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ON IP* SETS AND CENTRAL SETS

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IP* sets and central sets are subsets of $\mathbb{N}$ which are known to have rich combinatorial structure. We establish here that structure is significantly richer that was previously known. We also establish that multiplicatively central sets have rich additive structure. The relationship among IP* sets, central sets, and corresponding dynamical notions are also investigated.

1. Introduction

In the terminology of Furstenberg [7], an IP* set A is a subset of the set $\mathbb{N}$ of positive integers with the property that given any sequence $\langle x_n\rangle_{n=1}^{\infty}$ in $\mathbb{N}$, $A\cap FS(\langle x_n\rangle_{n=1}^{\infty})\neq\emptyset$. (Here $FS(\langle x_n\rangle_{n=1}^{\infty})=\{\sum_{n\in F} x_n : F$ is a finite nonempty subset of $\mathbb{N}\}$.) Extending the notion of "central" introduced in [7] to an arbitrary semigroup $(S,\cdot)$, we said [4] that a subset B of S is central provided B is a member of some minimal idempotent of $(\beta S,\cdot)$. (This correspondence was discovered by Glasner [10] in the case S is an abelian group.) Here βS is the Stone–Cech compactification of the discrete space S and $\cdot$ on βS denotes the extension of $\cdot$ on S which makes $(\beta S,\cdot)$ a left topological semigroup with S contained in its topological center. An element p of βS is a minimal idempotent provided $p=p\cdot p$ and p is a member of some minimal right ideal of βS. We will describe this structure in more detail later in this introduction. See [6] for unfamiliar algebraic terminology.) We shall use the term "central" here to refer to subsets B of $\mathbb{N}$ which are central in $(\mathbb{N},+)$. Subsets of $\mathbb{N}$ which are central in $(\mathbb{N},\cdot)$ will be called "multiplicatively central".

It is easy to see that all IP* sets are central. (For example, see Section 2.) Further, each central set is known to posses intricate combinatorial structure. (See [7, Proposition 8.21], using [4, Corollary 6.12] to observe that the notions of central defined in [7] and [4] agree on $\mathbb{N}$.) We show in Section 2 that IP* sets have even richter structure than this. For example, given an IP* set A and a sequence $\langle x_n\rangle_{n=1}^{\infty}$

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AMS subject classification code (1991): 05 D 10 there is a subsystem $\langle y_n\rangle_{n=1}^\infty$ of $FS(\langle x_n\rangle_{n=1}^\infty)$ with $FS(\langle y_n\rangle_{n=1}^\infty)\cup FP(\langle y_n\rangle_{n=1}^\infty)\subseteq A$. (Here $FP(\langle y_n\rangle_{n=1}^\infty)=\{\prod_{n\in F} y_n : F$ is a finite nonempty subset of $\mathbb{N}\}$. To say $\langle y_n\rangle_{n=2}^\infty$ is a subsystem of $FS(\langle x_n\rangle_{n=2}^\infty)$ means there is a pairwise disjoint sequence $\langle F(n)\rangle_{n=2}^\infty$ of finite nonempty subsets of $\mathbb{N}$ with each $y_n=\sum_{m\in F(N)} x_m$.)

In Section 3 we investigate the relationship between (additively) central sets and multiplicatively central sets. We see that central sets need have very little multiplicative structure but that multiplicatively central sets must have substantial additive structure.

We take the points of βN to be the ultrafilters on $\mathbb{N}$, the principal ultrafilters being identified with the points of $\mathbb{N}$. Given $A\subseteq\mathbb{N}$, $\overline{A}=\{p\in\beta\mathbb{N}:A\in p\}$. The set $\{\overline{A}:A\subseteq\mathbb{N}\}$ is a basis for the open sets (as well as a basis for the closed sets) of $\beta\mathbb{N}$. When we say $(\beta\mathbb{N},+)$ is a left topological semigroup we mean that for each $p\in\beta\mathbb{N}$ the function $\lambda_p:\beta\mathbb{N}\to\beta\mathbb{N}$, defined by $\lambda_p(q)=p+q$, is continuous. To say that $\mathbb{N}$ is contained in the topological center means that for each $x\in\mathbb{N}$ the function $\rho_x:\beta\mathbb{N}\to\beta\mathbb{N}$, defined by $\rho_x(p)=p+x$, is continuous. The operation $+$ and $\cdot$ on $\beta\mathbb{N}$ are characterized as follows. Given $A\subseteq\mathbb{N}$, $A\in p+q$ if and only if $\{x\in\mathbb{N}:A-x\in p\}\in q$. Likewise $A\in p\cdot q$ if and only if $\{x\in\mathbb{N}:A/x\in\}\in q$. (Here $A-x=\{y\in\mathbb{N}:y+x\in A\}$ and $A/x=\{y\in\mathbb{N}:y\cdot x\in A\}$.) See [15] for an elementary construction of $\beta\mathbb{N}$ and derivations of some of the basic algebraic facts.

2. IP* sets and central sets

We begin by establishing a simple characterization of IP* sets.

Lemma 2.1. *Given $A\subseteq\mathbb{N}$, A is an IP* set if and only if $\{p\in\beta\mathbb{N}:p+p=p\}\subseteq\overline{A}$. That is, A is an IP* set if and only if A is a member of every idempotent of $\beta\mathbb{N}$. Further $cl\{p\in\beta\mathbb{N}:p+p=p\}=\bigcap\{\overline{A}:A$ is an IP* set$\}$.*

Proof. It is an old result of Galvin (see [13, Theorem 2.3(b) and Theorem 2.5] for example) that there is a sequence $\langle x_n\rangle_{n=1}^\infty$ with $FS(\langle x_n\rangle_{n=1}^\infty)\subseteq A$ if and only if there is some $p\in\beta\mathbb{N}$ with $p+p=p$ and $A\in p$. Given an IP* set A and an idempotent p, if one had $p\notin\overline{A}$ one would obtain $FS(\langle x_n\rangle_{n=1}^\infty)\subseteq\mathbb{N}\backslash A$. Conversely if $\overline{A}$ contains all idempotents, then $\mathbb{N}\backslash A$ does not contain $FS(\langle x_n\rangle_{n=1}^\infty)$.

One now concludes immediately that $cl\{p\in\beta\mathbb{N}:p+p=p\}\subseteq\bigcap\{\overline{A}:A$ is an IP* set$\}$. For the reverse inclusion let $q\in\bigcap\{\overline{A}:A$ is an IP* set$\}$ and let $B\in q$. We claim that some $FS(\langle x_n\rangle_{n=1}^\infty)\subseteq B$ and hence that there is some idempotent $p\in\overline{B}$. Indeed if not one has $\mathbb{N}\backslash B$ is an IP* set from which we conclude $\mathbb{N}\backslash B\in q$, a contradiction. ∎

IP* sets were so named because they hit every "IP set" (which is the name used for sets of the form $FS(\langle x_n\rangle_{n=1}^\infty)$ in [7] and [9]). Analogously one defines central* sets, called C^* sets in [7].

Definition 2.2 A subset A of $\mathbb{N}$ is central* if and only if $A\cap B\neq\emptyset$ whenever B is a central set.

It is immediate from the following characterization (and Lemma 2.1) that every IP* set is central*, and of course every central* set is central. (See also [7, p. 186].)

Lemma 2.3. *Given $A \subseteq \mathbb{N}$, A is central* if and only if $\{p \in \beta\mathbb{N} : p$ is a minimal idempotent of $(\beta\mathbb{N},+)\} \subseteq \overline{A}$. That is A is central* if and only if A is a member of any minimal idempotent.*

Proof. This is immediate from the definition of a central set as one which is a member of a minimal idempotent. ∎

The notion of a combinatorially large ultrafilter was defined in [3.p. 44] (The definition is too complicated to include here.) It was shown there that any member of such an ultrafilter satisfies a long list of combinatorial conclusions. As one example, such a set must contain solutions to any partition regular system of homogeneous linear equations. As one of the consequences of the following theorem each central* set (and hence each IP* set) must satisfy these same conclusions.

Theorem 2.4. *Let A be a central*-set. There is a combinatorially large ultrafilter $q \in \overline{A}$ such that q is a minimal idempotent in $(\beta\mathbb{N},\cdot)$ and every member of q is additively central. In particular A is both additively and multiplicatively central.*

Proof. In [4, Theorem 5.6] it is shown that there is a combinatorially large ultrafilter q such that q is a minimal idempotent of $(\beta\mathbb{N},\cdot)$ and every member of q is additively central. This $q \in M = cl\{p \in \beta\mathbb{N} : p$ is a minimal idempotent of $(\beta\mathbb{N},+)\}$. By Lemma 2.3 $q \in \overline{A}$. ∎

The proof of the following lemma follows very closely the proof of [12, Lemma 3.3]. We use it to prove in Theorem 2.6 that IP* sets contain $FS(\langle x_n\rangle_{n=1}^{\infty}) \cup FPZ(\langle x_n\rangle_{n=1}^{\infty})$ for some sequence $\langle x_n\rangle_{n=1}^{\infty}$. When we write $x \equiv y \pmod{\{A_i\}_{i=1}^{r}}$, we mean x, $y \in A_i$ for some $i \in \{1,2,\ldots,r\}$.

Lemma 2.5. *Let $r \in \mathbb{N}$, let $\{A_i\}_{i=1}^{r}$ be a partition of $\mathbb{N}$ and let $\langle y_n\rangle_{n=1}^{\infty}$ be a sequence in $\mathbb{N}$. There is an increasing sequence $\langle x_n\rangle_{n=1}^{\infty}$ such that*

(a) $\langle x_n\rangle_{n=1}^{\infty}$ is a subsystem of $FS(\langle y_n\rangle_{n=1}^{\infty})$. In fact there is a sequence $\langle T(n)\rangle_{n=1}^{\infty}$ of nonempty finite subsets of $\mathbb{N}$ such that for each n $\max T(n) < \min T(n+1)$ and $x_n = \sum_{k \in T(n)} y_k$.

(b) If F and G are finite nonempty subsets of $\mathbb{N}$, $k \leq \min(F \cup G)$, and $t \leq \prod_{i=1}^{k-1} x_i$ (or $t=1$ if $k=1$) then $t \cdot \sum_{n \in F} x_n \equiv t \cdot \sum_{n \in G} x_n \pmod{\{A_i\}_{i=1}^{r}}$.

Proof. For each $n \in \mathbb{N}$, let $H(1,n) = \{n\}$. Let $q \in \mathbb{N}$ with $q > 1$ and assume we have chosen for each $p \in \{1,\ldots,q-1\}$ a sequence $\langle H(p,n)\rangle_{n=1}^{\infty}$ of finite non-empty subsets of $\mathbb{N}$ satisfying:

(1) For each $n \in \mathbb{N}$ $\max H(p,n) < \min H(p,n+1)$.

(2) If $p \in \mathbb{N}$ and $n \in \{1,\ldots,p-1\}$, then $H(p,n) = H(p-1,n)$.

(3) If $m \in \{1,2,\ldots,p\}$, F and G are finite nonempty subsets of $\mathbb{N}$, $m \leq \min(F \cup G)$, and $t \in \{1,2,\ldots,m-1\}$, then $t \cdot \sum_{n \in F} \sum_{j \in H(p,n)} y_j \equiv t \cdot \sum_{n \in G} \sum_{j \in H(p,n)} y_j \pmod{\{A_i\}_{i=1}^{r}}$.

Hypothesis (1) is trivially satisfied at $p=1$ and (2) and (3) are vacuous there. For finite nonempty F, $G \subseteq \mathbb{N}$ agree that $F \sim G$ if and only if for all $t \in \{1,2,\ldots,q-1\}$,

$t\cdot\sum_{n\in F}\sum_{j\in H(q-1,n)} y_j \equiv t\cdot\sum_{n\in G}\sum_{j\in H(q-1,n)} y_j \pmod{\{A_i\}_{i=1}^r}$. There are only finitely many equivalence classes mod $\sim$ (at most r^{q-1}) Pick by [11, Corollary 3.3] a sequence $\langle J(n)\rangle_{n=1}^\infty$ of finite nonempty subsets of $\mathbb{N}$ with for each n, $\max J(n) < \min J(n+1)$ and so that if H, K are finite nonempty subsets of $\mathbb{N}$ we have $\bigcup_{n\in H} J(n) \sim \bigcup_{n\in K} J(n)$.

Now for $n\in\{1,2,\ldots,q-1\}$, let $H(q,n)=H(q-1,n)$. For $n\in\mathbb{N}$ with $n\geq q$ let $H(q,n)=\bigcup_{k\in J(n)} H(q-1,k)$. Hypothesis (1) and (2) are easily verified. To verify hypothesis (3), let $m\in\{1,2,\ldots,q\}$, let F and G be finite nonempty subsets of $\mathbb{N}$ with $m\leq\min(F\cup G)$, and let $t\in\{1,2,\ldots,m-1\}$. Let $K=\bigcup\{J(n): n\in F$ and $q\leq n\}\cup\{n\in F: n<q\}$ and $L=\bigcup\{J(n): n\in G$ and $q\leq n\}\cup\{n\in G: n<q\}$. Then $\sum_{n\in F}\sum_{j\in H(q,n)} y_j = \sum_{n\in K}\sum_{j\in H(q-1,n)} y_j$ and $\sum_{n\in G}\sum_{j\in H(q,n)} y_j = \sum_{n\in L}\sum_{j\in H(q-1,n)} y_j$. Assume first that $m<q$. Then $m\leq q-1$. Since hypothesis (3) held at $q-1$ we have $t\cdot\sum_{n\in K}\sum_{j\in H(q-1,n)} y_j \equiv t\cdot\sum_{n\in L}\sum_{j\in H(q-1,n)} y_j \pmod{\{A_i\}_{i=1}^r}$. Now assume $m=q$ and note that in this case $K=\bigcup_{n\in F} J(n)$ and $L=\bigcup_{n\in G} J(n)$. Since $K\sim L$ we again get $t\cdot\sum_{n\in K}\sum_{j\in H(q-1,n)} \equiv t\cdot\sum_{n\in L}\sum_{j\in H(q-1,n)} y_j \pmod{\{A_i\}_{i=1}^r}$.

The construction being complete, let $f(1)=2$, let $T(1)=H(2,2)$, and let $x_1=\sum_{j\in T(1)} y_j$. Inductively given $\langle x_k\rangle_{k=1}^{n-1}$ let $f(n)=1+\prod_{k=1}^{n-1} x_k$, let $T(n)=H(f(n),f(n))$, and let $x_n=\sum_{j\in T(n)} y_j$. Now given $n>1$ we have $\min T(n)=\min H(f(n),f(n))\geq \min H(f(n-1),f(n)) > \max H(f(n-1),f(n-1)) = \max T(n-1)$, so conclusion (a) holds. To verify (b), let F and G be finite nonempty subsets of $\mathbb{N}$, let $k\leq \min(F\cup G)$, and let $t\leq\prod_{i=1}^{k-1} x_i$, or if $k=1$ let $t=1$. Let $K=\{f(n): n\in F\}$, let $L=\{f(n): n\in G\}$, and let $q=\max(K\cup L)$. Then by hypothesis (2) we have for each $n\in F\cup G$, $x_n=\sum_{j\in H(f(n),f(n))} y_j = \sum_{j\in H(q,f(n))} y_j$. Thus $\sum_{n\in F} x_n = \sum_{n\in F}\sum_{j\in H(q,f(n))} y_j = \sum_{n\in K}\sum_{j\in H(q,n)} y_j$ and $\sum_{n\in G} x_n = \sum_{n\in L}\sum_{j\in H(q,n)} y_j$. If $k=1$ we have $t=1<2=f(1)$. If $k>1$ we have $t\leq\prod_{i=1}^{k-1} x_i$ so $t<f(k)$. Thus in either case $t<\min(K\cup L)$ so by (3) we have $t\cdot\sum_{n\in F} x_n \equiv t\cdot\sum_{n\in G} x_n \pmod{\{A_i\}_{i=1}^r}$. ∎

Note that in following theorem, one is guaranteed that expressions like $(x_1\cdot(x_3+x_5)+x_6)\cdot(x_{10}+x_{12})$ must all be in the given IP*-set. (The requirement is simply that the term with the largest indices be added and the result multiplied by the rest.)

Theorem 2.6. *Let A be an IP*-set and let $\langle y_n\rangle_{n=1}^\infty$ be a sequence in $\mathbb{N}$. There is a subsystem $\langle x_n\rangle_{n=1}^\infty$ of $FS(\langle y_n\rangle_{n=1}^\infty)$ such that for any finite nonempty $F\subseteq\mathbb{N}$, if $k=\min F$ and $t\in\mathbb{N}$ with $t=1$ if $k=1$ and $t\le\prod_{n=1}^{k-1}x_n$ for $k>1$, then $t\cdot\sum_{n\in F}x_n\in A$. In particular $FS(\langle x_n\rangle_{n=1}^\infty)\cup FP(\langle x_n\rangle_{n=1}^\infty)\subseteq A$.*

Proof. Choose $\langle x_n\rangle_{n=1}^\infty$ as guaranteed by Lemma 2.5 for $\langle y_n\rangle_{n=1}^\infty$ and the partition $\{A,\mathbb{N}\backslash A\}$ of $\mathbb{N}$. Let F, k, and t be given. For each $n\in\mathbb{N}$ let $z_n=t\cdot x_{k+n}$. Since A is an IP* set pick finite nonempty $G\subseteq\mathbb{N}$ with $\sum_{n\in G}z_n\in A$. Let $H=G+k$. Then $t\cdot\sum_{n\in H}x_n=\sum_{n\in G}z_n$. Since $t\cdot\sum_{n\in F}x_n\equiv t\cdot\sum_{n\in H}x_n\ (\mathrm{mod}\{A,\mathbb{N}\backslash A\})$ we have $t\cdot\sum_{n\in F}x_n\in A$.

For the "in particular" conclusion note that $t=1$ yields $FS(\langle x_n\rangle_{n=1}^\infty)\subseteq A$. Let G be a finite nonempty subset of $\mathbb{N}$ and let $m=\max G$. If $G=\{m\}$, we already have $\prod_{n\in G}x_n=x_m\in A$. Assume $|G|>1$ and let $t=\prod_{n\in G\backslash\{m\}}x_n$. Then $t\le\prod_{i=1}^{m-1}$ so $t\cdot x_m=t\cdot\sum_{n\in\{m\}}x_n\in A$. ∎

3. Multiplicatively and additively central sets

Theorem 2.6 established that IP* sets, which are defined in terms of the additive structure of $\mathbb{N}$ nevertheless have substantial multiplicative structure. In this section we investigate further the relationship between additive and multiplicative structure, beginning with the following simple result.

Theorem 3.1. *Let $A\subseteq\mathbb{N}$ and let $n\in\mathbb{N}$: The following statements are equivalent.*
(a) *A is an IP* set.*
(b) *A/n is an IP* set.*
(c) *An is an IP* set.*

Proof. (a)⇒(b). Let $\langle x_m\rangle_{m=1}^\infty$ be given. Since A is an IP* set $A\cap FS(\langle n\cdot x_m\rangle_{m=1}^\infty)\neq\emptyset$ so pick finite nonempty $F\subseteq\mathbb{N}$ with $\sum_{m\in F}nx_m\in A$. Then $\sum_{m\in F}x_m\in A/n$.

(b)⇒(a). Let $\langle x_m\rangle_{m=1}^\infty$ be given and choose a subsytem $\langle y_m\rangle_{m=1}^\infty$ of $FS(\langle x_m\rangle_{m=1}^\infty)$ such that n divides y_m for each m. (Of any n^2-n+1 terms some n are congruent mod n and their sum is divisible by n.) Choose finite nonempty $F\subseteq\mathbb{N}$ with $\sum_{m\in F}(y_m/n)\in A/n$. Then $\sum_{m\in F}y_m\in A\cap FS(\langle x_m\rangle_{m=1}^\infty)$.

Since $A=(An)/n$ the equivalence of (a) with (c) follows from the equivalence of (a) with (b). ∎

Given the characterization of IP* sets in Lemma 2.1 and the fact that for any idempotent p and any $A\in p$ one has $\{n\in\mathbb{N}:A-n\in p\}\in p$, one might expect IP* sets to often translate to other IP* sets.

Theorem 3.2. *There si an IP* set A such that for all $n\in\mathbb{N}$ $A-n$ is not an IP* set and $A+n$ is not an IP* set.*

Proof. Let $\langle D(n)\rangle_{n\in\mathbb{Z}}$ be a sequence of pairwise disjoint infinite subsets of $\mathbb{N}2$ (the even natural numbers) with each $\min D(n) > |n|$. For each n enumerate $D(n)$ in increasing order as $\langle a(n,k)\rangle_{k=1}^{\infty}$ and for each $k\in\mathbb{N}$ let $y_{n,k}=2^{a(n,2k-1)}+2^{a(n,2k)}$. For each $n\in\mathbb{Z}\setminus\{0\}$ let $B_n=FS(\langle y_{n,k}\rangle_{k=1}^{\infty})$. Let $C=\bigcup\{B_n+n:n\in\mathbb{Z}\setminus\{0\}\}$ and let $A=\mathbb{N}\setminus C$. Given any $n\in\mathbb{Z}\setminus\{0\}$, $B_n\cap(A-n)=\emptyset$ so $A-n$ is not an IP* set.

We claim A is an IP* set. Suppose instead we have some $\langle x_n\rangle_{n=1}^{\infty}$ with $FS(\langle x_n\rangle_{n=1}^{\infty})\cap A=\emptyset$, that is $FS(\langle x_n\rangle_{n=1}^{\infty})\subseteq C$. By choosing a suitable subsystem we may presume $x_1<x_2<\ldots$.

Note that if u, $v\in B_n+n$ and $u>v$, then $u+v$ is not in any B_m+m where m, $n\in\mathbb{Z}\setminus\{0\}$. Indeed the largest power of 2 in the binary expansion of u is 2^{2a}, say, where $a\in\mathbb{N}$ and $2a\in D(n)$. (It is here that we use the fact that each $y_{n,k}$ has two 1's in its binary expansion so if $n<0$ the leftmost 1 is not disturbed.) Thus $u+v$ has largest power 2^{2a} or 2^{2a+1}. But the latter case is impossible for any number in any B_m+m. Thus $u+v$ is also in B_n+n. But this is impossible because $2^{\min D(n)}\geq|4n|$ and then cannot divide, simultaneously, $u-n$, $v-n$, and $u+v-n$, as it would have to if $\{u,v,u+v\}\subseteq B_n+n$.

Consider now x_1 and x_i, $i>1$. By assumption x_1, x_i, and x_1+x_i are all in C. Thus $x_i\in B_{n(i)}+n(i)$, where by the preceding paragraph $n(i)\neq n(j)$ if $i\neq j$. We can thus choose i with $|n(i)|>x_1$. Then the binary expansion of x_1+x_i has the same leading term as x_i so $x_1+x_i\in B_{n(i)}+n(i)$. But any two different numbers in $B_{n(i)}+n(i)$ differ by at least $2^{\min D(n(i))}>(x_1+x_i)-x_i$, which gives a contradiction. ∎

Given a central set A we have some minimal idempotent p with $A\in p$. Then $\{n\in\mathbb{N}:A-n\in p\}\in p$ so in particular $\{n\in\mathbb{N}:A-n$ is centrral$\}$. It is obvious fact that also $\{n\in\mathbb{N}:A+n$ is central$\}$ is central [5]. We do not know whether there is a central* set which never or seldom translates to a central* set.

As was shown in [4, Corollary 5.5], any finite partition of $\mathbb{N}$ includes one cell which is both additively and multiplicatively central. It is natural to ask whether all additively central sets must contain rich multiplicative structure and similarly whether all multiplicative central sets must contain rich additive structure. The answers turn out to be "no" (Theorem 3.4) and "yes" (Theorem 3.5) respectively.

Lemma 3.3. *Let $\langle x_n\rangle_{n=1}^{\infty}$ be a sequence in $\mathbb{N}$. Then $\bigcup_{n=1}^{\infty}\{x_n+1,x_n+2,\ldots,x_n+n\}$ is additively central and $\bigcup_{n=1}^{\infty}\{x_n\cdot 1,x_n\cdot 2,\ldots,x_n\cdot n\}$ is multiplicatively central.*

Proof. Since $\{x,x+1,\ldots\}$ is additively central and $\{x,2x,\ldots,\}$ is multiplicatively central, we may assume $\{x_n:n\in\mathbb{N}\}$ is infinite.

We do the multiplicative case, the other being nearly identical. Let $A=\bigcup_{n=1}^{\infty}\{x_n\cdot 1,x_n\cdot 2,\ldots,x_n\cdot n\}$ and pick $p\in\beta\mathbb{N}\setminus\mathbb{N}$ with $\{x_n:n\in\mathbb{N}\}\in p$. Then $p\cdot\beta\mathbb{N}\subseteq\overline{A}$. (Indeed for all $n\in\mathbb{N}$ $A/n\supseteq\{x_m:m\geq n\}$ so $A/n\in p$ so $\{n:A/n\in p\}=\mathbb{N}\in q$ for any $q\in\beta\mathbb{N}$.) Now $p\cdot\beta\mathbb{N}$ is a right ideal of $(\beta\mathbb{N},\cdot)$ so it contains a minimal right ideal R, which is necessarily closed. (This is well known and is at any rate an easy Zorn's Lemma argument.) Thus (see for example [6, Theorem 1.3.11]) R contains an idempotent q. Then $A\in q$ so A is multiplicatively central. ∎

Theorem 3.4. *There is an additively central* $A \subseteq \mathbb{N}$ *such that for no* x, $y \in \mathbb{N}$ *is* $\{x,y,x\cdot y\}\subseteq A$. *(In particular no* $\{x,x^2\}\subseteq A$.*)*

Proof. Let $x_1=2$ and for $n\geq 1$ choose $x_{n+1}>(x_n+n)^2$. Let $A=\bigcup_{n=1}^{\infty}\{x_n+1,x_n+2,\ldots,x_n+n\}$. By Lemma 3.3 A is central. Suppose we have $y\leq z$ with $\{y,z,y\cdot z\}\subseteq A$ and pick n with $z\in\{x_n+1,x_n+2,\ldots,x_n+n\}$. Now $y\geq 2$ so $y\cdot z\geq 2z\geq 2x_n+n$ so $y\cdot z\geq x_{n+1}+1>(x_n+n)^2\geq z\cdot z\geq y\cdot z$, a contradiction. ∎

Theorem 3.5. *Let* $A\subseteq\mathbb{N}$ *be a multiplicatively central set. Then for each* m *there exists* $\langle y_n\rangle_{n=1}^m$ *with* $FS(\langle y_n\rangle_{n=1}^m)\subseteq A$.

Proof. Let $T=\{p\in\beta\mathbb{N}:$ for all $B\in p$ and all $m\in\mathbb{N}$ there exists $\langle y_n\rangle_{n=1}^m$ with $FS(\langle y_n\rangle_{n=1}^m)\subseteq B\}$. Now all additive idempotents are in T so $T\neq\emptyset$. We calim T is a two sided ideal of $(\beta\mathbb{N},\cdot)$. To this end let $p\in T$ and let $q\in\beta\mathbb{N}$. To see that $p\cdot q\in T$, let $B\in p\cdot q$ and let $m\in\mathbb{N}$ be given. Then $\{n\in\mathbb{N}:B/n\in p\}\in q$ so pick $n\in\mathbb{N}$ with $B/n\in p$. Pick $\langle y_t\rangle_{t=1}^m$ with $FS(\langle y_t\rangle_{t=1}^m)\subseteq B/n$. Then $FS(\langle n\cdot y_t\rangle_{t=1}^m)\subseteq B$.

To see that $q\cdot p\in T$, let $B\in q\cdot p$ and let $m\in\mathbb{N}$ be given. Then $\{n\in\mathbb{N}:B/n\in q\}\in p$ so pick $\langle y_t\rangle_{t=1}^m$ with $FS(\langle y_t\rangle_{t=1}^m)\subseteq\{n\in\mathbb{N}:B/n\in q\}$. Since $FS(\langle y_t\rangle_{t=1}^m)$ is finite we have $\bigcap\{B/n:n\in FS(\langle y_t\rangle_{t=1}^m)\}\in q$ so pick $a\in\bigcap\{B/n:n\in FS(\langle y_t\rangle_{t=1}^m)\}$. Then $FS(\langle a\cdot y_t\rangle_{t=1}^m)\subseteq B$.

Now A is multiplicatively central so pick a minimal idempotent p of $(\beta\mathbb{N},\cdot)$ with $A\in p$. Pick a minimal right ideal R of$(\beta\mathbb{N},\cdot)$ with $p\in R$. Since T is a two sided ideal $R\subseteq T$. (Since T is a left ideal $T\cap R\neq\emptyset$ and hence $T\cap R$ is a right ideal so $T\cap R=R$.) Then $p\in T$. Since $A\in p$, we're done. ∎

In a similar fashion, since $\{p\in\beta\mathbb{N}:$ for all $B\in p$, b contains arbitrarily long arithmetic progressions$\}$ is a two-sided ideal of $(\beta\mathbb{N},\cdot)$ one sees that each multiplicatively central set contains arbitrarily long arithmetic progressions. In fact more generally we have from [3, Lemma 2.2] that $\{q\in\beta\mathbb{N}:$ for all $B\in q$ and all m, p, $c\in\mathbb{N}$, b contains an (m,p,c) set$\}$ is a two-sided ideal of $(\beta\mathbb{N},\cdot)$. Therefore multiplicatively central sets all contain solutions to any partition regular system of homogeneous linear equations with integer coefficients.

On the other hand, one cannot require multiplicatively central sets to contain finite sums from infinite sequences.

Theorem 3.6. *There is a multiplicatively central* $A\subseteq\mathbb{N}$ *such that for no* $\langle y_n\rangle_{n=1}^\infty$ *is* $FS(\langle y_n\rangle_{n=1}^\infty)\subseteq A$.

Proof. Let $x_1=1$ and for $n\geq 1$ pick $x_{n+1}>x_n\cdot(n+1)$. Let $A=\bigcup_{n=1}^{\infty}\{x_n,x_n\cdot 2,\ldots,x_n\cdot n\}$. By Lemma 3.3 A is multiplicatively central. Suppose we have $\langle y_n\rangle_{n=1}^\infty$ with $FS(\langle y_n\rangle_{n=1}^\infty)\subseteq A$. We may presume $\langle y_n\rangle_{n=1}^\infty$ is an increasing sequence. (If the original sequence repeats infinitely often one gets all multiples of some fixed a in $FS(\langle y_n\rangle_{n=1}^\infty)$ so if $z_n=a\cdot n$ one has $FS(\langle z_n\rangle_{n=1}^\infty)\subseteq FS(\langle y_n\rangle_{n=1}^\infty)$.) Pick m, n, and r in $\mathbb{N}$ with $y_m\in\{x_n,x_n\cdot 2,\ldots,x_n\cdot n\}$, $y_{m+1}\in\{x_r,x_r\cdot 2,\ldots,x_r\cdot r\}$, and $n<r$. Pick $k\in\{1,2,\ldots,r\}$ with $y_{m+1}=k\cdot x_r$. Then $k\cdot x_r<y_{m+1}+y_m\leq k\cdot x_r+n\cdot x_n<k\cdot x_r+x_r=(k+1)\cdot x_r$ so $y_{m+1}+y_m\notin A$. ∎

A measure space is a triple $(X,\mathcal{B},\mu)$ where X is a set, $\mathcal{B}$ is a σ-algebra of subsets of X, and μ is a countably additive measure on $\mathcal{B}$. A function T from

X to X is a measure preserving transformation provided whenever $B \in \mathscr{B}$ one has $T^{-1}B \in \mathscr{B}$ and $\mu(T^{-1}B) = \mu(B)$.

It is simple observation that given a measure space $(X, \mathscr{B}, \mu)$ with $\mu(X) < \infty$ and a measure preserving transformation T one has for any $A \in \mathscr{B}$ with $\mu(A) > 0$ that $\{n \in \mathbb{N} : \mu(A \cap T^{-n}A) > 0\}$ is an IP* set. (To see this let $\langle x_n \rangle_{n=1}^{\infty}$ be given and for each n let $a_n = \sum_{i=1}^{n} x_i$. Since $\mu(X) < \infty$ we have some $n < m$ in $\mathbb{N}$ with $\mu(T^{-a_n}A \cap T^{-a_m}A) > 0$. Let $b = \sum_{i=n+1}^{m} x_i$. Then $\mu(A \cap T^{-b}A) = \mu(T^{-a_n}[A \cap T^{-b}A]) = \mu(T^{-a_n}A \cap T^{-a_m}A) > 0$.) In the following definition "Po" stands for "Poincaré", because of our utilization of Poincaré recurrence. A Po* set is related to Furstenbers's Poincaré sets [7] in the same way as IP* sets are related to IP sets.

Definition 3.7 A set $B \subseteq \mathbb{N}$ is a Po* set if and only if there exists a measure space $(X, \mathscr{B}, \mu)$, some $A \in \mathscr{B}$ with $\mu(A) > 0$, and a measure preserving transformation T such that $\{n \in \mathbb{N} : \mu(A \cap T^{-n}A) > 0\} \subseteq B$.

Our observation before Definition 3.7 shows that any Po* set is an IP* set. We conclude this paper by showing in Corollary 3.9 that the converse fails. We are grateful to one of the referees for suggesting this simple proof.

Theorem 3.8. *Let B be a Po* set. There is a Po* set $C \subseteq B$ such that for every $n \in C$, $C - n$ is a Po* set (and hence $B - n$ is a Po* set).*

Proof. Pick a measure space $(X, \mathscr{B}, \mu)$, some $A \in B$ with $\mu(A) > 0$, and a measure preserving transformation T such that $\{n \in \mathbb{N} : \mu(A \cap T^{-n}A) > 0\} \subseteq B$. Let $C = \{n \in \mathbb{N} : \mu(A \cap T^{-n}A) > 0\}$ and let $n \in C$. Let $D = A \cap T^{-n}A$. We show that $\{m \in \mathbb{N} : \mu(D \cap T^{-m}D) > 0\} \subseteq C - n$. Indeed, given $n \in \mathbb{N}$ one has $D \cap T^{-m}D \subseteq A \cap T^{-n-m}A$ so if $\mu(D \cap T^{-m}D) > 0$ one has $\mu(A \cap T^{-(m+n)}A) > 0$. ∎

Corollary 3.9. *Not every IP* set is a Po* set.*

Proof. Pick A as guaranteed by Theorem 3.2. If A were a Po* set one would have by Theorem 3.8 that some $A - n$ would be a Po* set and hence an IP* set. ∎

In fact even stronger results are available. In recent work Bergelson and Furstenberg [2] have shown the following: Let $(X, \mathscr{B}, \mu)$ be a measure space with $\mu(X) < \infty$ and let T be a measure preserving transformation. Let p be a polynomial with $p(0) = 0$ and $p(n) \in \mathbb{Z}$ for all $n \in \mathbb{Z}$. Then for $A \in \mathscr{B}$ with $\mu(A) > 0 \{n \in \mathbb{N} : \mu(A \cap T^{p(n)}A) > \mu^2(A) - \epsilon\}$ is an IP* set for each $\epsilon > 0$. This result has numerous consequences. A simple sonsequence is that any Po* set contains squares. (Let $p(n) = -n^2$.) In particular, since $\mathbb{N} \setminus \{n^2 : n \in \mathbb{N}\}$ is trivially an IP* set, this shows again that not all IP* sets are Po* sets.

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