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MONOCHROMATIC SUMS EQUAL TO PRODUCTS IN $\mathbb{N}$

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Abstract

Csikvári, Gyarmati, and Sárközy asked whether, whenever the set $\mathbb{N}$ of positive integers is finitely colored, there must exist monochromatic a, b, c, and d such that $a + b = cd$ and $a \neq b$. We provide an affirmative answer, showing that a much stronger statement is true.

1. Introduction

In [9, Corollary 1], Sárközy established that if p is a prime, $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{D}$ are subsets of $\mathbb{Z}_p$, and $|\mathcal{A}| \cdot |\mathcal{B}| \cdot |\mathcal{C}| \cdot |\mathcal{D}| > p^3$, then there exist $a \in \mathcal{A}$, $b \in \mathcal{B}$, $c \in \mathcal{C}$, and $d \in \mathcal{D}$ such that $a + b = cd$. In [6] this result was extended to finite fields. That is, if q is a prime power, $\mathbf{F}_q$ is the field with q elements, $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{D}$ are subsets of $\mathbf{F}_q$, and $|\mathcal{A}| \cdot |\mathcal{B}| \cdot |\mathcal{C}| \cdot |\mathcal{D}| > q^3$, then there exist $a \in \mathcal{A}$, $b \in \mathcal{B}$, $c \in \mathcal{C}$, and $d \in \mathcal{D}$ such that $a + b = cd$.

These results led Csikvári, Gyarmati, and Sárközy to ask [5, Problem B] whether whenever the set $\mathbb{N}$ of positive integers is finitely colored, there must exist monochromatic a, b, c, and d with $a \neq b$ such that $a + b = cd$. We shall answer this question affirmatively, showing in addition that one can demand that a, b, c, and d are all distinct and that the color of $a+b$ is that same as that of a, b, c, and d. In fact, our main result is considerably stronger than this. In order to describe it, we introduce

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some notation. Given a set X, we write $\mathcal{P}_f(X)$ for the set of finite nonempty subsets of X.

Definition 1. Let $\langle x_n\rangle_{n=1}^\infty$ be an infinite sequence in $\mathbb{N}$, let $m \in \mathbb{N}$, and let $\langle y_n\rangle_{n=1}^m$ be a finite sequence in $\mathbb{N}$.

(a) $FS(\langle x_n\rangle_{n=1}^\infty) = \{\sum_{n\in F} x_n : F \in \mathcal{P}_f(\mathbb{N})\}$ and $FP(\langle x_n\rangle_{n=1}^\infty) = \{\prod_{n\in F} x_n : F \in \mathcal{P}_f(\mathbb{N})\}$.

(b) $FS(\langle y_n\rangle_{n=1}^m) = \big\{\sum_{n\in F} y_n : \emptyset \neq F \subseteq \{1,2,\ldots,m\}\big\}$ and $FP(\langle y_n\rangle_{n=1}^m) = \big\{\prod_{n\in F} y_n : \emptyset \neq F \subseteq \{1,2,\ldots,m\}\big\}$.

(c) The sequence $\langle x_n\rangle_{n=1}^\infty$ has *distinct finite sums* if and only if whenever $F, G \in \mathcal{P}_f(\mathbb{N})$ and $F \neq G$, one has $\sum_{n\in F} x_n \neq \sum_{n\in G} x_n$. The analogous definition applies to $\langle y_n\rangle_{n=1}^m$.

(d) The sequence $\langle x_n\rangle_{n=1}^\infty$ has *distinct finite products* if and only if whenever $F, G \in \mathcal{P}_f(\mathbb{N})$ and $F \neq G$, one has $\prod_{n\in F} x_n \neq \prod_{n\in G} x_n$. The analogous definition applies to $\langle y_n\rangle_{n=1}^m$.

(e) The sequence $\langle x_n\rangle_{n=1}^\infty$ is *strongly increasing* if and only if for each $n \in \mathbb{N}$, $\sum_{t=1}^n x_t < x_{n+1}$.

Notice that if $\langle x_n\rangle_{n=1}^\infty$ is strongly increasing, then it has distinct finite sums.

We shall establish in a straightforward manner the following generalization of the affirmative answer to the question of Csikvári, Gyarmati, and Sárközy.

Theorem 2. *Let $m, r \in \mathbb{N}$ with $m > 1$ and let $\mathbb{N} = \bigcup_{k=1}^r A_k$. There exist $k \in \{1, 2, \ldots, r\}$, $d \in \mathbb{N}$, and sequences $\langle x_t\rangle_{t=1}^m$ and $\langle y_t\rangle_{t=1}^m$ such that*

(1) $\langle x_t\rangle_{t=1}^m$ has distinct finite sums;

(2) $\langle y_t\rangle_{t=1}^m$ has distinct finite products;

(3) $\sum_{t=1}^m x_t = \prod_{t=1}^m y_t = d$;

(4) $FS(\langle x_t\rangle_{t=1}^m) \cup FP(\langle y_t\rangle_{t=1}^m) \subseteq A_k$; and

(5) $FS(\langle x_t\rangle_{t=1}^m) \cap FP(\langle y_t\rangle_{t=1}^m) = \{d\}$.

Notice that the strength of Theorem 2 increases as m increases. If one has

$\langle x_t\rangle_{t=1}^{m+1}$ and $\langle y_t\rangle_{t=1}^{m+1}$ as guaranteed for $m+1$, one may let

$$x'_t = \begin{cases} x_t & \text{if } t < m \\ x_m + x_{m+1} & \text{if } t = m \end{cases} \text{ and } y'_t = \begin{cases} y_t & \text{if } t < m \\ y_m \cdot y_{m+1} & \text{if } t = m\,, \end{cases}$$

and then $\langle x'_t\rangle_{t=1}^{m}$ and $\langle y'_t\rangle_{t=1}^{m}$ are as required for m.

We shall show in fact that one may get sequences as guaranteed by Theorem 2 for any finite set of values of m simultaneously, in such a way that all sums and all products from any of the sequences (except, of course, for the one on which they agree) are distinct.

Theorem 3. *Let $n, r \in \mathbb{N}$ with $n > 1$ and let $\mathbb{N} = \bigcup_{k=1}^{r} A_k$. There exist $k \in \{1, 2, \ldots, r\}$, $d \in \mathbb{N}$, and for each $i \in \{2, 3, \ldots, n\}$ sequences $\langle x_{i,t}\rangle_{t=1}^{i}$ and $\langle y_{i,t}\rangle_{t=1}^{i}$ such that*

(1) for each $i \in \{2, 3, \ldots, n\}$, $\langle x_{i,t}\rangle_{t=1}^{i}$ has distinct finite sums;

(2) for each $i \in \{2, 3, \ldots, n\}$, $\langle y_{i,t}\rangle_{t=1}^{i}$ has distinct finite products;

(3) for each $i \in \{2, 3, \ldots, n\}$, $\sum_{t=1}^{i} x_{i,t} = \prod_{t=1}^{i} y_{i,t} = d$;

(4) for each $i \in \{2, 3, \ldots, n\}$, $FS(\langle x_{i,t}\rangle_{t=1}^{i}) \cup FP(\langle y_{i,t}\rangle_{t=1}^{i}) \subseteq A_k$;

(5) for $i \neq j$ in $\{2, 3, \ldots, n\}$, $FS(\langle x_{j,t}\rangle_{t=1}^{j}) \cap FS(\langle x_{i,t}\rangle_{t=1}^{i}) = \{d\}$ and $FP(\langle y_{j,t}\rangle_{t=1}^{j}) \cap FP(\langle y_{i,t}\rangle_{t=1}^{i}) = \{d\}$; and

(6) for $i, j \in \{2, 3, \ldots, n\}$, $FS(\langle x_{j,t}\rangle_{t=1}^{j}) \cap FP(\langle y_{i,t}\rangle_{t=1}^{i}) = \{d\}$.

We shall establish these results in Section 2. The proof of the main tool, namely Theorem 5, uses the algebraic structure of $\beta\mathbb{N}$, the Stone-Čech compactification of $\mathbb{N}$. The points of $\beta\mathbb{N}$ are the ultrafilters on $\mathbb{N}$, with the principal ultrafilters being identified with the points of $\mathbb{N}$. The existence of nonprincipal ultrafilters depends unescapably on the Axiom of Choice. In Section 3 we show how our results can be proved without invoking the Axiom of Choice.

2. Finding Many Monochromatic Solutions to $\sum_{t=1}^{n} x_t = \prod_{t=1}^{n} y_t$

The only fact which one needs to know in this section about the algebraic structure of $\beta\mathbb{N}$ is that there is an ultrafilter $p \in \beta\mathbb{N}$ such that, for every $A \in p$ there exist sequences $\langle x_n\rangle_{n=1}^{\infty}$ and $\langle y_n\rangle_{n=1}^{\infty}$ with $FS(\langle x_n\rangle_{n=1}^{\infty}) \subseteq A$ and $FP(\langle y_n\rangle_{n=1}^{\infty}) \subseteq A$. This is shown in the proof of [8, Corollary 5.22]. Everything preceeding that point in [8] (except for Section 2.5 which is not needed for this fact) is established in a routine elementary fashion, so the naive reader is encouraged to investigate this proof.

In the following lemma, we demand that the terms of $\langle w_t\rangle_{t=1}^\infty$ not be equal to 1 only to forbid a sequence which is eventually equal to 1 (whose set of finite products would then be finite).

Lemma 4. *Let $\langle w_t\rangle_{t=1}^\infty$ be a sequence in $\mathbb{N}\setminus\{1\}$. There exist sequences $\langle x_t\rangle_{t=1}^\infty$ and $\langle y_t\rangle_{t=1}^\infty$ such that $\langle x_t\rangle_{t=1}^\infty$ is strongly increasing (and thus has distinct finite sums), $\langle y_t\rangle_{t=1}^\infty$ is increasing and has distinct finite products, $FS(\langle x_t\rangle_{t=1}^\infty)\subseteq FS(\langle w_t\rangle_{t=1}^\infty)$, and $FP(\langle y_t\rangle_{t=1}^\infty)\subseteq FP(\langle w_t\rangle_{t=1}^\infty)$.*

Proof. We first construct the sequence $\langle x_t\rangle_{t=1}^\infty$ and a sequence $\langle H_t\rangle_{t=1}^\infty$ in $\mathcal{P}_f(\mathbb{N})$ inductively. Let $x_1=w_1$ and $H_1=\{1\}$. Let $n\in\mathbb{N}$ and assume that we have chosen $\langle x_k\rangle_{k=1}^n$ and $\langle H_k\rangle_{k=1}^n$ so that for each $k\in\{1,2,\ldots,n\}$, $x_k=\sum_{t\in H_k} w_t$ and if $k<n$, then $\max H_k<\min H_{k+1}$ and $\sum_{t=1}^k x_t<x_{k+1}$. Let $l=\max H_n$ and pick $H_{n+1}\in\mathcal{P}_f(\mathbb{N})$ with $\min H_{n+1}>l$ such that $\sum_{t\in H_{n+1}} w_t>\sum_{k=1}^n x_k$ and let $x_{n+1}=\sum_{t\in H_{n+1}} w_t$.

The sequence $\langle x_n\rangle_{n=1}^\infty$ having been chosen, we have immediately that it is strongly increasing. To see that $FS(\langle x_n\rangle_{n=1}^\infty)\subseteq FS(\langle w_t\rangle_{t=1}^\infty)$, let $F\in\mathcal{P}_f(\mathbb{N})$ and let $K=\bigcup_{n\in F} H_n$. Then $\sum_{n\in F} x_n=\sum_{t\in K} w_t$.

Now we construct the sequence $\langle y_t\rangle_{t=1}^\infty$ and a sequence $\langle H_t\rangle_{t=1}^\infty$ in $\mathcal{P}_f(\mathbb{N})$ inductively. Let $y_1=w_1$ and $H_1=\{1\}$. Let $n\in\mathbb{N}$ and assume that we have chosen $\langle y_k\rangle_{k=1}^n$ and $\langle H_k\rangle_{k=1}^n$ so that for each $k\in\{1,2,\ldots,n\}$, $y_k=\prod_{t\in H_k} w_t$ and if $k<n$, then $\max H_k<\min H_{k+1}$. Assume further that if F and G are distinct finite nonempty subsets of $\{1,2,\ldots,n\}$, then $\prod_{k\in F} y_k\neq\prod_{k\in G} y_k$. Let $E=FP(\langle y_k\rangle_{k=1}^n)$, let $l=\max H_n$, and pick $H_{n+1}\in\mathcal{P}_f(\mathbb{N})$ with $\min H_{n+1}>l$ such that $\prod_{t\in H_{n+1}} w_t>y_n$ and $\prod_{t\in H_{n+1}} w_t\notin E\cup\{u^{-1}v: u,v\in E\}$. Let $y_{n+1}=\prod_{t\in H_{n+1}} w_t$. Now let F and G be distinct finite nonempty subsets of $\{1,2,\ldots,n+1\}$ and suppose that $\prod_{k\in F} y_k=\prod_{k\in G} y_k$. If $n+1\notin F\cup G$, this contradicts the induction hypothesis. If $n+1\in F\cap G$, then letting $F'=F\setminus\{n+1\}$ and $G'=G\setminus\{n+1\}$, one has $\prod_{k\in F'} y_k=\prod_{k\in G'} y_k$ (where we let $\prod_{k\in\emptyset} y_k=1$ if either $F=\{n+1\}$ or $G=\{n+1\}$). This contradicts either the induction hypothesis, or the fact that each $w_t>1$. So assume without loss of generality that $n+1\in F$ and $n+1\notin G$. Then $\prod_{k\in G} y_k\in E$. If $F=\{n+1\}$, then $y_{n+1}\in E$. Otherwise, $y_{n+1}\in\{u^{-1}v: u,v\in E\}$. In either case we get a contradiction.

One sees that $FP(\langle y_n\rangle_{n=1}^\infty)\subseteq FP(\langle w_t\rangle_{t=1}^\infty)$ in exactly the same fashion as we saw that $FS(\langle x_n\rangle_{n=1}^\infty)\subseteq FS(\langle w_t\rangle_{t=1}^\infty)$. □

Theorem 5. *Let $r\in\mathbb{N}$ and let $\mathbb{N}=\bigcup_{i=1}^r A_i$. There exists $i\in\{1,2,\ldots,r\}$ such that for each $m\in\mathbb{N}$,*

(1) *there exists an increasing sequence* $\langle y_n\rangle_{n=1}^{\infty}$ *with distinct finite products such that*
$FP(\langle y_n\rangle_{n=1}^{\infty}) \subseteq A_i$ *and whenever* $F \in \mathcal{P}_f(\mathbb{N})$, *there exists a strongly increasing sequence* $\langle x_n\rangle_{n=1}^{m}$ *such that* $FS(\langle x_n\rangle_{n=1}^{m}) \subseteq A_i$ *and* $\sum_{n=1}^{m} x_n = \prod_{n\in F} y_n$ *and*

(2) *there exists a strongly increasing sequence* $\langle x_n\rangle_{n=1}^{\infty}$ *such that* $FS(\langle x_n\rangle_{n=1}^{\infty}) \subseteq A_i$ *and whenever* $F \in \mathcal{P}_f(\mathbb{N})$, *there exists an increasing sequence* $\langle y_n\rangle_{n=1}^{m}$ *with distinct finite products such that* $FP(\langle y_n\rangle_{n=1}^{m}) \subseteq A_i$ *and* $\prod_{n=1}^{m} y_n = \sum_{n\in F} x_n$.

Proof. Pick $p \in \beta\mathbb{N}$ such that, for every $A \in p$ there exist sequences $\langle x_n\rangle_{n=1}^{\infty}$ and $\langle y_n\rangle_{n=1}^{\infty}$ with $FS(\langle x_n\rangle_{n=1}^{\infty}) \subseteq A$ and $FP(\langle y_n\rangle_{n=1}^{\infty}) \subseteq A$. Pick $i \in \{1, 2, \ldots, r\}$ such that $A_i \in p$.

Let $m \in \mathbb{N}$ be given. Let $B = \{z \in A_i$: there exists strongly increasing $\langle x_n\rangle_{n=1}^{m}$ such that $FS(\langle x_n\rangle_{n=1}^{m}) \subseteq A_i$ and $z = \sum_{n=1}^{m} x_n\}$. Let $C = \{z \in A_i$: there exists increasing $\langle y_n\rangle_{n=1}^{m}$ satisfying uniqueness of finite products such that $FP(\langle y_n\rangle_{n=1}^{m}) \subseteq A_i$ and $z = \prod_{n=1}^{m} y_n\}$.

We claim that $B \in p$. Suppose instead that $B \notin p$, in which case $A_i \setminus B \in p$. Pick a sequence $\langle x_n\rangle_{n=1}^{\infty}$ with $FS(\langle x_n\rangle_{n=1}^{\infty}) \subseteq A_i \setminus B$. By Lemma 4 we may assume that $\langle x_n\rangle_{n=1}^{\infty}$ is strongly increasing. But then $\sum_{n=1}^{m} x_n \in B$, a contradiction. Similarly $C \in p$.

For conclusion (1) pick an increasing sequence $\langle y_n\rangle_{n=1}^{\infty}$ with distinct finite products such that $FP(\langle y_n\rangle_{n=1}^{\infty}) \subseteq B$. For conclusion (2) pick a strongly increasing sequence $\langle x_n\rangle_{n=1}^{\infty}$ such that $FS(\langle x_n\rangle_{n=1}^{\infty}) \subseteq C$. □

Note that the $m = 2$ case of either part of Theorem 5 is already strong enough to answer the question as posed by Csikvári, Gyarmati, and Sárközy.

Lemma 6. *Let* $m, n \in \mathbb{N}$, *let* $\langle x_t\rangle_{t=1}^{n}$ *be a strongly increasing sequence in* $\mathbb{N}$, *and let* $\langle y_t\rangle_{t=1}^{m}$ *be a sequence in* $\mathbb{N} \setminus \{1\}$ *such that* $\sum_{t=1}^{n} x_t = \prod_{t=1}^{m} y_t$. *If* $\emptyset \neq F \subsetneq \{1, 2, \ldots, n\}$ *and* $n \in F$, *then* $\sum_{t\in F} x_t \notin FP(\langle y_t\rangle_{t=1}^{m})$.

Proof. Suppose we have $\emptyset \neq G \subseteq \{1, 2, \ldots, m\}$ such that $\sum_{t\in F} x_t = \prod_{t\in G} y_t$. Since $\sum_{t\in F} x_t < \prod_{t=1}^{m} y_t$, we have $G \neq \{1, 2, \ldots, m\}$. Let $H = \{1, 2, \ldots, n\} \setminus F$ and let $K = \{1, 2, \ldots, m\} \setminus G$. Then $\sum_{t\in H} x_t + \sum_{t\in F} x_t = (\sum_{t\in F} x_t) \prod_{t\in K} y_t$. Since $n \in F$ we have $\sum_{t\in F} x_t > \sum_{t\in H} x_t = (\sum_{t\in F} x_t)(\prod_{t\in K} y_t - 1) \geq \sum_{t\in F} x_t$, a contradiction. □

Lemma 7. *For each $k, l \in \mathbb{N}$, there exists $b \in \mathbb{N}$ such that whenever $\langle x_t \rangle_{t=1}^{b}$ is a strongly increasing sequence in $\mathbb{N}$ and $A \subseteq \mathbb{N}$ with $|A| = l$, there exist $0 = a_0 < a_1 < \ldots < a_k < b$ so that, if for each $j \in \{1, 2, \ldots, k\}$, $z_j = \sum_{t=a_{j-1}+1}^{a_j} x_t$, then $FS(\langle z_j \rangle_{j=1}^{k}) \cap A = \emptyset$.*

Proof. We proceed by induction on k with fixed l and A. For $k = 1$, let $b = l + 2$. For some $a \in \{1, 2, \ldots, l + 1\}$, $\sum_{t=1}^{a} x_t \notin A$.

Now let $k \in \mathbb{N}$ and assume that we have b_1 as guaranteed for k. Let $b = b_1 + l2^k + 1$ and let strongly increasing $\langle x_t \rangle_{t=1}^{b}$ be given. Pick $0 = a_0 < a_1 < \ldots < a_k < b_1$ as guaranteed by hypothesis. For $j \in \{1, 2, \ldots, k\}$, let $z_j = \sum_{t=a_{j-1}+1}^{a_j} x_t$ and let $B = FS(\langle z_j \rangle_{j=1}^{k})$. Then $|B| = 2^k - 1$. Let $C = A \cup \bigcup_{w \in B}(A - w)$. Then $|C| \le l + (2^k - 1)l = 2^k l$. Pick $a_{k+1} \in \{a_k + 1, a_k + 2, \ldots, a_k + 2^k l + 1\}$ such that $\sum_{t=a_k+1}^{a_{k+1}} x_t \notin C$. Then $a_{k+1} \le a_k + 2^k l + 1 < b_1 + 2^k l + 1 = b$. □

We are now prepared to prove Theorem 2.

Proof of Theorem 2. Let $l = 2^m - 2$. Pick b as guaranteed by Lemma 7 for l and $k - 1$. Pick $i \in \{1, 2, \ldots, r\}$ and increasing $\langle y_n \rangle_{n=1}^{\infty}$ satisfying uniqueness of finite products as guaranteed by conclusion (1) of Theorem 5. Pick strongly increasing $\langle x_t \rangle_{t=1}^{b}$ such that $\sum_{t=1}^{b} x_t = \prod_{t=1}^{m} y_t$. Let $A = FP(\langle y_t \rangle_{t=1}^{m}) \setminus \{\prod_{t=1}^{m} y_t\}$. Then $|A| = l$. Pick $0 = a_0 < a_1 < a_2 < \ldots < a_{k-1}$ as guaranteed by Lemma 7. Let $a_k = b$. For $j \in \{1, 2, \ldots, k\}$, let $z_j = \sum_{t=a_{j-1}+1}^{a_j} x_t$.

Let $w \in FS(\langle z_j \rangle_{j=1}^{k}) \cap FP(\langle y_t \rangle_{t=1}^{m})$. Then $w = \sum_{t \in F} z_t$ for some $F \subseteq \{1, 2, \ldots, k\}$. If $k \notin F$, then since $w < \prod_{t=1}^{m} y_t$ we have by Lemma 7 that $w \notin FP(\langle y_t \rangle_{t=1}^{m})$. So $k \in F$. Then by Lemma 6, $F = \{1, 2, \ldots, k\}$. □

In order to prove Theorem 3 we need another preliminary lemma. This lemma is valid in an arbitrary semigroup. We shall use it with the semigroups $(\mathbb{N}, +)$ and $(\mathbb{N}, \cdot)$. We only defined "distinct finite products" for $(\mathbb{N}, \cdot)$, but it has its obvious interpretation in any semigroup. Likewise, $FP(\langle w_t \rangle_{t=1}^{m})$ has its obvious meaning.

Lemma 8. *Let $n \in \mathbb{N}$ and let $m = \dfrac{n^2 - n + 2}{2}$. Let $(S, \cdot)$ be a semigroup and let $\langle w_t \rangle_{t=1}^{m}$ be a sequence in S with distinct finite products. For each $i \in \{2, 3, \ldots, n\}$ there is a sequence $\langle y_{i,t} \rangle_{t=1}^{i}$ such that*

(1) for each i, $FP(\langle y_{i,t} \rangle_{t=1}^{i}) \subseteq FP(\langle w_t \rangle_{t=1}^{m})$;

(2) for each i, $\prod_{t=1}^{i} y_{i,t} = \prod_{t=1}^{m} w_t$; and

(3) for $i \neq j$, $FP(\langle y_{i,t}\rangle_{t=1}^{i}) \cap FP(\langle y_{j,t}\rangle_{t=1}^{j}) = \{\prod_{t=1}^{m} w_t\}$.

Proof. Define $f : \mathbb{N} \to \mathbb{N}$ by $f(i) = \dfrac{i^2 - 3i + 4}{2}$. Then $f(2) = 1$ and for each $i \in \mathbb{N}$, $f(i+1) = f(i)+i-1$. Note that $m = f(n+1)$. For $i \in \{2, 3, \ldots, n\}$, define $\langle y_{i,t}\rangle_{t=1}^{i}$ as follows.

$$y_{i,1} = \prod_{t=1}^{f(i)} w_t\,,$$
$$y_{i,t} = w_{f(i)+t-1} \text{ for } t \in \{2, 3, \ldots, i-1\}, \text{ and}$$
$$y_{i,i} = \prod_{t=f(i)+i-1}^{m} w_t\,.$$

The first two conclusions are immediate. Assume that $i, j \in \{2, 3, \ldots, n\}$ with $i < j$ and we have some $c \in FP(\langle y_{i,t}\rangle_{t=1}^{i}) \cap FP(\langle y_{j,t}\rangle_{t=1}^{j})$. Pick F and G with $\emptyset \neq F \subseteq \{1, 2, \ldots, i\}$ and $\emptyset \neq G \subseteq \{1, 2, \ldots, j\}$ such that $c = \prod_{t\in F} y_{i,t} = \prod_{t\in G} y_{j,t}$. Since $c \in FP(\langle w_t\rangle_{t=1}^{m})$, pick H with $\emptyset \neq H \subseteq \{1, 2, \ldots, m\}$ such that $c = \prod_{t\in H} w_t$.

Assume first that $i \notin F$. Then $\max H \leq f(i) + i - 2$. Since $\emptyset \neq G \subseteq \{1, 2, \ldots, j\}$, we have that $\max H \geq f(j) > f(i) + i - 2$, a contradiction. So $i \in F$ and consequently $\{f(i) + i - 1, f(i) + i, \ldots, m\} \subseteq H$. Since $f(i) + i - 1 \in H$ and $f(i) + i - 1 \leq f(j)$, we have $1 \in G$ and so $\{1, 2, \ldots, f(i) + i - 1\} \subseteq H$. Therefore, $H = \{1, 2, \ldots, m\}$ so $c = \prod_{t=1}^{m} w_t$. □

Proof of Theorem 3. Let $m = \dfrac{n^2 - n + 2}{2}$. Pick by Theorem 5(2) $k \in \{1, 2, \ldots, r\}$ and a strongly increasing sequence $\langle v_t\rangle_{t=1}^{\infty}$ such that $FS(\langle v_t\rangle_{t=1}^{\infty}) \subseteq A_k$ and whenever $F \in \mathcal{P}_f(\mathbb{N})$, there exists an increasing sequence $\langle w_t\rangle_{t=1}^{m}$ with distinct finite products such that $FP(\langle w_t\rangle_{t=1}^{m}) \subseteq A_k$ and $\prod_{t=1}^{m} w_t = \sum_{t\in F} v_t$.

Pick b as guaranteed by Lemma 7 for $k = m-1$ and $l = 2^m - 2$. Let $d = \sum_{t=1}^{b} v_t$ and pick an increasing sequence $\langle w_t\rangle_{t=1}^{m}$ with distinct finite products such that $\prod_{t=1}^{m} w_t = d$. Let $B = FP(\langle w_t\rangle_{t=1}^{m}) \setminus \{d\}$. Pick $0 = a_0 < a_1 < \ldots < a_{m-1} < b$ so that, if for each $j \in \{1, 2, \ldots, m-1\}$, $z_j = \sum_{t=a_{j-1}+1}^{a_j} v_t$, one has $FS(\langle z_j\rangle_{j=1}^{m-1}) \cap B = \emptyset$. Let $z_m = \sum_{t=a_{m-1}+1}^{b} v_t$. Then $\langle z_t\rangle_{t=1}^{m}$ is strongly increasing. We claim that

$$(*) \qquad FS(\langle z_t\rangle_{t=1}^{m}) \cap FP(\langle w_t\rangle_{t=1}^{m}) = \{d\}\,.$$

Certainly $d = \sum_{t=1}^{m} z_t = \prod_{t=1}^{m} w_t$. Assume that $u \in FS(\langle z_t\rangle_{t=1}^{m}) \cap FP(\langle w_t\rangle_{t=1}^{m})$. Pick F such that $\emptyset \neq F \subseteq \{1, 2, \ldots, m\}$ such that $u = \sum_{t\in F} z_t$. If $m \notin F$, then $u \in FS(\langle z_j\rangle_{j=1}^{m-1})$ so $u \notin B$ and, since $u < d$, $u \notin FP(\langle w_t\rangle_{t=1}^{m})$. Thus $m \in F$. By Lemma 6, $F = \{1, 2, \ldots, m\}$.

Applying Lemma 8 to the semigroup $(\mathbb{N}, +)$, pick for each $i \in \{2, 3, \ldots, n\}$ a sequence $\langle x_{i,t}\rangle_{t=1}^{i}$ such that

(1) for each i, $FS(\langle x_{i,t}\rangle_{t=1}^{i}) \subseteq FS(\langle z_t\rangle_{t=1}^{m})$;

(2) for each i, $\sum_{t=1}^{i} x_{i,t} = \sum_{t=1}^{m} z_t$; and

(3) for $i \neq j$, $FS(\langle x_{i,t}\rangle_{t=1}^{i}) \cap FS(\langle x_{j,t}\rangle_{t=1}^{j}) = \{\sum_{t=1}^{m} z_t\}$.

Applying Lemma 8 to the semigroup $(\mathbb{N}, \cdot)$, pick for each $i \in \{2, 3, \ldots, n\}$ a sequence $\langle y_{i,t}\rangle_{t=1}^{i}$ such that

(1) for each i, $FP(\langle y_{i,t}\rangle_{t=1}^{i}) \subseteq FP(\langle w_t\rangle_{t=1}^{m})$;

(2) for each i, $\prod_{t=1}^{i} y_{i,t} = \prod_{t=1}^{m} w_t$; and

(3) for $i \neq j$, $FP(\langle y_{i,t}\rangle_{t=1}^{i}) \cap FP(\langle y_{j,t}\rangle_{t=1}^{j}) = \{\prod_{t=1}^{m} w_t\}$.

All conclusions are easily verified. □

3. Worrying About the Axiom of Choice

The proof of Theorem 5 uses a nonprincipal ultrafilter on $\mathbb{N}$, and such things depend on some version of the Axiom of Choice for their existence. Since the conclusions of Theorem 3 are finitistic in nature, one may very well wish to avoid an appeal to the Axiom of Choice. We describe here how this may be done.

The following theorem was proved in [2] without appeal to the axiom of choice. Further, with one proviso, the argument was reasonably simple. That proviso is that the Finite Sums Theorem was used. In [3] Blass, Hirst, and Simpson showed that both the original proof of the Finite Sums Theorem [7] as well as Baumgartner's simplification [1] could be modified so as to not invoke the axiom of choice. However, one could not say that it was particularly easy to do either one (especially the original). (It was known before the publication of [3] that the Finite Sums Theorem does not require the Axiom of Choice for its proof because of absoluteness considerations. See [4, Section 4.2] for a discussion of this point.)

Theorem 9. *Let $r \in \mathbb{N}$ and let $\mathbb{N} = \bigcup_{i=1}^{r} A_i$. There exist $i \in \{1, 2, \ldots, r\}$ and sequences $\langle x_n\rangle_{n=1}^{\infty}$ and $\langle y_n\rangle_{n=1}^{\infty}$ such that $FS(\langle x_n\rangle_{n=1}^{\infty}) \cup FP(\langle y_n\rangle_{n=1}^{\infty}) \subseteq A_i$.*

Proof. [2, Theorem 2.4]. □

Corollary 10. *Let $r \in \mathbb{N}$ and let $\mathbb{N} = \bigcup_{i=1}^{r} A_i$. There exist $i \in \{1, 2, \ldots, r\}$ such that whenever A_i is partitioned into finitely many cells, there exist one cell B and sequences $\langle x_n\rangle_{n=1}^{\infty}$ and $\langle y_n\rangle_{n=1}^{\infty}$ such that $FS(\langle x_n\rangle_{n=1}^{\infty}) \cup FP(\langle y_n\rangle_{n=1}^{\infty}) \subseteq B$.*

Proof. If each A_i could be partitioned into finitely many pieces, none of which contained $FS(\langle x_n\rangle_{n=1}^\infty)\cup FP(\langle y_n\rangle_{n=1}^\infty)$ for any sequences $\langle x_n\rangle_{n=1}^\infty$ and $\langle y_n\rangle_{n=1}^\infty$, the result is a finite partition of $\mathbb{N}$ with that same property. $\square$

We now show how to reprove Theorem 5 without using the Axiom of Choice.

Theorem 5. *Let $r\in\mathbb{N}$ and let $\mathbb{N}=\bigcup_{i=1}^r A_i$. There exists $i\in\{1,2,\ldots,r\}$ such that for each $m\in\mathbb{N}$,*

(1) there exists an increasing sequence $\langle y_n\rangle_{n=1}^\infty$ with distinct finite products such that
$FP(\langle y_n\rangle_{n=1}^\infty)\subseteq A_i$ and whenever $F\in\mathcal{P}_f(\mathbb{N})$, there exists a strongly increasing sequence $\langle x_n\rangle_{n=1}^m$ such that $FS(\langle x_n\rangle_{n=1}^m)\subseteq A_i$ and $\sum_{n=1}^m x_n=\prod_{n\in F} y_n$ and

(2) there exists a strongly increasing sequence $\langle x_n\rangle_{n=1}^\infty$ such that $FS(\langle x_n\rangle_{n=1}^\infty)\subseteq A_i$ and whenever $F\in\mathcal{P}_f(\mathbb{N})$, there exists an increasing sequence $\langle y_n\rangle_{n=1}^m$ with distinct finite products such that $FP(\langle y_n\rangle_{n=1}^m)\subseteq A_i$ and $\prod_{n=1}^m y_n=\sum_{n\in F} x_n$.

Proof. Pick $i\in\{1,2,\ldots,r\}$ as guaranteed by Corollary 10 and let $m\in\mathbb{N}$ be given. Let $B=\{z\in A_i:$ there exists strongly increasing $\langle x_n\rangle_{n=1}^m$ such that $FS(\langle x_n\rangle_{n=1}^m)\subseteq A_i$ and $z=\sum_{n=1}^m x_n\}$. Let $C=\{z\in A_i:$ there exists increasing $\langle y_n\rangle_{n=1}^m$ satisfying uniqueness of finite products such that $FP(\langle y_n\rangle_{n=1}^m)\subseteq A_i$ and $z=\prod_{n=1}^m y_n\}$.

Then $\{A_i\setminus(B\cup C), B\setminus C, C\setminus B, B\cap C\}$ is a partition of A_i and the only cell which contains $FS(\langle x_n\rangle_{n=1}^\infty)\cup FP(\langle y_n\rangle_{n=1}^\infty)$ for some sequences $\langle x_n\rangle_{n=1}^\infty$ and $\langle y_n\rangle_{n=1}^\infty$ is $B\cap C$. By Lemma 4 one may assume that $\langle x_n\rangle_{n=1}^\infty$ is strongly increasing and $\langle y_n\rangle_{n=1}^\infty$ is increasing with distinct finite products.

For conclusion (1) pick an increasing sequence $\langle y_n\rangle_{n=1}^\infty$ with distinct finite products such that $FP(\langle y_n\rangle_{n=1}^\infty)\subseteq B$. For conclusion (2) pick a strongly increasing sequence $\langle x_n\rangle_{n=1}^\infty$ such that $FS(\langle x_n\rangle_{n=1}^\infty)\subseteq C$. $\square$

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