Available online at www.sciencedirect.com
ScienceDirect
Indagationes Mathematicae 29 (2018) 293–312
indagationes mathematicae
www.elsevier.com/locate/indag

Virtual Special Issue - L.E.J. Brouwer after 50 years

Recurrence in the dynamical system $(X, \langle T_s \rangle_{s \in S})$ and ideals of βS

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Abstract

A *dynamical system* is a pair $(X, \langle T_s \rangle_{s \in S})$, where X is a compact Hausdorff space, S is a semigroup, for each $s \in S$, T_s is a continuous function from X to X, and for all $s, t \in S$, $T_s \circ T_t = T_{st}$. Given a point $p \in \beta S$, the Stone–Čech compactification of the discrete space S, $T_p : X \to X$ is defined by, for $x \in X$, $T_p(x) = p\text{-}\lim_{s \in S} T_s(x)$. We let βS have the operation extending the operation of S such that βS is a right topological semigroup and multiplication on the left by any point of S is continuous. Given $p, q \in \beta S$, $T_p \circ T_q = T_{pq}$, but T_p is usually not continuous. Given a dynamical system $(X, \langle T_s \rangle_{s \in S})$, and a point $x \in X$, we let $U(x) = \{p \in \beta S : T_p(x)$ be uniformly recurrent$\}$. We show that each $U(x)$ is a left ideal of βS and for any semigroup we can get a dynamical system with respect to which $K(\beta S) = \bigcap_{x \in X} U(x)$ and $c\ell K(\beta S) = \bigcap\{U(x) : x \in X$ and $U(x)$ is closed$\}$. And we show that weak cancellation assumptions guarantee that each such $U(x)$ properly contains $K(\beta S)$ and has $U(x) \setminus c\ell K(\beta S) \neq \emptyset$.

1. Introduction

We take the Stone–Čech compactification of a discrete semigroup $(S, \cdot)$ to be the set of ultrafilters on S, identifying the points of S with the principal ultrafilters. Given $A \subseteq S$, we set $\overline{A} = \{p \in \beta S : A \in p\}$. The set $\{\overline{A} : A \subseteq S\}$ is a basis for the open sets and a basis for the closed sets of βS. The operation on S extends uniquely to βS so that $(\beta S, \cdot)$ is a right topological

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http://dx.doi.org/10.1016/j.indag.2016.12.004

semigroup with S contained in its topological center, meaning that ρ_p is continuous for each $p \in \beta S$ and λ_x is continuous for each $x \in S$, where for $q \in \beta S$, $\rho_p(q) = q \cdot p$ and $\lambda_x(q) = x \cdot q$. So, for every $p, q \in \beta S$, $pq = \lim_{s \to p} \lim_{t \to q} st$, where s and t denote elements of S. If $A \subseteq S$, $A \in p \cdot q$ if and only if $\{x \in S : x^{-1}A \in q\} \in p$, where $x^{-1}A = \{y \in S : xy \in A\}$. (We are following the custom of frequently writing xy for $x \cdot y$.)

The algebraic structure of βS is interesting in its own right, and has had substantial applications, especially to that part of combinatorics known as *Ramsey Theory*. See the book [4] for an elementary introduction to the structure of βS and its applications.

We are concerned in this paper with the relationship between the algebraic structure of βS and recurrence in *dynamical systems*.

Definition 1.1. A *dynamical system* is a pair $(X, \langle T_s \rangle_{s \in S})$ such that

(1) X is a compact Hausdorff topological space (called the *phase space* of the system);
(2) S is a semigroup;
(3) for each $s \in S$, T_s is a continuous function from X to X; and
(4) for all $s, t \in S$, $T_s \circ T_t = T_{st}$.

Associated with any semigroup S are at least two interesting dynamical systems, namely $(\beta S, \langle \lambda_s \rangle_{s \in S})$, and $({}^S\{0,1\}, \langle T_s \rangle_{s \in S})$ where ${}^S\{0,1\}$ is the set of all functions from S to $\{0,1\}$ with the product topology and $T_s(x) = x \circ \rho_s$. (We shall verify that this latter example is a dynamical system shortly.)

It is common to assume that the phase space of a dynamical system is a metric space, but we make no such assumption. If S is infinite, then βS is not a metric space. Everything we do here is boring if S is finite so whenever we write “let S be a semigroup” we shall assume that S is infinite. The interested reader can amuse herself by determining which of our results remain valid if that assumption is dropped.

The system $(\beta S, \langle \lambda_s \rangle_{s \in S})$ has significant general properties as can be seen in [4, Section 19.1], but will not be used much in this paper.

Given a product space ${}^S\{0,1\}$, recall that the product topology has a subbasis consisting of sets of the form $\pi_t^{-1}[\{a\}]$ for $t \in S$ and $a \in \{0,1\}$, where, for $x \in {}^S\{0,1\}$, $\pi_t(x) = x(t)$.

Lemma 1.2. *Let R be a semigroup and let S be a subsemigroup of R. Let $Z = {}^R\{0,1\}$, the set of all functions from R to $\{0,1\}$ with the product topology. For $x \in Z$ and $s \in S$, define $T_s(x) = x \circ \rho_s$. Then $(Z, \langle T_s \rangle_{s \in S})$ is a dynamical system.*

Proof. It is routine to verify that for $s, t \in S$, $T_s \circ T_t = T_{st}$. To see that T_s is continuous for each $s \in S$, let $s \in S$ be given. It suffices to show that the inverse image of each subbasic open set is open, so let $t \in R$ and $a \in \{0,1\}$ be given. Then $T_s^{-1}\big[\pi_t^{-1}[\{a\}]\big] = \pi_{ts}^{-1}[\{a\}]$. □

Recall that, if T is any discrete space, $p \in \beta T$, $\langle x_t \rangle_{t \in T}$ is any indexed family in a Hausdorff topological space X, and $y \in X$, then $p-\lim_{t \in T} x_t = y$ if and only if for every neighborhood U of y, $\{t \in T : x_t \in U\} \in p$. In compact spaces p-limits always exist.

Definition 1.3. Let $(X, \langle T_s \rangle_{s \in S})$ be a dynamical system and let $p \in \beta S$. Then $T_p : X \to X$ is defined by, for $x \in X$, $T_p(x) = p - \lim_{s \in S} T_s(x)$. So $T_p(x) = \lim_{s \to p} T_s(x)$ where s denotes an element of S.

Using [4, Theorem 4.5] one easily sees that for $p, q \in \beta S$, $T_p \circ T_q = T_{pq}$. However, $(X, \langle T_s \rangle_{s \in \beta S})$ is not in general a dynamical system, since T_p is not likely to be continuous when $p \in \beta S \setminus S$. However, for each $x \in X$, the map $p \mapsto T_p(x) : \beta S \to X$ is continuous. To see this, define $f_x(p) = T_p(x)$. If U is a neighborhood of $f_x(p)$ and $A = \{s \in S : T_s(x) \in U\}$, then $U \in p$ and $f_x[\,\overline{A}\,] \subseteq U$. Alternatively, one may note that $p \mapsto T_p(x)$ is the continuous extension to βS of the function $s \mapsto T_s(x) : S \to X$.

As a compact Hausdorff right topological semigroup, βS has a number of important algebraic properties, and we list some of those that we shall use. (Proofs can be found in [4, Chapters 1 and 2]. Assume that T is a compact Hausdorff right topological semigroup. A non-empty subset V of T is a *left ideal* if $tV \subseteq V$ for every $t \in T$, a *right ideal* if $Vt \subseteq V$ for every $t \in T$, and an *ideal* if it is both a left and a right ideal.

(1) T contains an idempotent.
(2) T has a smallest ideal $K(T)$, which is the union of the minimal left ideals of T and the union of the minimal right ideals of T.
(3) For every $t \in K(T)$, Tt is a minimal left ideal of T and tT is a minimal right ideal of T.
(4) The intersection of any minimal left ideal and any minimal right ideal of T is a group.
(5) Every left ideal of T contains a minimal left ideal, and every right ideal of T contains a minimal right ideal.
(5) Every minimal left ideal of T is compact.
(6) If $\{t \in T : \lambda_t$ is continuous$\}$ is dense in T, then the closure of every ideal in T is also an ideal.

We introduce the main objects of study in this paper now. Given a set X, we let $\mathcal{P}_f(X)$ be the set of finite nonempty subsets of X.

Definition 1.4. Let S be a semigroup and let $A \subseteq S$. We say the set A is *syndetic* if and only if there exists $F \in \mathcal{P}_f(S)$ such that $S = \bigcup_{t \in F} t^{-1}A$.

In the semigroup $(\mathbb{N}, +)$ a set is syndetic if and only if it has bounded gaps.

Definition 1.5. Let $(X, \langle T_s \rangle s \in S)$ be a dynamical system and let $x \in X$.

(a) The point x is *uniformly recurrent* if and only if for every neighborhood V of x, $\{s \in S : T_s(x) \in V\}$ is syndetic.
(b) $U(x) = U_X(x) = \{p \in \beta S : T_p(x)$ is uniformly recurrent$\}$.

In Section 2 of this paper we present well known results about $U(x)$ that are valid in arbitrary dynamical systems as well as the few simple results that we have in the dynamical system $(\beta S, \langle \lambda_s \rangle_{s \in S})$.

In Section 3 we present results about the dynamical systems described in Lemma 1.2.

In Section 4 we consider the effect of slightly modifying the phase space in the dynamical systems described in Lemma 1.2.

In Section 5 we consider surjectivity of T_p and the set $NS = NS_X = \{p \in \beta S : T_p : X \to X$ is not surjective$\}$ which is a right ideal of βS whenever it is nonempty.

2. General results

We begin with some well known basic facts.

Lemma 2.1. *Let $(X, \langle T_s \rangle_{s\in S})$ be a dynamical system, let L be a minimal left ideal of βS, and let $x \in X$. The following are equivalent:*

(a) *x is uniformly recurrent.*
(b) *There exists $q \in L$ such that $T_q(x) = x$.*
(c) *There exists an idempotent $q \in L$ such that $T_q(x) = x$.*
(d) *There exist $y \in X$ and $q \in L$ such that $T_q(y) = x$.*
(e) *There exists $q \in K(\beta S)$ such that $T_q(x) = x$.*
(f) *There exist $y \in X$ and $q \in K(\beta S)$ such that $T_q(y) = x$.*

Proof. The equivalence of (a)–(d) is shown in [4, Theorem 19.23]. Since (c) implies (e), and (e) implies (f), we shall show (f) implies (c) and this will establish the equivalence of all six statements. So assume that (f) holds. Let u denote the identity of the group $L \cap q\beta S$. Since $uq = q$, it follows that $T_u(x) = T_u T_q(y) = T_q(y) = x$. □

Corollary 2.2. *Let $(X, \langle T_s \rangle_{s\in S})$ be a dynamical system and let $x \in X$.*

(1) *If x is uniformly recurrent, $U(x) = \beta S$.*
(2) *For each $x \in X$, $U(x)$ is a left ideal of βS.*
(3) *For every $x \in X$, $K(\beta S) \subseteq U(x)$.*
(4) *$\bigcap_{x\in X} U(x)$ is a two sided ideal of βS.*

Proof. (1) Suppose that x is uniformly recurrent. Then $T_u(x) = x$ for some $u \in K(\beta S)$. Thus for every $v \in \beta S$, $T_v(x) = T_v T_u(x) = T_{vu}(x)$; since $vu \in K(\beta S)$, by Lemma 2.1(f), $T_v(x)$ is uniformly recurrent.

(2) Let $x \in X$, let $p \in U(x)$, and let $r \in \beta S$. By Lemma 2.1(e), pick $q \in K(\beta S)$ such that $T_q\big(T_p(x)\big) = T_p(x)$. Then $T_{rp}(x) = T_r\big(T_q\big(T_p(x)\big)\big) = T_{rqp}(x)$. Now $rqp \in K(\beta S)$, so by Lemma 2.1(f), $T_{rp}(x)$ is uniformly recurrent.

(3) This is immediate from Lemma 2.1(f).

(4) By (3), $\bigcap_{x\in X} U(x)$ is nonempty, so by (2) $\bigcap_{x\in X} U(x)$ is a left ideal of βS, so it is enough to show that $\bigcap_{x\in X} U(x)$ is a right ideal of βS. So suppose that $x \in X$, $p \in \bigcap_{x\in X} U(x)$ and $q \in \beta S$. Since $p \in U(T_q(x))$, $T_{pq}(x)$ is uniformly recurrent and so $pq \in U(x)$. □

The statements of Lemma 2.3 are modifications of basic well known facts that are proved in [2]. (Furstenberg assumes that the phase space is metric, but the proofs given do not use this assumption.) We shall say that a subspace Z of X is *invariant* if $T_s[Z] \subseteq Z$ for every $s \in S$. Of course, if Z is closed and invariant, then $T_p[Z] \subseteq Z$ for every $p \in \beta S$. (Let $x \in Z$. Then $T_s(x) \in Z$ for each $s \in S$ so $p - \lim_{s\in S} T_s(x) \in Z$.)

Lemma 2.3. *Let $(X, \langle T_s \rangle_{s\in S})$ be a dynamical system. Let L be a minimal left ideal of βS.*

(1) *A subspace Y of X is minimal among all closed and invariant subsets of X if and only if there is some $x \in X$ such that $Y = \{T_p(x) : p \in L\}$.*
(2) *Let Y be a subspace of X which is minimal among all closed and invariant subsets of X. Then every element of Y is uniformly recurrent.*
(3) *If $x \in X$ is uniformly recurrent and $Y = \{T_p(x) : p \in \beta S\}$, then Y is minimal among all closed and invariant subsets of X.*
(4) *If $x \in X$ is uniformly recurrent, then $T_p(x)$ is uniformly recurrent for every $p \in \beta S$.*

Proof. (1) Suppose that Y is a subspace of X which is minimal among all closed and invariant subsets of X. Pick $x \in Y$ and let $Z = \{T_p(x) : p \in L\}$. We claim that Z is a closed and invariant subspace of Y and is therefore equal to Y. If $p \in L$ and $s \in S$, then $T_s\big(T_p(x)\big) = T_{sp}(x)$ and $sp \in L$, so Z is invariant and obviously $Z \subseteq Y$. To see that Z is closed, it suffices to show that any net in Z has a cluster point in Z. To this end, let $\langle p_\alpha \rangle_{\alpha \in D}$ be a net in L and pick a cluster point p in L of $\langle p_\alpha \rangle_{\alpha \in D}$. Then $T_p(x)$ is a cluster point of $\langle T_{p_\alpha}(x) \rangle_{\alpha \in D}$.

Conversely, let $x \in X$ and let $Y = \{T_p(x) : p \in L\}$. Then Y is invariant and one sees as above that Y is closed. We shall show that Y is minimal among all closed and invariant subsets of X. To see this, suppose that Z is a subset of Y which is closed and invariant. We shall show that $Y \subseteq Z$, so let $y \in Y$ be given. Pick $z \in Z$. Then $y = T_p(x)$ and $z = T_q(x)$ for some p and q in L. Since $Lq = L$, there exists $r \in L$ such that $rq = p$. It follows that $T_r(z) = y$ and hence that $y \in Z$ as required.

(2) Let Y be a subspace of X which is minimal among all closed and invariant subsets of X and let $x \in Y$. Pick $y \in X$ such that $Y = \{T_p(y) : p \in L\}$. Pick $p \in L$ such that $x = T_p(y)$. By Lemma 2.1(f), x is uniformly recurrent.

(3) Let x be a uniformly recurrent point of X and let $Y = \{T_p(x) : p \in \beta S\}$. By Lemma 2.1(b), pick $q \in L$ such that $T_q(x) = x$. By (1) it suffices that $Y = \{T_p(x) : p \in L\}$. To see this, let $y \in Y$ and pick $p \in \beta S$ such that $y = T_p(x)$. Then $y = T_p\big(T_q(x)\big) = T_{pq}(x)$ and $pq \in L$.

(4) Let x be a uniformly recurrent point of X and let $Y = \{T_p(x) : p \in \beta S\}$. By (3) Y is minimal among all closed and invariant subsets of X so (2) applies. □

We conclude this section with a few results about the dynamical system $(\beta S, \langle \lambda_s \rangle_{s \in S})$. We observe that, if we define $\lambda_p : \beta S \to \beta S$ in this system by $\lambda_p(q) = \lim_{s \to p} \lambda_s(q)$, where s denotes an element of S, then $\lambda_p(q) = pq$ for every p and q in βS. So this does not conflict with the previous definition of λ_p given in the introduction.

Theorem 2.4. *Let S be a semigroup and let $x \in \beta S$. Statements* (a) *and* (b) *are equivalent and imply* (c). *If βS has a left cancelable element, all three are equivalent.*

(a) $x \in K(\beta S)$.
(b) *x is uniformly recurrent in the dynamical system $(\beta S, \langle \lambda_s \rangle_{s \in S})$.*
(c) *$\beta S x$ is a minimal left ideal of βS.*

Proof. To see that (a) implies (b), let $x \in K(\beta S)$ and let u be the identity of the group in $K(\beta S)$ to which x belongs. Then $x = \lambda_u(x)$ so by Lemma uniform recurrence(e), x is uniformly recurrent.

To see that (b) implies (a), assume that x is uniformly recurrent. By Lemma 2.1(f) pick $y \in \beta S$ and $q \in K(\beta S)$ such that $\lambda_q(y) = x$. Then $x = qy \in K(\beta S)$.

To see that (a) implies (c), assume that $x \in K(\beta S)$ and pick the minimal left ideal L of βS such that $x \in L$. Then $\beta S x$ is a left ideal of βS contained in L and so $\beta S x = L$.

Now assume that βS has a left cancelable element z and that $\beta S x$ is a minimal left ideal of βS. Pick an idempotent $u \in \beta S x$. Then $zx \in \beta S x$ so by [4, Lemma 1.30], $zx = zxu$ and therefore $x = xu \in \beta S x \subseteq K(\beta S)$. □

Corollary 2.5. *Let S be an infinite semigroup and let $x \in K(\beta S)$. Then $U(x) = \beta S$ with respect to the dynamical system $(\beta S, \langle \lambda_s \rangle_{s \in S})$.*

Proof. By Theorem 2.4, x is uniformly recurrent, so by Lemma 2.3(4), $U(x) = \beta S$. □

Corollary 2.6. *Let S be a semigroup and let $p, q \in \beta S$. Statements* (a) *and* (b) *are equivalent and imply statement* (c). *If βS has a left cancelable element, then all three statements are equivalent.*

(a) $qp \in K(\beta S)$.
(b) $q \in U(p)$ *with respect to the dynamical system* $(\beta S, \langle \lambda_s \rangle_{s \in S})$.
(c) βSqp *is a minimal left ideal of* βS.

Proof. We have that $q \in U(p)$ if and only if $\lambda_q(p)$ is uniformly recurrent and $\lambda_q(p) = qp$ so Theorem 2.4 applies. □

It is an old and difficult problem to characterize when $K(\beta S)$ is prime or when $c\ell K(\beta S)$ is prime. There are trivial situations where the answer is known. For example if S is left zero or right zero, then so is βS and thus $K(\beta S) = \beta S$, and is necessarily prime. It is not known whether $K(\beta\mathbb{N}, +)$ is prime or $c\ell K(\beta\mathbb{N}, +)$ is prime. (Some partial results were obtained in [3].)

Corollary 2.7. *Let S be a semigroup. The following statements are equivalent.*

(a) *There exists* $p \in \beta S \setminus K(\beta S)$ *such that, with respect to the dynamical system* $(\beta S, \langle \lambda_s \rangle_{s \in S})$, $K(\beta S) \subsetneq U(p)$.
(b) $K(\beta S)$ *is not prime.*

Proof. This is an immediate consequence of Corollary 2.6. □

3. Dynamical systems with phase space ${}^{R}\{0, 1\}$

Throughout this section we assume that R is a semigroup, S a subsemigroup of R, and $(Z, \langle T_s \rangle_{s \in S})$ is the dynamical system of Lemma 1.2. While our results are valid in this generality, in practice we are interested in just two situations, one in which $R = S$ and the other in which $R = S \cup \{e\}$ where e is a two sided identity adjoined to S.

Our first results in this section are aimed at showing that for any semigroup S, there is a dynamical system such that both $K(\beta S)$ and $c\ell K(\beta S)$ are intersections of sets of the form $U(x)$.

Definition 3.1. Given $x \in Z$ we denote the continuous extension of x from βR to $\{0, 1\}$ by $\widetilde{x}$.

Of course, for each $x \in Z$, each $p \in \beta S$ and each $t \in R$, $T_p(x)(t) = p - \lim_{s \in S} T_s(x)(t) = p - \lim_{s \in S} x(ts)$ and so $T_p(x)(t) = \widetilde{x}(tp)$.

Lemma 3.2. *Let $x \in Z$, let $p \in \beta S$, and let L be a minimal left ideal of βS. The following statements are equivalent:*

(a) $p \in U(x)$.
(b) *There exists* $q \in L$ *such that* $\widetilde{x}(tp) = \widetilde{x}(tqp)$ *for all* $t \in R$.
(c) *There exists an idempotent* $q \in L$ *such that* $\widetilde{x}(tp) = \widetilde{x}(tqp)$ *for all* $t \in R$.

Proof. To see that (a) implies (c), assume that $T_p(x)$ is uniformly recurrent. By Lemma 2.1(c), pick an idempotent $q \in L$ such that $T_q\big(T_p(x)\big) = T(p)(x)$. Then $T_{qp}(x) = T_p(x)$ so as noted above, for all $t \in R$, $\widetilde{x}(tqp) = \widetilde{x}(tp)$.

Trivially (c) implies (b). To see that (b) implies (a), pick $q \in L$ such that $\widetilde{x}(tp) = \widetilde{x}(tqp)$ for all $t \in R$. Then $T_p(x) = T_{qp}(x) = T_q\big(T_p(x)\big)$, so by Lemma 2.1(b), $T_p(x)$ is uniformly recurrent. □

Lemma 3.3. *Let $x \in Z$ and let $p \in \beta S$. Then $p \in U(x)$ if and only if for every minimal left ideal L of βS and every $F \in \mathcal{P}_f(R)$, there exists $q_F \in L$ such that for all $t \in F$, $\widetilde{x}(tp) = \widetilde{x}(tq_F p)$.*

Proof. The necessity is an immediate consequence of Lemma 3.2(b).

For the sufficiency, let L be a minimal left ideal of βS. For each $F \in \mathcal{P}_f(R)$, pick $q_F \in L$ as guaranteed. Direct $\mathcal{P}_f(R)$ by agreeing that $F < G$ if and only if $F \subseteq G$. Pick a cluster point $q \in L$ of the net $\langle q_F \rangle_{F \in \mathcal{P}_f(R)}$. It is then routine to show that for all $t \in R$, $\widetilde{x}(tqp) = \widetilde{x}(tp)$ so that by Lemma 3.2(b), $p \in U(x)$. $\square$

Theorem 3.4.

(1) $K(\beta S) \subseteq \bigcap_{x \in Z} U(x)$.
(2) *If $p \in \bigcap_{x \in Z} U(x)$, then, for every minimal left ideal L of βS, $\beta Sp = Lp$ and so βSp is a minimal left ideal of βS.*
(3) *If R contains a left cancelable element, then $K(\beta S) = \bigcap_{x \in Z} U(x)$. In particular, if R has a left identity, then $K(\beta S) = \bigcap_{x \in Z} U(x)$.*

Proof. (1) $K(\beta S) \subseteq \bigcap_{x \in Z} U(x)$ by Corollary 2.2(3).

(2) Assume that $p \in \bigcap_{x \in Z} U(x)$. Let L be a minimal left ideal of βS. We shall show that, for every $t \in R$, $tp \in tLp$. To see this, assume the contrary. Then for some $t \in R$, there exists $A \subseteq R$ such that $A \in tp$ and $\overline{A} \cap tLp = \emptyset$. Let $x = \chi_A$. So $\widetilde{x}$ is the characteristic function of $\overline{A}$. Since $p \in U(x)$, it follows from Lemma 3.2 that $\widetilde{x}(tp) = \widetilde{x}(tqp)$ for some $q \in L$. However, $\widetilde{x}(tp) = 1$ and $\widetilde{x}(tqp) = 0$. This contradiction establishes that $tp \in tLp$ for every $t \in R$. In particular, $\beta Sp = c\ell_{\beta S} Sp \subseteq Lp$. So $\beta Sp \subseteq Lp$. By [4, Theorem 1.46], Lp is a minimal left ideal of βS, and so $\beta Sp = Lp$.

(3) Now suppose that R contains a left cancelable element t and let $p \in \bigcap_{x \in Z} U(x)$. Since t is left cancelable in βR by [4, Lemma 8.1] and $tp = tqp$ for some $q \in L$, it follows that $p = qp \in K(\beta S)$. $\square$

Recall that a subset A of a semigroup S is *piecewise syndetic* if and only if there is some $G \in \mathcal{P}_f(S)$ such that for every $F \in \mathcal{P}_f(S)$, there is some $x \in S$ with $Fx \subseteq \bigcup_{t \in G} t^{-1}A$. The important fact about piecewise syndetic sets is that they are the subsets of S whose closure meets $K(\beta S)$, [4, Theorem 4.40].

Definition 3.5. $\Omega = \Omega_Z = \{x \in Z : \overline{x^{-1}[\{1\}] \cap S} \cap K(\beta S) = \emptyset\}$.

Thus $\Omega = \{x \in Z : x^{-1}[\{1\}] \cap S$ is not piecewise syndetic in $S\}$. Note that, since $K(\beta S)$ is usually not topologically closed, we have by Theorem 3.4 that not all sets of the form $U(x)$ are closed.

Definition 3.6. Let $x \in Z$. $N(x) = \{p \in \beta S : (\forall t \in R)(T_p(x)(t) = 0)\}$.

Lemma 3.7. *Let $x \in Z$. Then $N(x)$ is closed and $N(x) \subseteq U(x)$. If $N(x) = U(x)$, then $x \in \Omega$. If S is a left ideal of R, then $N(x) = U(x)$ if and only if $x \in \Omega$.*

Proof. To see that $N(x)$ is closed, let $p \in \beta S \setminus N(x)$, pick $t \in R$ such that $T_p(x)(t) = 1$, and let $A = \{s \in S : T_s(x)(t) = 1\}$. Then $A \in p$ and $\overline{A} \cap N(x) = \emptyset$.

If $T_p(x)$ is constantly equal to 0 on R, then $T_p(x)$ is uniformly recurrent and thus $p \in U(x)$.

Let $A = x^{-1}[\{1\}] \cap S$.

First assume that $N(x) = U(x)$ and suppose that $x \notin \Omega$. Since $\overline{A} \cap K(\beta S) \neq \emptyset$, pick $p \in \overline{A} \cap K(\beta S)$. By Corollary 2.2(3), $p \in U(x)$ and so for all $t \in R$, $T_p(x)(t) = 0$. Since $K(\beta S)$ is a union of groups, there exists $q \in K(\beta S)$ such that $qp = p$. Pick $t \in S$ such that $t^{-1}A \in p$. Also $T_p(x)(t) = 0$ so $\{s \in S : x(ts) = 0\} \in p$. Pick $s \in t^{-1}A$ such that $x(ts) = 0$, a contradiction.

Now assume that S is a left ideal in R. Let $x \in \Omega$ and let $p \in U(x)$. We claim that $p \in N(x)$. To see this, suppose we have some $t \in R$ such that $T_p(x)(t) = 1$. By Lemma 3.2, there exists an idempotent $q \in K(\beta S)$ such that $\widetilde{x}(tqp) = 1$. By [4, Theorem 2.17], βS is a left ideal of βR so $tqp \in \beta S$ and so $A \in tqp = tqqp$. Thus there is some $s \in S$ such that $tsqp \in \overline{A}$. Since $ts \in S$, $tsqp \in K(\beta S)$, a contradiction. □

Lemma 3.8. *Let $p \in \bigcap_{x \in \Omega} U(x)$ and let $t \in R$. If $tp \in \beta S$, then $tp \in c\ell K(\beta S)$. In particular, $\beta S p \subseteq c\ell K(\beta S)$.*

Proof. Assume that $tp \in \beta S \setminus c\ell(K\beta S)$. We can choose $A \in tp$ such that $A \subseteq S$ and $\overline{A} \cap K(\beta S) = \emptyset$. Let x be the characteristic function of A in R, so that $x \in \Omega$ and hence $p \in U(x)$. Observe that $\widetilde{x}$ is the characteristic function of $c\ell_{\beta R}(A)$ in βR and that $c\ell_{\beta R}(A) \subseteq \beta S$, because βS is clopen in βR. Since $\widetilde{x}(tp) = 1$, it follows from Lemma 3.2(b) that there exists $q \in K(\beta S)$ such that $\widetilde{x}(tqp) = 1$, and so $A \in tqp$. Now $\{r \in \beta S : tqr \in \beta S\}$ is non-empty and is a right ideal of βS. There exists an idempotent u in the intersection of this right ideal with the left ideal $\beta S q$ of βS. Since $q \in \beta S u$, $qu = q$. So $tqp = tquup \in K(\beta S)$, because $tqu \in \beta S$ and $u \in \beta S q \subseteq K(\beta S)$. This contradicts the assumption that $\overline{A} \cap K(\beta S) = \emptyset$. □

Corollary 3.9. *Each of the following statements implies that $\bigcap_{x \in \Omega} U(x) \subseteq c\ell K(\beta S)$.*

(a) *There exists $e \in R$ such that $es = s$ for every $s \in S$.*
(b) *S contains a left cancelable element.*

Proof. It follows from Lemma 3.8 that (a) implies that $\bigcap_{x \in \Omega} U(x) \subseteq c\ell K(\beta S)$. So assume that s is a left cancelable element in S and let $p \in \bigcap_{x \in \Omega} U(x)$. By [4, Lemma 8.1], s is left cancelable in βS. By Lemma 3.8, $sp \in c\ell K(\beta S)$. Now $s\beta S = \overline{sS}$ is clopen in βS. So $sp \in c\ell(K(\beta S) \cap s\beta S)$. We claim that, if $q \in K(\beta S) \cap s\beta S$, then $q \in sK(\beta S)$. To see this, suppose that $q \in K(\beta S)$ and that $q = sv$ for some $v \in \beta S$. There is an idempotent $u \in K(\beta S)$ for which $qu = q$. This implies that $sv = svu$ and hence that $v = vu \in K(\beta S)$. So $sp \in c\ell\big(sK(\beta S)\big) = sc\ell K(\beta S)$ and hence $p \in c\ell K(\beta S)$. □

Corollary 3.10. *Assume that S is a left ideal of R. Then each of the hypotheses* (a) *and* (b) *of Corollary* 3.9 *implies that $\bigcap_{x \in \Omega} U(x) = c\ell K(\beta S)$.*

Proof. Assume that one of the hypotheses of Corollary 3.9 holds. Then

$$\bigcap_{x \in \Omega} U(x) \subseteq c\ell K(\beta S).$$

To see that $c\ell K(\beta S) \subseteq \bigcap_{x \in \Omega} U(x)$, let $x \in \Omega$ be given. By Lemma 3.7, $U(x) = N(x)$ and so $U(x)$ is closed. By Corollary 2.2(3), $K(\beta S) \subseteq U(x)$ and hence $c\ell K(\beta S) \subseteq U(x)$. □

For the statement of the following corollary we depart from our standing assumptions about R, S, and $(Z, \langle T_s \rangle_{s \in S})$.

Corollary 3.11. *Let S be a semigroup. There exist a dynamical system $(X, \langle T_s\rangle_{s\in S})$ and a subset M of X such that $K(\beta S) = \bigcap_{x\in X} U(x)$ and $c\ell K(\beta S) = \bigcap_{x\in M} U(x)$.*

Proof. If S has a left identity, let $R = S$. Otherwise, let $R = S \cup \{e\}$ where e is an identity adjoined to S. The conclusion then follows from Theorem 3.4 and Corollary 3.10. □

In the proof of the above corollary, we could have simply let $R = S\cup\{e\}$ where e is an identity adjoined to S, regardless of whether S has a left identity, as was done in [4, Theorem 19.27] to produce a dynamical system for any semigroup S establishing the equivalence of the notions of *central* and *dynamically central*. We shall investigate the relationship between the systems with phase space $X = {}^R\{0, 1\}$ and $Y = {}^S\{0, 1\}$ in the next section.

We note that it is possible that $\bigcap_{x\in Z} U(x) \neq K(\beta S)$ and there is no subset M of Z such that $\bigcap_{x\in M} U(x) = c\ell K(\beta S)$. To see this, let S be an infinite zero semigroup. That is, there is an element $0 \in S$ such that $st = 0$ for all s and t in S. Then $pq = 0$ for all p and q in βS and so $c\ell K(\beta S) = K(\beta S) = \{0\}$. Let $R = S$. Given $x \in T$, if $a = x(0)$, then for all $p \in \beta S$, $T_p(x)$ is constantly equal to a and so $T_p(x)$ is uniformly recurrent. That is, for any $x \in Z$, $U(x) = \beta S$.

In [1] it was shown that $c\ell K(\beta\mathbb{N})$ is the intersection of all of the closed two sided ideals that strictly contain it. In a similar vein, we would like to show that each $U(x)$ properly contains $K(\beta S)$. One cannot hope for this to hold in general. For example, as we have already noted, if S is either left zero or right zero then so is βS and then $K(\beta S) = \beta S$. Results establishing that $U(x)$ properly contains $K(\beta S)$ require some weak cancellation assumptions.

Definition 3.12. Let S be a semigroup and let $A \subseteq S$.

(a) A is a *left solution set* if and only if there exist u and v in S such that $A = \{x \in S : ux = v\}$.
(b) A is a *right solution set* if and only if there exist u and v in S such that $A = \{x \in S : xu = v\}$.

As is standard, we denote by ω the first infinite ordinal, which is also the first infinite cardinal. That is, $\omega = \aleph_0$.

Definition 3.13. Let S be a semigroup with $|S| = \kappa \geq \omega$.

(a) S is *weakly left cancellative* if and only if every left solution set in S is finite.
(b) S is *weakly right cancellative* if and only if every right solution set in S is finite.
(c) S is *weakly cancellative* if and only if S is both weakly left cancellative and weakly right cancellative.
(d) S is *very weakly left cancellative* if and only if the union of any set of fewer than κ left solution sets has cardinality less than κ.
(e) S is *very weakly right cancellative* if and only if the union of any set of fewer than κ right solution sets has cardinality less than κ.
(f) S is *very weakly cancellative* if and only if S is both very weakly left cancellative and very weakly right cancellative.

Given a set X and a cardinal κ, we let $U_\kappa(X)$ be the set of κ-uniform ultrafilters on X. That is, $U_\kappa(X) = \{p \in \beta X : (\forall A \in p)(|A| \geq \kappa)\}$.

Theorem 3.14. *Assume that $|R| = |S| = \kappa \geq \omega$, S is very weakly cancellative, and has the property that $|\{e \in S : (\exists s \in S)(es = s)\}| < \kappa$. Then for all $x \in Z$, $U(x)\cap U_\kappa(S)\setminus c\ell K(\beta S) \neq \emptyset$.*

Proof. Let $E = \{e \in S : (\exists s \in S)(es = s)\}$. Let $x \in Z$ and pick $q \in K(\beta S)$. Let $y = T_q(x)$. By Corollary 2.2(3), y is uniformly recurrent. For each $F \in \mathcal{P}_f(R)$, let $B_F = \big\{s \in S : (\forall t \in F)\big(x(ts) = y(t)\big)\big\}$. Since

$$B_F = \left\{ s \in S : T_s(x) \in \bigcap_{t \in F} \pi_t^{-1}[\{y(t)\}] \right\},$$

we have $B_F \in q$. By [4, Lemma 6.34.3], $K(\beta S) \subseteq U_\kappa(S)$ and so $|B_F| = \kappa$. Note that if $F \subseteq H$, then $B_H \subseteq B_F$.

Enumerate $\mathcal{P}_f(R)$ as $\langle F_\alpha \rangle_{\alpha<\kappa}$. Choose $t_0 \in B_{F_0} \setminus E$. Let $0 < \alpha < \kappa$ and assume that we have chosen $\langle t_\delta \rangle_{\delta<\alpha}$ satisfying the following inductive hypotheses.

(1) For each $\delta < \alpha$, $t_\delta \in B_{F_\delta}$.
(2) For each $\delta < \alpha$, $FP(\langle t_\beta \rangle_{\beta\leq\delta}) \cap E = \emptyset$.
(3) For each $\delta < \alpha$, if $\delta > 0$, then $t_\delta \notin FP(\langle t_\beta \rangle_{\beta<\delta})$.
(4) For each $\delta < \alpha$, if $\delta > 0$, $s \in FP(\langle t_\beta \rangle_{\beta<\delta})$, and $\gamma < \delta$, then $st_\delta \neq t_\gamma$.

The hypotheses are satisfied for $\delta = 0$. Let

$$M_0 = \left\{ t \in S : \big(\exists H \in \mathcal{P}_f(\alpha)\big)\Big(\Big(\prod_{\beta \in H} t_\beta\Big) t \in E\Big)\right\} \text{ and let}$$

$$M_1 = \{t \in S : \big(\exists s \in FP(\langle t_\beta \rangle_{\beta<\alpha})\big)(\exists \gamma < \alpha)(st = t_\gamma)\}\,.$$

Note that $|FP(\langle t_\beta \rangle_{\beta<\alpha})| \leq |\mathcal{P}_f(\alpha)| < \kappa$. Also, given $H \in \mathcal{P}_f(\alpha)$ and $s \in E$, $\{t \in S : (\prod_{\beta\in H} t_\beta)t = s\}$ is a left solution set so $|M_0| < \kappa$. Note also that, given $s \in FP(\langle t_\beta \rangle_{\beta<\alpha})$ and $\gamma < \alpha$, $\{t \in S : st = t_\gamma\}$ is a left solution set so $|M_1| < \kappa$. Thus we may choose $t_\alpha \in B_{F_\alpha} \setminus (E \cup FP(\langle t_\beta \rangle_{\beta<\alpha}) \cup M_0 \cup M_1)$. The induction hypotheses are satisfied for α.

Let $B = \{t_\alpha : \alpha < \kappa\}$ and let $C = \bigcap_{\alpha<\kappa} c\ell FP(\langle t_\beta \rangle_{\alpha<\beta<\kappa})$. By [4, Theorem 4.20], C is a compact subsemigroup of βS. We claim that $\overline{B} \cap K(C) = \emptyset$. Suppose instead that we have $p \in \overline{B} \cap K(C)$. Pick $r \in K(C)$ such that $p = pr$. (By [4, Lemma 1.30], an idempotent in the minimal left ideal L of C in which p lies will do.) Let $D = \underline{\{s \in S : s^{-1}B \in r\}}$. Then $D \in p$ so $D \cap B \neq \emptyset$ so pick $\alpha < \kappa$ such that $t_\alpha^{-1}B \in r$. Then $t_\alpha^{-1}B \cap FP(\langle t_\beta \rangle_{\alpha<\beta<\kappa}) \neq \emptyset$ so pick finite $H \subseteq \{\beta : \alpha < \beta < \kappa\}$ such that $\prod_{\beta\in H} t_\beta \in t_\alpha^{-1}B$. Pick $\gamma < \kappa$ such that $t_\alpha \prod_{\beta\in H} t_\beta = t_\gamma$. Let $\max H = \mu$ and let $K = H \setminus \{\mu\}$. If $K = \emptyset$, then $t_\alpha t_\mu = t_\gamma$. If $K \neq \emptyset$, then $t_\alpha(\prod_{\beta\in K} t_\beta)t_\mu = t_\gamma$. If $\gamma > \mu$ we get a contradiction to hypothesis (3). If $\mu = \gamma$ we either get $t_\alpha \in E$ or $t_\alpha \prod_{\beta\in K} t_\beta \in E$, contradicting hypothesis (2). If $\gamma < \mu$ we get a contradiction to hypothesis (4). Thus $\overline{B} \cap K(C) = \emptyset$ as claimed.

Now we claim that $\overline{B} \cap K(\beta S) = \emptyset$. Suppose instead we have $p \in \overline{B} \cap K(\beta S)$. By [4, Lemma 6.34.3] we have that $p \in U_\kappa(S)$ and consequently, $p \in C$. Thus $K(\beta S) \cap C \neq \emptyset$ and so, by [4, Theorem 1.65], $K(C) = K(\beta S) \cap C$, contradicting the fact that $\overline{B} \cap K(C) = \emptyset$. Since $\overline{B}$ is clopen, we thus have $\overline{B} \cap c\ell K(\beta S) = \emptyset$.

Now let $\mathcal{C} = \{B_F : F \in \mathcal{P}_f(S)\} \cup \{B\}$. We claim that $\mathcal{C}$ has the κ-uniform finite intersection property. To see this, let $\mathcal{F} \in \mathcal{P}_f\big(\mathcal{P}_f(S)\big)$ and let $H = \bigcup \mathcal{F}$. If $\delta < \kappa$ and $H \subseteq F_\delta$, then $t_\delta \in B \cap \bigcap_{F\in\mathcal{F}} B_F$. Since $|\{\delta < \kappa : H \subseteq F_\delta\}| = |\{F \in \mathcal{P}_f(S) : H \subseteq F\}| = \kappa$, we have that $|\bigcap \mathcal{C}| = \kappa$ as required. Pick by [4, Corollary 3.14] $p \in U_\kappa(S)$ such that $\mathcal{C} \subseteq p$.

Since $B_F \in p$ for each $F \in \mathcal{P}_f(R)$, we have $T_p(x) = y$ so $p \in U(x)$. Since $B \in p$, $p \notin c\ell K(\beta S)$. $\square$

Corollary 3.15. *Assume that $|R| = |S| = \kappa \geq \omega$ and that S is right cancellative and very weakly left cancellative. Then for all $x \in Z$, $U(x) \cap U_\kappa(S) \setminus c\ell K(\beta S) \neq \emptyset$.*

Proof. Let $E = \{e \in S : (\exists s \in S)(es = s)\}$. It suffices to show that $|E| < \kappa$. Pick $x \in S$. Given $e \in E$ and $s \in S$ such that $es = s$, we have that $xes = xs$ so $xe = x$. Thus E is contained in the left solution set $\{t \in S : xt = x\}$. □

Corollary 3.16. *Assume that $|R| = |S| = \kappa \geq \omega$, that S is very weakly cancellative, that S has a member e such that $es = s$ for all $s \in S$, and $|\{e \in S : (\exists s \in S)(es = s)\}| < \kappa$. Then $K(\beta S) = \bigcap_{x \in Z} U(x)$ and for each $x \in Z$, $U(x)$ properly contains $K(\beta S)$.*

Proof. This is an immediate consequence of Theorems 3.4 and 3.14. □

Corollary 3.17. *Assume that S is a left ideal of R, $|R| = |S| = \kappa \geq \omega$, S is very weakly cancellative, S has a member e such that $es = s$ for all $s \in S$, and $|\{e \in S : (\exists s \in S)(es = s)\}| < \kappa$. Then $c\ell K(\beta S) = \bigcap_{x \in \Omega} U(x)$ and for each $x \in \Omega$, $U(x)$ properly contains $c\ell K(\beta S)$.*

Proof. By Corollary 3.10 $c\ell K(\beta S) = \bigcap_{x \in \Omega} U(x)$. By Theorem 3.14, for each $x \in \Omega$, $U(x)$ properly contains $c\ell K(\beta S)$. □

4. Relations between systems with phase spaces X and Y

Throughout this section we will let S be an arbitrary semigroup and let $Q = S \cup \{e\}$, where e is an identity adjoined to S, even if S already has an identity. We will let $(X, \langle T_{X,s} \rangle_{s \in S})$ be the dynamical system of Lemma 1.2 determined by $R = Q$ let $(Y, \langle T_{Y,s} \rangle_{s \in S})$ be the dynamical system of Lemma 1.2 determined by $R = S$. For $x \in X$ we will let $U_X(x) = \{p \in \beta S : T_{X,p}(x)$ is uniformly recurrent$\}$ and let $U_Y(x) = \{p \in \beta S : T_{Y,p}(x)$ is uniformly recurrent$\}$.

We have from the results of the previous section that for any semigroup S, $K(\beta S) = \bigcap_{x \in X} U_X(x)$ and $c\ell K(\beta S) = \bigcap_{x \in \Omega_X} U_X(x)$. We are interested in determining when the corresponding assertions hold with respect to Y. Of course, the simplest situation in which they do is when for each $x \in X$, $U_X(x) = U_Y(x_{|S})$ so we address this problem first, beginning with the following simple observation.

Lemma 4.1. *Let $x \in X$. Then $U_X(x) \subseteq U_Y(x_{|S})$.*

Proof. Let $y = x_{|S}$ and note that $\widetilde{y}$ is the restriction of $\widetilde{x}$ to βS. Let L be a minimal left ideal of βS. By Lemma 3.2, $p \in U_X(x)$ if and only if there exists $q \in L$, such that $\widetilde{x}(tp) = \widetilde{x}(tqp)$ for all $t \in Q$. And $p \in U_Y(x_{|S})$ if and only if there exists $q \in L$ such that $\widetilde{y}(tp) = \widetilde{y}(tqp)$ for all $t \in S$. □

Theorem 4.2. *The following statements are equivalent.*

(a) *For all $x \in X$, $U_X(x) = U_Y(x_{|S})$.*
(b) *There do not exist $p \in \beta S$ and $x \in X$ such that $T_{X,p}(x)$ is the characteristic function of $\{e\}$ in X.*
(c) *For every $p \in \beta S$, $p \in \beta S p$.*

Proof. Assume that (a) holds and suppose we have $p \in \beta S$ and $x \in X$ such that $T_{X,p}(x)$ is the characteristic function of $\{e\}$ in X. Then $T_{Y,p}(x_{|S})$ is constantly 0 so $p \in U_Y(x_{|S})$. But $V = \{u \in X : w(e) = 1\}$ is a neighborhood of $w = T_{X,p}(x)$ in X, while $\{s \in S : T_{X,s}(w) \in V\} = \emptyset$, so $p \notin U_X(x)$.

To see that (b) implies (c), assume that (b) holds and suppose that we have some $p \in \beta S$ such that $p \notin \beta S p$. Since $\beta S p = \rho_p[\beta S]$, $\beta S p$ is closed. Pick $A \in p$ such that $\overline{A} \cap \beta S p = \emptyset$. Let x be the characteristic function of A in X. First let $s \in S$. Then $sp \notin \overline{A}$ so $s^{-1}(S \setminus A) \in p$ so to see that $T_{X,p}(s) = 0$, it suffices to observe that $s^{-1}(S \setminus A) \subseteq \{t \in S : T_{X,t}(x)(s) = 0\}$. Since $A \in p$ and for $t \in A$, $T_{X,t}(x)(e) = x(t) = 1$, we have that $T_{X,p}(x)(e) = 1$.

By Lemma 4.1, we have $U_X(x) \subseteq U_Y(x_{|S})$ for all $x \in X$, so to show that (c) implies (a), it suffices to let $x \in X$, let $p \in U_Y(x_{|S})$, assume that $p \in \beta S p$, and show that $p \in U_X(x)$. By Lemma 3.3, it suffices to let L be a minimal left ideal of βS and let $F \in \mathcal{P}_f(Q)$ and show that there is some $q \in L$ such that $\widetilde{x}(tp) = \widetilde{x}(tqp)$ for every $t \in F$. For $t \in F$, let $B_t = \{s \in S : x(ts) = \widetilde{x}(tp)\}$. Then $\bigcap_{t \in F} B_t \in p$ and $p \in \beta S p = c\ell(Sp)$ so pick $v \in S$ such that $\bigcap_{t \in F} B_t \in vp$. Let $y = x_{|S}$. Now $Fv \in \mathcal{P}_f(S)$ so pick by Lemma 3.3 $q \in L$ such that for all $t \in F$, $\widetilde{y}(tvp) = \widetilde{y}(tvqp)$. Let $q' = vq$ and note that $q' \in L$. Let $t \in F$ be given. Then $B_t \in vp$ so $\widetilde{x}(tvp) = \widetilde{x}(tp)$ and thus $\widetilde{x}(tp) = \widetilde{y}(tvp) = \widetilde{y}(tvqp) = \widetilde{x}(tq'p)$. □

Corollary 4.3. *If for all $p \in \beta S$, $p \in \beta S p$, then $K(\beta S) = \bigcap_{x \in Y} U_Y(x)$ and $c\ell K(\beta S) = \bigcap_{x \in \Omega_Y} U_Y(x)$.*

Proof. The first assertion is an immediate consequence of Theorems 3.4 and 4.2. The second assertion follows from Corollary 3.10 and Theorem 4.2. □

We have already mentioned the problem of determining whether $K(\beta S)$ or $c\ell K(\beta S)$ is prime. Recall that an ideal I in a semigroup is *semiprime* if and only if whenever $ss \in I$, one must have $s \in I$.

Corollary 4.4.

(1) *If $K(\beta S) \neq \bigcap_{x \in Y} U_Y(x)$, then $K(\beta S)$ is not semiprime.*
(2) *If $c\ell K(\beta S) \neq \bigcap_{x \in \Omega_Y} U_Y(x)$, then $c\ell K(\beta S)$ is not semiprime.*

Proof. (1) If $p \in \bigcap_{x \in Y} U_Y(x) \setminus K(\beta S)$, then $pp \in \beta S p$ and by Theorem 3.4, $\beta S p \subseteq K(\beta S)$.
(2) If $p \in \bigcap_{x \in \Omega_Y} U_Y(x) \setminus c\ell K(\beta S)$, then $pp \in \beta S p$ and by Lemma 3.8, $\beta S p \subseteq c\ell K(\beta S)$. □

By virtue of Theorem 4.2 we are interested in knowing when there is some $p \in \beta S$ such that $p \notin \beta S p$.

Lemma 4.5. *Let $p \in \beta S$. Then $p \notin \beta S p$ if and only if there exists $A \subseteq S$ such that for all $x \in S$, $x^{-1}A \in p$ and $A \notin p$.*

Proof. Let $C(p) = \{A \subseteq S : (\forall x \in S)(x^{-1}A \in p)\}$. By [4, Theorem 6.18], $p \in \beta S p$ if and only if $C(p) \subseteq p$. □

Theorem 4.6. *Assume that $|S| = \kappa \geq \omega$. There exists $p \in \beta S$ such that $p \notin \beta S p$ if and only if there exists $\langle y_F \rangle_{F \in \mathcal{P}_f(S)}$ in S such that $\{y_F : F \in \mathcal{P}_f(S)\} \cap \bigcup \{F y_F : F \in \mathcal{P}_f(S)\} = \emptyset$.*

Proof. Necessity. Pick $p \in \beta S$ such that $p \notin \beta S p$. By Lemma 4.5, pick $A \subseteq S$ such that for all $x \in S$, $x^{-1}A \in p$ and $A \notin p$. For $F \in \mathcal{P}_f(S)$ pick $y_F \in (S \setminus A) \cap \bigcap_{x \in F} x^{-1}A$.

Sufficiency. Let $A = \bigcup \{F y_F : F \in \mathcal{P}_f(S)\}$. Then $\{S \setminus A\} \cup \{x^{-1}A : x \in S\}$ has the finite intersection property so pick $p \in \beta S$ such that $\{S \setminus A\} \cup \{x^{-1}A : x \in S\} \subseteq p$. By Lemma 4.5, $p \notin \beta S p$. □

One of the assumptions in the following corollary is that $S^* = \beta S \setminus S$ is a right ideal of βS. A (not very simple) characterization of when S^* is a right ideal of βS is given in [4, Theorem 4.32]. By [4, Corollary 4.33 and Theorem 4.36] it is sufficient that S be either right cancellative or weakly cancellative.

Corollary 4.7. *Assume that $|S| = \kappa \geq \omega$ and assume that*

$$|S \setminus \{t \in S : (\exists s \in S)(st = t)\}| = \kappa.$$

If either S^* is a right ideal of βS or S is very weakly left cancellative, then there exists p in βS such that $p \notin \beta S p$.

Proof. Assume first that S^* is a right ideal of βS, and pick $t \in S$ such that there is no $s \in S$ with $st = t$. Then $t \notin St$ and $t \notin S^* t$.

Now assume that S is very weakly left cancellative. Enumerate $\mathcal{P}_f(S)$ as $\langle F_\alpha \rangle_{\alpha<\kappa}$. By Theorem 4.6, it suffices to produce $\langle t_\alpha \rangle_{\alpha<\kappa}$ in S such that $\{t_\alpha : \alpha < \kappa\} \cap \bigcup\{F_\alpha t_\alpha : \alpha < \kappa\} = \emptyset$.

Let $E = \{t \in S : (\exists s \in S)(st = t)\}$. Pick $t_0 \in S \setminus E$. Let $0 < \alpha < \kappa$ and assume we have chosen $\langle t_\delta \rangle_{\delta<\alpha}$ in $S \setminus E$ such that if $\delta > 0$, then $t_\delta \notin \bigcup_{\mu<\delta} F_\mu t_\mu$ and for each $x \in F_\delta$, $x t_\delta \notin \{t_\mu : \mu < \delta\}$.

For $x \in S$ and $\mu < \alpha$, let $H_{x,\mu} = \{t \in S : xt = t_\mu\}$. Then each $H_{x,\mu}$ is a left solution set so $|\bigcup\{H_{x,\mu} : x \in F_\alpha \text{ and } \mu < \alpha\}| < \kappa$. Pick

$$t_\alpha \in S \setminus \left(E \cup \bigcup\{H_{x,\mu} : x \in F_\alpha \text{ and } \mu < \alpha\} \cup \bigcup_{\mu<\alpha} F_\mu t_\mu \right).$$

Suppose we have some $\mu < \kappa$ such that $t_\mu \in \bigcup\{F_\alpha t_\alpha : \alpha < \kappa\}$ and pick $\alpha < \kappa$ and $x \in F_\alpha$ such that $t_\mu = x t_\alpha$. Then $\alpha \neq \mu$ because $t_\alpha \notin E$. If $\alpha < \mu$, we would have $t_\mu \in F_\alpha t_\alpha$. So we must have $\mu < \alpha$. But then $t_\alpha \in H_{x,\mu}$, a contradiction. □

We conclude this section by exhibiting a sufficient condition which guarantees $K(\beta S) = \bigcap_{x\in Y} U_Y(x)$. We shall see that this does not require equality between $U_X(x)$ and $U_Y(x_{|S})$ for all $x \in X$.

Theorem 4.8. *Assume that for all $p \in \bigcap_{x\in Y} U_Y(x)$ and all $A \in p$ the assumption that $\{t \in S : t^{-1}sA \in p\}$ is syndetic for every $s \in S$, implies that $\{t \in S : t^{-1}A \in p\} \neq \emptyset$. Then $K(\beta S) = \bigcap_{x\in Y} U_Y(x)$.*

Proof. Assume that $p \in \bigcap_{x\in Y} U_Y(x) \setminus K(\beta S)$. By Theorem 3.4(2), $\beta S p \subseteq K(\beta S)$ so $p \notin \beta S p$. Pick $A \in p$ such that $\overline{A} \cap \beta S p = \emptyset$. Thus $\{t \in S : t^{-1}A \in p\} = \emptyset$. We claim that for all $s \in S$, $\{t \in S : t^{-1}sA\}$ is syndetic. So let $s \in S$. By [4, Theorem 4.48] it suffices to let L be a minimal left ideal of βS and show that there is some $q \in L$ such that $\{t \in S : t^{-1}sA \in p\} \in q$. By Theorem 3.4(2), $sp \in Lp$ so pick $q \in L$ such that $sp = qp$. Then $sA \in qp$ so $\{t \in S : t^{-1}sA \in p\} \in q$ as required. □

Note that by Theorem 3.4(3), $K(\beta\mathbb{N}, +) = \bigcap_{x\in Y} U_Y(x)$ while $1 \notin \beta\mathbb{N}+1$ so by Theorem 4.2, it is not the case that for all $x \in X$, $U_X(x) = U_Y(x_{|S})$. On the other hand, given $p \in K(\beta\mathbb{N}, +)$ one has $p = q + p$ for some $p \in K(\beta\mathbb{N}, +)$ so automatically for any $A \in p$, $\{t \in \mathbb{N} : -t + A \in p\} \neq \emptyset$ so the hypotheses of Theorem 4.8 are valid.

5. Recurrence and surjectivity of T_p

So far in this paper we have been considering the notion of uniform recurrence. We now introduce a notion which is usually weaker.

Definition 5.1. Let $(X, \langle T_s\rangle_{s\in S})$ be a dynamical system. The point $x \in X$ is *recurrent* if and only if for each neighborhood V of x in X, $\{s \in S : T_s(x) \in V\}$ is infinite.

If all syndetic subsets of a semigroup S are infinite, then any uniformly recurrent point of X is recurrent. This is not always the case. For example, if S is a left zero semigroup and $x \in S$, then x is uniformly recurrent in the dynamical system $(\beta S, \langle\lambda_s\rangle_{s\in S})$ but is not recurrent. (We have that $\{x\}$ is a neighborhood of x and $\big\{s \in S : \lambda_s(x) \in \{x\}\big\} = \{x\}$, which is syndetic, but finite.)

The following characterization of recurrence is very similar to the characterization of uniform recurrence in [4, Theorem 19.23]. Part of the results depends on the assumption that S^* is a subsemigroup of βS. There is a characterization of S^* as a subsemigroup in [4, Theorem 4.28]. By [4, Corollary 4.29 and theorem 4.31] it is sufficient that S be right cancellative or weakly left cancellative.

Theorem 5.2. *Let $(X, \langle T_s\rangle_{s\in S})$ be a dynamical system. Statements* (a) *and* (b) *are equivalent and imply statements* (c) *and* (d)*, which are equivalent. If S^* is a subsemigroup of βS, then all four statements are equivalent.*

(a) *There exists an idempotent $p \in S^*$ such that $T_p(x) = x$.*
(b) *There exist $y \in X$ and an idempotent $p \in S^*$ such that $T_p(y) = x$.*
(c) *There exists $p \in S^*$ such that $T_p(x) = x$.*
(d) *x is recurrent.*

Proof. Trivially (a) implies (b) and (a) implies (c). To see that (b) implies (a), pick $y \in X$ and an idempotent $p \in S^*$ such that $T_p(y) = x$. Then $x = T_p(y) = T_{pp}(y) = T_p\big(T_p(y)\big) = T_p(x)$.

To see that (c) implies (d), pick $p \in S^*$ such that $T_p(x) = x$. Let V be a neighborhood of x. Then $\{s \in S : T_s(x) \in V\} \in p$ so $\{s \in S : T_s(x) \in V\}$ is infinite.

To see that (d) implies (c), assume that x is recurrent and for each neighborhood V of x, let $D_V = \{s \in S : T_s(x) \in V\}$. Then any finite subfamily of $\{D_V : V \text{ is a neighborhood of } x\}$ has infinite intersection so pick by [4, Corollary 3.14] some $p \in S^*$ such that $\{D_V : V \text{ is a neighborhood of } x\} \subseteq p$. Then $T_p(x) = x$.

Now assume that S^* is a semigroup. To see that (c) implies (a), pick $p \in S^*$ such that $T_p(x) = x$ and let $E = \{q \in S^* : T_q(x) = x\}$. Since S^* is a subsemigroup of βS, we have that E is a subsemigroup of βS. We claim that E is closed. To see this, let $q \in \beta S \setminus E$. If $q \in S$, then $\{q\}$ is a neighborhood of q missing E, so assume that $q \in S^*$. Pick an open neighborhood U of $T_q(x)$ such that $x \notin c\ell U$ and let $A = \{s \in S : T_s(x) \in U\}$. Then $\overline{A}$ is a neighborhood of q which misses E. Since E is a compact right topological semigroup, there is an idempotent in E. $\square$

Recall that in any dynamical system, $(X, \langle T_s\rangle_{s\in S})$, $K(\beta S) \subseteq \bigcap_{x\in X} U_X(x)$ and we have obtained sufficient conditions for equality.

Theorem 5.3. *Let $(X, \langle T_s\rangle_{s\in S})$ be a dynamical system, let $p \in \beta S$, and assume that $T_p : X \to X$ is surjective and $K(\beta S) = \bigcap_{x\in X} U_X(x)$. Then for any $q \in \beta S$, $qp \in K(\beta S)$ if and only if $q \in K(\beta S)$.*

Proof. Let $q \in \beta S$. The sufficiency is trivial, so assume that $qp \in K(\beta S)$. It suffices to show that $q \in \bigcap_{x \in X} U(x)$, so let $x \in X$ be given. Pick $y \in X$ such that $T_p(y) = x$. Then $T_q(x) = T_q\big(T_p(y)\big) = T_{qp}(y)$. Since $qp \in U(y)$ we have $T_{qp}(y)$ is uniformly recurrent, and so $T_q(x) \in U(x)$ as required. □

Definition 5.4. Let $(X, \langle T_s \rangle_{s \in S})$ be a dynamical system. Then $NS = NS_X = \{p \in \beta S : T_p \text{ is not surjective}\}$.

We have seen that $U(x)$ is always a left ideal of βS.

Lemma 5.5. *Let $(X, \langle T_s \rangle_{s \in S})$ be a dynamical system. If $NS \neq \emptyset$, then NS is a right ideal of βS.*

Proof. Given $p \in NS$ and $q \in \beta S$, the range of T_{pq} is contained in the range of T_p. □

Lemma 5.6. *Let $(X, \langle T_s \rangle_{s \in S})$ be a dynamical system. If there is some $x \in X$ such that x is not recurrent, then $\{p \in S^* : pp = p\} \subseteq NS$.*

Proof. Pick $x \in X$ such that x is not recurrent and let p be an idempotent in S^*. We claim that x is not in the range of T_p, so suppose instead we have $y \in X$ such that $T_p(y) = x$. Then by Theorem 5.2, x is recurrent. □

We shall establish a strong connection between the surjectivity of T_p and p being right cancelable in βS. The purely algebraic result in Theorem 5.8 will be useful.

Lemma 5.7. *Let S be a countable right cancellative and weakly left cancellative semigroup and let B be an infinite subset of S. There is an infinite subset D of B with the property that whenever s and t are distinct members of S, there is a finite subset F of D such that $sa \neq tb$ whenever $a, b \in D \setminus F$.*

Proof. Let $\Delta = \{(s, s) : s \in S\}$ and enumerate $(S \times S) \setminus \Delta$ as $\langle (s_n, t_n) \rangle_{n=1}^{\infty}$. Pick $a_1 \in B$. Assume $n \in \mathbb{N}$ and we have chosen $\langle a_i \rangle_{i=1}^{n}$. Let $W_n = \{b \in S : \text{there exist } i, j \in \{1, 2, \ldots, n\} \text{ such that } s_i a_j = t_i b \text{ or } s_i b = t_i a_j\}$. Then W_n is the union of finitely many left solution sets, so is finite. Pick $a_{n+1} \in B \setminus (W_n \cup \{a_1, a_2, \ldots, a_n\})$.

Let $D = \{a_n : n \in \mathbb{N}\}$. Let s and t be distinct members of S and pick n such that $(s, t) = (s_n, t_n)$. Let $F = \big\{a_i : i \in \{1, 2, \ldots, n\}\big\}$. To see that F is as required, let $a, b \in D \setminus F$ and suppose $sa = tb$. Then by right cancellation, $a \neq b$. Pick $m > n$ and $r > n$ such that $a = a_m$ and $b = a_r$. If $m < r$, then $a_r \in W_{r-1}$. If $r < m$, then $a_m \in W_{m-1}$. □

Theorem 5.8. *Let S be a countable cancellative semigroup. If $p \in \beta S \setminus K(\beta S)$, then there exists an infinite $D \subseteq S$ such that for every $r \in D^*$, rp is right cancelable in βS.*

Proof. Choose $q \in K(\beta S)$. We first claim that for each $s \in S$, $sp \notin K(\beta S)$ and in particular, $sp \notin \beta S q p$. So suppose we have $sp \in K(\beta S)$. Then sp is in a minimal left ideal L of βS. Pick an idempotent $r \in L$. By [4, Lemma 1.30], $sp = spr$. By [4, Lemma 8.1] s is left cancelable in βS so $p = pr$, and thus $p \in K(\beta S)$. This contradiction establishes the claim. For each $s \in S$, pick $U_s \in sp$ such that $\overline{U_s} \cap \beta S q p = \emptyset$. For each $s, t \in S$, there exists $V_{s,t} \in q$ such that $\overline{U_s} \cap t \overline{V_{s,t}} p = \emptyset$ because $\lambda_t \circ \rho_p(q) \in \beta S \setminus \overline{U_s}$.

By [4, Theorem 3.36], there exists an infinite subset B of S such that $B^* \subseteq \bigcap_{s,t\in S} \overline{V_{s,t}}$. Then for every $r \in B^*$ and every $s, t \in S$, $trp \notin \overline{U_s}$.

By Lemma 5.7 pick an infinite subset D of B such that, whenever s and t are distinct elements of S, there is a finite subset F of D such that $sa \neq tb$ whenever $a, b \in D \setminus F$. Enumerate D as $\langle d_n \rangle_{n=1}^{\infty}$ and for each distinct s and t in S, pick $n_{s,t} \in \mathbb{N}$ such that $sd_m \neq td_n$ whenever $m, n > n_{s,t}$.

We claim that, for every $r \in D^*$, rp is right cancelable in βS. We shall apply [4, Theorem 3.40] three times.

Assume that $q_1 rp = q_2 rp$, where q_1 and q_2 are distinct elements of βS. Let A_1 and A_2 be disjoint subsets of S which are members of q_1 and q_2 respectively. Since $q_1 rp \in cl(A_1 rp)$ and $q_2 rp \in cl(A_2 rp)$, an application of [4, Theorem 3.40] shows that either $A_1 rp \cap cl(A_2 rp) \neq \emptyset$ or $A_2 rp \cap cl(A_1 rp) \neq \emptyset$, and without loss of generality, we may assume that the former holds. Thus we have some $s \in A_1$ and $q' \in \overline{A_2}$ such that $srp = q'rp$. Now $srp \in c\ell(sDp)$ and $q'rp \in c\ell\big((S \setminus \{s\})rp\big)$, so either $sDp \cap c\ell\big((S \setminus \{s\})rp\big) \neq \emptyset$ or $(S \setminus \{s\})rp \cap c\ell(sDp) \neq \emptyset$. We thus have either

(i) $sDp \cap c\ell\big((S \setminus \{s\})rp\big) \neq \emptyset$, in which case we choose $d \in D$ and $y \in \beta S$ such that $sdp = yrp$; or

(ii) $sDp \cap c\ell\big((S \setminus \{s\})rp\big) = \emptyset$, in which case we pick $t \in S \setminus \{s\}$ and $r' \in \overline{D}$ such that $sr'p = trp$. Since $sDp \cap c\ell\big((S \setminus \{s\})rp\big) = \emptyset$, we have $r' \in D^*$.

Suppose that (i) holds. Then $U_{sd} \in sdp$ so $\{v \in S : v^{-1}U_{sd} \in rp\} \in y$, so pick $v \in S$ such that $U_{sd} \in vrp$. But $r \in V_{sd,v}$, so this is a contradiction. Thus (ii) holds.

Now $sr'p \in c\ell\{sd_m p : m > n_{s,t}\}$ and $trp \in c\ell\{td_m p : m > n_{s,t}\}$ so, essentially without loss of generality, we have $\{sd_m p : m > n_{s,t}\} \cap c\ell\{td_m p : m > n_{s,t}\} \neq \emptyset$. (We have distinguished between s and t at this stage, but the arguments below with s and t interchanged remain valid.) Thus either

(iii) there exist $m, n > n_{s,t}$ such that $sd_m p = td_n p$; or

(iv) there exist $m > n_{s,t}$ and $r'' \in D^*$ such that $sd_m p = tr''p$.

If (iii) holds, then by [4, Lemma 6.28], $sd_m = td_n$, contradicting the choice of $n_{s,t}$. So (iv) holds. But $r'' \in V_{sd_m,t}$ so $tr''p \notin U_{sd_m}$, a contradiction. □

We now present several results about the dynamical systems considered in Section 3.

Lemma 5.9. *Let S be a semigroup and let p be a right cancelable element of βS. Then for any clopen subset E of βSp, there is some $A \subseteq S$ such that $E = \overline{A} \cap \beta Sp$.*

Proof. Let E be a clopen subset of βSp. Let $\mathcal{D} = \{\overline{D} \cap \beta Sp : D \subseteq S \text{ and } \overline{D} \cap \beta Sp \subseteq E\}$. Since $\{\overline{D} \cap \beta Sp : D \subseteq S\}$ is a basis for the topology of βSp and E is open in βSp, we have that $E = \bigcup \mathcal{D}$. Since E is compact, pick finite $\mathcal{F} \subseteq \mathcal{P}(S)$ such that $E = \bigcup_{D\in\mathcal{F}} (\overline{D} \cap \beta Sp)$ and let $A = \bigcup \mathcal{F}$. □

Theorem 5.10. *Let S be a semigroup. Let $(Y, \langle T_s \rangle_{s\in S})$ be the dynamical system of Lemma 1.2 determined by $R = S$. Let $p \in \beta S$. If p is right cancelable in βS, then $T_p : Y \to Y$ is surjective.*

Proof. Note that since $\rho_p : \beta S \to \beta Sp$ is injective and takes closed sets to closed sets, it is a homeomorphism.

To see that T_p is surjective, let $z \in Y$, let $B = \{s \in S : z(s) = 1\}$, and let $E = \rho_p[\overline{B}\,]$. Then E is clopen in βSp so by Lemma 5.9 pick $A \subseteq S$ such that $E = \overline{A} \cap \beta Sp$. Let x be the characteristic function of A in Y. We claim that $T_p(x) = z$. For this, it suffices that for each $s \in S$, $\{t \in S : T_t(x)(s) = z(s)\} \in p$. So let $s \in S$. Note that $\{t \in S : T_t(x)(s) = 1\} = \{t \in S : x(st) = 1\} = s^{-1}A$. Also $s^{-1}A \in p$ if and only if $s \in \rho_p^{-1}[\overline{A} \cap \beta Sp]$ so $s \in B$ if and only if $s^{-1}A \in p$.

If $z(s) = 1$, then $s \in B$ so $s^{-1}A \in p$ so $\{t \in S : T_t(x)(s) = z(s)\} \in p$. If $z(s) = 0$, then $s \notin B$ so $s^{-1}A \notin p$ so $\{t \in S : T_t(x)(s) = z(s)\} \in p$. □

Notice that the hypotheses of the following corollary hold if S has any right cancelable element.

Corollary 5.11. *Let S be a semigroup. Let $(Y, \langle T_s\rangle_{s\in S})$ be the dynamical system of Lemma 1.2 determined by $R = S$. Let $p \in \beta S$. Assume that for whenever q and r are distinct elements of βS, there exists $s \in S$ such that $sq \neq sr$. Then $T_p : Y \to Y$ is surjective if and only if p is right cancelable in βS.*

Proof. The necessity is Theorem 5.10.

So assume that T_p is surjective and suppose that we have distinct q and r in βS such that $qp = rp$. We claim that $T_q = T_r$. To see this, let $x \in Y$ be given. Pick $z \in Y$ such that $T_p(z) = x$. Then $T_q(x) = T_q\big(T_p(z)\big) = T_{qp}(z) = T_{rp}(z) = T_r\big(T_p(z)\big) = T_r(x)$.

Pick $s \in S$ such that $sq \neq sr$, pick $A \in sq \setminus sr$, and let x be the characteristic function of A in Y. Then $A \subseteq \{t \in S : T_t(x)(s) = 1\}$ so $T_q(x)(s) = 1$ and $S \setminus A \subseteq \{t \in S : T_t(x)(s) = 0\}$ so $T_r(x)(s) = 1$. □

Theorem 5.12. *Let S be a semigroup and let $Q = S \cup \{e\}$ where e is an identity adjoined to S. Let $(X, \langle T_s\rangle_{s\in S})$ be the dynamical system of Lemma 1.2 determined by $R = Q$ and let $p \in \beta S$. Then $T_p : X \to X$ is surjective if and only if p is right cancelable in βQ.*

Proof. Sufficiency. Note that $\rho_p : \beta S \to \beta Sp$ is a homeomorphism. Note also that $p \notin \beta Sp$. (If we had $p = qp$ for some $q \in \beta S$, then we would have $ep = qp$.) Let $x \in X$ and let $B = \{s \in S : x(s) = 1\}$. By Lemma 5.9, pick $A \subseteq S$ such that $\rho_p[\overline{B}\,] = \overline{A} \cap \beta Sp$. Pick $P \in p$ such that $\overline{P} \cap \beta Sp = \emptyset$. If $x(e) = 1$, let $D = A \setminus P$. If $x(e) = 0$, let $D = A \cup B$. Let z be the characteristic function of D in X.

We claim that $T_p(z) = x$. As in the proof of Theorem 5.10, we see that for $s \in S$, $T_p(z)(s) = x(s)$. Regardless of the value of $x(e)$, we have that $P \subseteq \{s \in S : T_s(z)(e) = x(e)\}$, so $T_p(z)(e) = x(e)$.

Necessity. Suppose that T_p is surjective and we have $q \neq r$ in βQ such that $qp = rp$. Assume first that $e \in \{q, r\}$, so without loss of generality, $q = e$. Let x be the characteristic function of S in X and pick $z \in X$ such that $T_p(z) = x$. Then $0 = x(e) = T_p(z)(e) = T_{rp}(z)(e) = T_r\big(T_p(z)\big)(e) = T_r(x)(e) = 1$, a contradiction.

So we can assume that q and r are in βS. Pick $A \in q \setminus r$ and let A be the characteristic function of A in X. Pick $z \in X$ such that $T_p(z) = x$. Then $0 = T_r(x)(e) = T_{rp}(z)(e) = T_{qp}(z)(e) = T_q\big(T_p(z)\big)(e) = T_q(x)(e) = 1$, a contradiction. □

Theorem 5.13. *Let S be a countable semigroup which can be embedded in a group and assume that S can be enumerated as $\langle s_t\rangle_{t=0}^{\infty}$ so that if $u, v \in S$, $i, j \in \omega$ with $i < j$, and $s_i u = s_j v$,* *then $s_0 s_i^{-1} s_j \in S$. Let $(Y, \langle T_s \rangle_{s\in S})$ be the dynamical system of Lemma 1.2 determined by $R = S$ and let $p \in \beta S$. The T_p is surjective if and only if there exists $x \in Y$ such that $T_p(x)$ is the characteristic function of $\{s_0\}$ in Y.*

Proof. The necessity is trivial. Assume that we have $x \in Y$ such that $T_p(x)$ is the characteristic function of $\{s_0\}$ in Y. For $m \in \mathbb{N}$, let $D_m = \{s_0 s_i^{-1} s_j : i, j \in \{0, 1, \ldots, m\}\,,\ i < j$, and $s_0 s_i^{-1} s_j \in S\}$ and note that $s_0 \notin D_m$. For each $m \in \mathbb{N}$, let

$$B_m = \left\{ s \in S : T_s(x) \in \pi_{s_0}^{-1}[\{1\}] \cap \bigcap_{i=1}^{m} \pi_{s_i}^{-1}[\{0\}] \cap \bigcap_{r\in D_m} \pi_r^{-1}[\{0\}] \right\},$$

and note that $B_m \in p$. We claim that if $m, k \in \mathbb{N}$, $u \in B_m$, $v \in B_k$, $i \in \{0, 1, \ldots, m\}$, $j \in \{0, 1, \ldots, k\}$, and $s_i u = s_j v$, then $i = j$. Suppose instead we have such m, k, u, v, i, j with $i \neq j$ and assume without loss of generality that $i < j$. Then $u = s_i^{-1} s_j v$. By assumption $s_0 s_i^{-1} s_j \in S$ so $s_0 s_i^{-1} s_j \in D_k$. Since $u \in B_m$, $1 = T_u(x)(s_0) = x(s_0 u)$. Since $v \in B_k$ and $s_0 s_i^{-1} s_j \in D_k$, $0 = T_v(x)(s_0 s_i^{-1} s_j) = x(s_0 s_i^{-1} s_j v)$, a contradiction.

Now to see that T_p is surjective, let $y \in Y$ be given. Define $w \in Y$ as follows. If $m \in \mathbb{N}$, $u \in B_m$, and $i \in \{0, 1, \ldots, m\}$, then $w(s_i u) = y(s_i)$. For $s \in S$ which is not of the form $s_i u$ for some $m \in \mathbb{N}$, $u \in B_m$, and $i \in \{0, 1, \ldots, m\}$, define $w(s)$ at will. To see that $T_p(w) = y$, let U be a neighborhood of y. Pick $m \in \mathbb{N}$ such that $\bigcap_{i=0}^{m} \pi_i^{-1}[\{y(s_i)\}] \subseteq U$. Then $B_m \subseteq U$. □

The following is an immediate corollary of Theorem 5.13.

Corollary 5.14. *Let S be a countable group with identity e, let $(Y, \langle T_s \rangle_{s\in S})$ be the dynamical system of Lemma 1.2 determined by $R = S$, and let $p \in \beta S$. The following statements are equivalent.*

(a) *T_p is surjective.*
(b) *For each $s \in S$, there exists $x \in Y$ such that $T_p(x)$ is the characteristic function of $\{s\}$.*
(c) *There exists $x \in Y$ such that $T_p(x)$ is the characteristic function of $\{e\}$.*

Notice that the hypotheses of the following theorem hold if S is very weakly left cancellative and right cancellative. If κ is regular, the assumption that for any subset D of S with fewer than κ members, $|\{e \in S : (\exists s \in D)(\exists t \in D \setminus \{s\})(se = te)\}| < \kappa$ can be replaced by the assumption that for all distinct s and t in S, $|\{e \in S : se = te\}| < \kappa$.

Theorem 5.15. *Let S be a semigroup with $|S| = \kappa \geq \omega$ which is very weakly left cancellative and has the property that for any subset D of S with fewer than κ members, $|\{e \in S : (\exists s \in D)(\exists t \in D \setminus \{s\})(se = te)\}| < \kappa$. Let $(Y, \langle T_s \rangle_{s\in S})$ be the dynamical system of Lemma 1.2 determined by $R = S$. There is a dense open subset W of $U_\kappa(S)$ such that for every $p \in W$, p is right cancelable in βS and $T_p : Y \to Y$ is surjective.*

Proof. We show that for any $C \in [S]^\kappa$, there exists $B \in [C]^\kappa$ such that for every $p \in \overline{B} \cap U_\kappa(S)$, p is right cancelable in βS and $T_p : Y \to Y$ is surjective.

Enumerate S as $\langle s_\gamma \rangle_{\gamma<\kappa}$. Choose $t_0 \in C$. Let $0 < \alpha < \kappa$ and assume that we have chosen $\langle t_\delta \rangle_{\delta<\alpha}$ in C satisfying the following inductive hypotheses:

(1) If $\gamma < \delta$, then $t_\gamma \neq t_\delta$.
(2) If $\gamma < \delta$, $\mu < \beta \leq \delta$, and $\mu \neq \gamma$, then $s_\gamma t_\delta \neq s_\mu t_\beta$.

The hypotheses are satisfied for $\delta = 0$. Let $E = \{e \in S : (\exists \mu < \beta \leq \alpha)(s_\mu e = s_\beta e)\}$. For $\mu < \beta < \alpha$ and $\gamma < \alpha$ let $A_{\gamma,\mu,\beta} = \{t \in S : s_\gamma t = s_\mu t_\beta\}$. Then each $A_{\gamma,\mu,\beta}$ is a left solution set. Pick

$$t_\alpha \in C \setminus \left(\{t_\gamma : \gamma < \alpha\} \cup E \cup \bigcup_{\gamma<\alpha} \bigcup_{\beta<\alpha} \bigcup_{\mu<\beta} A_{\gamma,\mu,\beta} \right).$$

Hypothesis (1) is trivially satisfied and if $\mu < \beta < \alpha$ and $\gamma < \alpha$, then $t_\alpha \notin A_{\gamma,\mu,\beta}$ so $s_\gamma t_\alpha \neq s_\mu t_\beta$. If $\mu < \beta = \alpha$ and $\gamma < \alpha$, then $t_\alpha \notin E$ so $s_\gamma t_\alpha \neq s_\mu t_\beta$.

Let $B = \{t_\alpha : \alpha < \kappa\}$ and let $p \in \overline{B} \cap U_\kappa(S)$. To see that p is right cancelable in βS, let $q \neq r \in \beta S$ and suppose that $qp = rp$. Pick subsets C and D of S such that $C \cap D = \emptyset$ and $C \in q$ and $D \in r$. Then $H = \{s_\gamma t_\alpha : \gamma < \alpha \text{ and } s_\gamma \in C\} \in qp$. (To see this, let $s_\gamma \in C$. Then $\{t_\alpha : \gamma < \alpha < \kappa\} \subseteq s_\gamma^{-1} H$.) Similarly, $\{s_\mu t_\beta : \mu < \beta \text{ and } s_\mu \in D\} \in rp$. Since these sets are disjoint by hypothesis (2), we have a contradiction.

The fact that T_p is surjective follows from Theorem 5.10. □

Lemma 5.16. *Let S be a cancellative semigroup, let $a \in S$, and let $(Y, \langle T_s \rangle_{s\in S})$ be the dynamical system of Lemma 1.2 determined by $R = S$. If x is the characteristic function of $\{a\}$ in Y, then x is not a recurrent point.*

Proof. We claim that $|\{s \in S : T_s(x)(a) = 1\}| \leq 1$. Indeed, if $x(as) = 1$, then $as = a$ so by left cancellation, s is a left identity for S and then by right cancellation, s is a two sided identity for S. □

We have seen that $U(x)$ is always a left ideal of βS and that NS is a right ideal of βS provided it is nonempty.

Theorem 5.17. *Let S be a countable cancellative semigroup. Let $(Y, \langle T_s \rangle_{s\in S})$ be the dynamical system of Lemma 1.2 determined by $R = S$. Then NS_Y is not a left ideal of βS.*

Proof. By [4, Corollary 6.33] pick an idempotent $p \in \beta S \setminus K(\beta S)$. By Theorem 5.8 pick $r \in \beta S$ such that rp is right cancelable in βS. By Lemmas 5.16 and 5.6, $p \in NS$ and by Theorem 5.10, $rp \notin NS$. □

If S is commutative, then by [4, Exercise 4.4.9] and Lemma 5.5, if $NS \neq \emptyset$, then $c\ell NS$ is a two sided ideal of βS. The following theorem shows that this may fail if S is not commutative.

Theorem 5.18. *Let S be the free semigroup on the alphabet $\{a, b\}$ (where $a \neq b$). Let $(Y, \langle T_s \rangle_{s\in S})$ be the dynamical system of Lemma 1.2 determined by $R = S$. Then $NS \neq \emptyset$ and $c\ell NS$ is not a left ideal of βS.*

Proof. Let p be an idempotent in βS with $\{a^n : n \in \mathbb{N}\} \in p$. By Lemmas 5.16 and 5.6, $p \in NS$. We will show that bp$\notin c\ell NS$. Let $B = \{ba^n : n \in \mathbb{N}\}$. Then $B \in bp$. We shall show that $\overline{B} \cap NS = \emptyset$. So let $q \in \overline{B}$. Let $s_0 = a$ and let $\langle s_n \rangle_{n=1}^\infty$ enumerate $S \setminus \{a\}$ so that if the length of s_i is less than the length of s_j, then $i < j$. By Theorem 5.13, to see that T_q is surjective, it suffices to show that there is some $x \in Y$ such that $T_q(x)$ is the characteristic function of $\{a\}$.

Let x be the characteristic function of $\{aba^n : n \in \mathbb{N}\}$ in Y. Let U be a neighborhood of $\chi_{\{a\}}$ and pick $F \in \mathcal{P}_f(S \setminus \{a\})$ such that $\pi_a^{-1}[\{1\}] \cap \bigcap_{y\in F} \pi_y^{-1}[\{0\}] \subseteq U$. It suffices to show that $B \subseteq \{w \in S : T_w(x) \in \pi_a^{-1}[\{1\}] \cap \bigcap_{y\in F} \pi_y^{-1}[\{0\}]\}$. So let $ba^n \in B$. Then $T_{ba^n}(x)(a) = x(aba^n) = 1$ and for $y \in F$, $T_{ba^n}(x)(y) = x(yba^n) = 0$. □

We remark that Theorem 5.18 remains valid if S is the free semigroup on a countably infinite alphabet.

Acknowledgment

First author acknowledges support received from the National Science Foundation (USA) via Grant DMS-1460023.

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