IP-sets and Polynomial Recurrence

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Abstract. We combine recurrence properties of polynomials and IP-sets and show that polynomials evaluated along IP-sequences also give rise to Poincaré sets for measure preserving systems, that is, sets of integers along which the analog of the Poincaré recurrence theorem holds. This is done by applying to measure preserving transformations a limit theorem for products of appropriate powers of a commuting family of unitary operators.

0. *Introduction.*
One of the formulations of the classical Poincaré recurrence theorem is that for any measure preserving system $(X, \mathcal{B}, \mu, T)$ and any $A \in \mathcal{B}$, $\mu(A) > 0$, there exists $n \in \mathbf{N}$ such that $\mu(A \cap T^{-n}A) > 0$. In certain applications, refinements of this result are useful. Thus the same statement is valid with n replaced by n^2. Now a general principle discussed in [F1] and [F2] enables us to deduce from this that in a subset of integers of positive density, there are necessarily two distinct numbers whose difference is a perfect square. This type of result has served as an impetus to find examples of "sets of recurrence".

To be precise, let us say that a set $S \subset \mathbf{N}$ is a *set of recurrence* if for any measure preserving system $(X, \mathcal{B}, \mu, T)$ and any $A \in \mathcal{B}$, $\mu(A) > 0$, there exists $n \in S$ such that $\mu(A \cap T^{-n}A) > 0$. Furthermore, a subset $A \subset \mathbf{Z}^n$ will be said to have *positive upper Banach density* if there exists a sequence of rectangles $R_j = \prod_{i=1}^{n} [a_i^{(j)}, b_i^{(j)}]$, with $b_i^{(j)} - a_i^{(j)} \to \infty$, $1 \le i \le n$, such that $\limsup_{j\to\infty} \frac{|R_j \cap A|}{|R_j|} > 0$. The maximum value of this limit is achieved for some sequence of rectangles, and we write $d^*(A)$ for this value. One may show that S is a set of recurrence if and only if for every subset $A \subset \mathbf{Z}$ having positive upper Banach density, there exists $a, b \in A$ and $n \in S$ such that $n = a - b$ (i.e. $(A - A) \cap S \ne \phi$). Here are a few examples of sets of recurrence: (i) the set of differences $A - A = \{a - b : \ a, b \in A, a > b\}$ of any infinite set $A \subset \mathbf{N}$; (ii) $\{n^2 : \ n \in \mathbf{N}\}$; (iii) $\{p_n - 1 : n \in \mathbf{N}\}$ where p_n is the nth prime; (iv) $\{[[n\sqrt{2}]\sqrt{2}\,]^2 : \ n \in \mathbf{N}\}$, where $[\cdot]$ denotes the integer part. Not every infinite set is a set of recurrence (consider, for example, the odd integers). Squares of elements of a set of recurrence need not form a set of recurrence. For example, there are sets of differences $A - A$ such that $\{n^2 : \ n \in A - A\}$ is not a set of recurrence ([F2], p. 177).

Given an infinite sequence $G = \{g_i\}_{i \in \mathbf{N}} \subset \mathbf{N}$, the *IP-set generated by* G is the set

$$\Gamma = \{g_{i_1} + g_{i_2} + \cdots + g_{i_k} : \ k \in \mathbf{N}, \ i_1 < i_2 < \cdots < i_k\}$$

of all finite sums of elements with distinct indices from G. Note that $\{g_i + g_{i+1} + \cdots + g_j : \ i < j, \ i, j \in \mathbf{N}\} \subset \Gamma$ is a set of differences. It follows that any IP-set is a set of recurrence. It turns out, somewhat surprisingly, that the set of squares $\{n^2 : \ n \in \Gamma\}$ is also a set of recurrence (see Theorem A below).

We proceed to describe our results in greater detail. Let $\mathcal{F}$ denote the family of all non-empty finite subsets of $\mathbf{N}$. By an $\mathcal{F}$-sequence we mean a sequence indexed by $\mathcal{F}$. One type of $\mathcal{F}$-sequence of particular interest to us is a sequence $(n_\alpha)_{\alpha \in \mathcal{F}} \subset \mathbf{N}$ with the property that $n_\alpha + n_\beta = n_{\alpha \cup \beta}$ whenever $\alpha \cap \beta = \phi$. It is easy to see that the range of such a sequence is precisely an IP-set.

Conversely, given an IP-set Γ generated by $\{g_i\}_{i \in \mathbf{N}}$, we may realize Γ as the range of an $\mathcal{F}$-sequence with the above property by letting $n_\alpha = \sum_{i \in \alpha} g_i$. (Henceforth, in a small abuse of terminology, we will sometimes use the term IP-set for the $\mathcal{F}$-sequence itself, not merely its range.) It is easy to see that the aforementioned fact about squares of elements of IP-sets is a special case of the following theorem, which is in its turn a special case of Corollary 2.1 below.

THEOREM A. *For any polynomial* $p(x_1,\cdots,x_k)\in \mathbf{Z}[x_1,\cdots,x_k]$ *satisfying* $p(0,\cdots,0)=0$, *and for any* k *IP-sets* $(n_\alpha^{(1)})_{\alpha\in\mathcal{F}},\cdots,(n_\alpha^{(k)})_{\alpha\in\mathcal{F}}$, *the set*

$$\{p(n_\alpha^{(1)},\cdots,n_\alpha^{(k)}):\ \alpha\in\mathcal{F}\}$$

is a set of recurrence.

Theorem A is a refinement of the result of Sárközy, ([S]), Conze, and Furstenberg ([F2], p. 75) asserting that if $p(x)\in\mathbf{Z}[x]$, $p(0)=0$, then $\{p(n):\ n\in\mathbf{Z}\}$ is a set of recurrence. Suppose now that $\mathbf{T}=\{T_w:\ w\in W\}$ is an indexed family of measure-preserving transformations of a probability space $(X,\mathcal{B},\mu)$. We shall say that $\mathbf{T}$ possesses the *R-property* if for any $A\in\mathcal{B}$, $\mu(A)>0$, there exists $w\in W$ such that $\mu(A\cap T_w^{-1}A)>0$. Theorem A above asserts that the family $\{T^{p(n_\alpha^{(1)},\cdots,n_\alpha^{(k)})}:\ \alpha\in\mathcal{F}\}$ has the R-property. This is a special case of the following.

THEOREM B. *Suppose that* $p_i(x_1,\cdots,x_k)\in\mathbf{Z}[x_1,\cdots,x_k]$ *satisfy* $p_i(0,\cdots,0)=0$, $1\le i\le m$, *and that* $(n_\alpha^{(i)})_{\alpha\in\mathcal{F}}$ *are IP-sets,* $1\le i\le k$. *Let* $p_\alpha^{(j)}=p_j(n_\alpha^{(1)},\cdots,n_\alpha^{(k)})$, $1\le j\le m$. *Then for any commuting measure-preserving transformations* $T_1,\cdots,T_m$, *the family* $\{\prod_{i=1}^m T_i^{p_\alpha^{(i)}}:\ \alpha\in\mathcal{F}\}$ *has the R-property.*

Theorem B again follows from the (still stronger) statement contained in Corollary 2.1, which follows from a property of $\mathcal{F}$-sequences of unitary operators established in the main theorem of this paper, Theorem 1.8. The main tools employed in the proof of Theorem 1.8 are the notion of IP-convergence, the Milliken-Taylor Theorem, and Lemma 1.4, an "IP-van der Corput trick". These will be explained in detail in the next section. Finally we mention two more corollaries of Theorem 1.8.

THEOREM C. (Corollary 2.2) *Suppose* $E\subset\mathbf{Z}^t$, $d^*(E)>0$, $p_i(x_1,\cdots,x_k)\in\mathbf{Z}[x_1,\cdots,x_k]$ *satisfy* $p_i(0,\cdots,0)=0$, $1\le i\le t$, *and that* $(n_\alpha^{(i)})_{i=1}^k$ *are IP-sets in* $\mathbf{N}$. *Then for some* $x,y\in E$ *and* $\alpha\in\mathcal{F}$,

$$x-y=\big(p_1(n_\alpha^{(1)},\cdots,n_\alpha^{(k)}),\cdots,p_t(n_\alpha^{(1)},\cdots,n_\alpha^{(k)})\big).$$

THEOREM D. (Proposition 2.3) *Suppose* (Y,ν) *is a measure space with* $\nu(Y)\le\infty$ *and that* S *is a conservative (in case* $\nu(Y)=\infty$*) measure preserving transformation of* Y. *Suppose that* (X,μ) *is a probability space and that* T *is an invertible measure preserving tranformation of* X. *Then if* $A\subset Y\times X$ *with* $(\nu\times\mu)(A)>0$ *and* $p(x)\in\mathbf{Z}[x]$, $p(0)=0$, *there exists* $n\in\mathbf{N}$ *such that*

$$(\nu\times\mu)\big(A\cap(S^n\times T^{p(n)})^{-1}A\big)>0.$$

1. *$\mathcal{F}$-sequences of Unitary Operators.*

In this section we present the main theorem of this paper, Theorem 1.8, from which all of the applications of the next section follow. The theorem asserts that the weak IP-limits of certain classes of polynomially generated $\mathcal{F}$-sequences of unitary operators on an abstract Hilbert space are orthogonal projections. We now proceed to develop the tools and terminology needed for the formulation and proof of Theorem 1.8.

Definition 1.1. If $\alpha,\beta\in\mathcal{F}$ have the property that whenever $a\in\alpha$, $b\in\beta$ we have $a<b$, then we will write $\alpha<\beta$. If $(\alpha_i)_{i\in\mathbf{N}}\subset\mathcal{F}$ has the property that for every $i\in\mathbf{N}$, $\alpha_i<\alpha_{i+1}$, then $\mathcal{F}^{(1)}=\{\bigcup_{i\in\beta}\alpha_i:\ \beta\in\mathcal{F}\}$ will be called an *IP-ring*. In this case the map $\gamma:\mathcal{F}\to\mathcal{F}^{(1)}$, $\gamma(\beta)=\bigcup_{i\in\beta}\alpha_i$, is bijective and preserves the operation of union. In particular any sequence (y_α) indexed by $\mathcal{F}^{(1)}$ can be identified naturally with a unique $\mathcal{F}$-sequence. If (x_α) is an $\mathcal{F}$-sequence in a topological space and $\mathcal{F}^{(1)}$ is an IP-ring then we write

$$\operatorname*{IP-lim}_{\alpha\in\mathcal{F}^{(1)}} x_\alpha=z$$

if for every neighborhood W of z there exists $\beta\in\mathcal{F}$ such that for every $\alpha\in\mathcal{F}^{(1)}$ with $\alpha>\beta$, $x_\alpha\in W$.

Hindman's Theorem ([H]) states that if $\mathcal{F}^{(1)}$ is an IP-ring and $\mathcal{F}^{(1)}=\bigcup_{i=1}^r C_i$ then for some i, $1\le i\le r$, C_i contains an IP-ring $\mathcal{F}^{(2)}$. A natural consequence is the following.

PROPOSITION 1.2. ([FK], Theorem 1.5.) *Suppose that, for every $n \in \mathbf{N}$, X_n is a compact metric space and $\{x_\alpha^{(n)}\}_{\alpha\in\mathcal{F}}$ is an $\mathcal{F}$-sequence in X_n. Then there exists an IP-ring $\mathcal{F}^{(1)}$ such that*

$$\operatorname*{IP-lim}_{\alpha\in\mathcal{F}^{(1)}} x_\alpha^{(n)} = z_n$$

exists for each $n \in \mathbf{N}$.

We will find it convenient to use the following generalization of Hindman's Theorem.

THEOREM 1.3. ([M], [T]) *Suppose that $\mathcal{F}^{(1)}$ is an IP-ring, $l \in \mathbf{N}$, and*

$$\{(\alpha_1,\cdots,\alpha_l) \in (\mathcal{F}^{(1)})^l : \alpha_1 < \cdots < \alpha_l\} = \bigcup_{i=1}^{r} C_i.$$

Then there exists j, $1 \le j \le r$, and an IP-ring $\mathcal{F}^{(2)} \subset \mathcal{F}^{(1)}$ such that

$$\{(\alpha_1,\cdots,\alpha_l) \in (\mathcal{F}^{(2)})^l : \alpha_1 < \cdots < \alpha_l\} \subset C_j.$$

LEMMA 1.4. ([FK]) *Suppose $\{x_\alpha\}_{\alpha\in\mathcal{F}}$ is a bounded $\mathcal{F}$-sequence of vectors in a Hilbert space $\mathcal{H}$ and that $\mathcal{F}^{(1)}$ is an IP-ring. If*

$$\operatorname*{IP-lim}_{\beta\in\mathcal{F}^{(1)}} \operatorname*{IP-lim}_{\alpha\in\mathcal{F}^{(1)}} \langle x_\alpha,\, x_{\alpha\cup\beta}\rangle = 0$$

then for some IP-subring $\mathcal{F}^{(2)} \subset \mathcal{F}^{(1)}$, $\operatorname*{IP-lim}_{\alpha\in\mathcal{F}^{(2)}} x_\alpha = 0$ in the weak topology.

Finally we have the following proposition, for which we introduce the following notion: An $\mathcal{F}$-sequence $\{U_\alpha\}_{\alpha\in\mathcal{F}}$ of commuting unitary operators on a Hilbert space $\mathcal{H}$ satisfying $U_{\alpha\cup\beta} = U_\alpha U_\beta$ whenever $\alpha\cap\beta = \phi$ will be called an *IP-system.*

PROPOSITION 1.5. ([FK], Theorem 1.7.) *Suppose that $\mathcal{H}$ is a Hilbert space and $\{U_\alpha\}_{\alpha\in\mathcal{F}}$ is an IP-system of unitary operators on $\mathcal{H}$.*
If for some IP-ring $\mathcal{F}^{(1)}$

$$\operatorname*{IP-lim}_{\alpha\in\mathcal{F}^{(1)}} U_\alpha = P$$

weakly, then P is an orthogonal projection onto a subspace of $\mathcal{H}$.

We now give two lemmas which will help facilitate the proof of Theorem 1.8.

LEMMA 1.6. *Suppose that $s \in \mathbf{N}$ and that $\{v_\alpha\}$ is an $\mathcal{F}$-sequence in $\mathbf{Z}^s$. Then for any IP-ring $\mathcal{F}^{(1)}$ there exists $l \le s$, an $l-$dimensional subgroup $V \subset \mathbf{Z}^s$, and an IP-subring $\mathcal{F}^{(2)} \subset \mathcal{F}^{(1)}$, such that $\{v_\alpha : \alpha \in \mathcal{F}^{(2)}\} \subset V$ and such that (if $l > 0$) whenever $\alpha_1,\cdots,\alpha_l \in \mathcal{F}^{(2)}$ with $\alpha_1 < \cdots < \alpha_l$, the set $\{v_{\alpha_1},\cdots,v_{\alpha_l}\}$ is linearly independent.*

Proof. Let $l \ge 0$ be minimal with respect to the property that there exists an $l-$dimensional subgroup $V \subset \mathbf{Z}^s$ and an IP-ring $\mathcal{G} \subset \mathcal{F}^{(1)}$ (both of which we now fix) such that $\{v_\alpha : \alpha \in \mathcal{G}\} \subset V$. If $l = 0$ we are done, so we now assume that $l > 0$. Then $\{(\alpha_1,\cdots,\alpha_l) \in \mathcal{G}^l : \alpha_1 < \cdots < \alpha_l\}$ is the union of the two sets

$$S_1 = \{(\alpha_1,\cdots,\alpha_l) \in \mathcal{G}^l : \alpha_1 < \cdots < \alpha_l,\ \{v_{\alpha_1},\cdots,v_{\alpha_l}\} \text{ is linearly independent}\},$$

$$S_2 = \{(\alpha_1,\cdots,\alpha_l) \in \mathcal{G}^l : \alpha_1 < \cdots < \alpha_l,\ \{v_{\alpha_1},\cdots,v_{\alpha_l}\} \text{ is linearly dependent}\}.$$

By Theorem 1.3 there exists an IP-ring $\mathcal{F}^{(2)} \subset \mathcal{G}$ such that

$$\{(\alpha_1,\cdots,\alpha_l) \in (\mathcal{F}^{(2)})^l : \alpha_1 < \cdots < \alpha_l\} \subset S_i,$$

where either $i = 1$ or $i = 2$. Suppose $i = 2$. Let n be maximal with respect to the property that there exist $\alpha_1,\cdots,\alpha_n \in \mathcal{F}^{(2)}$ (which we now fix) with $\alpha_1 < \alpha_2 < \cdots < \alpha_n$ such that the set $\{v_{\alpha_1},\cdots,v_{\alpha_n}\}$ is linearly independent. If $n = 0$ then $v_\alpha = 0$ for all $\alpha \in \mathcal{F}^{(2)}$, contradicting the fact that $l > 0$. Therefore we may assume that $n > 0$. Let V' be the n dimensional subgroup consisting of all elements $v \in \mathbf{Z}^s$ for which, for some $k \in \mathbf{N}$, kv lies in the subgroup generated by $\{v_{\alpha_1}, \cdots, v_{\alpha_n}\}$. V' contains v_α for every α in the IP-ring $\{\alpha \in \mathcal{F}^{(2)} : \alpha > \alpha_n\}$, contradicting the minimality of l. Hence $i = 1$, completing the proof.

□

LEMMA 1.7. *Suppose that $l \in \mathbf{N}$, $\mathcal{F}^{(1)}$ is an IP-ring, $\mathcal{H}$ is a Hilbert space and $\{P(\alpha)\}_{\alpha\in\mathcal{F}}$ is an $\mathcal{F}$-sequence of commuting orthogonal projections on $\mathcal{H}$ such that whenever $\alpha_1, \cdots, \alpha_l \in \mathcal{F}^{(1)}$, $\alpha_1 < \cdots < \alpha_l$, and $f \in \mathcal{H}$ we have $\left(\prod_{i=1}^l P(\alpha_i)\right) f = 0$. Then $\operatorname*{IP-lim}_{\alpha\in\mathcal{F}^{(1)}} \|P(\alpha)f\| = 0$.*

Proof. It suffices to show that for an arbitrary sequence $\beta_1 < \beta_2 < \cdots$, $\beta_i \in \mathcal{F}^{(1)}$, and $f \in \mathcal{H}$, we have $\lim_{i\to\infty} \|P(\beta_i)f\| = 0$. Fix N. For non-empty $A \subset \{1, \cdots, N\}$ put

$$\mathcal{H}_A = \Big(\bigcap_{i\in A} P(\beta_i)\mathcal{H}\Big) \cap \Big(\bigcap_{i\in\{1,\cdots,N\}\setminus A} \big(P(\beta_i)\mathcal{H}\big)^\perp\Big).$$

Also put $\mathcal{H}_\emptyset = \bigcap_{i=1}^N \left(P(\beta_i)\mathcal{H}\right)^\perp$. Then $\mathcal{H} = \oplus_{A\subset\{1,\cdots,N\},\,|A|<l}\mathcal{H}_A$. Let P_A denote the orthogonal projection onto $\mathcal{H}_A$. Then for each $g \in \mathcal{H}$,

$$\|g\|^2 = \sum_{A\subset\{1,\cdots,N\},\,|A|<l} \|P_A g\|^2,$$

so that

$$\begin{aligned}\sum_{i=1}^N \|P(\beta_i)f\|^2 &= \sum_{i=1}^N \sum_{A\subset\{1,\cdots,N\}, i\in A} \|P_A f\|^2 \\ &= \sum_{A\subset\{1,\cdots,N\},\,|A|<l} |A|\,\|P_A f\|^2 \le l\|f\|^2.\end{aligned}$$

But this is true for any N, so $\sum_{i=1}^\infty \|P(\beta_i)f\|^2 < \infty$ and $\lim_{i\to\infty}\|P(\beta_i)f\| = 0$, as desired.

□

Our main result is the following.

THEOREM 1.8. *Suppose that $\mathcal{H}$ is a Hilbert space, $(U_i)_{i=1}^t$ is a commuting family of unitary operators on $\mathcal{H}$, $\left(p_i(x_1, \cdots, x_k)\right)_{i=1}^t \subset \mathbf{Z}[x_1, \cdots, x_k]$, $p_i(0, \cdots, 0) = 0$, $1 \le i \le t$, and that $(n_\alpha^{(j)})_{\alpha\in\mathcal{F}}$ are IP-sets, $1 \le j \le k$. Suppose $\mathcal{F}^{(1)}$ is an IP-ring such that for each $f \in \mathcal{H}$,*

$$\operatorname*{IP-lim}_{\alpha\in\mathcal{F}^{(1)}} \Big(\prod_{i=1}^t U_i^{p_i(n_\alpha^{(1)},\cdots,n_\alpha^{(k)})}\Big) f = P_{(p_1,\cdots,p_t)} f \tag{1.1}$$

exists in the weak topology. Then $P_{(p_1,\cdots,p_t)}$ is an orthogonal projection. Projections of this type commute, that is, if also $\left(q_i(x_1, \cdots, x_k)\right)_{i=1}^t \subset \mathbf{Z}[x_1, \cdots, x_k]$, $q_i(0, \cdots, 0) = 0$, $1 \le i \le t$, then $P_{(p_1,\cdots,p_t)}P_{(q_1,\cdots,q_t)} = P_{(q_1,\cdots,q_t)}P_{(p_1,\cdots,p_t)}$.

Remark. Let us note that for any IP-ring there exists an IP-subring such that the limit in (1.1) exists. Moreover, by Proposition 1.2, we may even require of this subring that limits of this type exist not only for a single choice of polynomials p_i, but for *any* choice of polynomials. (This is possible because the set of choices for these polynomials is countable.) We will incorporate this remark in the proof.

Proof. We use induction on $d = \max_{1\le j\le t} \deg p_j$. The case $d = 1$ is easily seen to follow from Proposition 1.5. Suppose now that the conclusion holds for families of polynomials of degree less than d. We will show presently that $P_{(p_1,\cdots,p_t)}$ is idempotent. ¿From (1.1) it is clear that $\|P_{(p_1,\cdots,p_t)}\| \le 1$. It is not difficult to show that any idempotent Hilbert space operator P with $\|P\| \le 1$ satisfies $P = P^*$ and is an orthogonal projection onto a subspace of $\mathcal{H}$. Finally $P_{(p_1,\cdots,p_t)}$ and $P_{(q_1,\cdots,q_t)}$ will commute because weak limits of commuting operators necessarily commute. Therefore the theorem will be proved once we establish idempotence. Since any $h \in \mathcal{H}$ is contained in a separable subspace containing its entire orbit under the U_i's, we may assume without loss of generality that $\mathcal{H}$ is separable. We now make some observations.

$W = \{(q_1, \cdots, q_t) \in (\mathbf{Z}[x_1, \cdots, x_k])^t : \deg q_i < d,\ q_i(0, \cdots, 0) = 0\}$ is isomorphic to $\mathbf{Z}^s$ for some s. For each $\alpha \in \mathcal{F}^{(1)}$ let

$$q_i^{(\alpha)}(x_1, \cdots, x_k) = p_i(n_\alpha^{(1)} + x_1, \cdots, n_\alpha^{(k)} + x_k) - p_i(x_1, \cdots, x_k) - p_i(n_\alpha^{(1)}, \cdots, n_\alpha^{(k)}).$$

Then $(q_1^{(\alpha)}, \cdots, q_t^{(\alpha)}) \in W$. By Lemma 1.6, there exists $l \leq s$, an $l-$dimensional subgroup $V \subset W$ and an IP-ring $\mathcal{F}^{(2)} \subset \mathcal{F}^{(1)}$ such that $\{(q_1^{(\alpha)}, \cdots, q_t^{(\alpha)}) : \alpha \in \mathcal{F}^{(2)}\} \subset V$ and such that whenever $\alpha_i \in \mathcal{F}^{(2)}$, $\alpha_1 < \cdots < \alpha_l$, the set $\{(q_1^{(\alpha_i)}, \cdots, q_t^{(\alpha_i)}) : 1 \leq i \leq l\}$ is linearly independent. As indicated in the above remark, we may further require of $\mathcal{F}^{(2)}$ that all limits of the type (1.1) exist along $\mathcal{F}^{(2)}$ for any choice of polynomials.

Now for any $f \in \mathcal{H}$, the set $\{(q_1, \cdots, q_t) \in V : P_{(q_1, \cdots, q_t)} f = f\}$ is a subgroup of V, as can be easily seen from (1.1). It follows that if Y is an $l-$dimensional subgroup of V generated by the linearly independent set $\{(q_1^{(\alpha_i)}, \cdots, q_t^{(\alpha_i)}) : 1 \leq i \leq l\}$, where $\alpha_1 < \alpha_2 < \cdots < \alpha_l$, we have

$$\{f : P_{(q_1, \cdots, q_t)} f = f \text{ for every } (q_1, \cdots, q_t) \in Y\}$$

$$= P_{(q_1^{(\alpha_1)}, \cdots, q_t^{(\alpha_1)})} \cdots P_{(q_1^{(\alpha_l)}, \cdots, q_t^{(\alpha_l)})} \mathcal{H} = P_Y \mathcal{H}.$$

P_Y is an orthogonal projection since by hypothesis the projections $P_{(q_1^{(\alpha_i)}, \cdots, q_t^{(\alpha_i)})}$ commute.

For every $n \in \mathbf{N}$ let $V_n = n!W \cap V$. Then V_n is an l-dimensional subgroup of V for each n and P_{V_n} is an increasing sequence of orthogonal projections, so that $P = \lim_{n \to \infty} P_{V_n}$ is an orthogonal projection. Furthermore, we have for every l-dimensional subgroup $Y \subset V$, $V_n \subset Y$ for all n large enough. Hence

$$P\mathcal{H} = \overline{\{f \in \mathcal{H} : P_{V_n} f = f \text{ for some } n \in \mathbf{N}\}},$$

$$(P\mathcal{H})^\perp = \{f \in \mathcal{H} : P_Y f = 0 \text{ for every } l\text{-dimensional subgroup } Y \subset V\}.$$

According to our earlier remarks, all we must show is that for an arbitrarily chosen $f \in \mathcal{H}$, which we now fix, $P_{(p_1, \cdots, p_t)} f = P^2_{(p_1, \cdots, p_t)} f$. We may assume that $||f|| < 1$. We have $f = g + h$ where $g \in P\mathcal{H}$ and $h \in (P\mathcal{H})^\perp$. Let $x_\alpha = \big(\prod_{i=1}^t U_i^{p_i(n_\alpha^{(1)}, \cdots, n_\alpha^{(k)})}\big) h$. We claim that $P_{(p_1, \cdots, p_t)} h = \underset{\alpha \in \mathcal{F}^{(1)}}{\text{IP-lim}}\ x_\alpha = 0$. By Lemma 1.4 it will suffice to show that

$$\underset{\alpha \in \mathcal{F}^{(2)}}{\text{IP-lim}}\ \underset{\beta \in \mathcal{F}^{(2)}}{\text{IP-lim}}\ \langle x_{\alpha \cup \beta}, x_\beta \rangle = 0.$$

Notice that by the properties ascribed to $\mathcal{F}^{(2)}$ earlier and the fact that $h \in (P\mathcal{H})^\perp$, we have that whenever $\alpha_i \in \mathcal{F}^{(2)}$, $\alpha_1 < \cdots < \alpha_l$, $\big(\prod_{i=1}^l P_{(q_1^{(\alpha_i)} \cdots q_t^{(\alpha_i)})}\big) h = 0$. Therefore by Lemma 1.7 we have

$$\underset{\alpha \in \mathcal{F}^{(2)}}{\text{IP-lim}}\ ||P_{(q_1^{(\alpha)}, \cdots, q_t^{(\alpha)})} h|| = 0$$

and

$$\underset{\alpha \in \mathcal{F}^{(2)}}{\text{IP-lim}}\ \underset{\beta \in \mathcal{F}^{(2)}}{\text{IP-lim}}\ \Big\langle \big(\prod_{i=1}^t U_i^{p_i(n_{\alpha\cup\beta}^{(1)}, \cdots, n_{\alpha\cup\beta}^{(k)})}\big) h,\ \big(\prod_{i=1}^t U_i^{p_i(n_\beta^{(1)}, \cdots, n_\beta^{(k)})}\big) h \Big\rangle$$

$$= \underset{\alpha \in \mathcal{F}^{(2)}}{\text{IP-lim}}\ \underset{\beta \in \mathcal{F}^{(2)}}{\text{IP-lim}}\ \Big\langle \big(\prod_{i=1}^t U_i^{q_i^{(\alpha)}(n_\beta^{(1)}, \cdots, n_\beta^{(k)})}\big) h,\ \big(\prod_{i=1}^t U_i^{-p_i(n_\alpha^{(1)}, \cdots, n_\alpha^{(k)})}\big) h \Big\rangle$$

$$= \underset{\alpha \in \mathcal{F}^{(2)}}{\text{IP-lim}}\ \Big\langle P_{(q_1^{(\alpha)}, \cdots, q_t^{(\alpha)})} h,\ \big(\prod_{i=1}^t U_i^{-p_i(n_\alpha^{(1)}, \cdots, n_\alpha^{(k)})}\big) h \Big\rangle$$

$$\leq ||h|| \big(\underset{\alpha \in \mathcal{F}^{(2)}}{\text{IP-lim}}\ ||P_{(q_1^{(\alpha)}, \cdots, q_t^{(\alpha)})} h||\big) = 0.$$

This establishes our claim. Next we show that $P_{(p_1, \cdots, p_t)} g = P^2_{(p_1, \cdots, p_t)} g$. Let $\epsilon > 0$ be arbitrary. Choose g', $||g'|| < 1$, and $n \in \mathbf{N}$, with $P_{V_n} g' = g'$ and $||g - g'|| < \epsilon$. Let ρ be a metric on the unit ball of $\mathcal{H}$ for the weak topology satisfying $\rho(x,y) \leq ||x-y||$. There exists $\alpha_0 \in \mathcal{F}^{(2)}$ such that for every $\alpha \in \mathcal{F}^{(2)}$, $\alpha > \alpha_0$, $\rho\big((\prod_{i=1}^t U_i^{p_i(n_\alpha^{(1)},\cdots,n_\alpha^{(k)})})g',\ P_{(p_1,\cdots,p_t)}g'\big) < \epsilon$ and

$$\rho\Big(\big(\prod_{i=1}^t U_i^{p_i(n_\alpha^{(1)},\cdots,n_\alpha^{(k)})}\big)P_{(p_1,\cdots,p_t)}g',\ P^2_{(p_1,\cdots,p_t)}g'\Big) < \epsilon.$$

Let $\alpha \in \mathcal{F}^{(2)}$ be chosen with $\alpha > \alpha_0$ and such that $n!$ divides $n_\alpha^{(i)}$, $1 \leq i \leq k$. This will ensure that $(q_1^{(\alpha)},\cdots,q_t^{(\alpha)}) \in V_n$. For every $\beta \in \mathcal{F}^{(2)}$, $\beta > \alpha$, we have $(\alpha \cup \beta) > \alpha_0$ as well, so that

$$\rho\Big(\big(\prod_{i=1}^t U_i^{p_i(n_\alpha^{(1)},\cdots,n_\alpha^{(k)})+p_i(n_\beta^{(1)},\cdots,n_\beta^{(k)})+q_i^{(\alpha)}(n_\beta^{(1)},\cdots,n_\beta^{(k)})}\big)g',\ P_{(p_1,\cdots,p_t)}g'\Big)$$

$$= \rho\Big(\big(\prod_{i=1}^t U_i^{p_i(n_{\alpha\cup\beta}^{(1)},\cdots,n_{\alpha\cup\beta}^{(k)})}\big)g',\ P_{(p_1,\cdots,p_t)}g'\Big) < \epsilon.$$

Since $(q_1^{(\alpha)},\cdots,q_t^{(\alpha)}) \in V_n$ there exists $\beta_0 \in \mathcal{F}^{(2)}$, $\beta_0 > \alpha$, such that for every $\beta \in \mathcal{F}^{(2)}$ with $\beta > \beta_0$, $||(\prod_{i=1}^t U_i^{q_i^{(\alpha)}(n_\beta^{(1)},\cdots,n_\beta^{(k)})})g' - g'|| < \epsilon$, which implies that

$$\rho\Big(\big(\prod_{i=1}^t U_i^{p_i(n_\alpha^{(1)},\cdots,n_\alpha^{(k)})+p_i(n_\beta^{(1)},\cdots,n_\beta^{(k)})+q_i^{(\alpha)}(n_\beta^{(1)},\cdots,n_\beta^{(k)})}\big)g',$$

$$\big(\prod_{i=1}^t U_i^{p_i(n_\alpha^{(1)},\cdots,n_\alpha^{(k)})+p_i(n_\beta^{(1)},\cdots,n_\beta^{(k)})}\big)g'\Big) < \epsilon.$$

We may now fix such a β with the further property that

$$\rho\Big(\big(\prod_{i=1}^t U_i^{p_i(n_\alpha^{(1)},\cdots,n_\alpha^{(k)})+p_i(n_\beta^{(1)},\cdots,n_\beta^{(k)})}\big)g',\ \big(\prod_{i=1}^t U_i^{p_i(n_\alpha^{(1)},\cdots,n_\alpha^{(k)})}\big)P_{(p_1,\cdots,p_t)}g'\Big) < \epsilon.$$

(We have used weak continuity of $\prod_{i=1}^t U_i^{p_i(n_\alpha^{(1)},\cdots,n_\alpha^{(k)})}$.) Notice now that the last four displayed inequalities together with the triangle inequality give us

$$\rho\big(P_{(p_1,\cdots p_t)}g',\ P^2_{(p_1,\cdots p_t)}g'\big) < 4\epsilon.$$

Recall that $||Px|| \leq ||x||$. Therefore

$$\rho\big(P_{(p_1,\cdots p_t)}g,\ P_{(p_1,\cdots p_t)}g'\big) < \epsilon; \quad \rho\big(P^2_{(p_1,\cdots p_t)}g,\ P^2_{(p_1,\cdots p_t)}g'\big) < \epsilon$$

which gives us finally $\rho\big(P_{(p_1,\cdots p_t)}g,\ P^2_{(p_1,\cdots p_t)}g\big) < 6\epsilon$. Since ϵ was arbitrary, we have

$$P_{(p_1,\cdots,p_t)}f = P_{(p_1,\cdots,p_t)}g = P^2_{(p_1,\cdots,p_t)}g = P^2_{(p_1,\cdots,p_t)}f.$$

This establishes that $P_{(p_1,\cdots,p_t)}$ is idempotent and completes the proof of Theorem 1.8.

□

2. *Applications.*

Theorem 1.8 is applicable to measure preserving systems due to the fact that measure preserving actions of $(X,\mathcal{B},\mu)$ induce unitary actions of $L^2(X)$. The jump to combinatorics (Corollary 2.2) is made via a correspondence principle (see [F1], [F2],) which arises out of natural measure-preserving systems induced by large subsets of $\mathbf{Z}^n$. Finally, Proposition 2.3 deals with those transformations of infinite measure spaces, so called *conservative transformations*, for which the Poincaré recurrence theorem is valid.

COROLLARY 2.1. *Suppose that* (X,μ) *is a measure space with* $\mu(X)=1$ *and that* $\{T_1,\cdots,T_t\}$ *is a collection of commuting invertible measure preserving transformations of* X. *Suppose* $(n_\alpha^{(i)})_{\alpha\in\mathcal{F}}\subset\mathbf{N}$ *are IP-sets,* $1\le i\le k$, *and that* $p_j(x_1,\cdots,x_k)\in\mathbf{Z}[x_1,\cdots,x_k]$ *satisfy* $p_j(0,\cdots,0)=0$, $1\le j\le t$. *Then for any measurable* $A\subset X$, *there exists an IP-ring* $\mathcal{F}^{(1)}\subset\mathcal{F}$ *such that*

$$\operatorname*{IP-lim}_{\alpha\in\mathcal{F}^{(1)}}\mu\Big(A\cap\big(\prod_{i=1}^t T_i^{p_i(n_\alpha^{(1)},\cdots,n_\alpha^{(k)})}\big)^{-1}A\Big)\ge\mu(A)^2.$$

Proof. For $f\in L^2(X)$ let $U_if(x)=f(T_ix)$, $1\le i\le t$. Then $\{U_1,\cdots,U_t\}$ is a commuting set of unitary operators on the Hilbert space $L^2(X)$. Let $(f_j)_{j\in\mathbf{N}}$ be dense in $L^2(X)$. Since closed norm-bounded sets in $L^2(X)$ are compact in the weak topology, by Proposition 1.2, there exists an IP-ring $\mathcal{F}^{(1)}\subset\mathcal{F}$ such that

$$\operatorname*{IP-lim}_{\alpha\in\mathcal{F}^{(1)}}\big(\prod_{i=1}^t U_i^{p_i(n_\alpha^{(1)},\cdots,n_\alpha^{(k)})}\big)f_j$$

exists weakly for all $j\in\mathbf{N}$. It follows that

$$\operatorname*{IP-lim}_{\alpha\in\mathcal{F}^{(1)}}\big(\prod_{i=1}^t U_i^{p_i(n_\alpha^{(1)},\cdots,n_\alpha^{(k)})}\big)f=Pf$$

exists for all $f\in L^2(X)$. By Theorem 1.8, P is an orthogonal projection. Notice that since constant functions are invariant under the transformations T_i, $1\le i\le t$, they will be contained in $P\big(L^2(X)\big)$. Let (g_i) be an orthonormal basis for $P\big(L^2(X)\big)$ with $g_1(x)=1$, $x\in X$. Let $h=1_A$. Then $Ph=\sum_i\langle h,g_i\rangle g_i$ so that

$$\operatorname*{IP-lim}_{\alpha\in\mathcal{F}^{(1)}}\mu\Big(A\cap\big(\prod_{i=1}^t T_i^{p_i(n_\alpha^{(1)},\cdots,n_\alpha^{(k)})}\big)^{-1}A\Big)$$

$$=\operatorname*{IP-lim}_{\alpha\in\mathcal{F}^{(1)}}\langle h,\ \big(\prod_{i=1}^t U_i^{p_i(n_\alpha^{(1)},\cdots,n_\alpha^{(k)})}\big)h\rangle=\langle h,\ Ph\rangle$$

$$=\big|\big|Ph\big|\big|^2\ge\big|\big|\langle h,\ g_1\rangle g_1\big|\big|^2=\mu(A)^2.$$

□

Remark. If $\mu(A)>0$ in the foregoing then we obtain a positive limit. In particular this proves Theorem B of the Introduction. Another consequence of Corollary 2.1 is Khintchine's recurrence theorem, which states that for any $\epsilon>0$, any measure preserving system $(X,\mathcal{B},\mu,T)$, and $A\in\mathcal{B}$, the set $\{n:\ \mu(A\cap T^{-n}A)>\mu(A)^2-\epsilon\}$ has bounded gaps (or, in modern terminology, is *syndetic*). The reason is that any subset of $\mathbf{N}$ intersecting every IP-set must be syndetic.

COROLLARY 2.2. *Assume that* $t\in\mathbf{N}$ *and that* $E\subset\mathbf{Z}^t$ *with* $d^*(E)>0$. *Suppose* $k\in\mathbf{N}$, $p_i(x_1,\cdots,x_k)\in\mathbf{Z}[x_1,\cdots,x_k]$, $p_i(0,\cdots,0)=0$, $1\le i\le t$, *and that* $(n_\alpha^{(i)})_{\alpha\in\mathcal{F}}$ *are IP-sets,* $1\le i\le k$. *Then for some* $x,y\in E$ *and* $\alpha\in\mathcal{F}$,

$$x-y=\big(p_1(n_\alpha^{(1)},\cdots,n_\alpha^{(k)}),\cdots,p_t(n_\alpha^{(1)},\cdots,n_\alpha^{(k)})\big).$$

Proof. Let $\Omega=\{0,1\}^{\mathbf{Z}^t}$ and give Ω the product topology; that is, $\gamma_n\in\Omega$ converges to $\gamma\in\Omega$ precisely when for every $v\in\mathbf{Z}^t$, $\gamma_n(v)=\gamma(v)$ eventually. For every $v\in\mathbf{Z}^t$ define the shift homeomorphism T_v on Ω by $T_v\gamma(v')=\gamma(v+v')$. We denote by e_i the element of $\mathbf{Z}^t$ with 1 for its ith coordinate and 0 for its other coordinates. If then we put $T_i=T_{e_i}$, $1\le i\le t$, then $\{T_i:\ 1\le i\le t\}$ is a commuting family of homeomorphisms.

Next let $\omega\in\Omega$ be defined by $\omega(v)=1$ if and only if $v\in E$. Let X be the orbit closure of ω, $X=\overline{\{T_v\omega:\ v\in\mathbf{Z}^t\}}$, and put $A=\{\gamma\in X:\ \gamma(0)=1\}$. Then X is compact, T_i-invariant, $1\le i\le t$, and (see [F], p. 152) there exists a Borel measure μ on X, invariant under each of the T_i's, such that (since $d^*(E) > 0$,) $\mu(A) > 0$. By Corollary 2.1, there exists $\alpha \in \mathcal{F}$ such that

$$\mu\Big(A \cap \big(\prod_{i=1}^{t} T_i^{p_i(n_\alpha^{(1)},\cdots,n_\alpha^{(k)})}\big)^{-1} A\Big) > 0.$$

Pick a point $\gamma \in \Big(A \cap \big(\prod_{i=1}^{t} T_i^{-p_i(n_\alpha^{(1)},\cdots,n_\alpha^{(k)})}\big) A\Big)$. Then $\gamma(0) = 1$ and

$$\gamma\Big(\big(p_1(n_\alpha^{(1)},\cdots,n_\alpha^{(k)}),\cdots,p_t(n_\alpha^{(1)},\cdots,n_\alpha^{(k)})\big)\Big) = 1.$$

Since γ is in the orbit closure of ω, there exists $y \in \mathbf{Z}^t$ such that $T_y\omega(0) = 1$ and

$$T_y\omega\Big(\big(p_1(n_\alpha^{(1)},\cdots,n_\alpha^{(k)}),\cdots,p_t(n_\alpha^{(1)},\cdots,n_\alpha^{(k)})\big)\Big) = 1.$$

That is, $\omega(y) = 1$ and

$$\omega\Big(y + \big(p_1(n_\alpha^{(1)},\cdots,n_\alpha^{(k)}),\cdots,p_t(n_\alpha^{(1)},\cdots,n_\alpha^{(k)})\big)\Big) = 1.$$

This means that $y \in E$ and $x = y + \big(p_1(n_\alpha^{(1)},\cdots,n_\alpha^{(k)}),\cdots,p_t(n_\alpha^{(1)},\cdots,n_\alpha^{(k)})\big) \in E$.

$\square$

PROPOSITION 2.3. *Suppose (Y,ν) is a measure space with $\nu(Y) \le \infty$ and that S is a conservative (in case $\nu(Y) = \infty$) measure preserving transformation of Y. Suppose that (X,μ) is a probability space and that T is an invertible measure preserving tranformation of X. Then if $A \subset Y \times X$ with $(\nu \times \mu)(A) > 0$ and $p(x) \in \mathbf{Z}[x]$, $p(0) = 0$, there exists $n \in \mathbf{N}$ such that*

$$(\nu \times \mu)\big(A \cap (S^n \times T^{p(n)})^{-1} A\big) > 0.$$

Proof. There exists $\beta > 0$ and a set $B_0 \subset Y$, $\nu(B_0) > 0$, such that for every $y \in B_0$ we have $\mu(A_y) = \mu(\{x : (y,x) \in A\}) \ge \beta$. By separability of $L^2(X)$, for some subset $B_1 \subset B_0$, $\nu(B_1) > 0$, and some $C \subset X$, $\mu(C) \ge \beta$, we have for every $y \in B_1$,

$$\mu(A_y \triangle C) = \int |1_{A_y} - 1_C|^2 \, d\mu = \big|\big|1_{A_y} - 1_C\big|\big|^2_{L^2(X)} < \frac{1}{6}\beta^2.$$

We claim that there exists an IP-set $(n_\alpha)_{\alpha\in\mathcal{F}}$ such that for every $\alpha \in \mathcal{F}$ we have $\nu(B_1 \cap S^{-n_\alpha} B_1) > 0$. The "atoms" $(n_{\{i\}})$ will be determined inductively. Since S is conservative there exists $n_{\{1\}} \in \mathbf{N}$ such that $B_2 = B_1 \cap S^{-n_{\{1\}}} B_1$ satisfies $\nu(B_2) > 0$. Having found $n_{\{1\}},\cdots,n_{\{k-1\}}$ and $B_1,\cdots,B_k$, $\nu(B_i) > 0$, choose $n_{\{k\}}$ such that $B_{k+1} = B_k \cap S^{-n_{\{k\}}} B_k$ satisfies $\nu(B_{k+1}) > 0$. For $\alpha \in \mathcal{F}$ let $n_\alpha = \sum_{i\in\alpha} n_{\{i\}}$. Then (n_α) is an IP-set, and for every α, if i is the largest element in α, $B_{i+1} \subset B_1 \cap S^{-n_\alpha} B_1$, so our claim is proved.

By Corollary 2.1 there exists $\alpha \in \mathcal{F}$ such that

$$\mu(C \cap T^{-p(n_\alpha)} C) > \frac{\mu(C)^2}{2} \ge \frac{\beta^2}{2}.$$

Suppose that $y \in B_1 \cap S^{-n_\alpha} B_1$. Then $y, S^{n_\alpha} y \in B_1$, so that

$$\mu(C \triangle A_y) < \frac{1}{6}\beta^2, \quad \mu(T^{-p(n_\alpha)} C \triangle T^{-p(n_\alpha)} A_{S^{n_\alpha} y}) = \mu(C \triangle A_{S^{n_\alpha} y}) < \frac{1}{6}\beta^2.$$

Therefore $\mu(A_y \cap T^{-p(n_\alpha)} A_{S^{n_\alpha} y}) > \frac{1}{6}\beta^2$. For $x \in A_y \cap T^{-p(n_\alpha)} A_{S^{n_\alpha} y}$, $(y,x) \in A$ and $(S^{n_\alpha} y, T^{p(n_\alpha)} x) \in A$, so that $(y,x) \in A \cap (S^{n_\alpha} \times T^{p(n_\alpha)})^{-1} A$. Therefore by Fubini's Theorem

$$(\nu \times \mu)\big(A \cap (S^{n_\alpha} \times T^{p(n_\alpha)})^{-1} A\big)$$

$$\geq \int_{B_1 \cap S^{-n_\alpha} B_1} \mu(A_y \cap T^{-p(n_\alpha)} A_{S^{n_\alpha} y})\, d\nu(y) > \frac{1}{6}\beta^2 \nu(B_1 \cap S^{-n_\alpha} B_1) > 0.$$

□

3. *VIP-systems.*
As mentioned earlier, the case $d = 1$ of Theorem 1.8, in which the unitary operator expressions $U_\alpha = \prod_{i=1}^{t} U_i^{\sum_{j=1}^{k} a_{ij} n_\alpha^{(j)}}$ form IP-systems, follows from Proposition 1.5. The more general expression

$$V_\alpha = \prod_{i=1}^{t} U_i^{p_i(n_\alpha^{(1)}, \cdots, n_\alpha^{(k)})} \tag{3.1}$$

can be seen to form a variant of an IP-system and it is tempting to try to establish a result similar to Proposition 1.5 for this variant. To be precise let us say that an $\mathcal{F}$-sequence $\{V_\alpha\}_{\alpha \in \mathcal{F}}$ of invertible operators (or elements of a multiplicative group) form a *VIP-system* if the V_α commute and for some d (the *degree* of the system,) we have for any $\alpha_0, \cdots, \alpha_d \in \mathcal{F}$, $\alpha_0 < \cdots < \alpha_d$,

$$\prod_{0 \leq i(1) < i(2) < \cdots < i(k) \leq d} V_{\alpha_{i(1)} \cup \cdots \cup \alpha_{i(k)}}^{(-1)^k} = I.$$

If the polynomials p_i in (3.1) are of degree $\leq d$, then one can see that the corresponding system $\{V_\alpha\}$ is a VIP-system of degree d.

We now show by an example that the analogue of Proposition 1.5 is not valid for arbitrary VIP-systems. Let (Ω, P) be a probability space on which two independent systems of Bernoulli variables $\{X_i\}_{i=1}^{\infty}$, $\{Y_i\}_{i=1}^{\infty}$ are defined with $P(X_i = 1) = P(X_i = 0) = P(Y_i = 1) = P(Y_i = 0) = \frac{1}{2}$. For each $\alpha \in \mathcal{F}$ set

$$Z_\alpha = (-1)^{\sum_{i,j \in \alpha} X_i Y_j}.$$

One checks that for $\alpha, \beta, \gamma \in \mathcal{F}$ disjoint we have

$$Z_{\alpha \cup \beta \cup \gamma} Z_{\alpha \cup \beta}^{(-1)} Z_{\alpha \cup \gamma}^{(-1)} Z_{\beta \cup \gamma}^{(-1)} Z_\alpha Z_\beta Z_\gamma = 1.$$

It follows that if we let V_α be the operator on $L^2(\Omega, P)$ of multiplication by Z_α, $\{V_\alpha\}$ will form a VIP-system of degree 2. However, for disjoint α, β, Z_α and Z_β are independent and it follows that the weak IP-limit of Z_α is $E(Z_\alpha) = \frac{1}{2}$. Hence the weak operator limit of V_α is $\frac{1}{2}I$, which is not a projection operator. Therefore it appears that the conclusion of Theorem 1.8 is restricted to special VIP-systems.

REFERENCES

[F1] H. Furstenberg, Ergodic behavior of diagonal measures and a theorem of Szemerédi on arithmetic progressions, *J. d'Analyse Math.* **31** (1977) 204-256.

[F2] H. Furstenberg, *Recurrence in Ergodic Theory and Combinatorial Number Theory*, Princeton University Press, 1981.

[FK] H. Furstenberg and Y. Katznelson, An ergodic Szemerédi theorem for IP-systems and combinatorial theory, *J. d'Analyse Math.* **45** (1985) 117-168.

[H] N. Hindman, Finite sums from sequences within cells of a partition of **N**, *J. Combinatorial Theory* (Series A) **17** (1974) 1-11.

[M] K. Milliken, Ramsey's Theorem with sums or unions, *J. Combinatorial Theory* (Series A) **18** (1975), 276-290.

[S] A. Sárközy, On difference sets of integers III, *Acta. Math. Acad. Sci. Hungar.*, **31** (1978) 125-149.

[T] A. Taylor, A canonical partition relation for finite subsets of ω, *J. Combinatorial Theory* (Series A) **21** (1976), 137-146.