Algebra in the Space of Ultrafilters and Ramsey Theory

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Abstract. We survey developments in the algebraic theory of the Stone-Čech compactification βS of a discrete semigroup S and its applications to Ramsey Theory that have occurred since the publication of [**43**].

1. Introduction

If S is a discrete topological space, we view its Stone-Čech compactification, βS, as the set of all ultrafilters on S, the points of S being identified with the principal ultrafilters. Given $A \subseteq S$, $\overline{A} = c\ell_{\beta S} A = \{p \in \beta S : A \in p\}$. The set $\{\overline{A} : A \subseteq S\}$ is a basis for the open sets of βS as well as a basis for the closed sets of βS.

If $(S, \cdot)$ is a discrete semigroup, then the operation extends uniquely to βS making $(\beta S, \cdot)$ a right topological semigroup with S contained in its topological center. To say that (βS) is right topological is to say that for each $p \in \beta S$, the function $\rho_p : \beta S \to \beta S$ defined by $\rho_p(q) = q \cdot p$ is continuous. The topological center of βS is $\{q \in \beta S : \lambda_q \text{ is continuous}\}$, where $\lambda_q : \beta S \to \beta S$ is defined by $\lambda_q(p) = q \cdot p$. The fact that this extension can be done was implicitly established by M. Day [**21**] using a multiplication on the second conjugate of a Banach algebra, in this case $l_1(S)$, first introduced by R. Arens [**5**] for arbitrary Banach algebras. P. Civin and B. Yood [**18**, Theorem 3.4] explicitly stated that if S is a discrete group, then the above operation produced an operation on the Stone-Čech compactification of S, viewed as a subspace of that second dual. R. Ellis [**26**] carried out the extension in βS viewed as a space of ultrafilters, again assuming that S is a group. He also proved the important fact that any compact Hausdorff right topological semigroup has an idempotent [**26**, Corollary 2.10].

The fact that $(\beta\mathbb{N}, +)$ has an idempotent provided the first application of the algebra of βS to Ramsey Theory, namely a very simple proof due to F. Galvin and S. Glazer of the Finite Sums Theorem. (Given a set X, $\mathcal{P}_f(X)$ is the set of finite nonempty subsets of X.)

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THEOREM 1.1 (Finite Sums Theorem). *Let* $r \in \mathbb{N}$ *and let* $\mathbb{N} = \bigcup_{i=1}^{r} C_i$. *There exist* $i \in \{1, 2, \ldots, r\}$ *and a sequence* $\langle x_n \rangle_{n=1}^{\infty}$ *in* $\mathbb{N}$ *such that for each* $F \in \mathcal{P}_f(\mathbb{N})$, $\sum_{n \in F} x_n \in C_i$.

This proof, while never published by either of the originators, is widely available. For their original proof and a newer one see [**42**, Theorem 5.8]. For a detailed discussion of the history of the discovery of this proof see [**37**].

Any compact right topological semigroup T has a smallest two sided ideal, $K(T)$, which is the union of all of the minimal left ideals of T and is also the union of all of the minimal right ideals of T. The intersection of any minimal left ideal and any minimal right ideal is a group, and any two such groups are isomorphic. If S is a discrete semigroup, then it may happen that $K(\beta S)$ has only trivial structure. For example, if S is a right zero semigroup, (i.e., $st = t$ for all s and t in S), then so is βS. In this case the minimal left ideals are precisely the singletons. But it often happens that the structure of $K(\beta S)$ is very rich. For example, in $(\beta\mathbb{N}, +)$ the groups in the smallest ideal all contain a copy of the free group on $2^{\mathfrak{c}}$ generaters.

We shall be concerned in this paper with results about the algebraic structure of βS and with continued application of that structure to the branch of combinatorics known as Ramsey Theory. We will concentrate on results obtained since the preparation of the last survey [**43**] which we wrote on this subject.

Section 2 will be concerned with results establishing that there are many objects (such as minimal left or right ideals in βS) or that certain such objects can be very large or very small.

Section 3 will deal with more general results about the algebra of βS for either arbitrary semigroups S, or particular semigroups other than $(\mathbb{N}, +)$ or $(\mathbb{N}, \cdot)$ while Section 4 will deal with results about $\beta\mathbb{N}$.

Section 5 will deal with *central* sets as well as other notions of size such as *syndetic* or *piecewise syndetic* that had their origins in topological dynamics.

Section 6 will consist of new applications of the algebra of βS to Ramsey Theory.

Of course we will necessarily not mention most of the results in the papers we are surveying. We will choose sample results which are reasonably easy to state without introducing too much notation, and the reader should decide whether she is interested in consulting the original papers.

2. Number and cardinality of substructures of βS

We remark that the following theorem was known previously for countable cancellative semigroups [**42**, Corollary 6.41]. However, this result was completely new for uncountable groups.

THEOREM 2.1 (Zelenyuk). *Let* G *be an infinite discrete abelian group and let* $\kappa = |G|$. *Then* βG *contains* 2^{2^κ} *minimal right ideals.*

PROOF. [**74**, Theorem 1]. □

THEOREM 2.2 (Zelenyuk and Zelenyuk). *Let* G *be an infinite group and let* $\kappa = |G|$. *If* G *can be embedded into a direct sum of countable groups, in particular if* G *is abelian, then every maximal group in* $K(\beta G)$ *contains a free group on* 2^{2^κ} *generators.*

PROOF. [**77**, Theorem 1.1 and Corollary 1.2]. □

The "in particular" part of Theorem 2.2 was obtained independently in [**27**].

Definition 2.3. A *digital representation* of a semigroup $(S,\cdot)$ is a family $\langle F_t\rangle_{t\in I}$, where I is a linearly ordered set, each F_t is a finite non-empty subset of S and every element of S is uniquely representable in the form $\prod_{t\in H} x_t$ where H is a finite subset of I, each $x_t \in F_t$, and products are taken in increasing order of indices.

A *strong digital representation* of S is a digital representation with the additional property that, for each $t \in I$, $F_t = \{x_t, x_t^2, \cdots, x_t^{m_t-1}\}$ for some $x_t \in S$ and some $m_t > 1$ in $\mathbb{N}$, where $m_t = 2$ if x_t has infinite order and m_t is a prime, with the order of x_t being a power of m_t, if x_t has finite order.

Theorem 2.4 (Ferri, Hindman and Strauss). *Every Abelian group has a strong digital representation.*

Proof. [**27**, Theorem 3.9]. □

Theorem 2.5 (Ferri, Hindman, and Strauss). *Let S be a left cancellative semigroup with identity of cardinality κ which has a digital representation $\langle F_t\rangle_{t\in\kappa}$ such that whenever $s < t < \kappa$, $x \in F_s$, and $y \in F_t$, one has $xy = yx$. Then every maximal group in the smallest ideal of βS contains a free group on 2^{2^κ} generators. In particular if S is an abelian group, then every maximal group in $K(\beta S)$ contains a free group on 2^{2^κ} generators.*

Proof. [**27**, Theorem 3.9 and Corollary 4.16]. □

Recall that a set theoretic (or Boolean algebra) *ideal* is the dual of a filter. That is an ideal $\mathcal{I}$ on a set X is a nonempty set of subsets of X which does not include X and is closed under finite unions and subsets. Thus $\mathcal{I}$ is an ideal on X if and only if $\{X \setminus A : A \in \mathcal{I}\}$ is a filter on X. Interestingly enough, a special kind of ideal in a discrete group G has consequences for algebraic ideals in βG.

Definition 2.6. Let G be an infinite group. A *Boolean group ideal* is an ideal $\mathcal{I}$ on G such that

(a) every finite subset of G is a member of $\mathcal{I}$;
(b) if $A \in \mathcal{I}$, then $A^{-1} \in \mathcal{I}$; and
(c) if $A, B \in \mathcal{I}$, then $AB \in \mathcal{I}$.

Recall that, given a filter $\mathcal{F}$ on a set X, $\widehat{\mathcal{F}} = \{p \in \beta X : \mathcal{F} \subseteq p\}$.

Theorem 2.7 (Protasov and Protasova). *Let G be an infinite group, let $\mathcal{I}$ be a Boolean group ideal on G, and let $\mathcal{F} = \{G \setminus A : A \in \mathcal{I}\}$. Then $\widehat{\mathcal{F}}$ is a closed two sided ideal of the semigroup βG.*

Proof. [**64**, Theorem 1]. □

Corollary 2.8. Let G be an infinite abelian group with $|G| = \kappa$. There are 2^{2^κ} distinct closed two sided ideals in βG.

Proof. If $\mathcal{F}$ and $\mathcal{G}$ are distinct filters on G, then $\widehat{\mathcal{F}} \neq \widehat{\mathcal{G}}$. (If say $A \in \mathcal{F} \setminus \mathcal{G}$, then $\mathcal{G} \cup \{G \setminus A\}$ has the finite intersection property so there is an ultrafilter p on G such that $\mathcal{G} \cup \{G \setminus A\} \subseteq p$. Then $p \in \widehat{\mathcal{G}} \setminus \widehat{\mathcal{F}}$.) By [**63**, Theorem 3] any infinite abelian group of cardinality κ has 2^{2^κ} distinct Boolean group ideals. □

The following theorem drops the commutativity requirement from Corollary 2.8, but applies only when $\kappa = \omega$.

THEOREM 2.9 (Filali, Lutsenko, and Protasov). *Let G be a countably infinite group. There are 2^{2^ω} distinct Boolean group ideals on G. Consequently, there are 2^{2^ω} distinct closed left ideals of βG.*

PROOF. Theorem 2.7 and [**30**, Theorem 1.1]. □

DEFINITION 2.10. Let S be an infinite semigroup with cardinality κ.

(a) A subset A of S is a *left solution set of S* if and only if there exist $w, z \in S$ such that $A = \{x \in S : w = zx\}$.
(b) S is *very weakly left cancellative* if the union of fewer than κ left solution sets of S must have cardinality less than κ.

Note that if κ is regular, S is very weakly left cancellative if and only if every left solution set of S has cardinality less than κ. If κ is singular, S is very weakly left cancellative if and only if there is a cardinal less than κ which is an upper bound for the cardinalities of all left solution sets of S.

THEOREM 2.11 (Carlson, Hindman, McLeod, and Strauss). *Let S be an infinite very weakly left cancellative semigroup with cardinality κ. There is a collection of 2^{2^κ} pairwise disjoint left ideals of βS. In particular, βS has 2^{2^κ} minimal idempotents.*

PROOF. [**17**, Theorem 1.7]. □

The subsemigroup $\mathbb{H}$ of $(\beta\mathbb{N}, +)$ contains much of the known algebraic structure of $\beta\mathbb{N}$. (See [**42**, Section 6.1].)

DEFINITION 2.12. $\mathbb{H} = \bigcap_{n=1}^{\infty} c\ell_{\beta\mathbb{N}}(\mathbb{N}2^n)$.

DEFINITION 2.13. Let κ be an infinite cardinal.

(a) $G_\kappa = \bigoplus_{\sigma<\kappa} \mathbb{Z}_2$.
(b) For $x \in G_\kappa \setminus \{0\}$, $\mathrm{supp}(x) = \{\sigma < \kappa : x_\sigma \neq 0\}$.
(c) $\mathbb{H}_\kappa = \bigcap_{\sigma<\kappa} c\ell_{G_\kappa}(\{x \in G_\kappa \setminus \{0\} : \min \mathrm{supp} x > \sigma\}$.

It is an easy fact that if $f : G_\omega \to \mathbb{N}$ is defined by $f(x) = \sum_{n\in\omega} 2^n$ and $\widetilde{f} : \beta G_\omega \to \beta\mathbb{N}$ is its continuous extension, then the restriction of $\widetilde{f}$ to $\mathbb{H}_\omega$ is both an isomorphism and a homeomorphism onto $\mathbb{H}$. More generally, we have the following.

THEOREM 2.14 (Hindman, Strauss, and Zelenyuk). *Let S be an infinite cancellative discrete semigroup with $|S| = \kappa$. Then S^* contains a topological and algebraic copy of $\mathbb{H}_\kappa$.*

PROOF. [**52**, Theorem 2.7]. □

We remark that the following theorem follows immediately from Zelenyuk's Theorem and from a previously known theorem due to I. Protasov, in the case in which $\kappa = \omega$ [**62**], but that it is completely new if $\kappa > \omega$.

THEOREM 2.15 (Zelenyuk). *Let κ be an infinite cardinal. Then $\mathbb{H}_\kappa$ contains no notrivial finite group.*

PROOF. [**73**, Theorem 1.2]. □

However, we see that $\mathbb{H}_\kappa$ has plenty of large semigroups.

THEOREM 2.16 (Hindman, Strauss, and Zelenyuk). *Let κ be an infinite cardinal. Let L be a left zero semigroup and let R be a right zero semigroup with $|L| = |R| = 2^{2^\kappa}$. There is an algebraic copy of $L \times R$ contained in $\mathbb{H}_\kappa$. If $\kappa = \omega$, there is an algebraic copy of $L \times R$ contained in $K(\mathbb{H})$. In particular, there is an algebraic copy of $L \times R$ contained in $K(\beta\mathbb{N})$.*

PROOF. [**52**, Corollaries 3.10 and 3.11]. $\square$

DEFINITION 2.17. Let G be a group with topology τ, let e be the identity of G, and let G_d be G with the discrete topology. Then $\mathrm{Ult}_\tau(G) = \{p \in \beta G_d : p$ converges to e with respect to $\tau\}$.

By a *left topological group* we mean a group with a topology making it a left topological semigroup. If (G, τ) is a left topological group, then $\mathrm{Ult}_\tau(G)$ is a compact subsemigroup of $G^* = \beta G_d \setminus G$ [**42**, Exercise 9.2.3].

THEOREM 2.18 (Zelenyuk). *Let $(G, +, \tau)$ be a countable abelian nondiscrete topological group and let $B = \{x \in G : x + x = 0\}$. If B is not open, then* $\mathrm{Ult}_\tau(G)$ *contains $2^\mathfrak{c}$ minimal right ideals. In particular, $\big|K\big(\mathrm{Ult}_\tau(G)\big)\big| = 2^\mathfrak{c}$.*

PROOF. [**70**, The Theorem and Corollary]. $\square$

On the other hand, it is possible for $\mathrm{Ult}_\tau(G)$ to be small. A *chain of idempotents* is a set of idempotents linearly ordered by the relation $p \leq q$ if and only if $p = pq = qp$.

THEOREM 2.19 (Zelenyuk). *Let G be a countable group and let $n \in \mathbb{N}$. There is a topology τ on G such that (G, τ) is a regular Hausdorff left topological group without isolated points and* $\mathrm{Ult}_\tau(G)$ *is an n-element chain of idempotents.*

PROOF. [**71**, Theorem 6.1]. $\square$

Recall that the *Bohr compactification* of $\mathbb{Z}$, also known as $\mathcal{SAP}(\mathbb{Z}) = \mathcal{AP}(\mathbb{Z})$, is the largest compact topological group containing an algebraic copy of $\mathbb{Z}$ as a dense subgroup. Recall also that a P-point in a topological space is a point with the property that any countable intersection of neighborhoods is again a neighborhood. While the existence of P-points in $\mathbb{N}^*$ follows from Martin's axiom, it is consistent that there are none [**67**, VI, §].

THEOREM 2.20 (Zelenyuk). *Assume that there are no P-points in $\mathbb{N}^*$ and let (G, τ) be a countable nondiscrete topological group. Then* $\mathrm{Ult}_\tau(G)$ *can be partitioned into closed right ideals, each of which admits a continuous homomorphism onto the Bohr compactification of $\mathbb{Z}$.*

PROOF. [**69**, The Theorem]. $\square$

THEOREM 2.21 (Zelenyuk). *Let (G, τ) be a coutable regular Hausdorff left topological group and assume that G contains a discrete subset with exactly one accumulation point. There exists a continuous surjective homomorphism from* $\mathrm{Ult}_\tau(G)$ *onto $\beta\mathbb{N}$.*

PROOF. [**72**, Lemma 1.1 and Corollary 2.3]. $\square$

DEFINITION 2.22. Let S be an infinite semigroup and let p be an idempotent in βS. Then $H(p)$ is the largest subgroup of βS with p as its identity.

If S is cancellative and $|S| = \kappa$, then by [**42**, Corollary 7.39] there exists an idempotent $p \in \beta S$ such that $H(p)$ contains a copy of the free group on 2^{2^κ} generators. We see here that it is consistent that maximal groups in such semigroups are as small as possible.

THEOREM 2.23 (Legette). *Let S and G be respectively the free semigroup and the free group on a countably infinite set of generators. For an idempotent $p \in \beta S$, let $H_S(p)$ and $H_G(p)$ be the maximal groups associated with p in βS and βG respectively. Assume Martin's Axiom. Then there is an idempotent $p \in \beta S$ such that $H_S(p) = H_G(p) = \{p\}$.*

PROOF. [**54**, Theorem 4.3 and Corollary 4.4]. □

The ultrafilters which Legette produces for the proof of Theorem 2.23 are essentially equivalent to *ordered union ultrafilters* introduced in [**12**], and the existence of ordered union ultrafilters is known to be independent of ZFC.

3. General algebra and topology of βS

Recall that for any ultrafilter p, $||p|| = \min\{|A| : A \in p\}$.

DEFINITION 3.1. Let S be an infinite semigroup and let $\omega \leq \kappa \leq |S|$. Then $P_\kappa(S) = \{p \in \beta S : ||p|| = \kappa\}$.

If $\kappa = |S|$, then $P_\kappa(S) = U_\kappa(S)$, the space of κ-uniform ultrafilters on S. $P_\kappa(S)$ need not be a subsemigroup of βS, but if S is weakly left cancellative it is [**29**, Proposition 2.3].

THEOREM 3.2 (Filali). *Let S be an infinite discrete semigroup and let $\omega \leq \kappa \leq |S|$. If either S is right cancellative and weakly left cancellative and $\kappa = |S|$ or S is a subset of a group, then $\{p \in P_\kappa(S) : p$ is right cancelable in $\beta S\}$ has dense interior in $P_\kappa(S)$.*

PROOF. [**29**, Theorems 3.2 and 3.3]. □

THEOREM 3.3 (Hindman and Strauss). *Let S be a discrete semigroup. The following statements are equivalent:*

(a) *βS is simple.*
(b) *S is a simple semigroup with a minimal left ideal containing an idempotent. Furthermore, the structure group of S is finite and S has only a finite number of minimal left ideals or only a finite number of minimal right ideals.*
(c) *S contains a finite group G, a left zero semigroup X and a right zero semigroup Y such that S is isomorphic to the semigroup $X \times G \times Y$ with the semigroup operation defined by $(x, g, y)(x', g', y') = (x, gyx'g', y')$ for every $x, x' \in X, g, g' \in G, y, y' \in Y$. Furthermore, either X or Y is finite.*

PROOF. [**46**, Theorem 4]. □

In [**46**, Theorems 5 and 6] several equivalent conditions to βS being right or left cancellative are obtained. As a consequence, one has the following.

THEOREM 3.4 (Hindman and Strauss). *Let S be a discrete semigroup. If $K(\beta S)$ contains an element left cancelable in βS and an element right cancelable in βS, then S is a finite group.*

PROOF. [**46**, Corollary 7]. □

DEFINITION 3.5. Let S be a semigroup.

(a) Let $f : S \to \mathbb{C}$, let $\epsilon > 0$, and let $F \in \mathcal{P}_f(S)$. Then $S(f, \epsilon, F) = \{t \in S : \mathrm{diam} f[Ft \cup \{t\}] < \epsilon\}$.

(b) Let $f : S \to \mathbb{C}$. Then $\mathrm{so}(f) = \{A \subseteq S : (\exists \epsilon > 0)\big(\exists F \in \mathcal{P}_f(S)\big)(S(f, \epsilon, F) \subseteq A)\}$.

Notice that $\mathrm{so}(f)$ is a filter. The following theorem characterizes closed left ideals of βS for certain countable semigroups. (Notice that these include any right cancellative semigroup which has a left identity.)

THEOREM 3.6 (Alaste and Filali). *Let S be a countable semigroup, assume that there exists $s \in S$ such that λ_s has no fixed point in S, and let L be a closed left ideal of βS. There exists a family $\mathcal{F}$ of bounded functions from S to $\mathbb{C}$ such that $L = \bigcap_{f \in \mathcal{F}} \{p \in \beta S : \mathrm{so}(f) \subseteq p\}$.*

PROOF. [**4**, Corollary 2.1]. □

THEOREM 3.7 (Protasov). *Let G be a countable discrete group, let $E = \{(p, q) \in G^* \times G^* : (\exists g \in G)(gp = q)\}$, let $\check{E} = \bigcap\{R : R$ is an equivalence relation on G^*, R is closed in $G^* \times G^*$, and $E \subseteq R\}$, and for $p \in G^*$, let $\check{p} = \{q \in G^* : (p, q) \in \check{E}\}$. For any $p \in G^*$, $\beta G p \subseteq \check{p}$. If p is a P-point in G^*, then $\beta G p = \check{p}$.*

PROOF. [**60**, Theorem 4.1]. □

Protasov then asked [**60**, Question 4.2] whether one can prove in ZFC that there exist a countable group G and $p \in G^*$ such that $\beta G p = \check{p}$.

Recall that ultrafilters p and q on a discrete space X are *of the same type* if and only if there exists $f : X \xrightarrow[\text{onto}]{\text{1-1}} X$ such that $\widetilde{f}(p) = q$, where $\widetilde{f} : \beta X \to \beta X$ is the continuous extension of f.

THEOREM 3.8 (Protasov). *Let G be a countable discrete group and for $p \in G^*$, let $G(p)$ be the finest topology on G with respect to which G is a left topological group and p converges to the identity of G, and let p and q be right cancelable members of G^*. Then $G(p)$ is homeomorphic to $G(q)$ if and only if p and q are of the same type.*

PROOF. [**61**, Theorem 2]. □

DEFINITION 3.9. Let G be an infinite group with identity e and let p be an idempotent in G^*. Then $\mathcal{T}_p$ is the topology on G such that $\{A \cup \{e\} : A \in p\}$ is the filter of neighborhoods of e and $(G, \mathcal{T}_p)$ is a left topological group.

DEFINITION 3.10. Let S be a semigroup and let p be an idempotent in S^*. Then p is *strongly right maximal in* S^* if and only if $\{q \in S^* : qp = p\} = \{p\}$.

The existence of strongly right maximal idempotents in $\mathbb{N}^*$ is a ZFC theorem [**42**, Theorem 9.10] due to Protasov. It has been an open question as to whether there exists a uniform strongly right maximal idempotent on an uncountable semigroup.

Similarly, it has been an open question as to whether there exists an uncountable homogeneous regular space with topology which is (1) maximal among topologies with no isolated points and (2) has uncountable dispersion character. (Recall that the *dispersion character* of a space is the minimum cardinality of a nonempty open subset.)

THEOREM 3.11 (Protasov). *Let G be an infinite group and let p be an idempotent in G^*. Then $\mathcal{T}_p$ is Hausdorff. Also, $\mathcal{T}_p$ is regular if and only if p is strongly right maximal in G^*.*

PROOF. [**42**, Theorem 9.15]. $\square$

THEOREM 3.12 (Zelenyuk). *Let S be an infinite cancellative semigroup. There is a uniform strongly right maximal idempotent in S^*. Consequently, if G is an infinite group with cardinality κ, then $(G, \mathcal{T}_p)$ is a homogeneous regular space with dispersion character κ such that $\mathcal{T}_p$ is maximal among topologies with no isolated points.*

PROOF. [**75**, Corollary 1.4]. $\square$

Given an infinite set J, let $\mathcal{I} = \mathcal{P}_f(J)$ and consider the semigroup $(\mathcal{I}, \cup)$. Grainger characterized the closure of the smallest ideal of $\mathcal{I}$.

THEOREM 3.13 (Grainger). *Let J be an infinite set and let $\mathcal{I} = \mathcal{P}_f(J)$. Then $c\ell K(\beta\mathcal{I}) = \{p \in \beta\mathcal{I} : (\forall F \in \mathcal{I})(\{G \in \mathcal{I} : F \subseteq G\} \in p)\}$.*

PROOF. [**35**, Theorem 7.5]. $\square$

Koppleberg showed that in fact the smallest ideal of $\beta\mathcal{I}$ is closed, and extended the result of Theorem 3.13 to *semilattices*.

DEFINITION 3.14. A *upper semilattice* is a commutative semigroup $(S, +)$, every member of which is an idempotent. If S is an upper semilattice, and $x, y \in S$, then $x \leq y$ if and only if $x + y = y$.

THEOREM 3.15 (Koppleberg). *Let S be an upper semilattice. Then $K(\beta S) = \{p \in \beta S : (\forall x \in S)(\{y \in S : x \leq y\} \in p)\}$.*

PROOF. [**53**, Theorem 2.4]. $\square$

She also showed that two distinct notions of size are equivalent for an upper semilattice.

THEOREM 3.16 (Koppelberg). *Let S be an upper semilattice and let $A \subseteq S$. Then A is central if and only if A is piecewise syndetic.*

PROOF. [**53**, Corollary 2.5]. $\square$

Notice that, if J is an infinite set and $\mathcal{I} = \mathcal{P}_f(J)$, then $\{p \in \beta\mathcal{I} : (\forall F \in \mathcal{I})(\{G \in \mathcal{I} : F \subseteq G\} \in p)\} = \{p \in \beta\mathcal{I} : (\forall j \in J)(\{G \in \mathcal{I} : j \in G\} \in p)\}$.

DEFINITION 3.17. Let J be an infinite set, let $\mathcal{I} = \mathcal{P}_f(J)$, and let $A \subseteq J$. Then $\beta_A(\mathcal{I}) = \{p \in \mathcal{I} : (\forall j \in A)(\{G \in \mathcal{I} : j \in G\} \in p)$ and $(\forall j \in J \setminus A)(\{G \in \mathcal{I} : j \notin G\} \in p)\}$.

Notice that by Theorem 3.15, $K(\beta\mathcal{I}) = \beta_J(\mathcal{I})$.

THEOREM 3.18 (Grainger). *Let J be an infinite set, let $\mathcal{I} = \mathcal{P}_f(J)$, and let $A \subseteq J$. Let $V = \bigcup\{\beta_B(\mathcal{I}) : B \subseteq A\}$. If A and $J \setminus A$ are infinite, then $\beta_A(\mathcal{I})$ and V are subsemigroups of $\beta(\mathcal{I})$, $c\ell K(V)$ is a proper subset of $\beta_A(\mathcal{I})$, and $K(V) = K\big(\beta_A(\mathcal{I})\big)$.*

PROOF. [**35**, Proposition 5.4 and Theorem 5.14] and [**36**, Remark 4.3 and Proposition 4.8]. □

It has been known for some years that, if S is a countable cancellative semigroup, there are elements in the closure of the set of minimal idempotents in βS which are not in S^*S^* [**42**, Theorem 8.22]. The proof in [**42**] depends very essentially on countability. Y. Zelenyuk has obtained an analogous result for a large class of uncountable semigroups.

THEOREM 3.19 (Zelenyuk). *Let S be an infinite discrete semigroup which can be embedded algebraically in a compact topological group. Then $K(\beta S)$ and $E(K(\beta S))$ are not closed.*

PROOF. [**76**, Theorem 1]. □

The topological properties of minimal left ideals and minimal right ideals in compact right topological semigroups can be very different. Every minimal left ideal is compact; but this statement is far from being true in general for minimal right ideals. E. Glasner has recently shown that the minimal right ideals of $\beta\mathbb{Z}$ are not Borel measurable [**33**].

I. Protasov and J. Pym have obtained results about the discontinuity of the mappings λ_x on the remainder spaces $G^* = G^{LUC} \setminus G$, where G denotes a locally compact non-compact group and G^{LUC} denotes its largest semigroup compactification. In the case in which G is discrete, $G^{LUC} = \beta G$. They have shown that, for every $q \in G^*$, λ_q is discontinuous at some point $p \in G^*$. If G is σ-compact, there is one element p which will serve for every q [**65**, Theorem 1].

If S is a discrete semigroup, the dual of $C(\beta S)$ is $M(\beta S)$, the Banach space of complex valued regular Borel measures defined on βS. Since $C(\beta S)$ can be identified with $l_\infty(S)$, the dual of $l_1(S)$, $M(\beta S)$ is the second dual of a Banach algebra and, as such, it is a compact right topological semigroup for the Arens product $\Box$. This is a semigroup in which βS is embedded as a compact subsemigroup. H. G. Dales, A. T.-M. Lau and D. Strauss have shown that, for a class of semigroups S significantly larger than the class of cancellative semigroups, there are two points $p, q \in S^*$ such that, for every non-zero $\mu \in M(\beta S)$, λ_μ cannot be continuous at both p and q [**19**]. This result was obtained independently by M. Neufang for cancellative semigroups. Note that this claim cannot be made in general for fewer than two points, since, if S is commutative, λ_μ is continuous at μ for every $\mu \in M(\beta S)$.

There is another interesting compact right topological semigroup in which βS can be embedded as a compact subsemigroup. If X is a topological space, an inclusion hyperspace of X is a family $\mathcal{F}$ of closed non-empty subsets of X which is closed in the Vietoris topology and has the property that, for any two non-empty closed subsets A and B of X, $A \in \mathcal{F}$ and $A \subseteq B$, implies that $B \in \mathcal{F}$. $G(X)$ denotes the space of all inclusion hyperspaces of X endowed with the topology defined by choosing the sets of the form U^+ and U^- as a subbase, where U is an open subset of X, $U^+ = \{\mathcal{A} \in G(X) : \exists B \in \mathcal{A} \text{ with } B \subseteq U\}$ and $U^- = \{\mathcal{A} \in G(X) : \forall B \in \mathcal{A}, B \cap U \neq \emptyset\}$. V. Gavrilkiv has shown that, if S is a discrete semigroup, the semigroup operation of S can be extended to $G(S)$ so that $G(S)$ becomes a compact right topological semigroup [**32**]. $G(S)$ then contains βS, as well as many other interesting spaces, as compact subsemigroups.

4. Algebra of $\beta\mathbb{N}$

One of the most challenging open questions about the algebra of $(\beta\mathbb{N},+)$ is whether there are any nontrivial continuous homomorphisms from $\beta\mathbb{N}$ into $\mathbb{N}^* = \beta\mathbb{N}\setminus\mathbb{N}$. It is known that if ϕ is such a homomorphism, then $\phi[\beta\mathbb{N}]$ is finite, and $|\phi[\mathbb{N}^*]| = 1$ [**42**, Theorem 18.10]. There are also two simple equivalent conditions to the existence of such a homomorphism.

THEOREM 4.1. *The following statements are equivalent.*

(a) *There is a nontrivial continuous homomorphism from $\beta\mathbb{N}$ into $\mathbb{N}^*$.*
(b) *There exist $p \neq q$ in $\mathbb{N}^*$ such that $p+p = p+q = q+p = q+q = q$.*
(c) *There is a finite subsemigroup of $\mathbb{N}^*$ whose elements are not all idempotents.*

PROOF. [**42**, Corollary 10.20]. □

DEFINITION 4.2. For $k \in \mathbb{N}$, define $\phi_k : \mathbb{N} \to \mathbb{N}$ by $\phi_k(n) = k \cdot n$, let $\widetilde{\phi_k} : \beta\mathbb{N} \to \beta\mathbb{N}$ be its continuous extension, and let ϕ_k^* be the restriction of $\widetilde{\phi_k}$ to $\mathbb{N}^*$.

Each ϕ_k^* is a continuous homomorphism from $\mathbb{N}^*$ to $\mathbb{N}^*$. It is not known whether these are the only nontrivial continuous homomorphisms from $\mathbb{N}^*$ to $\mathbb{N}^*$.

THEOREM 4.3 (Adams and Strauss). *Let $\phi : \mathbb{N}^* \to \mathbb{N}^*$ be a continuous homomorphism which is not equal to ϕ_k^* for any $k \in \mathbb{N}$ and let $C = \phi[\mathbb{N}^*]$. Then*

(1) $|K(C)| = 1$;
(2) *C has a unique idempotent q and $C + C = \{q\}$; and*
(3) $|\phi^2[\mathbb{N}^*]| = 1$.

PROOF. [**3**, Theorems 3.10 and 3.11 and Corollary 3.12] □

The list from Theorem 4.1 can be extended.

THEOREM 4.4 (Adams and Strauss). *The following statements are equivalent.*

(a) *There is a nontrivial continuous homomorphism from $\mathbb{N}^*$ to $\mathbb{N}^*$ which is not equal to ϕ_k^* for any $k \in \mathbb{N}$.*
(b) *There is a nontrivial continuous homomorphism from $\beta\mathbb{N}$ into $\mathbb{N}^*$.*
(c) *There exist $p \neq q$ in $\mathbb{N}^*$ such that $p+p = p+q = q+p = q+q = q$.*
(d) *There is a finite subsemigroup of $\mathbb{N}^*$ whose elements are not all idempotents.*

PROOF. [**3**, Theorem 3.14]. □

This investigation was extended in [**28**] to consideration of continuous homomorphisms from countable semigroups to countable groups.

THEOREM 4.5 (Ferri and Strauss). *Let S be a countably infinite commutative semigroup and let T be a countably infinite group. If $\varphi : \beta S \to T^*$ is a continuous homomorphism, then $K(\varphi[\beta S])$ is a finite group.*

PROOF. [**28**, Corollary 1.13]. □

Given any idempotent $p = p + p$ in $\beta\mathbb{N}$, $\{n + p : n \in \mathbb{Z}\}$ is an algebraic copy of $\mathbb{Z}$. However it cannot be discrete – indeed it has no isolated points.

THEOREM 4.6 (Hindman and Strauss). *Let H be a maximal group in the smallest ideal of $\beta\mathbb{N}$. There are $2^{\mathfrak{c}}$ discrete copies of $\mathbb{Z}$ contained in H, any two of which intersect only at the identity. There are $2^{\mathfrak{c}}$ discrete copies of the free group on 2 generators contained in H, any two of which intersect only at the identity.*

PROOF. [**45**, Corollaries 2.2 and 3.4]. □

The identity function $\iota : \mathbb{N}\times\mathbb{N} \to \mathbb{N}\times\mathbb{N} \subseteq \beta\mathbb{N}\times\beta\mathbb{N}$ has a continuous extension $\widetilde{\iota} : \beta(\mathbb{N}\times\mathbb{N}) \to \beta\mathbb{N}\times\beta\mathbb{N}$, and $\widetilde{\iota}\big[K\big(\beta(\mathbb{N}\times\mathbb{N})\big)\big] = K(\beta\mathbb{N})\times K(\beta\mathbb{N}) = K(\beta\mathbb{N}\times\beta\mathbb{N})$. It has been known since the early 1970's that there are points $(p,q) \in \beta\mathbb{N}\times\beta\mathbb{N}$ such that $|\widetilde{\iota}^{-1}[\{(p,q)\}]| = 2^{\mathfrak{c}}$ and that it follows from the Continuum Hypothesis that there are points $(p,q) \in \beta\mathbb{N}\times\beta\mathbb{N}$ such that $|\widetilde{\iota}^{-1}[\{(p,q)\}]| = 2$.

THEOREM 4.7 (Moche). *Let $(p,q) \in K(\beta\mathbb{N})\times K(\beta\mathbb{N})$. Then $\{r \in K\big(\beta(\mathbb{N}\times\mathbb{N})\big) : \widetilde{\iota}(r) = (p,q)\}$ is infinite.*

PROOF. [**56**, Corollary 3.10]. □

It has been known for some time that the smallest ideals of $(\beta\mathbb{N},+)$ and $(\beta\mathbb{N},\cdot)$ are disjoint while $K(\beta\mathbb{N},\cdot) \cap c\ell K(\beta\mathbb{N},+) \neq \emptyset$. (See [**42**, Corollaries 13.15 and 16.25].)

THEOREM 4.8 (Strauss). *$c\ell K(\beta\mathbb{N},\cdot)\cap(\mathbb{N}^*+\mathbb{N}^*) = \emptyset$. In particular, $c\ell K(\beta\mathbb{N},\cdot)\cap K(\beta\mathbb{N},+) = \emptyset$.*

PROOF. [**68**, Corollary 2.3]. □

DEFINITION 4.9. For $r \in \mathbb{N}$,

$$\Sigma_r = \{p \in \beta\mathbb{N} : (\forall A \in p)(\exists \langle x_t\rangle_{t=1}^r)(FS(\langle x_t\rangle_{t=1}^r) \subseteq A)\}\,.$$

Also $\Sigma = \bigcap_{r=1}^\infty \Sigma_r$.

It is easy to see that Σ is a compact subsemigroup of $(\beta\mathbb{N},+)$ and an ideal of $(\beta\mathbb{N},\cdot)$ and all of the idempotents of $(\beta\mathbb{N},+)$ are in Σ. Bergelson suggested in a personal communication that perhaps Σ is the smallest such object.

THEOREM 4.10 (Hindman and Strauss). *There is a compact subsemigroup of $(\beta\mathbb{N},+)$ which contains the idempotents of $(\beta\mathbb{N},+)$, is a two sided ideal of $(\beta\mathbb{N},\cdot)$, and is properly contained in Σ.*

PROOF. [**47**, Theorem 3.1]. □

THEOREM 4.11 (Maleki). *Let $p \in \mathbb{N}^*$.*

(1) *If $a,b,c,d \in \mathbb{N}$ and $a\cdot p + b\cdot p = c\cdot p + d\cdot p$, then $a = c$ and $b = d$.*
(2) *If $n,m,a_1,a_2,\ldots,a_n,b_1,b_2,\ldots,b_m \in \mathbb{N}$, $a_1\cdot p+\ldots a_n\cdot p = b_1\cdot p+\ldots b_m\cdot p$ and p is right cancelable in $\beta\mathbb{N}$, then $m = n$ and for each $i \in \{1,2,\ldots,n\}$, $a_i = b_i$.*

PROOF. [**55**, Theorems E and H]. □

Given $f : \mathbb{N} \to \mathbb{N}$, let $\widetilde{f} : \beta\mathbb{N} \to \beta\mathbb{N}$ be its continuous extension.

DEFINITION 4.12. An *ultrafilter semiring* is a triple $(A,\oplus,\odot)$ such that

(1) $\emptyset \neq A \subseteq \beta\mathbb{N}$;
(2) $(\forall\mathcal{U} \in A)(\forall f : \mathbb{N} \to \mathbb{N})(\widetilde{f}(\mathcal{U}) \in A)$;
(3) $\oplus$ is associative and commutative;

(4) $\odot$ is associative and distributive with respect to $\oplus$;
(5) $(\forall \mathcal{U}, \mathcal{V} \in A)(\exists! \mathcal{W} \in A)(\mathcal{U} \oplus \mathcal{W} = \mathcal{V}$ or $\mathcal{V} \oplus \mathcal{W} = \mathcal{U})$;
(6) $(\forall \mathcal{U}, \mathcal{V}, \mathcal{W} \in A)(\mathcal{U} \oplus \mathcal{W} = \mathcal{V}$ and $\mathcal{V} \oplus \mathcal{W} = \mathcal{U} \Rightarrow \mathcal{U} = \mathcal{V})$; and
(7) $(\forall \mathcal{U} \in A)(\forall f, g : \mathbb{N} \to \mathbb{N})(\widetilde{f}(\mathcal{U}) \oplus \widetilde{g}(\mathcal{U}) = \widetilde{f+g}(\mathcal{U})$ and $\widetilde{f}(\mathcal{U}) \odot \widetilde{g}(\mathcal{U}) = \widetilde{f \cdot g}(\mathcal{U}))$.

Definition 4.13. A *Hausdorff* ultrafilter is an ultrafilter $\mathcal{U}$ on $\mathbb{N}$ such that $(\forall f, g \colon \mathbb{N} \to \mathbb{N})(\widetilde{f}(\mathcal{U}) = \widetilde{g}(\mathcal{U}) \Rightarrow \{x \in \mathbb{N} : f(x) = g(x)\} \in \mathcal{U})$.

Theorem 4.14 (DiNasso and Forti). *Let $\emptyset \neq A \subseteq \beta\mathbb{N}$. Then A is an ultrafilter semiring if and only if*

(a) $(\forall \mathcal{U} \in A)(\forall f \colon \mathbb{N} \to \mathbb{N})(\widetilde{f}(\mathcal{U}) \in A)$;
(b) $(\forall \mathcal{U}, \mathcal{V} \in A)(\exists f, g : \mathbb{N} \to \mathbb{N})(\exists \mathcal{W} \in A)(\widetilde{f}(\mathcal{W}) = \mathcal{U}$and $g^*(\mathcal{W}) = \mathcal{V})$; *and*
(c) $(\forall \mathcal{U} \in A)(\mathcal{U}$ *is Hausdorff*$)$.

Proof. [**25**, Theorem 1.6]. □

Since $(\mathbb{N}, \cdot)$ is a semigroup, $(\beta\mathbb{N}, \cdot)$ is a compact right topological semigroup. Further, the fact that $(\mathbb{N}, +, \cdot)$ is distributive leads to some strong combinatorial applications of the algebra of $(\beta\mathbb{N}, +)$ and $(\beta\mathbb{N}, \cdot)$. (See [**42**, Sections 5.3 and 17.1].) However, there is very limited interaction in $\mathbb{N}^*$. For example, by [**42**, Corollary 13.27], $\{p \in \mathbb{N}^* : (\forall q, r \in \mathbb{N}^*)(q \cdot p + r \cdot p \neq (q + r) \cdot p$ and $r \cdot (q + p) \neq r \cdot q + r \cdot p)\}$ has dense interior in $\mathbb{N}^*$. There is another way to define a multiplication on $\beta\mathbb{N}$ which is in some respects better behaved. On the other hand, that multiplication is not associative.

Theorem 4.15 (Hindman, Pym, and Strauss). *For $q \in \beta\mathbb{N}$, define $r_q : \mathbb{N} \to \beta\mathbb{N}$ by $r_q(n) = q + q + \ldots + q$, n times and let $\widetilde{r_q} : \beta\mathbb{N} \to \beta\mathbb{N}$ be its continuous extension. Define for $p, q \in \beta\mathbb{N}$, $p * q = \widetilde{r_q}(p)$. Then*

(1) *1 is a two sided identity for $(\beta\mathbb{N}, *)$;*
(2) *for $p, q \in \beta\mathbb{N}$ and $x \in \mathbb{N}$, $(p + q) * x = p * x + q * x$;*
(3) *for $p \in \beta\mathbb{N}$ and $x, y \in \mathbb{N}$, $p * (x + y) = p * x + p * y$;*
(4) *the topological and algebraic centers of $(\beta\mathbb{N}, *)$ are both equal to $\{1\}$;*
(5) *$*$ is not associative on $\beta\mathbb{N}$; and*
(6) *neither the left nor right distributive laws hold in $(\beta\mathbb{N}, +, *)$.*

Proof. [**41**, Theorems 2.1, 6.1, 6.2, and 6.4]. □

5. Central sets and other large subsets of S

In [**31**] Furstenberg defined a *central* subset of the set $\mathbb{N}$ of positive integers in terms of some notions from topological dynamics. He showed that if $\mathbb{N}$ is partitioned into finitely many classes, one of these classes contains a central set. Then he proved the following theorem.

Theorem 5.1 (Furstenberg). *Let $l \in \mathbb{N}$ and for each $i \in \{1, 2, \ldots, l\}$, let $f_i : \mathbb{N} \to \mathbb{Z}$. Let C be a central subset of $\mathbb{N}$. Then there exist sequences $\langle a_n \rangle_{n=1}^\infty$ in $\mathbb{N}$ and $\langle H_n \rangle_{n=1}^\infty$ in $\mathcal{P}_f(\mathbb{N})$ such that*

(1) *for all n, $\max H_n < \min H_{n+1}$ and*
(2) *for all $F \in \mathcal{P}_f(\mathbb{N})$ and all $i \in \{1, 2, \ldots, l\}$, $\sum_{n \in F} \big(a_n + \sum_{t \in H_n} f_i(t)\big) \in C$.*

Proof. [**31**, Proposition 8.21]. □

This theorem was strong enough to show that central subsets of $\mathbb{N}$ have remarkably strong combinatorial properties such as containing solutions to any partition regular system of homogeneous linear equations.

Subsequently Vitaly Bergelson had the idea that one might be able to derive the conclusion of the Central Sets Theorem for a set $C \subseteq \mathbb{N}$ which is a member of an idempotent in $K(\beta\mathbb{N})$. This suggested the following definition of central, which makes sense in any semigroup. It turned out that in fact this definition agrees with Furstenberg's original definition.

DEFINITION 5.2. Let S be a discrete semigroup and let C be a subset of S.

(a) The set C is *central* if and only there exists an idempotent $p \in K(\beta S) \cap \overline{C}$.
(b) The set C is *quasi-central* if and only there exists an idempotent $p \in c\ell K(\beta S) \cap \overline{C}$.

In [**13**, Theorem 3.4] a characterization of quasi-central sets in terms of dynamical notions similar to Furstenberg's original definition of central sets was obtained.

Theorem 5.1 dealt with finitely many sequences at a time and sums from one of those sequences at a time. In [**42**, Theorems 14.11 and 14.15] versions for commutative and noncommutative semigroups respectively were proved that dealt with countably many sequences at a time and allowed the sequence whose sums were taken to change as n changed. The following is the currently strongest version of the Central Sets Theorem for commutative semigroups. (For the general version for arbitrary semigroups see [**23**, Corollary 3.10].)

THEOREM 5.3 (De, Hindman, and Strauss). *Let $(S,+)$ be a commutative semigroup and let $\mathcal{T} = {}^{\mathbb{N}}S$, the set of sequences in S. Let C be a central subset of S. There exist functions $\alpha : \mathcal{P}_f(\mathcal{T}) \to S$ and $H : \mathcal{P}_f(\mathcal{T}) \to \mathcal{P}_f(\mathbb{N})$ such that*

(1) *if $F, G \in \mathcal{P}_f(\mathcal{T})$ and $F \subsetneq G$, then $\max H(F) < \min H(G)$ and*
(2) *whenever $m \in \mathbb{N}$, $G_1, G_2, \ldots, G_m \in \mathcal{P}_f(\mathcal{T})$, $G_1 \subsetneq G_2 \subsetneq \ldots \subsetneq G_m$, and for each $i \in \{1, 2, \ldots, m\}$, $\langle y_{i,n}\rangle_{n=1}^{\infty} \in G_i$, one has $\sum_{i=1}^{m}(\alpha(G_i) + \sum_{t\in H(G_i)} y_{i,t}) \in C$.*

PROOF. [**23**, Theorem 2.2]. □

The following version is superficially stronger than Theorem 5.3, replacing $\mathbb{N}$ by an arbitrary directed set with no largest element. However, as is shown in [**24**, Theorem 3.6], any set C satisfying the conclusion of Theorem 5.3 also satisfies the conclusion of Theorem 5.4.

THEOREM 5.4 (De, Hindman, and Strauss). *Let $(S,+)$ be a commutative semigroup, let $(D, \leq)$ be a directed set with no largest element, and let $\mathcal{T} = {}^{D}S$. Let C be a central subset of S. There exist functions $\alpha : \mathcal{P}_f(\mathcal{T}) \to S$ and $H : \mathcal{P}_f(\mathcal{T}) \to \{K \in \mathcal{P}_f(D) : K$ is linearly ordered$\}$ such that*

(1) *if $F, G \in \mathcal{P}_f(\mathcal{T})$ and $F \subsetneq G$, then $\max H(F) < \min H(G)$ and*
(2) *whenever $m \in \mathbb{N}$, $G_1, G_2, \ldots, G_m \in \mathcal{P}_f(\mathcal{T})$, $G_1 \subsetneq G_2 \subsetneq \ldots \subsetneq G_m$, and for each $i \in \{1, 2, \ldots, m\}$, $\langle y_{i,n}\rangle_{n=1}^{\infty} \in G_i$, one has $\sum_{i=1}^{m}(\alpha(G_i) + \sum_{t\in H(G_i)} y_{i,t}) \in C$.*

PROOF. [**24**, Corollary 3.5]. □

From the point of view of combinatorial applications, what one cares most about central sets is that they satisfy the conclusion of Theorem 5.3. We call such sets C-sets. We give now the definition of such sets for commutative semigroups. We observe that J-sets and C-sets are defined in arbitrary semigroups. We only state the definitions for commutative semigroups here because the general definitions are significantly more complicated. They can be found in [**48**].

DEFINITION 5.5. Let $(S,+)$ be a commutative semigroup and let $\mathcal{T} = {}^{\mathbb{N}}S$.

(a) Let $A \subseteq S$. Then A is a C-set if and only if there exist functions $\alpha : \mathcal{P}_f(\mathcal{T}) \to S$ and $H : \mathcal{P}_f(\mathcal{T}) \to \mathcal{P}_f(\mathbb{N})$ such that
  (1) if $F, G \in \mathcal{P}_f(\mathcal{T})$ and $F \subsetneq G$, then $\max H(F) < \min H(G)$ and
  (2) whenever $m \in \mathbb{N}$, $G_1, G_2, \ldots, G_m \in \mathcal{P}_f(\mathcal{T})$, $G_1 \subsetneq G_2 \subsetneq \ldots \subsetneq G_m$, and for each $i \in \{1, 2, \ldots, m\}$, $\langle y_{i,n} \rangle_{n=1}^{\infty} \in G_i$, one has $\sum_{i=1}^{m} (\alpha(G_i) + \sum_{t \in H(G_i)} y_{i,t}) \in A$.

(b) Let $A \subseteq S$. Then A is a J-set if and only if whenever $F \in \mathcal{P}_f(\mathcal{T})$ there exist $a \in S$ and $H \in \mathcal{P}_f(S)$ such that for each $f \in F$, $a + \sum_{t \in H} f(t) \in A$.

(c) $J(S) = \{p \in \beta S : (\forall A \in p)(A \text{ is a } J\text{-set})\}$.

The following four theorems are in fact valid for arbitrary semigroups (with the more general definitions of C-sets and J-sets).

THEOREM 5.6 (De, Hindman and Strauss). *Let S be a commutative semigroup. Then $J(S)$ is an ideal of βS.*

PROOF. [**23**, Theorem 3.5]. □

THEOREM 5.7 (Hindman and Strauss). *Let $(S,+)$ be a commutative semigroup and let $A \subseteq S$. Then A is a C-set if and only if there is an idempotent in $J(S) \cap \overline{A}$.*

PROOF. [**50**, Theorem 1.13]. □

THEOREM 5.8 (Hindman and Strauss). *Let S be a commutative semigroup and let $A \subseteq S$ be a J-set in S. If $A = A_1 \cup A_2$ then A_1 is a J-set in S or A_2 is a J-set in S.*

PROOF. [**48**, Theorem 2.14]. □

THEOREM 5.9 (Hindman and Strauss). *Let S and T be commutative semigroups, let $A \subseteq S$ and let $B \subseteq T$. If A and B are central, so is $A \times B$. If A and B are C-sets, so is $A \times B$. If A and B are J-sets, so is $A \times B$.*

PROOF. [**48**, Corollary 2.2 and Theorems 2.11 and 2.16]. □

In [**48**] a characterization of when the arbitrary product of central sets is central was also obtained.

While the following notion of density was introduced by Polya in [**58**], it is commonly called “Banach density”.

DEFINITION 5.10. Let $A \subseteq \mathbb{N}$. Then

$$\begin{aligned} d^*(A) &= \sup\{\alpha \in [0,1] : \ (\forall k \in \mathbb{N})(\exists n \geq k)(\exists a \in \mathbb{N}) \\ &\qquad (|A \cap \{a, a+1, \ldots, a+n-1\}| \geq \alpha \cdot n)\} \text{ and} \\ \Delta^* &= \{p \in \beta\mathbb{N} : \ (\forall A \subseteq \mathbb{N})(p \in c\ell(A) \Rightarrow d^*(A) > 0)\}\,. \end{aligned}$$

Since Δ^* is a two sided ideal of $(\beta\mathbb{N}, +)$, one has that if C is a central subset of $\mathbb{N}$, then $d^*(C) > 0$. And the following result establishes that a set need not be central in order to satisfy the conclusion of the original Central Sets Theorem. (It is a consequence of Theorems 5.18 and 5.19, that there are idempotents in $\Delta^* \setminus c\ell\big(K(\beta\mathbb{N})\big)$.)

THEOREM 5.11 (Beiglböck, Bergelson, Downarowicz, and Fish). *Let $C \subseteq \mathbb{N}$ and assume that there is an idempotent in $\Delta^* \cap c\ell(C)$. Let $l \in \mathbb{N}$ and for each $i \in \{1, 2, \ldots, l\}$, let f_i be a sequence in $\mathbb{Z}$. Then there exist sequences $\langle a_n \rangle_{n=1}^{\infty}$ in $\mathbb{N}$ and $\langle H_n \rangle_{n=1}^{\infty}$ in $\mathcal{P}_f(\mathbb{N})$ such that*

(1) *for all n, $\max H_n < \min H_{n+1}$ and*
(2) *for all $F \in \mathcal{P}_f(\mathbb{N})$ and all $i \in \{1, 2, \ldots, l\}$, $\sum_{n \in F} \big(a_n + \sum_{t \in H_n} f_i(t)\big) \in C$.*

PROOF. [**8**, Theorem 11]. □

The question then naturally arose as to whether any subset C of $\mathbb{N}$ which satisfies the conclusion of Theorem 5.11 must satisfy $d^*(C) > 0$. This question was answered in the negative in [**39**, Theorem 2.1], where it was shown that there is a C-set $C \subseteq \mathbb{N}$ with $d^*(C) = 0$.

Banach density extends naturally to any semigroup satisfying the *Strong Følner condition*.

DEFINITION 5.12. Let $(S, \cdot)$ be a semigroup.

(a) S satisfies the *Strong Følner Condition* (SFC) if and only if $\big(\forall H \in \mathcal{P}_f(S)\big)\big(\forall \epsilon > 0\big)\big(\exists K \in \mathcal{P}_f(S)\big)\big(\forall s \in H\big)\big(|K \bigtriangleup sK| < \epsilon \cdot |K|\big)$.
(b) For $A \subseteq S$, $d(A) = \sup\{\alpha \in [0,1] : \big(\forall H \in \mathcal{P}_f(S)\big)(\forall \epsilon > 0)\big(\exists K \in \mathcal{P}_f(S)\big) \big((\forall s \in H)(|K \setminus sK| < \epsilon \cdot |K|) \text{ and } |A \cap K| \geq \alpha \cdot |K|\big)\}$.
(c) $\Delta^*(S) = \{p \in \beta S : (\forall A \subseteq S)(p \in c\ell(A) \Rightarrow d(A) > 0)\}$.

Sets satisfying SFC include all commutative semigroups [**6**, Theorem 4]. By [**50**, Theorem 1.9] for $A \subseteq \mathbb{N}$, the densities as defined in Definitions 5.10 and 5.12 are the same, and so $\Delta^*(\mathbb{N}) = \Delta^*$.

THEOREM 5.13 (Hindman and Strauss). *Let S and T be left cancellative semigroups satisfying SFC, let $A \subseteq S$, and let $B \subseteq T$. Then $S \times T$ satisfies SFC and $d(A \times B) = d(A) \cdot d(B)$.*

PROOF. [**51**, Lemma 3.1 and Theorems 2.12 and 3.4]. □

A stronger version of Theorem 5.11 holds.

THEOREM 5.14 (Hindman and Strauss). *Let $(S, +)$ be a commutative semigroup and let $A \subseteq S$. If A is central, then $d(A) > 0$. If $d(A) > 0$, then A is a J-set in S. If there is an idempotent in $\Delta^*(S) \cap \overline{A}$, then A is a C-set in S.*

PROOF. [**50**, Theorem 5.12]. □

In view of Theorem 5.14 and the fact already mentioned that there are C-sets contained in $\mathbb{N}$ with density zero, one asks how common are semigroups that contain C-sets with density zero.

THEOREM 5.15 (Hindman and Strauss). *If S is a subsemigroup of $(\mathbb{R}, +)$ such that $\mathbb{Z} \subseteq S$ or S is the direct sum of countably many finite abelian groups, then there is a C-set $C \subseteq S$ such that $d(C) = 0$.*

PROOF. [**50**, Theorems 3.5 and 4.3]. □

Elementary characterizations of central sets have been known for some time. (See [**42**, Section 14.5].) These characterizations have, however, limited utility because they all involve showing that some collection of sets is *collectionwise piecewise* *syndetic*, a notion with a very complicated definition. The corresponding characterization of C-sets, only recently discovered, involves the much simpler notion of a J-set.

THEOREM 5.16 (Hindman and Strauss). *Let S be an infinite semigroup and let $A \subseteq S$. Statements (a) and (b) are equivalent and are implied by statement (c). If S is countable, then all three statements are equivalent.*

(a) *A is a C-set.*
(b) *There is a downward directed family $\langle C_F \rangle_{F\in I}$ of subsets of A such that*
  (i) *for all $F \in I$ and all $x \in C_F$, there exists $G \in I$ such that $C_G \subseteq x^{-1}C_F$ and*
  (ii) *for each $F \in I$, C_F is a J-set.*
(c) *There is a decreasing sequence $\langle C_n \rangle_{n=1}^{\infty}$ of subsets of A such that*
  (i) *for all $n \in \mathbb{N}$ and all $x \in C_n$, there exists $m \in \mathbb{N}$ such that $C_m \subseteq x^{-1}C_n$ and*
  (ii) *for all $n \in \mathbb{N}$, C_n is a J-set.*

PROOF. [**49**, Theorem 2.6]. $\square$

The notions of *syndetic* and *piecewise syndetic* have reasonably simple combinatorial characterizations. (See [**42**, Definition 4.38].) We take the following even simpler algebraic characterizations as the definitions here.

DEFINITION 5.17. Let S be a semigroup and let $A \subseteq S$.

(a) A is *syndetic* if and only if for every left ideal L of βS, $\overline{A} \cap L \neq \emptyset$.
(b) A is *piecewise syndetic* if and only if $\overline{A} \cap K(\beta S) \neq \emptyset$.

If $\langle x_n \rangle_{n=1}^{\infty}$ is a sequence in $\mathbb{N}$ such that for each $n \in \mathbb{N}$, $x_{n+1} > \sum_{t=1}^{n} x_t$, then $\bigcap_{m=1}^{\infty} c\ell_{\beta S} FS(\langle x_n \rangle_{n=m}^{\infty})$ contains much of the known algebraic structure of $K(\beta\mathbb{N})$. The following result says that several notions of size are equivalent for such nicely behaved sequences in $\mathbb{N}$.

THEOREM 5.18 (Adams, Hindman, and Strauss). *Let $\langle x_n \rangle_{n=1}^{\infty}$ be a sequence in $\mathbb{N}$ such that for each $n \in \mathbb{N}$, $x_{n+1} > \sum_{t=1}^{n} x_t$. The following statements are equivalent:*

(a) *For all $m \in \mathbb{N}$, $FS(\langle x_n \rangle_{n=m}^{\infty})$ is central.*
(b) *$FS(\langle x_n \rangle_{n=1}^{\infty})$ is central.*
(c) *For all $m \in \mathbb{N}$, $FS(\langle x_n \rangle_{n=m}^{\infty})$ is piecewise syndetic.*
(d) *$FS(\langle x_n \rangle_{n=1}^{\infty})$ is piecewise syndetic.*
(e) *$\{x_{n+1} - \sum_{t=1}^{n} x_t : n \in \mathbb{N}\}$ is bounded.*
(f) *$FS(\langle x_n \rangle_{n=1}^{\infty})$ is syndetic.*
(g) *For all $m \in \mathbb{N}$, $FS(\langle x_n \rangle_{n=m}^{\infty})$ is syndetic.*
(h) *$\bigcap_{m=1}^{\infty} c\ell_{\beta S} FS(\langle x_n \rangle_{n=m}^{\infty}) \cap K(\beta\mathbb{N}) \neq \emptyset$.*

PROOF. [**2**, Theorem 2.8 and Corollary 4.2] $\square$

As a consequence of the following theorem, one has much of the algebraic structure of $K(\beta\mathbb{N})$, specifically all of the structure of $K(\mathbb{H})$, close to, but disjoint from, $K(\beta\mathbb{N})$.

THEOREM 5.19 (Adams). *Let $\epsilon > 0$. There exists a sequence $\langle x_n \rangle_{n=1}^{\infty}$ in $\mathbb{N}$ such that for each $n \in \mathbb{N}$, $x_{n+1} > \sum_{t=1}^{n} x_t$, $\{x_{n+1} - \sum_{t=1}^{n} x_t : n \in \mathbb{N}\}$ is unbounded, and the density $d\big(FS(\langle x_n \rangle_{n=1}^{\infty})\big) > 1 - \epsilon$.*

PROOF. [**1**, Theorem 2.15] □

Recall that a set $\mathcal{A}$ of subsets of a set X is said to be a *set of almost disjoint subsets of* X if and only if for each $A \in \mathcal{A}$, $|A| = |X|$ and if A and B are distinct members of $\mathcal{A}$, then $|A \cap B| < |X|$. As is well known, there is a set $\mathcal{A}$ of $\mathfrak{c}$ almost disjoint subsets of $\mathbb{N}$. If $|S| = \kappa > \omega$, there may not exist *any* set of 2^κ almost disjoint subsets of S. (Baumgartner proved [**7**, Theorem 2.8] that there is always a family of κ^+ almost disjoint subsets of S, and also showed that it is consistent with ZFC that if $\kappa = \omega_1$, there is no family of 2^κ almost disjoint subsets of S.)

THEOREM 5.20 (Carlson, Hindman, McLeod, and Strauss). *Let S be an infinite very weakly left cancellative semigroup with cardinality κ. Assume that κ has a set of μ almost disjont sets. Then every central set in S has a set of μ almost disjoint central subsets. Furthermore, every central set in S contains κ pairwise disjoint central subsets.*

PROOF. [**17**, Theorem 3.4]. □

6. Combinatorial applications

DEFINITION 6.1. Let $u, v \in \mathbb{N}$ and let A be a $u \times v$ matrix with entries from $\mathbb{Q}$. Let S be a subsemigroup of $(\mathbb{R}, +)$.

(a) A is *kernel partition regular over* S (KPR/S) if and only if, whenever $S \setminus \{0\}$ is finitely colored, there must exist monochromatic $\vec{x} \in S^v$ such that $A\vec{x} = \overline{0}$.

(b) A is *image partition regular over* S (IPR/S) if and only if, whenever $S \setminus \{0\}$ is finitely colored, there must exist $\vec{x} \in S^v$ such that the entries of $A\vec{x}$ are monochromatic.

A survey of results on image and kernel partition regular matrices can be found in [**38**].

A particularly simple class of image partition regular matrices is the class of *first entries* matrices.

DEFINITION 6.2. Let $u, v \in \mathbb{N}$ and let A be a $u \times v$ matrix. Then A is a *first entries matrix* if and only if

(1) the entries of A are from $\mathbb{Q}$;
(2) no row of A is $\overline{0}$;
(3) the first (leftmost) nonzero entry of each row is positive; and
(4) the first nonzero entries of any two rows are equal if they occur in the same column.

The first non-zero entry in any row of A is called a *first entry* of A.

The following theorem, which extends Theorem 15.5 in [**42**], illustrates the combinatorial richness of C-sets.

THEOREM 6.3 (Hindman and Strauss). *Let $u, v \in \mathbb{N}$ and let A be a $u \times v$ first entries matrix with entries from ω. Let $(S, +)$ be a commutative semigroup with an identity 0. For $n \in \mathbb{N}$ and $s \in S$, let ns denote the sum $s + s + \cdots + s$ with n terms and let $0s = 0$. Let C be a C-set in S and let p be an idempotent in $J(S) \cap \overline{C}$ such that $nS \in p$ for every first entry n of A. Then there exist sequences $\langle x_{1,n}\rangle_{n=1}^\infty, \langle x_{2,n}\rangle_{n=1}^\infty, \ldots, \langle x_{v,n}\rangle_{n=1}^\infty$ such that for every $F \in \mathcal{P}_f(\mathbb{N})$,*

$\vec{x}_F \in (S \setminus \{0\})^v$ and $A\vec{x}_F \subseteq C^u$, where $\vec{x}_F = \sum_{n\in F} \begin{pmatrix} x_{1,n} \\ x_{2,n} \\ \vdots \\ x_{v,n} \end{pmatrix}$. In the case in which S is a commutative group, this statement holds if A is a first entries matrix with entries from $\mathbb{Z}$.

PROOF. [**50**, Theorem 2.8]. □

In his 1933 paper [**66**] Rado characterized the kernel partition regularity of matrices, which of course correspond to linear transformations. In that same paper he also characterized the kernel partition regularity of affine transformations. These characterizations are not as well known as his linear characterizations, probably because, with the exception of Theorem 6.4(b)(ii), the answer is that the affine transformation is kernel partition regular if and only if it is trivially so, that is it has a constant solution. (Given a number k we write $\overline{k}$ for a vector with all terms equal to k.)

THEOREM 6.4 (Rado). *Let $u, v \in \mathbb{N}$, let A be a $u \times v$ matrix with entries from $\mathbb{Q}$, and let $\vec{b} \in \mathbb{Q}^u \setminus \{\overline{0}\}$.*

(a) *Whenever $\mathbb{Z}$ is finitely colored, there exists a monochromatic $\vec{x} \in \mathbb{Z}^v$ such that $A\vec{x} + \vec{b} = \overline{0}$ if and only if there exists $k \in \mathbb{Z}$ such that $A\overline{k} + \vec{b} = \overline{0}$.*

(b) *Whenever $\mathbb{N}$ is finitely colored, there exists a monochromatic $\vec{x} \in \mathbb{N}^v$ such that $A\vec{x} + \vec{b} = \overline{0}$ if and only if either*

  (i) *there exists $k \in \mathbb{N}$ such that $A\overline{k} + \vec{b} = \overline{0}$ or*

  ii) *there exists $k \in \mathbb{Z}$ such that $A\overline{k} + \vec{b} = \overline{0}$ and the linear mapping $\vec{x} \mapsto A\vec{x}$ is kernel partition regular.*

PROOF. (a) [**66**, Satz VIII].
(b) [**66**, Satz V]. □

The following characterization of image partition regularity of an affine transformation over $\mathbb{Z}$ is nearly identical to Rado's characterization of kernel partition regularity of affine transformations.

THEOREM 6.5 (Moshesh). *Let $u, v \in \mathbb{N}$, let A be a $u \times v$ matrix with entries from $\mathbb{Q}$, and let $\vec{b} \in \mathbb{Q}^u \setminus \{\overline{0}\}$. Whenever $\mathbb{Z}$ is finitely colored, there exists $\vec{x} \in \mathbb{Z}^v$ such that the entries of $A\vec{x} + \vec{b}$ are monochromatic if and only if there exist $\vec{x} \in \mathbb{Z}^v$ and $k \in \mathbb{Z}$ such that $A\vec{x} + \vec{b} = \overline{k}$.*

PROOF. [**57**, Theorem 4.8]. □

The characterization in the following is significantly more interesting. (Note in particular the appearance of central sets.)

THEOREM 6.6 (Hindman and Moshesh). *Let $u, v \in \mathbb{N}$, let A be a $u \times v$ matrix with entries from $\mathbb{Q}$, and let $\vec{b} \in \mathbb{Q}^u \setminus \{\overline{0}\}$. Whenever $\mathbb{N}$ is finitely colored there exists $\vec{x} \in \mathbb{Z}^v$ such that the entries of $A\vec{x} + \vec{b}$ are monochromatic if and only if either*

(i) *there exists $k \in \mathbb{N}$ and $\vec{x} \in \mathbb{Z}^v$ such that $A\vec{x} + \vec{b} = \overline{k}$ or,*

(ii) *there exists $k \in \mathbb{Z}$ and $\vec{x} \in \mathbb{Z}^v$ such that $A\vec{x} + \vec{b} = \overline{k}$ and for every central set C in $\mathbb{N}$, there exists $\vec{x} \in \mathbb{Z}^v$ such that $A\vec{x} \in C^u$.*

PROOF. [**40**, Theorem 4.5]. □

DEFINITION 6.7. Let S be a subsemigroup of $(\mathbb{R},+)$ with $0 \in c\ell S$, let $u, v \in \mathbb{N}$, and let A be a $u \times v$ matrix with entries from $\mathbb{Q}$. Then A is *image partition regular over S near zero* (abbreviated IPR/S_0) if and only if, whenever $S \setminus \{0\}$ is finitely colored and $\delta > 0$, there exists $\vec{x} \in S^v$ such that the entries of $A\vec{x}$ are monochromatic and lie in the interval $(-\delta, \delta)$.

$\mathbb{D}$ denotes the set of dyadic rational numbers.

THEOREM 6.8. *Let $u, v \in \mathbb{N}$ and let A be a $u \times v$ matrix with entries from $\mathbb{Q}$. The seven statements in (I) below are equivalent and are strictly stronger than the seven equivalent statements in (II).*

(I)
- (a) *A is IPR/$\mathbb{N}$.*
- (b) *A is IPR/$\mathbb{D}^+$.*
- (c) *A is IPR/$\mathbb{Q}^+$.*
- (d) *A is IPR/$\mathbb{R}^+$.*
- (e) *A is IPR/$\mathbb{D}^+_0$.*
- (f) *A is IPR/$\mathbb{Q}^+_0$.*
- (g) *A is IPR/$\mathbb{R}^+_0$.*

(II)
- (a) *A is IPR/$\mathbb{Z}$.*
- (b) *A is IPR/$\mathbb{D}$.*
- (c) *A is IPR/$\mathbb{Q}$.*
- (d) *A is IPR/$\mathbb{R}$.*
- (e) *A is IPR/$\mathbb{D}_0$.*
- (f) *A is IPR/$\mathbb{Q}_0$.*
- (g) *A is IPR/$\mathbb{R}_0$.*

PROOF. [**22**, Theorem 2.6]. □

In [**22**] it is also demonstrated that there are many distinct notions of image partion regularity near zero for infinite matrices.

Let A denote a nonempty finite set (the *alphabet*). We choose a set $V = \{v_n : n \in \omega\}$ (of *variables*) such that $A \cap V = \emptyset$ and define W to be the semigroup of words over the alphabet $A \cup V$ (including the empty word), with concatenation as the semigroup operation. (Formally a *word* w is a function with domain $k \in \omega$ to the alphabet and the length $\ell(w)$ of w is k. We shall need to resort to this formal meaning, so that if $i \in \{0, 1, \ldots, \ell(w) - 1\}$, then $w(i)$ denotes the $(i+1)^{\text{st}}$ letter of w.)

DEFINITION 6.9. Let $n \in \omega$ and let $k \in \{0, 1, \ldots, n\}$. Then $[A]\binom{n}{k}$ is the set of all words w over the alphabet $A \cup \{v_0, v_1, \ldots, v_{k-1}\}$ of length n such that

(1) for each $i \in \{0, 1, \ldots, k-1\}$, if any, v_i occurs in w and
(2) for each $i \in \{0, 1, \ldots, k-2\}$, if any, the first occurrence of v_i in w precedes the first occurrence of v_{i+1}.

Let $k \in \mathbb{N}$. Then the set of *k-variable words* is $S_k = \bigcup_{n=k}^{\infty} [A]\binom{n}{k}$. Also S_0 is the semigroup of words over A. Given $w \in S_n$ and $u \in W$ with $\ell(u) = n$, we define $w\langle u\rangle$ to be the word with length $\ell(w)$ such that for $i \in \{0, 1, \ldots, \ell(w) - 1\}$

$$w\langle u\rangle(i) = \begin{cases} w(i) & \text{if } w(i) \in A \\ u(j) & \text{if } w(i) = v_j\,. \end{cases}$$

That is, $w\langle u\rangle$ is the result of substituting $u(j)$ for each occurrence of v_j in w. For example, let $A=\{a,b\}$, let $w=av_0bv_1v_0abv_2bv_0$, and let $u=bv_0v_1$. Then $w\langle u\rangle=abbv_0babv_1bb$.

The following theorem is commonly known as the Graham-Rothschild Parameter Sets Theorem. The original theorem [**34**] (or see [**59**]) is stated in a significantly stronger fashion. However this stronger version is derivable from the version stated here in a reasonably straightforward manner. (See [**16**, Theorem 5.1].)

THEOREM 6.10 (Graham-Rothschild). *Let $m,n\in\omega$ with $m<n$, and let S_m be finitely colored. There exists $w\in S_n$ such that $\left\{w\langle u\rangle : u\in[A]\binom{n}{m}\right\}$ is monochromatic.*

Section 9 of [**34**] contains 13 corollaries. Included among these are four results that were known at the time (namely the Hales-Jewett Theorem, van der Waerden's Theorem, Ramsey's Theorem, and the finite version of the Finite Sums Theorem). We believe that the other nine were new at the time. (These include the finite version of the Finite Unions Theorem. While the infinite version of the Finite Unions Theorem is obviously derivable from the infinite version of the Finite Sums Theorem, the finite version of the Finite Unions Theorem is not obviously derivable from the finite version of the Finite Sums Theorem.)

DEFINITION 6.11. For $r,n\in\mathbb{N}$ with $r>n$ and $u\in[A]\binom{r}{n}$ define $h_u:S_r\to S_n$ by, for $w\in S_r$, $h_u(w)=w\langle u\rangle$, and let $\widetilde{h_u}:\beta S_r\to\beta S_n$ be the continuous extension of h_u.

The following algebraic result was used in [**16**] to derive an infinitary extension of Theorem 6.10.

THEOREM 6.12 (Carlson, Hindman, and Strauss). *Let A be a nonempty alphabet. Let p be a minimal idempotent in βS_0. There is a sequence $\langle p_n\rangle_{n=0}^{\infty}$ such that*

(1) *$p_0=p$;*
(2) *for each $n\in\mathbb{N}$, p_n is a minimal idempotent in βS_n;*
(3) *for each $n\in\mathbb{N}$, $p_n\le p_{n-1}$; and*
(4) *for each $n\in\mathbb{N}$ and each $u\in[A]\binom{n}{n-1}$, $\widetilde{h_u}(p_n)=p_{n-1}$.*

Further, p_1 can be any minimal idempotent of βS_1 such that $p_1\le p_0$.

PROOF. [**16**, Theorem 2.12]. □

THEOREM 6.13 (Carlson, Hindman, and Strauss). *Let A be a nonempty alphabet, let $n\in\mathbb{N}$, and let*

$$T_n=\left\{p\in\beta S_n:(\forall r>n)(\exists q\in\beta S_r)(\forall u\in[A]\tbinom{r}{n})(\widetilde{h_u}(q)=p)\right\}.$$

Let $\kappa=\max\{|A|,\omega\}$. Then T_n is a subsemigroup of βS_n, $K(T_n)=T_n\cap K(\beta S_n)$, each minimal right ideal and each minimal left ideal of T_n contains 2^{2^κ} idempotents, and each maximal group in $K(T_n)$ contains a free group on 2^{2^κ} generators.

PROOF. [**14**, Theorems 2.3, 2.13, and 2.18] and [**42**, Theorem 1.65]. □

We shall not state here the infinitary generalization of Theorem 6.10 obtained using Theorem 6.12 because of the additional notation needed, stating instead the following consequence of that generalization.

Theorem 6.14 (Carlson, Hindman, and Strauss). *Let $u, v, k \in \mathbb{N}$, let M be a $u \times v$ first entries matrix with entries from $\mathbb{Z}$, let C be a central subset of $\mathbb{N}$, and let G be a K_k-free graph on $\mathbb{N}$. There is a sequence $\langle \vec{x_n} \rangle_{n=1}^{\infty}$ in $\mathbb{N}^v$ such that for every $F \in \mathcal{P}_f(\mathbb{N})$, the set of entries of $M(\sum_{n \in F} \vec{x_n})$ is an independent subset of C.*

Proof. [**16**, Theorem 4.8]. □

Another application of Theorem 6.12 is the following.

Theorem 6.15 (Carlson, Hindman, and Strauss). *Let $\mathbb{N}$ be finitely colored. Then there is a color class D which is central in $\mathbb{N}$ and*

(1) *there exists a pairwise disjoint collection $\{D_{i,j} : i, j \in \omega\}$ of central subsets of D and for each $i \in \omega$ there exists a sequence $\langle x_{i,n} \rangle_{n=i}^{\infty}$ in $D_{i,i}$ such that whenever F is a finite nonempty subset of ω and $f : F \to \{1, 2, \ldots, \min F\}$ one has that $\Sigma_{n \in F}\, x_{f(n),n} \in D_{i,j}$ where $i = f(\min F)$ and $j = f(\max F)$; and*
(2) *at stage n when one is chosing $(x_{0,n}, x_{1,n}, \ldots, x_{n,n})$, each $x_{i,n}$ may be chosen as an arbitrary element of a certain central subset of $D_{i,i}$, with the choice of $x_{i,n}$ independent of the choice of $x_{j,n}$.*

Proof. [**15**, Corollary 2.9]. □

In [**10**] several combined additive and multiplicative combinatorial structures were shown to exist in any multiplicatively large set. As an example consider the following.

Theorem 6.16 (Beiglböck, Bergelson, Hindman, and Strauss). *Let C be a central subset of $(\mathbb{N}, \cdot)$ and let $k \in \mathbb{N}$. There exist $a, b, d \in \mathbb{N}$ such that*

$$\begin{aligned}\big\{b(a+id)^j : i, j \in \{0, 1, \ldots, k\}\big\} &\cup \big\{bd^j : j \in \{0, 1, \ldots, k\}\big\} \\ \cup \big\{a + id : i \in \{0, 1, \ldots, k\}\big\} &\cup \{d\} \subseteq C\,.\end{aligned}$$

Proof. [**10**, Corollary 4.3]. □

By way of contrast, comparatively little multiplicative structure is guaranteed to additively large sets as is demonstrated by several purely combinatorial results in [**9**]. (We do not address these here as we are concerned with applications of the algebra of βS.) However, if a set A is very large additively, there must be significant multiplicative structure.

Theorem 6.17 (Beiglböck, Bergelson, Hindman, and Strauss). *Let $A \subseteq \mathbb{N}$ and assume that $K(\beta\mathbb{N}, +) \subseteq \overline{A}$. Then for all $t \in \mathbb{Z}$, $c\ell K(\beta\mathbb{N}, +) \subseteq \overline{(t + A) \cap \mathbb{N}}$ and in particular, $(t + A) \cap \mathbb{N}$ is central in $(\mathbb{N}, +)$ and in $(\mathbb{N}, \cdot)$.*

Proof. [**9**, Theorem 3.11]. □

In [**20**, Theorem 2.5], Davenport presented a proof using the algebraic structure of $\beta(\mathbb{N}^k)$ of the *multidimensional van der Waerden theorem*, otherwise known as Grünwald's theorem or Gallai's theorem. (Grünwald and Gallai were the same person.)

Theorem 6.18 (Grünwald=Gallai). *Let $k, r \in \mathbb{N}$ and asssume that $\mathbb{N}^k = \bigcup_{i=1}^{r} A_i$. Then there exist $l, d, a_1, a_2, \ldots, a_k \in \mathbb{N}$ and $j \in \{1, 2, \ldots, r\}$ such that $\big\{(a_1 + n_1 d, a_2 + n_2 d, \ldots, a_k + n_k d) : \{n_1, n_2, \ldots, n_k\} \subseteq \{1, 2, \ldots, l\}\big\} \subseteq A_j$.*

It is an old result of Bergelson [**11**] that if $\mathbb{N}$ is partitioned into finitely many cells, then one cell C satisfies $\overline{d}(\{n \in C : \overline{d}\big(C \cap (-n + C)\big) > 0\}) > 0$, where $\overline{d}$ is upper asymptotic density. In a similar vein we have the following, where d is the Følner density as defined in Definition 5.12.

THEOREM 6.19 (Hindman and Strauss). *Let $k \in \mathbb{N}$. Let S be a left cancellative semigroup satisfying SFC and assume that for all $x, y \in S$, $|\{s \in S : sx = y\}| \leq k$. Let $s \in S$ and let $A \subseteq S$. If $d(A) > 0$, then for each $l \in \mathbb{N}$ there exists $d \in \mathbb{N}$ such that $d(\{b \in S : \{s^d b, s^{2d} b, \ldots, s^{ld} b\} \subseteq A) > 0$.*

PROOF. [**44**, Theorems 4.16 and 5.5]. □

By [**44**, Theorems 2.4, 2.7, and 4.11] if S is as in Theorem 6.19 and A is a piecewise syndetic subset of S, then $d(A) > 0$, so the conclusion of Theorem 6.19 applies to A. One may wonder why the geometric progression is written in the form $\{s^d b, s^{2d} b, \ldots, s^{ld} b\}$ rather than $\{bs^d, bs^{2d}, \ldots, bs^{ld}\}$. It is shown in [**44**, Theorem 5.8] that there is a countable cancellative semigroup S satisfying SFC and a subset A such that $\overline{A}$ contains a left ideal of βS (so in particular A is central and hence piecewise syndetic) but there is some $s \in S$ such that $\{bs^n : n \in \mathbb{N}$ and $b \in S\} \cap A = \emptyset$.

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