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THE HALES-JEWETT THEOREM VIA RETRACTIONS

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ABSTRACT. Working in the Stone-Čech compactification of an arbitrary semigroup, we prove an abstract version of the Hales-Jewett theorem. We easily obtain the classical Hales-Jewett theorem, van der Waerden's theorem and Gallai's theorem as special cases. We observe that our abstract version of the Hales-Jewett theorem can be derived from the classical one.

In Section 2 of this paper, we will formulate and prove a, seemingly more general, abstract version (2.2) of the classical Hales-Jewett theorem 3.5. It has virtually the same proof as the classical version and allows to deduce, in Section 3, the Hales-Jewett theorem as well as van der Waerden's theorem 3.2 and Gallai's theorem as very easy special cases. In the final section 4, however, following a proof by N. Hindman, we explain that our abstract version follows easily from the Hales-Jewett theorem.

1. PRELIMINARIES

The proof of our abstract version of the Hales-Jewett theorem requires some minimal knowledge about compact right-topological semigroups. More precisely, we shall work in the Stone-Čech compactification βV of an arbitrary semigroup V. All definitions and results in this section can be found in [3].

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For a set V with the discrete topology, βV is the set of all ultrafilters on V. It is a compact zero-dimensional space with the basis consisting of the sets $\widehat{A} = \{p \in \beta V : A \in p\}$ for $A \subseteq V$. Identifying each $v \in V$ with the principal ultrafilter $\{A \subseteq V : v \in A\}$, we identify V with the set of isolated points of βV; thus V is a dense subspace of βV. For $A \subseteq V$, the closure of A in βV is simply $\widehat{A}$; it is canonically homeomorphic to the Stone-Čech compactification βA of A. Thus we write sometimes βA for $\widehat{A}$. If $V = A \cup B$ is a partitition of V, then $\beta V = \beta A \cup \beta B$ is a partition of βV.

For an arbitrary mapping $\sigma : V \to W \subseteq \beta W$ between discrete spaces V and W, $\beta\sigma : \beta V \to \beta W$ is the unique continuous extension of σ to βV. By abuse of notation, we write σ instead of $\beta\sigma$. Under this notation, $\sigma(p) = \{B \subseteq W : \sigma^{-1}[B] \in p\}$.

Now assume $(V, \cdot)$ is a semigroup, i.e. $\cdot$ is an associative binary operation on V. There is a unique extension of $\cdot$ to βV such that the map $x \mapsto x \cdot p$ (from βV into itself) is continuous for every $p \in \beta V$ and the map $x \mapsto v \cdot x$ is continuous for every $v \in V \subseteq \beta V$. Under this operation, $(\beta V, \cdot)$ becomes a compact right-topological semigroup.

It follows by continuity that, if $A, B, C \subseteq V$ and $A \cdot B \subseteq C$, then $\beta A \cdot \beta B \subseteq \beta C$; e.g. if A is a subsemigroup of V, then βA is a subsemigroup of βV. Moreover if $\sigma : V \to W$ is a semigroup homomorphism, then so is $\sigma(= \beta\sigma) : \beta V \to \beta W$.

We will use the following well-known facts about compact right-topological semigroups S.

Fact 1. Every left-ideal of S includes a minimal one.

Fact 2. Every minimal left-ideal of S contains an idempotent element.

Fact 3. If I is a minimal left ideal and K is a two-sided ideal of S, then $I \subseteq K$.

Fact 4. If I is a minimal left ideal of S and $p \in I$, then $I = S \cdot p$.

Fact 5. If p is an idempotent element of S and $x \in S \cdot p$, then $xp = x$.

2. The general theorem

Definition 2.1. Assume W is a subsemigroup of V.

We call W a nice subsemigroup of V if $R = V \setminus W$ is a two-sided ideal in V, i.e. a product $x \cdot y$ of elements x, y of V is in W iff $x \in W$ and $y \in W$.

A semigroup homomorphism $\sigma : V \to W$ is called a retraction (from V to W) if $\sigma \restriction W$ is the identity on W.

The following proof of Theorem 2.2 is a straightforward generalization of that in [1].

Theorem 2.2. (the Hales-Jewett theorem, abstract version) *Assume V is a semigroup and W is a proper nice subsemigroup of V. Let Σ be a finite set of retractions from V to W and $W = B_1 \cup \cdots \cup B_r$ a partition of W into finitely many pieces. Then there is some $j \in \{1, \ldots, r\}$ and some $v \in R = V \setminus W$ such that $\{\sigma(v) : \sigma \in \Sigma\} \subseteq B_j$.*

Proof. Note first that βW is a subsemigroup and βR a two-sided ideal of βV, i.e. βW is a nice subsemigroup of βV. Moreover for $\sigma \in \Sigma$, $\sigma = \beta\sigma$ is a retraction from βV to βW.

Choose a minimal left ideal J in βW and an idempotent element $q \in J$. Next, choose a minimal left ideal I in βV which is contained in the left ideal $\beta V \cdot q$ of βV and an idempotent element $i \in I$ and put $p = qi$. Thus $p \in I$.

Note that $R \in p$ because I (a minimal left ideal of βV) is contained in βR (a two- sided ideal of βV), so $p \in I \subseteq \beta R = \widehat{R}$.

Moreover it follows from $i \in I \subseteq \beta V \cdot q$ and Fact 5 that $iq = i$, and this implies $p = p^2 = pq = qp$.

Claim. For each $\sigma \in \Sigma$, $\sigma(p) = q$. - To see this, write $\sigma(p) = u$. Applying the retraction $\sigma : \beta V \to \beta W$ to the equation $p = p^2 = pq = qp$ gives $u = u^2 = uq = qu$, in particular, $u = uq \in J$. But $q \in J = \beta W \cdot u$, by Fact 4, thus also $qu = q$, by Fact 5, which proves the Claim.

To finish the proof of the theorem, let $j \in \{1, \ldots, r\}$ be such that $B_j \in q$ (an ultrafilter on W). For $\sigma \in \Sigma$, we have $B_j \in q = \sigma(p)$ and thus $\sigma^{-1}[B_j] \in p$. It follows that the set $D = R \cap \bigcap_{\sigma \in \Sigma} \sigma^{-1}[B_j]$ is in p, thus non-empty. Every $v \in D$ works for the theorem. $\square$

Remark 2.3. Readers somewhat familiar with the interplay of semigroups and combinatorics will expect a more precise version of 2.2: given V, W and Σ as in 2.2 and a piecewise syndetic subset B of W, there is some $v \in R = V \setminus W$ such that $\{\sigma(v) : \sigma \in \Sigma\} \subseteq B$. This is proved as follows: Theorem 4.43 in [3] says that there is some $x \in W$ such that the subset $A = \{s \in W : xs \in B\}$ is central in W. I.e. there are an idempotent $q \in \beta W$ and a minimal left ideal J of βW such that $A \in q$ and $q \in J$. Starting from these q and J, take I and p as in the proof of 2.2. Then $A \in q = \sigma(p)$, for all $\sigma \in \Sigma$, gives an $a \in R$ such that $\{\sigma(a) : \sigma \in \Sigma\} \subseteq A$. Now $v = x \cdot a$ is in R, and, for $\sigma \in \Sigma$, $\sigma(v) = x \cdot \sigma(a) \in B$.

3. Special cases

We derive from 2.2 some results of Ramsey theory which are well-known to follow from the Hales-Jewett theorem 3.5. They deal with commutative resp. free semigroups.

Corollary 3.1. *Assume $(S, +)$ is a commutative semigroup, $S = A_1 \cup \cdots \cup A_r$ is a partition of S into finitely many pieces and E is a finite subset of S. Then there are $j \in \{1, \ldots, r\}$, $a \in S$ and a natural number $d > 0$ such that $\{a + d \cdot e : e \in E\} \subseteq A_j$.*

Proof. Consider $V = S \times \omega$ with coordinatewise addition (where ω is the set of natural numbers, including 0) and $W = S \times \{0\}$; clearly V and W satisfy the assumptions of Theorem 2.2. Write pr for the projection map from W to the first coordinate and B_j for $pr^{-1}[A_j]$; so $W = B_1 \cup \cdots \cup B_r$. For every $s \in S$, we define a retraction σ_s from V to W by letting $\sigma_s(a, d) = (a + ds, 0)$. Take by Theorem 2.2 some $v = (a, d) \in V \setminus W$ (thus $d \neq 0$) and some $j \in \{1, \ldots, r\}$ such that $\{\sigma_e(v) : e \in E\} \subseteq B_j$. I.e. $pr(\sigma_e(v)) = a + de \in A_j$ for each $e \in E$. □

In the special case where S is the additive semigroup ω of natural numbers resp. the product ω^k of k copies of ω (where $k \in \omega \setminus \{0\}$), it is most natural to take $E = \{0, \cdots, n\}$ for some $n \in \omega$ resp. $E = \{0, \cdots, n\}^k$. This gives the following results.

Theorem 3.2. (van der Waerden's theorem) *Assume $\omega = A_1 \cup \cdots \cup A_r$ is a partition of ω into finitely many pieces and $n \in \omega$. Then there are $j \in \{1, \ldots, r\}$ and natural numbers a and $d > 0$ such that $\{a, a + d, a + 2d, \ldots, a + nd\} \subseteq A_j$, i.e. A_j contains an arithmetic progression of length $n + 1$.*

Theorem 3.3. (the k-dimensional van der Waerden theorem) *Assume $k \geq 1$ is a natural number, $\omega^k = A_1 \cup \cdots \cup A_r$ is a partition of ω^k into finitely many pieces, and $n \in \omega$. Then there are $j \in \{1, \ldots, r\}$, $a \in \omega^k$ and a natural mumber $d > 0$ such that $a + dx \in A_j$ holds for each vector $x \in \{0, \ldots, n\}^k$.*

The special case of 3.1 where S is the additive group of a vector space is Gallai's theorem (see e.g. [2]).

In the Hales-Jewett theorem, we deal with the following semigroup.

Definition 3.4. For an arbitrary set M (an alphabet), M^* is the set of all finite sequences (words) over M, a semigroup under concatenation of words. M^* is simply the free semigroup over the set M of free generators (where we identify each letter $m \in M$ with the word (m) of length one).

We will have $M = L \cup X$ where L and X are disjoint; here we consider the elements of L as "constant" letters and those of X as "variable" letters. For $v \in M^*$, $x \in X$ and $u \in L^*$, $v(x/u)$ denotes the result of substituting u for x everywhere in v. More generally, if $f : X \to L^*$ and $v \in M^*$, $v(X/f)$ denotes the result of simultaneously substituting $f(x)$ for x everywhere in v, for all $x \in X$ occurring in v.

Theorem 3.5. (the Hales-Jewett theorem, classical version) *Assume L is a finite alphabet and $L^* = A_1 \cup \cdots \cup A_r$ is a partition of L^* into finitely many pieces. Let x be a variable letter not in L. Then there are $j \in \{1, \ldots, r\}$ and a word $v \in (L \cup \{x\})^* \setminus L^*$ such that $\{v(x/a) : a \in L\} \subseteq A_j$.*

Proof. This is the special case of 2.2 for $V = (L \cup \{x\})^*$, $W = L^*$ and, for each $a \in L$, the retraction σ_a from V to W mapping v to $v(x/a)$. □

Seemingly more general versions of the Hales-Jewett theorem can be obtained by considering V and W as in the proof of 3.5 and, for arbitrary $u \in L^*$, the retraction σ_u from V to W mapping v to $v(x/u)$, resp. $V = (L \cup X)^*$, $W = L^*$ and, for arbitrary $f : X \to L^*$, the retraction σ_f from V to W mapping v to $v(X/f)$.

Let us remark that, for $V = (L \cup \{x\})^*$ and $W = L^*$, the retractions σ_u as defined above are the most general ones. I.e. every retraction σ from V to W coincides with σ_u, for some $u \in L^*$. This is because V is free over $L \cup \{x\}$ and $\sigma(a) = a$ for $a \in L$; thus σ is determined by $u = \sigma(x)$, and it follows that $\sigma = \sigma_u$. Similarly in the situation where $V = M^* = (L \cup X)^*$ and $W = L^*$, every retraction σ from V to W coincides with σ_f where $f = \sigma \restriction X$.

4. A sobering remark

Our abstract version 2.2 of the Hales-Jewett theorem looks very general, and it is certainly appropriate to give particularly straightforward proofs for the results in Section 3. It turns out, however, that the seemingly most general result 2.2 follows quite easily from the classical version 3.5 of the Hales-Jewett theorem. We include Hindman's proof of this fact with his kind permission; in a preliminary version of this paper, we had only shown how to derive 2.2 from the more general versions of the Hales-Jewett theorem mentioned at the end of Section 3.

Remark 4.1. We show how to derive 2.2 from 3.5. Thus assume V, W and Σ are given as in in 2.2, plus a partition $W = B_1 \cup \cdots \cup B_r$.

Fix a finite alphabet $L = \{a_\sigma : \sigma \in \Sigma\}$ in which the letters a_σ, $\sigma \in \Sigma$, are pairwise distinct. For every $\sigma \in \Sigma$, we have the substitution homomorphism

$$s_\sigma : (L \cup \{x\})^* \to L^*$$

mapping v to $v(x/a_\sigma)$.

Moreover, fix an arbitrary $u \in V \setminus W$ and the homomorphism

$$f : (L \cup \{x\})^* \to V$$

mapping x to u and, for $\sigma \in \Sigma$, a_σ to $\sigma(u)$. Then for each $\sigma \in \Sigma$, we obtain

$$\sigma \circ f = f \restriction L^* \circ s_\sigma$$

since this commutativity condition holds on $L \cup \{x\}$, hence on the whole of $(L \cup \{x\})^*$. And the preimage of W under f is L^* (since $f(x) = u \notin W$ and $W \subseteq V$ is a nice subsemigroup).

Putting $A_j = f^{-1}[B_j]$ for $1 \leq j \leq r$, we obtain a partition $L^* = A_1 \cup \cdots \cup A_r$ of L^*; the classical Hales-Jewett theorem 3.5 gives an element w of $(L \cup \{x\})^* \setminus L^*$ and some $j \in \{1, \ldots, r\}$ such that $\{w(x/a_\sigma) : \sigma \in \Sigma\} \subseteq A_j$, i.e. $\{f(w(x/a_\sigma)) : \sigma \in \Sigma\} \subseteq B_j$. Thus $v = f(w)$ is an element of $V \setminus W$. And for $\sigma \in \Sigma$, $\sigma(v) = \sigma(f(w)) = f(w(x/a_\sigma)) \in B_j$, q.e.d.

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