (1994), DS1.

[Ram] F. Ramsey, On a problem of formal logic, *Proc. London Math. Soc.* (2) **30** (1930), 264-286.

[Ro] K. Roth, Sur quelques ensembles d'entiers, *C.R. Acad. Sci. Paris* **234** (1952), 388-390.

[Rud] D. Rudolph, *Fundamentals of Measurable Dynamics. Ergodic Theory on Lebesgue Spaces*, Oxford Science Publications, 1990.

[Rus] Rusza, Difference sets and the Bohr topology. Preprint.

[S] Schur, Über die Kongruenz $x^m + y^m \equiv z^m \pmod{p}$. *Jahresbericht der Deutschen Math.-Ver.* **25** (1916), 114-117.

[Sá] A. Sárközy, On difference sets of integers III, *Acta. Math. Acad. Sci. Hungar.*, **31** (1978) 125-149.

[Sz1] E. Szemerédi, On sets of integers containing no four elements in arithmetic progression, *Acta Math. Acad. Sci. Hungar.* **20** (1969), 89-104.

[Sz2] E. Szemerédi, On sets of integers containing no k elements in arithmetic progression, *Acta Arith.* **27** (1975), 199-245.

[T] A. Taylor, A canonical partition relation for finite subsets of ω, *J. Combinatorial Theory* (Series A) **21** (1976), 137-146.

[vdW] B. L. van der Waerden, Beweis einer Baudetschen Vermutung, *Nieuw Arch. Wisk.* **15** (1927) 212-216.

[Z] R. J. Zimmer, Extensions of ergodic actions and generalized discrete spectrum, *Ill. J. Math.* **20** (1976), 555-588.

List of Symbols

Index