Journal of Combinatorial Theory, Series A **93**, 18–36 (2001)
doi:10.1006/jcta.2000.3061, available online at http://www.idealibrary.com on IDEAL®

Partition Regular Structures Contained in Large Sets Are Abundant

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Communicated by the Managing Editors

Received September 22, 1999

Furstenberg and Glasner have shown that for a particular notion of largeness in a group, namely *piecewise syndeticity*, if a set B is a large subset $\mathbb{Z}$, then for any $l \in \mathbb{N}$, the set of length l arithmetic progressions lying entirely in B is large among the set of all length l aritmetic progressions. We extend this result to apply to infinitely many notions of largeness in arbitrary semigroups and to partition regular structures other than arithmetic progressions. We obtain, for example, similar results for the Hales–Jewett theorem. © 2001 Academic Press

1. INTRODUCTION

A typical result of Ramsey theory states that for any finite partition of a certain kind of an infinite structure, one of the cells of the partition contains an arbitrarily large structure of the same kind. For instance, the celebrated van der Waerden's theorem [18] says that, given any finite partition of an infinite arithmetic progression, there is one cell containing arbitrarily long (finite) arithmetic progressions. Another well-known result

[1] This author acknowledges support received from the National Science Foundation via Grant DMS 9706057.

0097-3165/01 $35.00

is the geometric ramsey theorem, due to Graham *et al.* ([10]; see also [11, p. 45]), a special case of which says that for any finite partition of an infinite dimensional vector space over a finite field one of the cells must contain affine subspaces of arbitrarily large finite dimension.

A closer look reveals that the set of configurations obtained in one cell is usually large in one sense or another. For example, in van der Waerden's theorem, if $\mathbb{N} = \bigcup_{i=1}^{r} C_i$, then there is one C_i such that for any k,

$$R = \{d \in \mathbb{N} : \{a, a+d, ..., a+(k-1)\, d\} \subseteq C_i\}$$

has bounded gaps (or, in the terminology to be introduced below, is *syndetic*). In fact R is large in some other senses, as shall be explained below. (These facts illustrate what one of us has called the *third principle of Ramsey theory* [1].)

Recently, Furstenberg and Glasner [8] showed that for a particular notion of largeness (*piecewise syndeticity*, which will be introduced below), whenever B is a large subset of $\mathbb{N}$ and $l \in \mathbb{N}$, the set of length l arithmetic progressions lying entirely within B is large in the same sense among all arithmetic progressions of length l. The goal of this paper is to more closely examine this phenomenon.

In a semigroup S, there are several natural notions of largeness. (See [3] for a discussion of some of these.) One of these notions is the concept of syndeticity. This notion is not one of those for which our main result is valid. (See Theorem 3.9.) This notion appears here because we use it to introduce a notion for which that result does hold.

1.1. Definition. Let $(S, \cdot)$ be a semigroup. A set $A \subseteq S$ is *syndetic* if and only if there exists some $G \in \mathcal{P}_f(S)$ such that $S = \bigcup_{t \in G} t^{-1}A$.

(Given a set X, $\mathcal{P}_f(X)$ is the set of finite nonempty subsets of X. In a semigroup $(S, \cdot)$, if $A \subseteq S$ and $s \in S$, then $s^{-1}A = \{t \in S : s \cdot t \in A\}$.)

Throughout this paper $\mathbb{N} = \{1, 2, 3, ...\}$ and $\omega = \mathbb{N} \cup \{0\}$.

Notice that in the semigroup $(\mathbb{N}, +)$, a set is syndetic if and only if it has bounded gaps. Notice also that this notion is not partition regular as can be seen by considering the partition $\{A, B\}$ of $\mathbb{N}$, where $A = \bigcup_{n=0}^{\infty} \{2^{2n}, 2^{2n}+1, ..., 2^{2n+1}-1\}$ and $B = \bigcup_{n=1}^{\infty} \{2^{2n-1}, 2^{2n-1}+1, ..., 2^{2n}-1\}$.

The following notion of largeness is the first of our promised notions for which our main result holds.

1.2. Definition. Let $(S, \cdot)$ be a semigroup. A set $A \subseteq S$ is *piecewise syndetic* if and only if there exists some $G \in \mathcal{P}_f(S)$ such that for every $F \in \mathcal{P}_f(S)$ there exists $x \in S$ such that $F \cdot x \subseteq \bigcup_{t \in G} t^{-1}A$.

In $\mathbb{Z}$, a piecewise syndetic set is the intersection of a syndetic set with a set containing arbitrarily long intervals.

One can establish by elementary combinatorial methods that whenever a piecewise syndetic set is divided into finitely many parts, one of these parts must be piecewise syndetic. (See, for example, [3, Thm. 2.5]. In the case of the semigroup $(\mathbb{N}, +)$ this fact is apparently originally due to T. Brown. See [5] or [6].) Given that one is going to be involved in the algebraic structure of the Stone–Čech compactification βS of S (as we shall be later), one may also see this by the fact that, for any piecewise syndetic $A \subseteq S$, there are ultrafilters on S with A as a member, each of whose members is piecewise syndetic [15, Thm. 4.40 and Cor. 4.41].

The set of length l arithmetic progressions (including the constant ones) in $\mathbb{Z}$ forms a subgroup AP_l of $\mathbb{Z}^l$. As we have previously indicated, Furstenberg and Glasner showed that whenever B is a piecewise syndetic subset of $\mathbb{Z}$, the set of length l arithmetic progressions lying entirely within B (i.e., $AP_l \cap B^l$) is a piecewise syndetic subset of AP_l. In this paper we extend this result in two ways. First, we extend the result to arbitrary semigroups. As a consequence we shall see for example that an analogous statement applies to the Hales–Jewett theorem, a common generalization of the geometric Ramsey theorem and van der Waerden's theorem.

Second, we establish that it is valid for a large class of notions of largeness in addition to piecewise syndeticity.

A second notion of largeness which is good for us and, like piecewise syndeticity, is partition regular is that of "central." Central sets were introduced by Furstenberg [7], who defined them in terms of notions of topological dynamics. These sets enjoy very strong combinatorial properties. (See [7, Prop. 8.21] or [15, Chap. 14].) They have a nice characterization in terms of the algebraic structure of βS, the Stone–Čech compactification of the discrete semigroup S. We shall present this characterization below, after introducing the necessary background information.

We take the points of βS to be the ultrafilters on S, the principal ultrafilters being identified with the points of S. Given a set $A \subseteq S$, $\bar{A} = \{p \in \beta S : A \in p\}$. The set $\{\bar{A}: A \subseteq S\}$ is a basis for the open sets (as well as a basis for the closed sets) of βS. (We shall restrict our use of the notation $\bar{A}$ to the closure of a set in βS, writing $c\ell_Y(A)$, for example, for the closure of A in the space Y.)

There is a natural extension of the operation of S to βS, customarily denoted by the same symbol, making βS a compact right topological semigroup with S contained in its topological center. (If the operation is "$\cdot$" this says that for each $p \in \beta S$ the function $\rho_p: \beta S \to \beta S$ is continuous and for each $x \in S$, the function $\lambda_x: \beta S \to \beta S$ is continuous, where $\rho_p(q) = q \cdot p$ and $\lambda_x(q) = x \cdot q$.) See [15] for an elementary introduction to the semigroup βS as well as for any unfamiliar algebraic assertions encountered here.

Any compact Hausdorff right topological semigroup T has a smallest two sided ideal $K(T)$ which is the union of all of the minimal left ideals of T, each of which is closed [15, Thm. 2.8] and any compact right topological semigroup contains idempotents. Since the minimal left ideals are themselves compact right topological semigroups, this says that there are idempotents in the smallest ideal.

1.3. DEFINITION. Let $(S, \cdot)$ be a semigroup and let $A \subseteq S$. The set A is *central* if and only if there is some idempotent $p \in K(\beta S)$ such that $A \in p$.

See [15, Thm. 19.27] for a proof of the equivalence of the definition above with the original dynamical definition. (In [9, Prop. 4.6] Glasner anticipated this result by showing that, if S is a countable Abelian group, then a subset of S is central as defined above if and only if it satisfies conditions similar to Furstenberg's dynamical definition of "central.")

We shall now introduce two more partition regular notions (Δ and IP) which in certain settings are themselves appropriate notions of largeness, but, like syndetic, are not good for us because, as we shall see in Theorem 3.8, our main result is not valid for these notions. Some other notions which we shall consider arise as duals of these (and are good for us).

Given a sequence $\langle x_n \rangle_{n=1}^{\infty}$ in a semigroup S, let $FP(\langle x_n \rangle_{n=1}^{\infty}) = \{\Pi_{n \in F}\, x_n : F \in \mathcal{P}_f(\mathbb{N})\}$ where the product $\Pi_{n \in F}\, x_n$ is taken in increasing order of indices. (If the operation in S is denoted by "$+$" then $FS(\langle x_n \rangle_{n=1}^{\infty})$ is defined analogously.)

1.4. DEFINITION. Let $(S, \cdot)$ be a semigroup and let $A \subseteq S$. The set A is an *IP set* if and only if there exists a sequence $\langle x_n \rangle_{n=1}^{\infty}$ in S such that $FP(\langle x_n \rangle_{n=1}^{\infty}) \subseteq A$.

In $\mathbb{N}$, a Δ set can be defined as one containing the set of differences $\{s_m - s_n : m, n \in \mathbb{N} \text{ and } n < m\}$ for some sequence $\langle s_n \rangle_{n=1}^{\infty}$. This notion can be extended in an obvious fashion to any semigroup which can be embedded in a group, namely that the set contains $\{s_n{}^{-1} s_m : m, n \in \mathbb{N}$ and $n < m\}$. A slight adjustment yields what we believe is the appropriate notion in an arbitrary semigroup. (Notice that the definition agrees with the one above if S is embeddable in a group.)

1.5. DEFINITION. Let $(S, \cdot)$ be a semigroup and let $A \subseteq S$. Then A is a *Δ set* if and only if there exists a sequence $\langle s_n \rangle_{n=1}^{\infty}$ in S such that for every $n, m \in \mathbb{N}$ with $n < m$, $s_m \in s_n \cdot A$.

Notice that any IP set is a Δ set. (If $FP(\langle x_n \rangle_{n=1}^{\infty}) \subseteq A$, $s_n = \Pi_{t=1}^{n}\, x_t$, and $n < m$, then $s_m = s_n \cdot \Pi_{t=n+1}^{m}\, x_t \in s_n \cdot A$.)

The following notion will be used to algebraically characterize Δ sets.

1.6. Definition. Let $(S, \cdot)$ be a semigroup and let $p \in \beta S$. Then $D(p) = \{q \in \beta S : \text{for all } A \in q, \{x \in S : x \cdot A \in p\} \in p\}$.

If G is a group, $p \in \beta G$, and $p^{-1} = \{A^{-1} : A \in p\}$, where $A^{-1} = \{x^{-1} : x \in A\}$, then $p^{-1} \cdot p = \{A \subseteq G : \{x \in G : x \cdot A \in p\} \in p\}$, and consequently, $D(p) = \{p^{-1} \cdot p\}$.

1.7. Lemma. *Let $(S, \cdot)$ be a semigroup, let $A \subseteq S$, and let $p \in \beta S$. Then $\bar{A} \cap D(p) \neq \emptyset$ if and only if $\{x \in S : x \cdot A \in p\} \in p$. (In particular, $D(p) \neq \emptyset$ if and only if $\{x \in S : x \cdot S \in p\} \in p$.)*

Proof. The necessity is trivial.

For the sufficiency, let $\mathcal{A} = \{A\} \cup \{B \subseteq S : \{x \in S : x \cdot (S \setminus B) \in p\} \notin p\}$. We claim that $\mathcal{A}$ has the finite intersection property. To see this let $\mathcal{F} \in \mathcal{P}_f(\mathcal{P}(S))$ such that for each $B \in \mathcal{F}$, $\{x \in S : x \cdot (S \setminus B) \in p\} \notin p$. Then for each $B \in \mathcal{F}$, $\{x \in S : x \cdot (S \setminus B) \notin p\} \in p$. Since also $\{x \in S : x \cdot A \in p\} \in p$, pick $x \in S$ such that $x \cdot A \in p$ and for all $B \in \mathcal{F}$, $x \cdot (S \setminus B) \notin p$. Pick $z \in x \cdot A \cap \bigcap_{B \in \mathcal{F}} (S \setminus x \cdot (S \setminus B))$ and pick $y \in A$ such that $z = x \cdot y$. Then $y \in A \cap (\bigcap \mathcal{F})$.

Since $\mathcal{A}$ has the finite intersection property, pick $q \in \beta S$ such that $\mathcal{A} \subseteq q$. Then $A \in q$ and $q \in D(p)$. ∎

Given any property $\mathcal{E}$ of subsets of a set X, there is a dual property $\mathcal{E}^*$ defined by specifying that a subset B of X is an $\mathcal{E}^*$ set if and only if $B \cap A \neq \emptyset$ for every $\mathcal{E}$ set A.

1.8. Definition. Let $(S, \cdot)$ be a semigroup and let $B \subseteq S$. Then B is a *central** set if and only if $B \cap A \neq \emptyset$ for every central set A in S. Also, B is a *PS** set if and only if $B \cap A \neq \emptyset$ for every piecewise syndetic set A in S, B is an *IP** set if and only if $B \cap A \neq \emptyset$ for every IP set A in S, B is a *syndetic** set if and only if $B \cap A \neq \emptyset$ for every syndetic set A in S, and B is a Δ^* set if and only if $B \cap A \neq \emptyset$ for every Δ set A in S.

The concept of "syndetic*" is more commonly referred to as "thick", and we shall follow this practice.

1.9. Lemma. *Let $(S, \cdot)$ be a semigroup and let $A \subseteq S$. Let $P = \{p \in \beta S : p \cdot p = p\}$ and let $Q = \{p \in K(\beta S) : p \cdot p = p\}$.*

(a) *A is a Δ set if and only if there is some $p \in \beta S$ such that $\bar{A} \cap D(p) \neq \emptyset$.*

(b) *A is piecewise syndetic if and only if $\bar{A} \cap K(\beta S) \neq \emptyset$.*

(c) *A is IP if and only if* $\bar{A} \cap P \neq \varnothing$.

(d) *A is syndetic if and only if for every left ideal L of* βS, $\bar{A} \cap L \neq \varnothing$.

(e) *A is central if and only if* $\bar{A} \cap Q \neq \varnothing$.

(f) *A is central* if and only if* $Q \subseteq \bar{A}$.

(g) *A is thick if and only if* $\bar{A}$ *contains a left ideal of* βS.

(h) *A is IP* if and only if* $P \subseteq \bar{A}$.

(i) *A is PS* if and only if* $K(\beta S) \subseteq \bar{A}$.

(j) *A is a* Δ^* *set if and only if whenever* $p \in \beta S$, $D(p) \subseteq \bar{A}$.

Proof. (a). Necessity. Choose a sequence $\langle s_n \rangle_{n=1}^{\infty}$ in S such that for all $n, m \in \mathbb{N}$ with $n < m$, $s_m \in s_n \cdot A$. Pick $p \in \beta S$ such that

$$\{\{s_m : m > n\} : n \in \mathbb{N}\} \subseteq p.$$

Then $\{s_n : n \in \mathbb{N}\} \subseteq \{x \in S : x \cdot A \in p\}$ and so $\bar{A} \cap D(p) \neq \varnothing$ by Lemma 1.7.

Sufficiency. By Lemma 1.7, $B = \{x \in S : x \cdot A \in p\} \in p$. Choose $s_1 \in B$ and inductively, given $n \in \mathbb{N}$, choose $s_{n+1} \in B \cap \bigcap_{t=1}^{n} s_t \cdot A$.

Statement (b) is [15, Thm. 4.40], (c) is [15, Thm. 5.12], (d) is [3, Thm. 2.9(d)], and (e) is the definition of central. Statements (f), (g), (h), (i), and (j) follow easily from statements (e), (d), (c), (b), and (a), respectively. ▮

As a consequence of Lemma 1.9, and the observation already made that any IP set is a Δ set, we see that the pattern of implications given below holds.

$$\begin{tikzcd}
\Delta^* \arrow[d] & & \\
\mathrm{IP}^* \arrow[dr] & & \mathrm{PS}^* \arrow[dl] \arrow[d] \\
& \text{central}^* \arrow[dl] \arrow[d] & \text{thick}=\text{syndetic}^* \arrow[dl] \\
\text{syndetic} \arrow[d] & \text{central} \arrow[dl] \arrow[dr] & \\
\text{piecewise syndetic} & & \mathrm{IP} \arrow[d] \\
& & \Delta
\end{tikzcd}$$

That none of the missing implications is valid in general can be seen by considering the following table. Next to each property is listed a subset of $\mathbb{N}$ which has that property in the semigroup $(\mathbb{N}, +)$, but has only those of the other properties that it is forced to have by the implications in the above diagram.

Property	Set with no extra properties
Δ	$\{2^n - 2^m : n, m \in \mathbb{N}$ and $m < n\}$
IP	$\{\Sigma_{n \in F} 2^{2n} : F \in \mathscr{P}_f(\mathbb{N})\}$
Piecewise syndetic	$\{2^n + 2m - 1 : n, m \in \mathbb{N}$ and $m < n\}$
Central	$\{2^n + 2m : n, m \in \mathbb{N}$ and $m < n\}$
Syndetic	$\{2n + 1 : n \in \omega\}$
Thick	$\{2^n + m : n, m \in \mathbb{N}$ and $m < n\}$
Central*	$\{2n : n \in \mathbb{N}\} \setminus \{\Sigma_{n \in F} 2^{2n} : F \in \mathscr{P}_f(\mathbb{N})\}$
PS*	$\mathbb{N} \setminus \{\Sigma_{n \in F} 2^{2n} : F \in \mathscr{P}_f(\mathbb{N})\}$
IP*	$\{2n : n \in \mathbb{N}\} \setminus \{2^n - 2^m : n, m \in \mathbb{N}$ and $m < n\}$
Δ*	$\{2n : n \in \mathbb{N}\}$

To see, for example, that $\{2^n + 2m : n, m \in \mathbb{N}$ and $m < n\}$ is central, note that it is the intersection of $\{2^n + m : n, m \in \mathbb{N}$ and $m < 2n\}$ (which is thick, so that its closure contains a minimal left ideal, hence a minimal idempotent) with $\{2m : m \in \mathbb{N}\}$ which is IP*.

Finally, we introduce an infinite sequence of partition regular notions (none of which is good for our purposes, but all of whose duals are).

1.10. Definition. Let $n \in \mathbb{N} \setminus \{1\}$ and let S be a semigroup. A set $A \subseteq S$ is an *IP_n set* if and only if whenever $\mathscr{F}$ is a finite partition of A, there exist $F \in \mathscr{F}$ and $x_1, x_2, ..., x_n \in S$ such that $FP(\langle x_t \rangle_{t=1}^n) \subseteq F$. A set $B \subseteq S$ is an *IP_n* set* if and only if $B \cap A \neq \emptyset$ for every IP_n set A.

1.11. Definition. Let S be a semigroup. A set $A \subseteq S$ is an *$IP_{<\omega}$ set* if and only if whenever $\mathscr{F}$ is a finite partition of A and $n \in \mathbb{N}$, there exist $F \in \mathscr{F}$ and $x_1, x_2, ..., x_n \in S$ such that $FP(\langle x_t \rangle_{t=1}^n) \subseteq F$. A set $B \subseteq S$ is an *$IP_{<\omega}$* set* if and only if $B \cap A \neq \emptyset$ for every $\mathrm{IP}_{<\omega}$ set A.

The following pattern of implications holds among the properties just introduced, where the dashed arrows indicate an implication valid in any left cancellative semigroup. (We leave it to the reader to amuse himself or herself by showing that these implications are not valid in general.)

$$
\begin{tikzcd}
& \mathrm{IP} \arrow[d] \arrow[ddl] & \mathrm{IP}_2^* \arrow[d] \arrow[ddr, dashed] & \\
& \mathrm{IP}_{<\omega} \arrow[d] & \mathrm{IP}_3^* \arrow[d] & \\
\Delta \arrow[dddr, dashed] & \vdots \arrow[d] & \mathrm{IP}_4^* \arrow[d] & \Delta^* \arrow[dddl] \\
& \mathrm{IP}_4 \arrow[d] & \vdots \arrow[d] & \\
& \mathrm{IP}_3 \arrow[d] & \mathrm{IP}_{<\omega}^* \arrow[d] & \\
& \mathrm{IP}_2 & \mathrm{IP}^* &
\end{tikzcd}
$$

The validity of each of the implications is clear from the definitions except possibly the fact that any Δ set A in a left cancellative semigroup is an IP_2 set. To verify this, choose a sequence $\langle s_n \rangle_{n=1}^{\infty}$ in S with $s_m \in s_n \cdot A$ whenever $n < m$. For each such $n < m$ let $t_{n,m}$ be the unique member of A such that $s_m = s_n \cdot t_{n,m}$. Given $F \subseteq A$, let $B(F) = \{\{n, m\} : n < m$ and $t_{n,m} \in F\}$. Given a finite partition $\mathscr{F}$ of A, one has that $\{B(F) \colon F \in \mathscr{F}\}$ is a finite partition of the set of two element subsets of $\mathbb{N}$, so pick by Ramsey's theorem $k < n < m$ and $F \in \mathscr{F}$ with $\{k, n\}, \{k, m\}, \{n, m\} \in B(F)$. Then $s_m = s_n \cdot t_{n,m} = s_k \cdot t_{k,n} \cdot t_{n,m}$ and $s_m = s_k \cdot t_{k,m}$ and so $t_{k,m} = t_{k,n} \cdot t_{n,m}$.

It is a consequence of a theorem of Nešetřil and Rödl [16, Thm. 1.1] that in $(\mathbb{N}, +)$ there is for any $n \in \mathbb{N} \setminus \{1\}$ an IP_n set which is not an IP_{n+1} set. (See [14, Cor. 3.8] for a derivation of this consequence.) Also, $\{2^{2m} - 2^{2n} : n < m\}$ is a Δ set which is not an IP_3 set. That none of the other missing implications is valid is a consequence of the following result.

THEOREM 1.12. *There is an* $IP_{<\omega}$ *set in* $(\mathbb{N}, +)$ *which is not a* Δ *set.*

Proof. As a consequence of Folkman's Theorem [11, Thm. 3.11], given any $k, r \in \mathbb{N}$, there is some $m \in \mathbb{N}$ so that, whenever $FS(\langle x_t \rangle_{t=1}^{m}) = \bigcup_{i=1}^{r} B_i$, there exist $i \in \{1, 2, ..., r\}$ and $\langle y_t \rangle_{t=1}^{k}$ such that $FS(\langle y_t \rangle_{t=1}^{k}) \subseteq B_i$. (This fact can also be seen by applying a compactness argument to [15, Cor. 5.15].) Let

$$A = \left\{ \sum_{t \in F} 2^t : \text{there exists } m \in \mathbb{N} \text{ with } \emptyset \neq F \subseteq \{2^m + 1, 2^m + 2, ..., 2^{m+1}\} \right\}.$$

Then for any m, $FS(\langle 2^t \rangle_{t=2^m+1}^{2^{m+1}}) \subseteq A$ so, using the above fact, A is an $\mathrm{IP}_{<\omega}$ set.

Now suppose that A is a Δ set and pick a sequence $\langle s_t\rangle_{t=1}^{\infty}$ in $\mathbb{N}$ such that whenever $n<m$, $s_m-s_n\in A$ and note that necessarily the sequence $\langle s_t\rangle_{t=1}^{\infty}$ is increasing.

Pick r such that $s_1\leqslant 2^{r+1}$ and pick n such that $s_n>2^{r+2}$. Pick $H\in\mathcal{P}_f(\omega)$ such that $s_1=\sum_{t\in H}2^t$. Since $s_n-s_1\in A$ and $s_n-s_1>2^{r+1}$, there exist $l>r$ and $F\subseteq\{2^l+1, 2^l+2, ..., 2^{l+1}\}$ such that $s_n-s_1=\sum_{t\in F}2^t$. Since $\min F>\max H$, $s_n=\sum_{t\in F\cup H}2^t$.

Pick m such that $s_m>2^{l+1}+s_n$. Since $s_m-s_n\in A$ and $s_m-s_n>2^{l+1}$, there exist $k>l$ and $G\subseteq\{2^k+1, 2^k+2, ..., 2^{k+1}\}$ such that $s_m-s_n=\sum_{t\in G}2^t$. Since $\min G>\max F=\max(F\cup H)$, $s_m=\sum_{t\in G\cup F\cup H}2^t$. But then $s_m-s_1=\sum_{t\in G\cup F}2^t\notin A$, a contradiction. ∎

In Section 2 we present some results about preserving the "*" versions of partition regular notions in subsemigroups of a product of semigroups. The proofs as applied to the notions of "IP*," "$\mathrm{IP}_{<\omega}$*," "IP_n*," and "Δ*" are completely elementary, while the proofs for the notions of "PS*" and "central*" are a combination of elementary and algebraic methods. In Section 3 we present algebraic proofs establishing that the notions of "piecewise syndetic," "central," and "thick" are also often preserved in subsemigroups of a product of semigroups. In Section 4 we present some combinatorial consequences of these results.

2. PRESERVING THE "*" NOTIONS OF PARTITION REGULAR PROPERTIES IN PRODUCTS

Recall that a property is said to be "partition regular" provided that, whenever a set with that property is partitioned into finitely many parts, one of these parts must have the specified property. Of the properties that we have considered so far, the ones that are partition regular in any semigroup are "central," "piecewise syndetic," "IP," "$\mathrm{IP}_{<\omega}$," "IP_n," and "Δ". (That these must be partition regular is clear from the characterizations in Lemma 1.9 and Definitions 1.10 and 1.11. We have already noted that $\mathbb{N}$ can be divided into two sets neither of which is syndetic in $(\mathbb{N},+)$, and consequently none of the properties "Δ*," "IP*," "$\mathrm{IP}_{<\omega}$*," "IP_n*," "PS*," or "central*" is partition regular in $(\mathbb{N},+)$. The partition $\{2\mathbb{N}, 2\mathbb{N}-1\}$ of $\mathbb{N}$ shows that "thick" is not partition regular in $(\mathbb{N},+)$.)

In the following we write $I^{\diamond}$ (rather than simply I) for a subsemigroup of S^l for consistency of notation with the next section. When we say that $\mathscr{E}$ is a property which may be possessed by subsets of a semigroup, we mean properties, such as those we have been considering, whose definition depends on the particular semigroup in which the sets reside. By π_i we mean the projection onto the i^{th} coordinate.

2.1. LEMMA. *Let $\mathscr{E}$ be a partition regular property which may be possessed by subsets of a semigroup, let $(S, \cdot)$ be a semigroup, let $l \in \mathbb{N}$, and let $I^\diamond$ be a subsemigroup of S^l. Assume that for every $\mathscr{E}$ set A in $I^\diamond$ and every $i \in \{1, 2, ..., l\}$ $\pi_i[A]$ is an $\mathscr{E}$ set in S. For every $\mathscr{E}$ set A in $I^\diamond$, every $i \in \{1, 2, ..., l\}$, and every $\mathscr{E}^*$ set B in S, there exists an $\mathscr{E}$ set C in $I^\diamond$ such that $C \subseteq A$ and $\pi_i[C] \subseteq B$.*

Proof. Let such A, i, and B be given. Let $C_1 = \{x \in A : \pi_i(x) \in B\}$ and let $C_2 = \{x \in A : \pi_i(x) \notin B\}$. Since $\mathscr{E}$ is a partition regular property, pick $j \in \{1, 2\}$ such that C_j is an $\mathscr{E}$ set in $I^\diamond$. Then, by assumption $\pi_i[C_j]$ is an $\mathscr{E}$ set in S. Since B is an $\mathscr{E}^*$ set in S, $B \cap \pi_i[C_j] \neq \emptyset$ and thus $j = 1$. ∎

2.2. THEOREM. *Let $\mathscr{E}$ be a partition regular property which may be possessed by subsets of a semigroup, let $(S, \cdot)$ be a semigroup, let $l \in \mathbb{N}$, and let $I^\diamond$ be a subsemigroup of S^l. Statement* (a) *implies statement* (b). *If each superset of an $\mathscr{E}$ set in S is an $\mathscr{E}$ set, then statements* (a) *and* (b) *are equivalent.*

(a) *For every $\mathscr{E}$ set A in $I^\diamond$ and every $i \in \{1, 2, ..., l\}$, $\pi_i[A]$ is an $\mathscr{E}$ set in S.*

(b) *Whenever B is an $\mathscr{E}^*$ set in S, $B^l \cap I^\diamond$ is an $\mathscr{E}^*$ set in $I^\diamond$.*

Proof. (a) implies (b). Let A be an $\mathscr{E}$ set in $I^\diamond$. We need to show that $B^l \cap I^\diamond \cap A \neq \emptyset$. Pick by Lemma 2.1 an $\mathscr{E}$ set $C_1 \subseteq A$ in $I^\diamond$ such that $\pi_1[C_1] \subseteq B$. Inductively, let $i \in \{1, 2, ..., l-1\}$ be given and assume that we have chosen an $\mathscr{E}$ set C_i in $I^\diamond$. Pick by Lemma 2.1 an $\mathscr{E}$ set $C_{i+1} \subseteq C_i$ in $I^\diamond$ such that $\pi_{i+1}[C_{i+1}] \subseteq B$. Having chosen C_l, one has then that for each $i \in \{1, 2, ..., l\}$, $\pi_i[C_l] \subseteq B$. Pick $x = (x_1, x_2, ..., x_l) \in C_l$. Then $x \in B^l \cap I^\diamond \cap A$.

Now assume that each superset of an $\mathscr{E}$ set in S is an $\mathscr{E}$ set. To see that (b) implies (a), let A be an $\mathscr{E}$ set in $I^\diamond$, let $i \in \{1, 2, ..., l\}$, and suppose that $\pi_i[A]$ is not an $\mathscr{E}$ set in S. Let $B = S \backslash \pi_i[A]$. Since supersets of $\mathscr{E}$ sets are $\mathscr{E}$ sets, B is an $\mathscr{E}^*$ set in S. ∎

2.3. COROLLARY. *Let $(S, \cdot)$ be a semigroup, let $l \in \mathbb{N}$, and let $I^\diamond$ be a subsemigroup of S^l.*

(a) *If B is an IP* set in S, then $B^l \cap I^\diamond$ is an IP* set in $I^\diamond$.*

(b) *If B is an $IP_{<\omega}$* set in S, then $B^l \cap I^\diamond$ is an $IP_{<\omega}$* set in $I^\diamond$.*

(c) *If $n \in \mathbb{N}\backslash\{1\}$ and B is an $IP_n{}^*$ set in S, then $B^l \cap I^{\diamond}$ is an $IP_n{}^*$ set in $I^{\diamond}$.*

(d) *If B is a Δ^* set in S, then $B^l \cap I^{\diamond}$ is a Δ^* set in $I^{\diamond}$.*

Proof. One only needs to note, as is immediate from the definitions, that whenever A is an IP set in $I^{\diamond}$ and $i \in \{1, 2, ..., l\}$, then $\pi_i[A]$ is an IP set in S; whenever A is an $\mathrm{IP}_{<\omega}$ set in $I^{\diamond}$ and $i \in \{1, 2, ..., l\}$, then $\pi_i[A]$ is an $\mathrm{IP}_{<\omega}$ set in S; whenever $n \in \mathbb{N}$, A is an IP_n set in $I^{\diamond}$, and $i \in \{1, 2, ..., l\}$, then $\pi_i[A]$ is an IP_n set in S; and whenever A is a Δ set in $I^{\diamond}$ and $i \in \{1, 2, ..., l\}$, then $\pi_i[A]$ is a Δ set in S. ∎

There are many other notions of largeness which are partition regular in many semigroups for which the conclusions of Corollary 2.3 apply to their duals. Consider for example the following property of some subsets of $\mathbb{N}$, which makes sense in any commutative semigroup $(S, +)$. Given $n \in \mathbb{N}$, let $\mathscr{A}_n = \{A \subseteq S$: whenever $\mathscr{F}$ is a finite partition of A, some $F \in \mathscr{F}$ contains a length n arithmetic progression$\}$. It is a result of Spencer [17] that for any $k, n \in \mathbb{N}$ there is a subset of $\mathbb{N}$ which contains no length $n+1$ arithmetic progression but, whenever it is partitioned into k cells, one cell contains a length n arithmetic progression. It is then not hard to show that in $(\mathbb{N}, +)$, $\mathscr{A}_n \nsubseteq \mathscr{A}_{n+1}$. (See [2].) It is immediate that statement (a) of Theorem 2.2 holds for members of $\mathscr{A}_n$ defined in terms of $I^{\diamond}$.

The conclusions of Corollary 2.3 are very strong, applying to *any* subsemigroup of S^l. The following result shows that such strong conclusions are not valid for any of the other notions that we have been considering, even when S is commutative, cancellative, and finitely generated. (We shall, however, see in Corollary 2.7 that a version of Corollary 2.3 does hold for PS* sets and for central* sets in $\mathbb{N}^l$.) Recall that $\omega = \mathbb{N} \cup \{0\}$. In the following theorem, the omission of $\{(0, 0)\}$ is not essential. We do it so that S will be the free commutative semigroup with two generators. (The same result, with the same $I^{\diamond}$, is in fact valid in $\mathbb{N} \times \mathbb{N}$.)

2.4. THEOREM. *Let $S = (\omega \times \omega)\backslash\{(0, 0)\}$ (under addition), let $I^{\diamond} = \{(n, 2n) : n \in \mathbb{N}\}$, let $l = 1$, and let $B = S\backslash I^{\diamond}$. Then $I^{\diamond}$ is a subsemigroup of S^l and B is PS* in S, but $B^l \cap I^{\diamond} = \varnothing$.*

Proof. We need only show that $I^{\diamond}$ is not piecewise syndetic in S, so that B is PS*. So suppose instead that one has some $G \in \mathscr{P}_f(S)$ such that for every $F \in \mathscr{P}_f(S)$ there exists $(x, y) \in S$ with $F + (x, y) \subseteq \bigcup_{(s,t) \in G} (-(s, t) + I^{\diamond})$. Let $m = \max\{|2s - t| : (s, t) \in G\}$. Let $F = \{(1, 1), (m+2, 1)\}$ and pick $(x, y) \in S$ such that $F + (x, y) \subseteq \bigcup_{(s,t) \in G} (-(s, t) + I^{\diamond})$. Pick (s, t) and (u, v) in G such that $(s + 1 + x,\ t + 1 + y) \in I^{\diamond}$ and

$(u+m+2+x,\ v+1+y)\in I^{\diamond}$. Thus $t+1+y=2s+2+2x$ so that $y=1+2x+2s-t\leqslant 1+2x+m$. But also $v+1+y=2u+2m+4+2x$ so that $y=3+2x+2m+2u-v\geqslant 3+2x+m$, a contradiction. ∎

By requiring a little more of $I^{\diamond}$, we shall see that we can extend the conclusions of Corollary 2.3 to the "*" versions of our other partition regular notions.

2.5. LEMMA. *Let $(S,\cdot)$ and $(T,\cdot)$ be discrete semigroups and let $\varphi\colon S\to T$ be a surjective homomorphism.*

(a) *If A is piecewise syndetic in S, then $\varphi[A]$ is piecewise syndetic in T.*

(b) *If A is central in S, then $\varphi[A]$ is central in T.*

Proof. Let $\tilde{\varphi}\colon \beta S\to\beta T$ be the continuous extension of φ and note that by [15, Lemma 2.14], $\tilde{\varphi}$ is a homomorphism. We know by [15, Exercise 1.7.3] that $\tilde{\varphi}[K(\beta S)]=K(\beta T)$.

(a) Pick by Lemma 1.9(b) some $p\in\bar{A}\cap K(\beta S)$. Then $\tilde{\varphi}(p)\in\overline{\varphi[A]}\cap K(\beta T)$.

(b) Pick by Lemma 1.9(e) some idempotent $p\in A\cap K(\beta S)$. Then $\tilde{\varphi}(p)$ is an idempotent in $\overline{\varphi[A]}\cap K(\beta T)$. ∎

A consideration of the proof of Theorem 2.4 shows that the added hypothesis to Theorem 2.6 is exactly what is required.

2.6. THEOREM. *Let $(S,\cdot)$ be a semigroup, let $l\in\mathbb{N}$, and let $I^{\diamond}$ be a subsemigroup of S^{l}. Assume that for each $i\in\{1,2,...,l\}$, $\pi_i[I^{\diamond}]$ is piecewise syndetic in S.*

(a) *If B is PS^* in S, then $B^{l}\cap I^{\diamond}$ is PS^* in $I^{\diamond}$.*

(b) *If B is central* in S, then $B^{l}\cap I^{\diamond}$ is central* in $I^{\diamond}$.*

Proof. Let $i\in\{1,2,...,l\}$. By Theorem 2.2 it suffices to show that whenever A is piecewise syndetic in $I^{\diamond}$, $\pi_i[A]$ is piecewise syndetic in S and whenever A is central in $I^{\diamond}$, $\pi_i[A]$ is central in S. Since $\pi_i[I^{\diamond}]$ is piecewise syndetic in S, we have that $\overline{\pi_i[I^{\diamond}]}\cap K(\beta S)\neq\emptyset$. Consequently, by [15, Theorem 1.65] $K(\beta(\pi_i[I^{\diamond}]))=K(\overline{\pi_i[I^{\diamond}]})=\overline{\pi_i[I^{\diamond}]}\cap K(\beta S)$.

Now assume that A is piecewise syndetic in $I^{\diamond}$. Then by Lemma 2.5, $\pi_i[A]$ is piecewise syndetic in $\pi_i[I^{\diamond}]$. Thus $\overline{\pi_i[A]}\cap K(\beta(\pi_i[I^{\diamond}]))\neq\emptyset$ and consequently $\overline{\pi_i[A]}\cap K(\beta S)\neq\emptyset$.

Finally assume that A is central in $I^{\diamond}$. Then by Lemma 2.5, $\pi_i[A]$ is central in $\pi_i[I^{\diamond}]$. Pick an idempotent $p\in\overline{\pi_i[A]}\cap K(\beta(\pi_i[I^{\diamond}]))$. Then $p\in\overline{\pi_i[A]}\cap K(\beta S)$. ∎

Theorem 2.4 established that, even for a countable commutative semigroup, restrictions need to be placed on $I^\diamond$ in order to get the PS* and central* conclusions. We see now, however, that in $(\mathbb{N}, +)$ no restrictions are needed.

2.7. COROLLARY. *Let $l \in \mathbb{N}$, and let $I^\diamond$ be a subsemigroup of $\mathbb{N}^l$.*

(a) *If B is a PS* set in S, then $B^l \cap I^\diamond$ is a PS* set in $I^\diamond$.*

(b) *If B is a central* set in S, then $B^l \cap I^\diamond$ is a central* set in $I^\diamond$.*

Proof. By Theorem 2.6, it suffices to note that for any $i \in \{1, 2, ..., l\}$, $\pi_i[I^\diamond]$ is piecewise syndetic. In fact, picking $(x_1, x_2, ..., x_l) \in I^\diamond$, one has $\mathbb{N}x_i \subseteq \pi_i[I^\diamond]$, so in fact $\pi_i[I^\diamond]$ is IP*. (See [15, Lemma 16.13].) ∎

Corollary 2.7 raises its own questions. What about arbitrary semigroups of $\mathbb{N}^l$ with respect to thick, syndetic, central, piecewise syndetic, IP, and Δ sets in $\mathbb{N}$? Another question is raised by Theorem 2.6. Namely, if $\pi_i[I^\diamond]$ is syndetic in S for each $i \in \{1, 2, ..., l\}$ and B is thick in S, must $B^l \cap I^\diamond$ be thick in $I^\diamond$? (The proof of Theorem 2.7, which invokes Theorem 2.2, does not work because in Theorem 2.2, $\mathscr{E}$ needed to be a partition regular property, which "syndetic" is not.)

The following example (whose routine proof we omit) answers all but one of these questions, namely whether $B^l \cap I^\diamond$ must be syndetic in $I^\diamond$ whenever B is syndetic in $\mathbb{N}$. That question is answered by an even more trivial example, wherein $I^\diamond = \{(2n, 2m) : n, m \in \mathbb{N}\}$ and $B = \mathbb{N}2 + 1$.

2.8. THEOREM. *Let $I^\diamond = \{(a, 2a) \colon a \in \mathbb{N}\}$ and let $B = \{2^{2n} + i : n \in \mathbb{N}$ and $i \in \{1, 2, ..., n\}\}$. Then $I^\diamond$ is a subsemigroup of $\mathbb{N}^2$, $\pi_1[I^\diamond]$ and $\pi_2[I^\diamond]$ are syndetic in $\mathbb{N}$, and B is thick in $\mathbb{N}$, but $B^2 \cap I^\diamond = \varnothing$.*

3. PRESERVING PIECEWISE SYNDETIC, CENTRAL, AND THICK SETS IN PRODUCTS

Somewhat more delicate machinery is required to show that the notions of "piecewise syndetic," "central," and "thick," none of which is the dual of a partition regular property, are preserved in products.

Throughout this section we shall assume that we have a fixed semigroup S, a fixed $l \in \mathbb{N}$, a subsemigroup $E^\diamond$ of S^l with $\{(a, a, ..., a) : a \in S\} \subseteq E^\diamond$, and a two sided ideal $I^\diamond$ of $E^\diamond$.

3.1. DEFINITION. Let $Y = (\beta S)^l$ with the product topology and the coordinatewise operation. Then $E = c\ell_Y\, E^\diamond$ and $I = c\ell_Y\, I^\diamond$. Given $p \in \beta S$, $\bar{p} = (p, p, ..., p) \in Y$.

3.2. LEMMA. *Y is a compact right topological semigroup, $\lambda_{\vec{x}}$ is continuous for each $\vec{x} \in S^l$, E is a subsemigroup of Y, I is an ideal of E, and $K(Y) = (K(\beta S))^l$.*

Proof. [15, Ths. 2.22, 2.23, and 4.17]. ∎

3.3. LEMMA. *Let $p \in K(\beta S)$. Then $\bar{p} = (p, p, ..., p) \in K(I) = (K(\beta S))^l \cap E$.*

Proof. Observe first that for any $q \in \beta S$, $q = (q, q, ..., q) \in E$. To see this let U be a neighborhood of $\bar{q}$ in Y and for each $i \in \{1, 2, ..., l\}$, pick $A_i \in q$ such that $\times_{i=1}^{l} \overline{A_i} \subseteq U$. Then $\bigcap_{i=1}^{l} A_i \in q$ so pick $a \in \bigcap_{i=1}^{l} A_i$. Then $\bar{a} = (a, a, ..., a) \in E^{\diamond} \cap U$.

Thus we have $\bar{p} \in (K(\beta S))^l \cap E$. Since, by Lemma 3.2, $K(Y) = (K(\beta S))^l$, we thus have that $K(Y) \cap E \neq \emptyset$. Thus, by [15, Thm. 1.65], $K(E) = K(Y) \cap E = (K(\beta S))^l \cap E$.

Since I is an ideal of E, $K(E) \subseteq I$. Consequently, again by [15, Thm. 1.65], $K(I) = K(E) = (K(\beta S))^l \cap E$. Thus $\bar{p} \in K(I)$ as required. ∎

Now $I^{\diamond}$ itself is a discrete semigroup, and thus $\beta I^{\diamond}$ is a compact right topological semigroup with a smallest ideal $K(\beta I^{\diamond})$.

3.4. DEFINITION. $\iota: I^{\diamond} \to I^{\diamond} \subseteq I$ is the identity function and $\tilde{\iota}: \beta I^{\diamond} \to I$ is its continuous extension.

3.5. LEMMA. *The function $\tilde{\iota}$ is a homomorphism and $\tilde{\iota}\,[K(\beta I^{\diamond})] = K(I)$.*

Proof. That $\tilde{\iota}$ is a homomorphism follows from [15, Lemma 2.14]. Since $\tilde{\iota}$ is surjective, it is then an easy exercise (which is [15, Exercise 1.7.3]) to show that $\tilde{\iota}[K(\beta I^{\diamond})] = K(I)$. ∎

3.6. LEMMA. *If $r \in \beta I^{\diamond}$ and $\tilde{\iota}(r) \in B^l$, then $\bar{B}^l \cap I^{\diamond} \in r$.*

Proof. One has that $\bar{B}^l \cap I$ is a neighborhood of $\tilde{\iota}(r)$ so pick $C \in r$ such that $\tilde{\iota}[\bar{C}] \subseteq \bar{B}^l \cap I$. Then $C = \iota[C] \subseteq B^l \cap I^{\diamond}$. ∎

The following is the major result of this section. For the convenience of the reader, we restate therein our standing assumptions.

3.7. THEOREM. *Let $(S, \cdot)$ be a semigroup, let $l \in \mathbb{N}$, let $E^{\diamond}$ be a subsemigroup of S^l with $\{(a, a, ..., a) : a \in S\} \subseteq E^{\diamond}$, and let $I^{\diamond}$ be an ideal of $E^{\diamond}$. Let $B \subseteq S$.*

(a) *If B is piecewise syndetic in S, then $B^l \cap I^{\diamond}$ is piecewise syndetic in $I^{\diamond}$.*

(b) *If B is central in S, then $B^l \cap I^{\diamond}$ is central in $I^{\diamond}$.*

(c) *If B is thick in S, then $B^l \cap I^{\diamond}$ is thick in $I^{\diamond}$.*

(d) *If B is central* in S, then $B^l \cap I^\diamond$ is central* in $I^\diamond$.*

(e) *If B is PS* in S, then $B^l \cap I^\diamond$ is PS* in $I^\diamond$.*

(f) *If B is IP* in S, then $B^l \cap I^\diamond$ is IP* in $I^\diamond$.*

(g) *If B is $IP_{<\omega}$* in S, then $B^l \cap I^\diamond$ is $IP_{<\omega}$* in $I^\diamond$.*

(h) *If $n \in \mathbb{N}$ and B is IP_n* in S, then $B^l \cap I^\diamond$ is IP_n* in $I^\diamond$.*

(i) *If B is Δ* in S, then $B^l \cap I^\diamond$ is Δ* in $I^\diamond$.*

Proof. (a). Pick by Lemma 1.9, some $p \in K(\beta S)$ such that $B \in p$. Let $\bar{p} = (p, p, ..., p)$. By Lemma 3.3, $\bar{p} \in K(I)$. Pick by Lemma 3.5 some $r \in K(\beta I^\diamond)$ such that $\tilde{\imath}(r) = \bar{p}$. By Lemma 3.6, $B^l \cap I^\diamond \in r$.

(b) Pick by Lemma 1.9, some $p \in K(\beta S)$ such that $p = p \cdot p$ and $B \in p$. Let $\bar{p} = (p, p, ..., p)$. By Lemma 3.3, $\bar{p} \in K(I)$. Pick by Lemma 3.5 some $s \in K(\beta I^\diamond)$ such that $\tilde{\imath}(s) = \bar{p}$. Pick a minimal left ideal L of $\beta I^\diamond$ such that $s \in L$. Let $T = \{r \in L : \tilde{\imath}(r) = \bar{p}\}$. Then T is a compact subsemigroup of $\beta I^\diamond$ so pick an idempotent $r \in T$. By Lemma 3.6, $B^l \cap I^\diamond \in r$.

(c) Pick a left ideal L of βS such that $L \subseteq \bar{B}$. Since each left ideal contains a minimal left ideal [15, Corollary 2.6], we may presume that L is a minimal left ideal, and consequently $L \subseteq K(\beta S)$. Pick $p \in L$ and let $\bar{p} = (p, p, ..., p)$. By Lemma 3.3, $\bar{p} \in K(I)$. Pick by Lemma 3.5 some $r \in K(\beta I^\diamond)$ such that $\tilde{\imath}(r) = \bar{p}$. We claim that $(\beta I^\diamond) \cdot r \subseteq \overline{B^l \cap I^\diamond}$ for which it suffices by Lemma 3.6 to let $q \in (\beta I^\diamond) \cdot r$ and show that $\tilde{\imath}(q) \in \bar{B}^l$. Pick $v \in \beta I^\diamond$ such that $q = v \cdot r$. Then for some $s_1, s_2, ..., s_l \in \beta S$ we have $\tilde{\imath}(v) = (s_1, s_2, ..., s_l)$. Thus $\tilde{\imath}(q) = \tilde{\imath}(v) \cdot \tilde{\imath}(r) = (s_1, s_2, ..., s_l) \cdot (p, p, ..., p) = (s_1 \cdot p, s_2 \cdot p, ..., s_l \cdot p) \in L^l \subseteq \bar{B}^l$.

To establish statements (d) and (e), it suffices by Theorem 2.6 to let $i \in \{1, 2, ..., l\}$ and show that $\pi_i[I^\diamond]$ is piecewise syndetic in S. Pick $x = (x_1, x_2, ..., x_l) \in I^\diamond$. Then given any $a \in S$, $(a \cdot x_1, a \cdot x_2, ..., a \cdot x_l) \in I^\diamond$ so $\pi_i[I^\diamond]$ is in fact thick.

Statements (f), (g), (h), and (i) follow immediately from Corollary 2.3. ∎

Conspicuously absent from Theorem 3.7 are the analogous statements about Δ sets, IP sets, $\mathrm{IP}_{<\omega}$ sets, IP_n sets, and syndetic sets. In Theorems 3.8 and 3.9 we see why.

3.8. THEOREM. *Let $I^\diamond = \{(a, a+d, a+2d) : a, d \in \mathbb{N}\}$, let $E^\diamond = I^\diamond \cup \{(a, a, a) : a \in \mathbb{N}\}$, and let $B = FS(\langle 2^{2n} \rangle_{n=1}^{\infty})$. Then $E^\diamond$ is a subsemigroup of $\mathbb{N}^3$ (under addition), $I^\diamond$ is an ideal of $E^\diamond$, B is an IP set, hence an $IP_{<\omega}$ set, an IP_n set for each $n > 1$, and a Δ set in $\mathbb{N}$, but $B^3 \cap I^\diamond = \emptyset$.*

Proof. Suppose we have $a, d \in \mathbb{N}$ with $\{a, a+d, a+2d\} \subseteq B$. Pick $F, G, H, L \in \mathcal{P}_f(\omega)$ such that $a = \Sigma_{n \in F} 2^n$, $d = \Sigma_{n \in G} 2^n$, $a + d = \Sigma_{n \in H} 2^n$, and $a + 2d = \Sigma_{n \in L} 2^n$. Then $F \cup H \cup L \subseteq 2\mathbb{N}$. Consider $k = \min G$. If k is odd, then $k \in H$, while if k is even, then $k+1 \in L$. In either case we have a contradiction. ▮

It is remarked in [8] that there is a syndetic subset B of $\mathbb{Z}$ for which $B^3 \cap AP_3$ is not syndetic in AP_3. We see, in fact, that one can require also that this set be thick.

3.9. THEOREM. *Let* $I^\diamond = \{(a, a+d, a+2d) : a, d \in \mathbb{N}\}$, *let* $E^\diamond = I^\diamond \cup \{(a, a, a) : a \in \mathbb{N}\}$, *and let*

$$B = \mathbb{N} \setminus (\{2^{2n} + 2t - 1 : n \in \mathbb{N} \text{ and } t \in \{1, 2, ..., n\}\}$$
$$\cup \{2^{2n} + 2^{n+1} + 2t : n \in \mathbb{N} \text{ and } t \in \{1, 2, ..., 2n\}\}).$$

Then $E^\diamond$ *is a subsemigroup of* $\mathbb{N}^3$ *(under addition),* $I^\diamond$ *is an ideal of* $E^\diamond$, B *is a thick syndetic set in* $\mathbb{N}$, *but* $B^3 \cap I^\diamond$ *is not a syndetic set in* $I^\diamond$. *(Though, of course,* $B^3 \cap I^\diamond$ *is thick in* $I^\diamond$.*)*

Proof. Trivially B is syndetic (having no gaps longer than 1, in fact) and thick. Suppose that $B^3 \cap I^\diamond$ is syndetic set in $I^\diamond$ and pick $H \in \mathcal{P}_f(I^\diamond)$ such that $I^\diamond = \bigcup_{x \in H} -x + (B^3 \cap I^\diamond)$. Pick $n \in \mathbb{N}$ such that $H \subseteq \{(a, a+d, a+2d) : a, d \in \{1, 2, ..., n\}\}$ and let $y = (2^{2n}, 2^{2n} + 2^n, 2^{2n} + 2^{n+1})$. Pick $x = (a, a+d, a+2d) \in H$ such that $x + y = (2^{2n} + a, 2^{2n} + 2^n + a + d, 2^{2n} + 2^{n+1} + a + 2d) \in B^3$. Then $2^{2n} + a \in B$ so a is even. But then $2^{2n} + 2^{n+1} + a + 2d \notin B$, a contradiction. ▮

4. COMBINATORIAL APPLICATIONS

We first observe that a strengthening of the result of Furstenberg and Glasner cited in the Introduction is a corollary of Theorem 3.7.

4.1. COROLLARY. *Let* $B \subseteq \mathbb{Z}$, *let* $l \in \mathbb{N}$, *and let* $AP_l = \{(a, a+d, a+2d, ..., a+(l-1)d) : a, d \in \mathbb{Z}\}$. *Let "large" be any of "piecewise syndetic," "central," "central*," "thick," "PS*," "IP*," "*$IP_{<\omega}$**," "*IP_n**," or "*Δ**." If* B *is large in* $(\mathbb{Z}, +)$, *then* $B^l \cap AP_l$ *is large in* AP_l.

Proof. Let S be the group $(\mathbb{Z}, +)$, let $E^\diamond = I^\diamond = AP_l$, and apply Theorem 3.7. ▮

It is probably not surprising that we also obtain the corresponding result about the set of length l arithmetic progressions in $\mathbb{N}$ (where the constant arithmetic progressions are not included).

4.2. COROLLARY. *Let* $B \subseteq \mathbb{N}$, *let* $l \in \mathbb{N}$, *and let* $AP_l = \{(a, a+d, a+2d, ..., a+(l-1)d) : a, d \in \mathbb{N}\}$. *Let "large" be any of "piecewise syndetic,"* *"central," "central*," "thick," "PS*," "IP*," "$IP_{<\omega}$*," "IP_n*," or "Δ*." If B is large in $(\mathbb{N}, +)$, then $B^l \cap AP_l$ is large in AP_l.*

Proof. Let S be the semigroup $(\mathbb{N}, +)$, let $I^\diamond = AP_l$, let $E^\diamond = I^\diamond \cup \{(a, a, ..., a) : a \in \mathbb{N}\}$, and apply Theorem 3.7. ∎

Perhaps less obvious is the fact that the corresponding results about the Hales–Jewett theorem are also valid. Given an alphabet A with l letters and a "variable" $v \notin A$, a *variable word* $w(v)$ is a word over the alphabet $A \cup \{v\}$ in which v actually occurs. Given a variable word $w(v)$ and $a \in A$, $w(a)$ has its obvious meaning, namely the word in which all occurrences of v are replaced by a. The Hales–Jewett theorem [12] says that if the free semigroup on A is divided into finitely many cells, then there is some variable word $w(v)$ such that $\{w(a): a \in A\}$ is contained in one of these cells. (The set $\{w(a): a \in A\}$ is often called a *combinatorial line*.)

4.3. COROLLARY. *Let $l \in \mathbb{N}$, let $A = \{a_1, a_2, ..., a_l\}$ be an alphabet on l letters, let S be the free semigroup on the alphabet A, let $HJ_l = \{(w(a_1), w(a_2), ..., w(a_l)) : w(v)$ is a variable word$\}$, and let $B \subseteq S$. Let "large" be any of "piecewise syndetic," "central," "central*," "thick," "PS*," "IP*," "$IP_{<\omega}$*," "IP_n*," or "Δ*." If B is large in S, then $B^l \cap HJ_l$ is large in HJ_l.*

Proof. Let $I^\diamond = HJ_l$, let $E^\diamond = I^\diamond \cup \{(w, w, ..., w) : w \in S\}$, and apply Theorem 3.7. ∎

We saw in the proof of Theorem 3.8 that one could not let "large" be "IP," "$\text{IP}_{<\omega}$," "IP_n," or "Δ" in the extension of van der Waerden's theorem (Corollary 4.2). We observe that the same situation is true with respect to the extension of the Hales–Jewett theorem. That is, let $A = \{a, b\}$ and let $B = \{a^n: n \in \mathbb{N}\}$. Then B is an IP set but $B^2 \cap HJ_2 = \emptyset$.

We also saw in the proof of Theorem 3.9 that "large" could not be taken to be "syndetic." We see here that a similar conclusion applies to the Hales–Jewett theorem.

4.4. THEOREM. *Let S be the free semigroup on the alphabet $\{a, b\}$ and let $HJ_2 = \{(w(a), w(b)) : w(v)$ is a variable word over $\{a, b\}\}$. Let $B = \{u_1 u_2 ... u_t : t \in \mathbb{N} \setminus \{1\}, u_1, u_2, ..., u_t \in \{a, b\}$, and either $u_t = u_{t-1} = a$ or $u_1 \neq u_t\}$. Then B is both thick and syndetic in S but $B^2 \cap HJ_2$ is not syndetic in HJ_2.*

Proof. Since $Saa \subseteq B$ we have that B is thick and $S = a^{-1}B \cup b^{-1}B$ so B is syndetic.

Now suppose that we have $H \in \mathcal{P}_f(HJ_2)$ such that $HJ_2 = \bigcup_{x \in H} x^{-1}(B^2 \cap HJ_2)$. Let $w(v) = bv$. Then $(ba, bb) = (w(a), w(b)) \in HJ_2$ so pick $x \in H$ such that $x(ba, bb) \in B^2 \cap HJ_2$. Since $x \in H \subseteq HJ_2$ pick a variable word $z(v)$ such that $x = (z(a), z(b))$. Then $z(a)\,ba \in B$ so the leftmost letter of $z(v)$ is b (not v) and thus the leftmost letter of $z(b)\,bb$ is b, contradicting the fact that $z(b)\,bb \in B$. ▮

Other related results also follow. As a sample, consider the following kind of subset of an arithmetic progression: $\{a, a+2d, a+3d, a+5d, a+7d, a+11d\}$. Since such a configuration is contained in a length 12 arithmetic progression, we know that we can find such in any piecewise syndetic set. But Theorem 3.7 tells us more. It tells us that if $I^{\diamond} = \{(a, a+2d, a+3d, a+5d, a+7d, a+11d) : a, d \in \mathbb{N}\}$ and $B \subseteq \mathbb{N}$ is large in any of the senses we have been discussing, then $B^6 \cap I^{\diamond}$ is large in the same sense in $I^{\diamond}$.

We close with a discussion of a problem that we can't solve. Consider the following result.

4.5. THEOREM. *Let A be a piecewise syndetic subset of $\mathbb{N}$, let $l \in \mathbb{N}$, and let $p_1, p_2, ..., p_l$ be polynomials such that for each $i \in \{1, 2, ..., l\}$ and each $n \in \mathbb{Z}$, $p_i(n) \in \mathbb{Z}$ and $p_i(0) = 0$. Then there exist a and n in $\mathbb{N}$ such that $\{a + p_1(n), a + p_2(n), ..., a + p_l(n)\} \subseteq A$. In fact, $\{n \in \mathbb{N} : \{a \in \mathbb{N} : \{a + p_1(n), a + p_2(n), ..., a + p_l(n)\} \subseteq A\}$ is piecewise syndetic$\}$ is an IP* set.*

Proof. This is a consequence of [4, Cor. 1.12]. For a proof using the algebra of $\beta\mathbb{N}$ see [13]. ▮

One would like to obtain analogues of Corollaries 4.1, 4.2, and 4.3 for polynomial progressions of the form $a + p_1(n), a + p_2(n), ..., a + p_l(n)$. An obstacle to obtaining such analogues is the fact that $\{(a + p_1(n), a + p_2(n), ..., a + p_l(n)) : a \in \mathbb{N}$ and $n \in \mathbb{N} \cup \{0\}\}$ is not a semigroup (unless each p_i is linear). Consider, however the following result.

4.6. THEOREM. *Let $B \subseteq \mathbb{N}$ and let $l \in \mathbb{N}$. Let "large" be any of "piecewise syndetic," "central," "central*," "thick," "PS*," "IP*," "$IP_{<\omega}$*," "IP_n*," or "Δ*." If B is large in $\mathbb{N}$, then $\{(a, d) \in \mathbb{N} \times \mathbb{N} : \{a, a+d, a+2d, ..., a+(l-1)\,d\} \subseteq B\}$ is large in $\mathbb{N} \times \mathbb{N}$.*

Proof. Let $AP_l = \{(a, a+d, a+2d, ..., a+(l-1)\,d) : a, d \in \mathbb{N}\}$. Then the function $\psi\colon \mathbb{N} \times \mathbb{N} \to AP_l$ defined by $\psi(a, d) = (a, a+d, a+2d, ..., a+(l-1)\,d)$ is an isomorphism so the conclusion follows from Corollary 4.2. ▮

4.7. QUESTION. *Let $l \in \mathbb{N}$, and let $p_1, p_2, ..., p_l$ be polynomials such that for each $i \in \{1, 2, ..., l\}$ and each $n \in \mathbb{Z}$, $p_i(n) \in \mathbb{Z}$ and $p_i(0) = 0$. For which, if any, of the notions of "piecewise syndetic," "central," "central*," "thick," "PS*," "IP*," "$IP_{<\omega}$*," "IP_n*," or "Δ*," is it true that whenever B is a large subset of $\mathbb{N}$, $\{(a, n) \in \mathbb{N} \times \mathbb{N} : \{a + p_1(n), a + p_2(n), ..., a + p_l(n)\} \subseteq B\}$ is large in the same sense?*

ACKNOWLEDGMENTS

The authors thank H. Furstenberg and S. Glasner for some helpful conversations.

REFERENCES

1. V. Bergelson, Ergodic Ramsey theory–An update, *in* "Ergodic Theory of $\mathbb{Z}^d$-actions," (M. Pollicott and K. Schmidt, Eds.), pp. 1–61, London Math. Soc. Lecture Note Series, Vol. 228, London Math. Soc., London, 1996.
2. V. Bergelson, N. Hindman, and I. Leader, Sets partition regular for n equations need not solve $n+1$, *Proc. London Math. Soc.* **73** (1996), 481–500.
3. V. Bergelson, N. Hindman, and R. McCutcheon, Notions of size and combinatorial properties of quotient sets in semigroups, *Topology Proc.*, to appear.
4. V. Bergelson and A. Leibman, Polynomial extensions of van der Waerden's and Szemerédi's theorems, *J. Amer. Math. Soc.* **9** (1996), 725–753.
5. T. Brown, On locally finite semigroups, *Ukrainian Math. J.* **20** (1968), 732–738. [in Russian]
6. T. Brown, An interesting combinatorial method in the theory of locally finite semigroups, *Pacific J. Math.* **36** (1971), 285–289.
7. H. Furstenberg, "Recurrence in Ergodic Theory and Combinatorial Number Theory," Princeton Univ. Press, Princeton, 1981.
8. H. Furstenberg and E. Glasner, Subset dynamics and van der Waerden's theorem, *Contemp. Math.* **215** (1998), 197–203.
9. S. Glasner, Divisibility properties and the Stone–Čech compactification, *Canad. J. Math.* **32** (1980), 993–1007.
10. R. Graham, K. Leeb, and B. Rothschild, Ramsey's theorem for a class of categories, *Adv. Math.* **8** (1972), 417–433.
11. R. Graham, B. Rothschild, and J. Spencer, "Ramsey Theory," Wiley, New York, 1990.
12. A. Hales and R. Jewett, Regularity and positional games, *Trans. Amer. Math. Soc.* **106** (1963), 222–229.
13. N. Hindman, Problems and new results in the algebra of βS and Ramsey theory, *in* "Unsolved Problems on Mathematics for the 21st Century" (J. Abe and S. Tanaka, Eds.), IOS Press, Amsterdam, to appear.
14. N. Hindman and D. Strauss, Compact subsemigroups of $(\beta\mathbb{N}, +)$ containing the idempotents, *Proc. Edinburgh Math. Soc.* **39** (1996), 291–307.
15. N. Hindman and D. Strauss, "Algebra in the Stone–Čech Compactification: Theory and Applications," de Gruyter, Berlin, 1998.
16. J. Nešetřil and V. Rödl, Finite union theorem with restrictions, *Graphs Combin.* **2** (1986), 357–361.
17. J. Spencer, Restricted Ramsey configurations, *J. Combin. Theory Ser. A* **19** (1975), 278–286.
18. B. van der Waerden, Beweis einer Baudetschen Vermutung, *Nieuw Arch. Wisk.* **19** (1927), 212–216.