TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 360, Number 2, February 2008, Pages 819–847
S 0002-9947(07)04370-X
Article electronically published on May 16, 2007

SOME NEW RESULTS IN MULTIPLICATIVE AND ADDITIVE RAMSEY THEORY

MATHIAS BEIGLBÖCK, VITALY BERGELSON, NEIL HINDMAN, AND DONA STRAUSS

ABSTRACT. There are several notions of largeness that make sense in any semigroup, and others such as the various kinds of density that make sense in sufficiently well-behaved semigroups including $(\mathbb{N},+)$ and $(\mathbb{N},\cdot)$. It was recently shown that sets in $\mathbb{N}$ which are *multiplicatively large* must contain arbitrarily large *geoarithmetic progressions*, that is, sets of the form $\big\{r^j(a+id) : i,j \in \{0,1,\ldots,k\}\big\}$, as well as sets of the form $\big\{b(a+id)^j : i,j \in \{0,1,\ldots,k\}\big\}$. Consequently, given a finite partition of $\mathbb{N}$, one cell must contain such configurations. In the partition case we show that we can get substantially stronger conclusions. We establish some combined additive and multiplicative Ramsey theoretic consequences of known algebraic results in the semigroups $(\beta\mathbb{N},+)$ and $(\beta\mathbb{N},\cdot)$, derive some new algebraic results, and derive consequences of them involving geoarithmetic progressions. For example, we show that given any finite partition of $\mathbb{N}$ there must be, for each k, sets of the form $\big\{b(a+id)^j : i,j \in \{0,1,\ldots,k\}\big\}$ together with d, the arithmetic progression $\big\{a+id : i \in \{0,1,\ldots,k\}\big\}$, and the geometric progression $\big\{bd^j : j \in \{0,1,\ldots,k\}\big\}$ in one cell of the partition. More generally, we show that, if S is a commutative semigroup and $\mathcal{F}$ a partition regular family of finite subsets of S, then for any finite partition of S and any $k \in \mathbb{N}$, there exist $b,r \in S$ and $F \in \mathcal{F}$ such that $rF \cup \{b(rx)^j : x \in F, j \in \{0,1,2,\ldots,k\}\}$ is contained in a cell of the partition. Also, we show that for certain partition regular families $\mathcal{F}$ and $\mathcal{G}$ of subsets of $\mathbb{N}$, given any finite partition of $\mathbb{N}$ some cell contains structures of the form $B \cup C \cup B \cdot C$ for some $B \in \mathcal{F}, C \in \mathcal{G}$.

1. INTRODUCTION

Our starting point is the famous theorem of van der Waerden [22] which says that whenever the set $\mathbb{N}$ of positive integers is divided into finitely many classes, one of these classes contains arbitrarily long arithmetic progressions. The analogous statement for geometric progressions is easily seen to be equivalent via the homomorphisms $b : (\mathbb{N},+) \to (\mathbb{N},\cdot)$ and $\ell : (\mathbb{N} \setminus \{1\},\cdot) \to (\mathbb{N},+)$ where $b(n) = 2^n$ and $\ell(n)$ is the length of the prime factorization of n.

Received by the editors October 21, 2005.

2000 *Mathematics Subject Classification.* Primary 05D10; Secondary 22A15.

Key words and phrases. Ramsey theory, central sets, piecewise syndetic.

The first author thanks the Austrian Science Foundation FWF for its support through Projects nos. S8312 and P17627-N12. He also thanks Ohio State University for its hospitality in the spring of 2004 while much of this research was being conducted.

The second author acknowledges support received from the National Science Foundation via grant DMS-0345350.

The third author acknowledges support received from the National Science Foundation via grants DMS-0243586 and DMS-0554803.

In 1975 Szemerédi [21] showed that any set with positive upper asymptotic density contains arbitrarily long arithmetic progressions. (Ergodic theoretic proofs of Szemerédi's Theorem can be found in [6], [7] or [9]. Also Gowers [10] has a proof which provides very good bounds.) It has recently been shown [1, Theorem 1.3] that any set which is multiplicatively large (see Definition 2.1 below) must contain substantial combined additive and multiplicative structure; in particular it must contain arbitrarily large *geoarithmetic progressions*, that is, sets of the form $\{r^j(a+id) : i,j \in \{0,1,\ldots,k\}\}$.

As we shall see below, the corresponding partition theorem (i.e., for any finite partition of the positive integers, some cell contains arbitrarily large geoarithmetic progressions) can be derived fairly simply from well-known Ramsey theoretic results.

We thank Imre Leader for providing us with an elementary proof of the following theorem, and we thank the referee for suggesting the more general version in statement (a). A family $\mathcal{A}$ of subsets of a set X is *partition regular* provided that whenever X is partitioned into finitely many classes, one of these classes contains a member of $\mathcal{A}$.

Theorem 1.1. *(a) Let S and T be sets, let $\mathcal{F}$ be a partition regular family of finite subsets of S, and let $\mathcal{G}$ be a partition regular family of subsets of T. Let $m \in \mathbb{N}$ and let $S \times T = \bigcup_{k=1}^m A_k$. Then there exist $k \in \{1,2,\ldots,m\}$, $B \in \mathcal{F}$, and $C \in \mathcal{G}$ such that $B \times C \subseteq A_k$.*

(b) Let $(S,\cdot)$ be a set with some binary operation and let $\mathcal{F}$ and $\mathcal{G}$ be partition regular families of subsets of S with all members of $\mathcal{F}$ finite. Let $m \in \mathbb{N}$ and let $S = \bigcup_{k=1}^m A_k$. Then there exist $k \in \{1,2,\ldots,m\}$, $B \in \mathcal{F}$, and $C \in \mathcal{G}$ such that $B \cdot C \subseteq A_k$.

Proof. (a) By a standard compactness argument (see, for example, [16, Section 5.5], or [11, Section 1.5]) pick a finite subfamily $\mathcal{H}$ of $\mathcal{F}$ such that whenever $S = \bigcup_{k=1}^m D_k$, there exist $k \in \{1,2,\ldots,m\}$ and $B \in \mathcal{H}$ such that $B \subseteq D_k$. For each $x \in T$, $S = \bigcup_{k=1}^m \{t : (t,x) \in A_k\}$; thus we may pick $B(x) \in \mathcal{H}$ and $k(x) \in \{1,2,\ldots,m\}$ such that $B(x) \times \{x\} \subseteq A_{k(x)}$. Define $\tau : S \to \mathcal{H} \times \{1,2,\ldots,m\}$ by $\tau(x) = \big(B(x),k(x)\big)$. Pick $B \in \mathcal{H}$, $C \in \mathcal{G}$, and $k \in \{1,2,\ldots,m\}$ such that for all $x \in C$, $\tau(x) = (B,k)$.

(b) For $k \in \{1,2,\ldots,m\}$ let $E_k = \{(x,y) \in S \times S : x \cdot y \in A_k\}$ and apply conclusion (a). □

Note that one cannot drop the assumption that all members of $\mathcal{F}$ are finite: Consider $S = T = \mathbb{N}$ and let $\mathcal{F} = \mathcal{G} = \{B \subseteq \mathbb{N} : B \text{ is infinite}\}$. Let $A_1 = \{(x,y) \in \mathbb{N} \times \mathbb{N} : x \geq y\}$ and let $A_2 = \{(x,y) \in \mathbb{N} \times \mathbb{N} : x < y\}$. Then the conclusion of Theorem 1.1(a) fails in this situation. For conclusion (b), consider the group $(\mathbb{Z},+)$ and let $\mathcal{F} = \{B \subseteq \mathbb{N} : B \text{ is infinite}\}$ and $\mathcal{G} = \{C \subseteq \mathbb{Z} \setminus \mathbb{N} : C \text{ is infinite}\}$. Given $B \in \mathcal{F}$ and $C \in \mathcal{G}$ the set $B + C$ contains positive and negative integers. Thus the partition $\mathbb{Z} = \mathbb{N} \cup (\mathbb{Z} \setminus \mathbb{N})$ shows that the conclusion of Theorem 1.1(b) also fails.

Theorem 1.1 applied to the semigroup $(\mathbb{N},\cdot)$, the family of all $(k+1)$-term arithmetic progressions and the family of all $(k+1)$-term geometric progressions yields that for any finite partition of $\mathbb{N}$ there exist $a,b,d,r \in \mathbb{N}$ with $r \neq 1$ and some cell A such that

$$\big\{(ba+ibd)r^j : i,j \in \{0,1,\ldots,k\}\big\} = \{a,a+d,\ldots,a+kd\} \cdot \{b,br,\ldots,br^k\} \subseteq A\,.$$

In particular we see that some cell contains arbitrarily large geoarithmetic progressions.

Of course Theorem 1.1 can be applied iteratively to different kinds of partition regular families and binary operations. For example, for each finite partition of $\mathbb{N}$ and each $k \in \mathbb{N}$ there exist k-term geometric progressions G_1, G_2 and a k-term arithmetic progression A such that $\{g_1 + a^{g_2} : g_1 \in G_1\,,\, g_2 \in G_2, \text{ and } a \in A\}$ is entirely contained in one cell.

In Section 3 we present some combined additive and multiplicative results that can be obtained from known algebraic results or easy extensions thereof and are stronger than Theorem 1.1. These results appear to be unlikely to be easily obtainable by elementary methods.

For example we show in Theorem 3.7 that for certain partition regular families $\mathcal{F}$ and $\mathcal{G}$ one can strengthen the conclusion of Theorem 1.1 and prove that for any finite partition of $\mathbb{N}$ some cell contains structures of the form $B \cup C \cup B \cdot C$ for some $B \in \mathcal{F}, C \in \mathcal{G}$.

As a special case of Corollary 3.10 we obtain, for example, the following easy extension of the geoarithmetic result about partitions stated above:

Corollary 1.2. *Let $k, m \in \mathbb{N}$ and let $\mathbb{N} = \bigcup_{s=1}^{m} A_s$. Then there exist $s \in \{1, 2, \ldots, m\}$, $a, d \in A_s$ and $r \in A_s \setminus \{1\}$ such that*

$$\big\{r^j(a + id) : i, j \in \{0, 1, \ldots, k\}\big\} \cup \big\{dr^j : j \in \{0, 1, \ldots, k\}\big\} \subseteq A_s\,.$$

In Section 4 we derive several new algebraic results and new combinatorial consequences thereof.

Consider the following result, which is [1, Theorem 3.13]. Given a set X, $\mathcal{P}_f(X)$ is the set of finite nonempty subsets of X. (We shall give a precise definition of "multiplicatively large" in Definition 2.1.)

Theorem 1.3. *Let $k \in \mathbb{N}$. For each $i \in \{0, 1, \ldots, k\}$ let $\langle x_{i,t}\rangle_{t=1}^{\infty}$ and $\langle y_{i,t}\rangle_{t=1}^{\infty}$ be sequences in $\mathbb{N}$, and let A be a multiplicatively large subset of $\mathbb{N}$. Then there exist $F, G \in \mathcal{P}_f(\mathbb{N})$ and $a, b \in \mathbb{N}$ such that $\big\{b(a + \sum_{t\in F} x_{i,t}) \cdot (\prod_{t\in G} y_{j,t}) : i, j \in \{0, 1, \ldots, k\}\big\} \subseteq A$.*

Corollary 1.4. *Let $m, k \in \mathbb{N}$. For each $i \in \{0, 1, \ldots, k\}$ let $\langle x_{i,t}\rangle_{t=1}^{\infty}$ and $\langle y_{i,t}\rangle_{t=1}^{\infty}$ be sequences in $\mathbb{N}$. Let $\mathbb{N} = \bigcup_{s=1}^{m} A_s$. Then there exist $s \in \{1, 2, \ldots, m\}$, $F, G \in \mathcal{P}_f(\mathbb{N})$, and $a, b \in \mathbb{N}$ such that $\big\{b(a + \sum_{t\in F} x_{i,t}) \cdot (\prod_{t\in G} y_{j,t}) : i, j \in \{0, 1, \ldots, k\}\big\} \subseteq A_s$.*

Notice that a particular consequence of Corollary 1.4 is that one cell of each finite partition of $\mathbb{N}$ must contain arbitrarily long geoarithmetic progressions. Furthermore, the common ratio r can be taken from $FP(\langle y_n\rangle_{n=1}^{\infty})$ for any prescribed $\langle y_n\rangle_{n=1}^{\infty}$ and the additive increment d can be guaranteed to be a multiple of some member of $FS(\langle x_n\rangle_{n=1}^{\infty})$ for any prescribed $\langle x_n\rangle_{n=1}^{\infty}$. (In a semigroup $(S, \cdot)$, $FP(\langle y_n\rangle_{n=1}^{\infty}) = \{\prod_{n\in F} y_n : F \in \mathcal{P}_f(\mathbb{N})\}$ where the products are taken in increasing order of indices. If the operation is denoted by $+$, the corresponding notion is denoted $FS(\langle y_n\rangle_{n=1}^{\infty})$. The notations stand for *finite products* and *finite sums* respectively.) To see this, for $i \in \{1, 2, \ldots, k\}$ and $t \in \mathbb{N}$, let $x_{i,t} = ix_t$ and $y_{i,t} = (y_t)^i$. Given F and G as guaranteed by Corollary 1.4, let $d = b \cdot \sum_{t\in F} x_t$ and $r = \prod_{t\in G} y_t$.

We show in Theorem 4.12 that one may take $F = G$ in Theorem 1.3 and in Corollary 4.15 that one may eliminate b from Corollary 1.4 (and in particular, that the additive increment for the geoarithmetic progressions described above can be taken from $FS(\langle x_n\rangle_{n=1}^\infty)$ for any $\langle x_n\rangle_{n=1}^\infty)$. We show also that one may not in general simultaneously take $F = G$ and eliminate b. The example of Theorem 4.20 shows also that one cannot eliminate the multiplier b in Theorem 1.3.

Another simply stated result from [1] is that any multiplicatively large set contains geometric progressions in which the common ratios form an arithmetic progression, that is, a set of the form $\{b(a+id)^j : i,j \in \{0,1,\ldots,k\}\}$ [1, Theorem 3.15]. From this one concludes that one cell of any finite partition of $\mathbb{N}$ must satisfy this property. Of course one might hope for a theorem with stronger conclusions in the partition case.

A well-known extension of van der Waerden's Theorem allows one to get the additive increment of the arithmetic progression in the same cell as the arithmetic progression. Similarly for any finite partition of $\mathbb{N}$ there exist some cell A and $b, r \in \mathbb{N}$ such that $\{b, br^2, \ldots, br^k, r\} \subseteq A$. One naturally wonders whether one can intertwine these two facts. Indeed, this is achieved in the following theorem, which is a consequence of Corollary 4.7. (See Definition 4.6 for the definition of an (m,p,c)-set. These sets were introduced by Deuber [5] and are known to have rich combinatorial structure.)

Theorem 1.5. *Let $r, k \in \mathbb{N}$ and let $\mathbb{N} = \bigcup_{s=1}^r A_s$. Then there exist $s \in \{1,2,\ldots,r\}$ and $a, d, b \in A_i$ such that*

$$\begin{aligned}\{b(a+id)^j : i,j \in \{0,1,\ldots,k\}\} \cup \{bd^j : j \in \{0,1,\ldots,k\}\}\\ \cup \{a+id : i \in \{0,1,\ldots,k\}\} \subseteq A_s\,.\end{aligned}$$

More generally for all $m, p, c \in \mathbb{N}$ there exist $b \in \mathbb{N}$, some (m,p,c)-set F and $s \in \{1,2,\ldots,r\}$ such that

$$F \cup \{bx^j : x \in F, j \in \{0,1,\ldots,k\}\} \subseteq A_s.$$

In Section 5 we establish some limitations on the algebraic approach. We also prove a theorem which, for countable commutative semigroups, is even stronger than the powerful Central Sets Theorem. (The Central Sets Theorem for the semigroup $(\mathbb{N},+)$ is [7, Proposition 8.21]. Central subsets of any semigroup are guaranteed substantial combinatorial structure; see [16, Part III] for numerous examples.) Several earlier results in the paper follow immediately from this theorem. However, we prove these earlier results directly instead of stating them as corollaries, because the direct proofs are reasonably simple, while the theorem proved in Section 5 might be considered a little daunting.

2. Preliminaries

We shall be concerned with several notions of largeness, both additive and multiplicative. Among these are various notions of density. The notion $\overline{d}$ defined below is referred to as *upper asymptotic density*.

Definition 2.1. Let $A \subseteq \mathbb{N}$.

(a) $\overline{d}(A) = \limsup_{n\to\infty} \dfrac{|A \cap \{1,2,\ldots,n\}|}{n}$.

(b) A *Følner sequence* in $(\mathbb{N},\cdot)$ is a sequence $\langle F_n\rangle_{n=1}^\infty$ of finite nonempty subsets of $\mathbb{N}$ such that for each $x \in \mathbb{N}$, $\lim_{n\to\infty} \dfrac{|xF_n \triangle F_n|}{|F_n|} = 0$.

(c) If $\mathcal{F} = \langle F_n \rangle_{n=1}^{\infty}$ is a Følner sequence in $(\mathbb{N}, \cdot)$, then

$$\overline{d}_{\mathcal{F}}(A) = \limsup_{n\to\infty} \frac{|A \cap F_n|}{|F_n|} .$$

(d) If $\mathcal{F} = \langle F_n \rangle_{n=1}^{\infty}$ is a Følner sequence in $(\mathbb{N}, \cdot)$, then

$$d^*_{\mathcal{F}}(A) = \limsup_{k\to\infty} \left\{ \frac{|A \cap (m \cdot F_n)|}{|F_n|} : m \in \mathbb{N} \text{ and } n \geq k \right\} .$$

(e) The set A is *multiplicatively large* if and only if there is some Følner sequence $\langle F_n \rangle_{n=1}^{\infty}$ in $(\mathbb{N}, \cdot)$ such that $\overline{d}_{\mathcal{F}}(A) > 0$.

An example of a Følner sequence in $(\mathbb{N}, \cdot)$ is given by $F_n = \big\{ \prod_{i=1}^{n} {p_i}^{\alpha_i} :$ for each $i \in \{1, 2, \ldots, n\}$, $\alpha_i \in \{0, 1, \ldots, n\} \big\}$, where $\langle p_i \rangle_{i=1}^{\infty}$ is the sequence of primes in any fixed order. It is an easy exercise to show that a subset A of $\mathbb{N}$ is multiplicatively large if and only if there is some Følner sequence $\langle F_n \rangle_{n=1}^{\infty}$ in $(\mathbb{N}, \cdot)$ such that $d^*_{\mathcal{F}}(A) > 0$.

Other notions of largeness with which we shall be concerned originated in topological dynamics and make sense in any semigroup. Four of these, namely *thick*, *syndetic*, *piecewise syndetic* and *IP-set*, have simple elementary descriptions and we introduce them now. The fifth, *central*, while originally defined by Furstenberg in dynamical terms [7], is most simply described in terms of the algebraic structure of βS, which we shall describe shortly. Given a semigroup $(S, \cdot)$, a subset A of S, and $x \in S$, we let $x^{-1}A = \{y \in S : xy \in A\}$.

Definition 2.2. Let $(S, \cdot)$ be a semigroup and let $A \subseteq S$.

(a) A is *thick* if and only if whenever $F \in \mathcal{P}_f(S)$ there exists $x \in S$ such that $Fx \subseteq A$.
(b) A is *syndetic* if and only if there exists $G \in \mathcal{P}_f(S)$ such that $S = \bigcup_{t\in G} t^{-1}A$.
(c) A is *piecewise syndetic* if and only if there exists $G \in \mathcal{P}_f(S)$ such that for every $F \in \mathcal{P}_f(S)$ there exists $x \in S$ such that $Fx \subseteq \bigcup_{t\in G} t^{-1}A$.
(d) A is an *IP-set* if and only if there exists a sequence $\langle x_n \rangle_{n=1}^{\infty}$ in S such that $FP(\langle x_n \rangle_{n=1}^{\infty}) \subseteq A$.

Each of the above notions is one-sided. So, for example, A could be said to be "right thick" if it satisfies the definition above and "left thick" if for each $F \in \mathcal{P}_f(S)$ there exists $x \in S$ such that $xF \subseteq A$. (On the other hand, "right" and "left" can be, and have been, interchanged.)

Notice that a set $A \subseteq S$ is *thick* if for each finite subset F of S, A contains some (multiplicative) right translate of F. If $S = \mathbb{Z}$, and our operation is addition, then $A \subseteq S$ is called *syndetic* if a finite number of translates of A cover S. If we generalize this to an arbitrary semigroup S with some operation $\cdot$, then a good definition, rich enough for most purposes, is that A is syndetic if S is covered by a finite number of sets of the form $t^{-1}A$. If S were a group, then note that $t^{-1}A$ is a (multiplicative) translate of A. A set $A \subseteq S$ is *piecewise syndetic* if there is a finite union of translates of A which is thick.

Notice that each of thick and syndetic imply piecewise syndetic and that thick sets are IP-sets. It is easy to construct examples in $(\mathbb{N}, +)$ showing that no other implications among these notions is valid in general.

The following lemma gives a hint why piecewise syndetic sets will be interesting for our purposes.

Lemma 2.3. *Let $(S, \cdot)$ be a semigroup, let $\mathcal{F}$ be a partition regular family of finite subsets of S, and let A be a piecewise syndetic subset of S. Then there exist $t, x \in S$ and $F \in \mathcal{F}$ such that $tFx \subseteq A$. If $(S, \cdot)$ is commutative, then there exist $t \in S$ and $F \in \mathcal{F}$ such that $tF \subseteq A$.*

Proof. Pick $G \in \mathcal{P}_f(S)$ such that $\bigcup_{t \in G} t^{-1}A$ is thick. By a standard compactness argument pick a finite subfamily $\mathcal{H} \subseteq \mathcal{F}$ such that for each partition of S into $|G|$ sets some cell contains a member of $\mathcal{H}$. Since all elements of $\mathcal{H}$ are finite, $\bigcup \mathcal{H}$ is finite as well. Thus there exists some $x \in S$ such that $(\bigcup \mathcal{H})x \subseteq \bigcup_{t \in G} t^{-1}A$. Equivalently, all members of $\mathcal{H}$ are subsets of $\bigcup_{t \in G} t^{-1}Ax^{-1}$. We conclude that for some $t \in G$ and $F \in \mathcal{H}$, $F \subseteq t^{-1}Ax^{-1}$. $\square$

Notice that if $(S, \cdot)$ is not commutative, then both multipliers in Lemma 2.3 may be required. For example, let S be the free semigroup on the letters a and b. Then $\mathcal{F} = \{bF : F \in \mathcal{P}_f(S)\}$ and $\mathcal{G} = \{Fb : F \in \mathcal{P}_f(S)\}$ are partition regular, aS and Sa are piecewise syndetic, but there do not exist $F \in \mathcal{F}$ and $x \in S$ with $Fx \subseteq aS$, and there do not exist $F \in \mathcal{G}$ and $t \in S$ with $tF \subseteq Sa$. (In fact, aS is syndetic in S.)

We now present a very brief review of basic facts about $(\beta S, \cdot)$. For additional information, including historical notes about the discovery of these facts, see [16].

Given a discrete semigroup $(S, \cdot)$ we take the points of the Stone-Čech compactification βS of S to be the ultrafilters on S, the principal ultrafilters being identified with the points of S. Given $A \subseteq S$, $\overline{A} = \{p \in \beta S : A \in p\}$ and the set $\{\overline{A} : A \subseteq S\}$ is a basis for the open sets (and a basis for the closed sets) of βS. Given $p, q \in \beta S$ and $A \subseteq S$, $A \in p \cdot q$ if and only if $\{x \in S : x^{-1}A \in q\} \in p$. In particular, the operation $\cdot$ on βS extends the operation $\cdot$ on S.

With this operation, $(\beta S, \cdot)$ is a compact Hausdorff right topological semigroup with S contained in its topological center. That is, for each $p \in \beta S$, the function $\rho_p : \beta S \to \beta S$ defined by $\rho_p(q) = q \cdot p$ is continuous and for each $x \in S$, the function $\lambda_x : \beta S \to \beta S$ defined by $\lambda_x(q) = x \cdot q$ is continuous. A subset I of a semigroup T is a *left ideal* provided $T \cdot I \subseteq I$, a *right ideal* provided $I \cdot T \subseteq I$, and a *two sided ideal* (or simply an *ideal*) provided it is both a left ideal and a right ideal.

Any compact Hausdorff right topological semigroup T has a smallest two sided ideal $K(T) = \bigcup\{L : L \text{ is a minimal left ideal of } T\} = \bigcup\{R : R \text{ is a minimal right ideal of } T\}$. Given a minimal left ideal L and a minimal right ideal R, $L \cap R$ is a group, and in particular contains an idempotent. An idempotent in $K(T)$ is a *minimal* idempotent. If p and q are idempotents in T we write $p \leq q$ if and only if $pq = qp = p$. An idempotent is minimal with respect to this relation if and only if it is a member of the smallest ideal.

A subset of S is an IP-set (Definition 2.2(d)) if and only if it is a member of some idempotent in βS. It is piecewise syndetic (Definition 2.2(c)) if and only if it is a member of an element of $K(\beta S)$.

Definition 2.4. Let S be a semigroup and let $A \subseteq S$. Then A is *central* if and only if there is a minimal idempotent p in βS such that $A \in p$.

A central set is in particular a piecewise syndetic IP-set. Given a minimal idempotent p and a finite partition of S, one cell must be a member of p; hence at least one cell of any finite partition of S must be central. Central sets are fundamental to the Ramsey theoretic applications of the algebra of βS.

We shall need the Hales-Jewett Theorem. Given the free semigroup S over an alphabet L, a *variable word* w is a word over $L \cup \{v\}$ in which v occurs, where v is a "variable" not in L. Given a variable word w and $a \in L$, $\theta_a(w)$ is the word in S obtained by replacing each occurrence of v by a.

Theorem 2.5 (Hales-Jewett)**.** *Let L be a finite alphabet, let S be the free semigroup over L, let $m \in \mathbb{N}$, and let $S = \bigcup_{i=1}^m A_i$. Then there exist $i \in \{1, 2, \ldots, m\}$ and a variable word w such that $\{\theta_a(w) : a \in L\} \subseteq A_i$.*

Proof. See [12, Theorem 1], or see [11, Theorem 2.3] or [16, Theorem 14.7]. □

The following application of the Hales-Jewett Theorem will be used later. This result is well known among afficionados.

Theorem 2.6. *Let $(S, \cdot)$ be a commutative semigroup, let A be a piecewise syndetic subset of S, let $k \in \mathbb{N}$, and for $i \in \{1, 2, \ldots, k\}$ let $\langle y_{i,n}\rangle_{n=1}^\infty$ be a sequence in S. There exist $F \in \mathcal{P}_f(\mathbb{N})$ and $b \in S$ such that $\{b\} \cup \big\{b \prod_{t\in F} y_{i,t} : i \in \{1, \ldots, k\}\big\} \subseteq A$.*

Proof. By virtue of Lemma 2.3 it is sufficient to show that the family

$$\big\{\{b\} \cup \big\{b \cdot \textstyle\prod_{t\in F} y_{i,t} : i \in \{1, \ldots, k\}\big\} : b \in S, F \in \mathcal{P}_f(\mathbb{N})\big\}$$

is partition regular.

Let $L = \{0, 1, \ldots, k\}$ and let T be the free semigroup on the alphabet L. Let $b_0 \in S$ be an arbitrary, fixed element. Given a word $w = l_1 l_2 \cdots l_n$ of length n in S, define $f(w) = b_0 \prod_{t\in\{1,2,\ldots,n\}, l_t \neq 0} y_{l_t,t}$ if there exists some $t \in \{1, 2, \ldots, n\}$ such that $l_t \neq 0$ and $f(w) = b_0$ otherwise.

Consider a partition $\{A_1, A_2, \ldots, A_m\}$ of S. Then $T = \bigcup_{s=1}^m f^{-1}[A_s]$, so pick by Theorem 2.5, $s \in \{1, 2, \ldots, m\}$ and a variable word $w = l_1 l_2 \cdots l_n$ (with each $l_t \in L \cup \{v\}$) such that $\{\theta_i(w) : i \in L\} \subseteq f^{-1}[A_s]$.

Let $F = \{t \in \{1, 2, \ldots, n\} : l_t = v\}$, let $G = \{1, 2, \ldots, n\} \setminus F$ and let $b = f\big(\theta_0(w)\big)$. Then $b \prod_{t\in F} y_{i,t} = f\big(\theta_i(w)\big)$ for $i \in \{1, 2, \ldots, k\}$, and thus $\{b\} \cup \big\{b \prod_{t\in F} y_{i,t} : i \in \{1, \ldots, k\}\big\} \subseteq A_s$. □

Corollary 2.7. *Let $(S, \cdot)$ be a commutative semigroup, let A be a piecewise syndetic subset of S, let B be an IP-set in S, and let $k \in \mathbb{N}$. There exist $b \in S$ and $r \in B$ such that $\{b, br, br^2, \ldots, br^k\} \subseteq A$. If A is central we may in particular take $A = B$ so that $\{r, b, br, br^2, \ldots, br^k\} \subseteq A$.*

Proof. Let $\langle x_n\rangle_{n=1}^\infty$ be a sequence in S such that $FP(\langle x_n\rangle_{n=1}^\infty) \subseteq B$. For $i \in \{1, 2, \ldots, k\}$ and $n \in \mathbb{N}$, let $y_{i,n} = (x_n)^i$. Pick b and F as guaranteed by Theorem 2.6 and let $r = \prod_{t\in F} x_t$.

Any central set is a piecewise syndetic IP-set and thus the "in particular" statement follows. □

In fact a stronger version of Theorem 2.6, presented below as Theorem 2.9, is a simple consequence of the following version of the Central Sets Theorem.

Theorem 2.8. *Let $(S, \cdot)$ be a commutative semigroup, let A be a central subset of S, and for each $i \in \mathbb{N}$, let $\langle y_{i,n}\rangle_{n=1}^\infty$ be a sequence in S. There exist a sequence $\langle a_n\rangle_{n=1}^\infty$ with $FP(\langle a_n\rangle_{n=1}^\infty) \subseteq A$ and a sequence $\langle H_n\rangle_{n=1}^\infty$ in $\mathcal{P}_f(\mathbb{N})$ such that $\max H_n < \min H_{n+1}$ for each $n \in \mathbb{N}$ and such that for each $f : \mathbb{N} \to \mathbb{N}$ with $f(n) \leq n$ for each $n \in \mathbb{N}$, $FP(\langle a_n \cdot \prod_{t\in H_n} y_{f(n),t}\rangle_{n=1}^\infty) \subseteq A$.*

Proof. The case $(S,\cdot) = (\mathbb{Z},+)$ is [7, Proposition 8.21]. For the general case, let $S' = S \cup \{1\}$ where 1 is a two sided identity and let $y_{0,n} = 1$ for every $n \in \mathbb{N}$. By [16, Exercise 15.1.1] A is central in S', so apply [16, Theorem 14.11] to the sequences $\langle y_{i,n}\rangle_{n=1}^{\infty}$ for $i \in \mathbb{N} \cup \{0\}$. Since for each $n \in \mathbb{N}$, $a_n = a_n \cdot \prod_{t\in H_n} y_{0,t}$, we have that each $a_n \in A$. □

Theorem 2.9. *Let $(S,\cdot)$ be a commutative semigroup, let A be a piecewise syndetic subset of S, and for each $i \in \{1,2,\ldots,k\}$ let $\langle y_{i,n}\rangle_{n=1}^{\infty}$ be a sequence in S. There exist $b \in A$, a sequence $\langle a_n\rangle_{n=1}^{\infty}$ in S, and a sequence $\langle H_n\rangle_{n=1}^{\infty}$ in $\mathcal{P}_f(\mathbb{N})$ such that $\max H_n < \min H_{n+1}$ for each $n \in \mathbb{N}$ and such that for each $f : \mathbb{N} \to \mathbb{N}$ with $f(n) \le n$ for each $n \in \mathbb{N}$, $b \cdot FP(\langle a_n\rangle_{n=1}^{\infty}) \subseteq A$ and $b \cdot FP(\langle a_n \cdot \prod_{t\in H_n} y_{f(n),t}\rangle_{n=1}^{\infty}) \subseteq A$. In particular there exist $F \in \mathcal{P}_f(\mathbb{N})$ and $b \in S$ such that $\{b\} \cup \big\{b\prod_{t\in F} y_{i,t} : i \in \{1,\ldots,k\}\big\} \subseteq A$.*

Proof. Pick $p \in K(\beta S)$ such that $A \in p$. Pick a minimal left ideal L of βS such that $p \in L$ and let e be an idempotent in L. Then $p = pe$, so $\{x \in S : x^{-1}A \in e\} \in p$. Pick $b \in A$ such that $b^{-1}A \in e$. Then $b^{-1}A$ is central, so apply Theorem 2.8. □

3. New wine from old wineskins

All of the results about the algebraic structure of $\beta\mathbb{N}$ that are used in this section have been known for several years.

There is a long list of configurations which are known to be present in any central subset of $(\mathbb{N},+)$ and a somewhat shorter, but still lengthy, list of structures which can be found in any central subset of $(\mathbb{N},\cdot)$. Some of these involve special subsets of $\beta\mathbb{N}$ defined by various notions of density.

Definition 3.1. (a) $\Delta = \{q \in \beta\mathbb{N} : (\forall A \in q)(\overline{d}(A) > 0)\}$.

(b) If $\mathcal{F} = \langle F_n\rangle_{n=1}^{\infty}$ is a Følner sequence in $(\mathbb{N},\cdot)$, then

$$\Delta_{\mathcal{F}} = \{q \in \beta\mathbb{N} : (\forall A \in q)(\overline{d}_{\mathcal{F}}(A) > 0)\}\,.$$

(c) If $\mathcal{F} = \langle F_n\rangle_{n=1}^{\infty}$ is a Følner sequence in $(\mathbb{N},\cdot)$, then

$$\Delta^*_{\mathcal{F}} = \{q \in \beta\mathbb{N} : (\forall A \in q)(d^*_{\mathcal{F}}(A) > 0)\}\,.$$

Lemma 3.2. *Let $\mathcal{F} = \langle F_n\rangle_{n=1}^{\infty}$ be a Følner sequence in $(\mathbb{N},\cdot)$. Then $\Delta^*_{\mathcal{F}}$ is a two sided ideal of $(\beta\mathbb{N},\cdot)$.*

Proof. Let $q \in \Delta^*_{\mathcal{F}}$ and let $p \in \beta\mathbb{N}$. To see that $p\cdot q \in \Delta^*_{\mathcal{F}}$ let $A \in p\cdot q$ and pick $x \in \mathbb{N}$ such that $x^{-1}A \in q$. Let $\alpha = d^*_{\mathcal{F}}(x^{-1}A)$, let $k \in \mathbb{N}$, and let $\epsilon > 0$. Pick $m \in \mathbb{N}$ and $n \ge k$ such that $|x^{-1}A \cap mF_n| \ge (\alpha-\epsilon)\cdot|F_n|$. Then $|A \cap xmF_n| \ge (\alpha-\epsilon)\cdot|F_n|$.

To see that $q\cdot p \in \Delta^*_{\mathcal{F}}$ let $A \in q\cdot p$ and let $B = \{x \in \mathbb{N} : x^{-1}A \in p\}$. Let $\alpha = d^*_{\mathcal{F}}(B)$, let $k \in \mathbb{N}$, and let $\epsilon > 0$. Pick $m \in \mathbb{N}$ and $n \ge k$ such that $|B \cap mF_n| \ge (\alpha-\epsilon)\cdot|F_n|$. Pick $t \in \bigcap\{x^{-1}A : x \in B \cap mF_n\}$. Then $|A \cap tmF_n| \ge |B \cap mF_n| \ge (\alpha-\epsilon)\cdot|F_n|$. □

In [20], Rado proved that a $u\times v$ matrix C is *kernel partition regular* over $(\mathbb{N},+)$ (meaning that whenever $r \in \mathbb{N}$ and $\mathbb{N} = \bigcup_{i=1}^{r} A_i$, there exist $i \in \{1,2,\ldots,r\}$ and $\vec{x} \in {A_i}^v$ such that $C\vec{x} = \vec{0}$) if and only if C satisfies a computable requirement called the *columns condition*.

A $u\times v$ matrix C with entries from $\mathbb{Q}$ is *image partition regular* over $(\mathbb{N},+)$ if and only if whenever $r \in \mathbb{N}$ and $\mathbb{N} = \bigcup_{i=1}^{r} A_i$, there exist $i \in \{1,2,\ldots,r\}$ and $\vec{x} \in \mathbb{N}^v$

such that all entries of $C\vec{x}$ are in A_i. We shall use the custom of denoting the entries of a matrix by the lower case of the same letter whose upper case denotes the matrix, so that the entry in row i and column j of C is denoted by $c_{i,j}$.

Definition 3.3. Let $u, v \in \mathbb{N}$ and let C be a $u \times v$ matrix with entries from $\mathbb{Q}$.

(a) C is a *first entries matrix* if and only if no row of C is $\vec{0}$ and for all $i, j \in \{1, 2, \ldots, u\}$ and all $k \in \{1, 2, \ldots, v\}$, if $k = \min\{t : c_{i,t} \neq 0\} = \min\{t : c_{j,t} \neq 0\}$, then $c_{i,k} = c_{j,k} > 0$.

(b) The number b is a *first entry* of C if and only if b is the first nonzero entry in some row of C.

Each first entries matrix is image partition regular over $(\mathbb{N}, +)$, and image partition regular matrices can be characterized in terms of first entries matrices. (See [16, Theorem 15.24].)

We first summarize some of the structures guaranteed to be present in any multiplicatively central set. See [16, Chapter 14] for a formal definition of the notion of *tree* in a set as well as the set of *successors* to a node. (Informally, there is a good chance it means what you think it means.)

Theorem 3.4. *Let A be a central subset of $(\mathbb{N}, \cdot)$.*

(a) *For any sequence $\langle x_n \rangle_{n=1}^{\infty}$ in $\mathbb{N}$ and any $k \in \mathbb{N}$, there exist $b \in \mathbb{N}$ and $r \in FP(\langle x_n \rangle_{n=1}^{\infty})$ such that $\{b, br, br^2, \ldots, br^k\} \subseteq A$.*

(b) *There is a tree T in A such that for any path g through T and any Følner sequence $\mathcal{F} = \langle F_n \rangle_{n=1}^{\infty}$, $FP(\langle g(n) \rangle_{n=1}^{\infty}) \subseteq A$ and for every node $f \in T$, the set B_f of successors to f satisfies $d^*_{\mathcal{F}}(B_f) > 0$.*

(c) *If $u, v \in \mathbb{N}$ and C is a $u \times v$ matrix with entries from $\mathbb{Z}$ which satisfies the columns condition over $\mathbb{Z}$, then there exists $\vec{x} \in A^v$ such that for all $i \in \{1, 2, \ldots, u\}$, $\prod_{j=1}^{v} {x_j}^{c_{i,j}} = 1$.*

(d) *If $u, v \in \mathbb{N}$ and C is a $u \times v$ first entries matrix with entries from $\mathbb{Z}$ and all first entries equal to 1, then there exists $\vec{x}$ in $\mathbb{N}^v$ such that for all $i \in \{1, 2, \ldots, u\}$, $\prod_{j=1}^{v} {x_j}^{c_{i,j}} \in A$.*

Proof. (a) This follows from Corollary 2.7.

(b) Pick a minimal idempotent q of $(\beta\mathbb{N}, \cdot)$ such that $A \in q$. By Lemma 3.2, $\Delta^*_{\mathcal{F}}$ is an ideal of $(\beta\mathbb{N}, \cdot)$, so $q \in \Delta^*_{\mathcal{F}}$ and [16, Lemma 14.24] applies.

(c) See [16, Theorem 15.16(a)].

(d) See [16, Lemma 15.14 and Theorem 15.5]. □

The conditions of Theorem 3.4(c) and (d) are stronger than those required for kernel and image partition regularity over $(\mathbb{N}, \cdot)$. (And necessarily so. The set $A = \mathbb{N} \setminus \{x^2 : x \in \mathbb{N}\}$ is central in $(\mathbb{N}, \cdot)$ [16, Exercise 15.1.2], the matrix $\begin{pmatrix} 2 & -2 & 1 \end{pmatrix}$ is kernel partition regular over $(\mathbb{N}, \cdot)$, and the matrix (2) is image partition regular over $(\mathbb{N}, \cdot)$. But one cannot get $x, y, z \in A$ with $x^2 y^{-2} z = 1$ and one cannot get $x \in \mathbb{N}$ with $x^2 \in A$.) By contrast, in $(\mathbb{N}, +)$, kernel partition regularity of C corresponds to solutions to $C\vec{x} = \vec{0}$ in any central set and image partition regularity of C corresponds to obtaining all entries of $C\vec{x}$ in any central set.

We shall be interested in a property stronger than central for our additive results. By [16, Theorem 6.79], Δ is a compact left ideal of $(\beta\mathbb{N}, +)$, so it contains a minimal idempotent of $(\beta\mathbb{N}, +)$. Consequently, any finite partition of $\mathbb{N}$ will have one cell satisfying the hypothesis of the following theorem.

Theorem 3.5. *Let $A \subseteq \mathbb{N}$ and assume that there is a minimal idempotent q of $(\beta\mathbb{N}, +)$ in $\overline{A} \cap \Delta$.*

(a) *For any sequence $\langle x_n \rangle_{n=1}^{\infty}$ in $\mathbb{N}$ and any $k \in \mathbb{N}$, there exist $a \in \mathbb{N}$ and $d \in FS(\langle x_n \rangle_{n=1}^{\infty})$ such that $\{a, a+d, \ldots, a+kd\} \subseteq A$.*

(b) *There is a tree T in A such that for any path g through T, $FS(\langle g(n) \rangle_{n=1}^{\infty}) \subseteq A$ and for every node $f \in T$, the set B_f of successors to f satisfies $\overline{d}(B_f) > 0$.*

(c) *If $u, v \in \mathbb{N}$ and C is a $u \times v$ matrix with entries from $\mathbb{Q}$ which is kernel partition regular over $(\mathbb{N}, +)$ (that is, C satisfies the columns condition over $\mathbb{Q}$), then there exists $\vec{x} \in A^v$ such that $C\vec{x} = \vec{0}$.*

(d) *If $u, v \in \mathbb{N}$ and C is a $u \times v$ matrix with entries from $\mathbb{Q}$ which is image partition regular over $(\mathbb{N}, +)$, (in particular, if C is a first entries matrix), then there exists $\vec{x}$ in $\mathbb{N}^v$ with all entries of $C\vec{x}$ in A.*

(e) *Let R be a finite set of polynomials which take integer values at integers and have zero constant term, and let $\langle z_i \rangle_{i=1}^{\infty}$ be a sequence in $\mathbb{Z}$. Then there exists $F \in \mathcal{P}_f(\mathbb{N})$ such that $\{a \in A : \{a + p(\Sigma_{i \in F} z_i) : p \in R\} \subseteq A\}$ is piecewise syndetic.*

Proof. (a) This follows from Corollary 2.7.

(b) See [16, Lemma 14.24].

(c) See [16, Theorem 15.16(b)].

(d) See [15, Theorem 2.10].

(e) In [3, Theorem C], it was shown that the conclusion follows from the assumption that A is piecewise syndetic. For an algebraic proof, see [14, Corollary 3.7]. $\square$

Lemma 3.6. *Let $D = \{q \in \Delta : q$ is a minimal idempotent of $(\beta\mathbb{N}, +)\}$. Then $c\ell D$ is a left ideal of $(\beta\mathbb{N}, \cdot)$.*

Proof. We have already observed that $D \neq \emptyset$. Let $r \in c\ell D$. To see that $\beta\mathbb{N} \cdot r \subseteq c\ell R$ it suffices by the continuity of ρ_r in $(\beta\mathbb{N}, \cdot)$ to show that $\mathbb{N} \cdot r \subseteq c\ell D$. So let $x \in \mathbb{N}$ and let $A \in x \cdot r$. Then $x^{-1}A \in r$, so pick $q \in D \cap \overline{x^{-1}A}$. Then $A \in x \cdot q$. By [16, Theorem 6.79], $x \cdot q \in \Delta$. By [15, Lemma 2.1], $x \cdot q$ is a minimal idempotent of $(\beta\mathbb{N}, +)$. $\square$

Plentiful examples of candidates for the sets $\mathcal{F}$ and $\mathcal{G}$ of Theorem 3.7 are provided by Theorems 3.4 and 3.5. Notice in particular that $\mathcal{G}$ could be any family of subsets of $\mathbb{N}$ such that any additively central set must contain a member of $\mathcal{G}$.

Theorem 3.7. *Let $D = \{q \in \Delta : q$ is a minimal idempotent of $(\beta\mathbb{N}, +)\}$. Let $\mathcal{F}$ be a set of subsets of $\mathbb{N}$ with the property that any multiplicatively central subset of $\mathbb{N}$ contains a member of $\mathcal{F}$ and let $\mathcal{G}$ be a set of subsets of $\mathbb{N}$ with the property that, whenever $A \subseteq \mathbb{N}$ and $\overline{A} \cap D \neq \emptyset$, some member of $\mathcal{G}$ is contained in A. Assume further that $\mathcal{F}$ or $\mathcal{G}$ consists of finite sets. Let $\mathcal{H} = \langle H_n \rangle_{n=1}^{\infty}$ be a Følner sequence in $(\mathbb{N}, \cdot)$. Whenever $r \in \mathbb{N}$ and $\mathbb{N} = \bigcup_{i=1}^{r} A_i$, there exists $i \in \{1, 2, \ldots, r\}$ such that $\overline{d}(A_i) > 0$, $d^*_{\mathcal{H}}(A_i) > 0$, and there exist $B \in \mathcal{F}$ and $C \in \mathcal{G}$ such that $B \cup C \cup B \cdot C \subseteq A_i$.*

Proof. By Lemma 3.6, $c\ell D$ is a left ideal of $(\beta\mathbb{N}, \cdot)$, so pick a minimal idempotent q of $(\beta\mathbb{N}, \cdot)$ in $c\ell D$. Pick $i \in \{1, 2, \ldots, r\}$ such that $A_i \in q$. Since $q \in c\ell D \subseteq \Delta$, $\overline{d}(A_i) > 0$. By Theorem 3.4(b), $d^*_{\mathcal{H}}(A_i) > 0$. Since $q = q \cdot q$, $\{x \in A_i : x^{-1}A_i \in q\} \in q$. Assume first that $\mathcal{F}$ consists of finite sets. Since $\{x \in A_i : x^{-1}A_i \in q\} \in q$, $\{x \in A_i : x^{-1}A_i \in q\}$ is multiplicatively central, so pick $B \in \mathcal{F}$ such that $B \subseteq \{x \in A_i : x^{-1}A_i \in q\}$. Since B is finite, $A_i \cap \bigcap_{x \in B} x^{-1}A_i \in q$ and thus $\overline{(A_i \cap \bigcap_{x \in B} x^{-1}A_i)} \cap D \neq \emptyset$. Pick $C \in \mathcal{G}$ such that $C \subseteq A_i \cap \bigcap_{x \in B} x^{-1}A_i$.

Now assume that $\mathcal{G}$ consists of finite sets. Since $\{x \in A_i : x^{-1}A_i \in q\} \in q$, $\overline{\{x \in A_i : x^{-1}A_i \in q\}} \cap D \neq \emptyset$, so pick $C \in \mathcal{G}$ such that $C \subseteq \{x \in A_i : x^{-1}A_i \in q\}$. Since C is finite, $A_i \cap \bigcap_{x \in C} x^{-1}A_i \in q$, so there exists some $B \in \mathcal{F}$ such that $B \subseteq A_i \cap \bigcap_{x \in C} x^{-1}A_i$. □

By adding the requirement that the members of both $\mathcal{F}$ and $\mathcal{G}$ are finite, we obtain an infinitary extension of Theorem 3.7 along the lines of the Central Sets Theorem.

Theorem 3.8. *Let $D = \{q \in \Delta : q \text{ is a minimal idempotent of } (\beta\mathbb{N},+)\}$. For each $n \in \mathbb{N}$, let $\mathcal{F}_n$ be a set of finite subsets of $\mathbb{N}$ with the property that any multiplicatively central subset of $\mathbb{N}$ contains a member of $\mathcal{F}_n$, and let $\mathcal{G}_n$ be a set of finite subsets of $\mathbb{N}$ with the property that, whenever $A \subseteq \mathbb{N}$ and $\overline{A} \cap D \neq \emptyset$, some member of $\mathcal{G}_n$ is contained in A. Let $\mathcal{H} = \langle H_n \rangle_{n=1}^\infty$ be a Følner sequence in $(\mathbb{N},\cdot)$. Whenever $r \in \mathbb{N}$ and $\mathbb{N} = \bigcup_{i=1}^r A_i$, there exists $i \in \{1,2,\ldots,r\}$ such that $\overline{d}(A_i) > 0$, $d^*_{\mathcal{H}}(A) > 0$, and there exist sequences $\langle B_n \rangle_{n=1}^\infty$ and $\langle C_n \rangle_{n=1}^\infty$ such that $B_n \in \mathcal{F}_n$ and $C_n \in \mathcal{G}_n$ for each n and for any $F \in \mathcal{P}_f(\mathbb{N})$ and any $f \in \bigtimes_{n \in F}(B_n \cup C_n \cup B_n \cdot C_n)$, $\prod_{n \in F} f(n) \in A_i$.*

Proof. Pick a minimal idempotent q of $(\beta\mathbb{N},\cdot)$ in $c\ell D$, and pick $i \in \{1,2,\ldots, r\}$ such that $A_i \in q$. Then $\overline{d}(A_i) > 0$ and $d^*_{\mathcal{H}}(A) > 0$. For any $X \in q$, let $X^\star = \{x \in X : x^{-1}X \in q\}$. Then by [16, Lemma 4.14], $X^\star \in q$ and for any $x \in X^\star$, $x^{-1}X^\star \in q$.

Choose $B_1 \in \mathcal{F}_1$ such that $B_1 \subseteq {A_1}^\star$ and choose $C_1 \in \mathcal{G}_1$ such that $C_1 \subseteq {A_1}^\star \cap \bigcap_{x \in B_1} x^{-1}{A_1}^\star$.

Inductively, let $n \in \mathbb{N}$ and assume we have chosen $B_t \in \mathcal{F}_t$ and $C_t \in \mathcal{G}_t$ for each $t \in \{1,2,\ldots,n\}$ with the property that for all nonempty $F \subseteq \{1,2,\ldots,n\}$ and all $f \in \bigtimes_{t \in F}(B_t \cup C_t \cup B_t \cdot C_t)$, $\prod_{t \in F} f(t) \in {A_i}^\star$. Let

$$X = {A_i}^\star \cap \bigcap \Big\{ \big(\textstyle\prod_{t \in F} f(t) \big)^{-1} {A_i}^\star : \begin{array}{l} \emptyset \neq F \subseteq \{1,2,\ldots,n\} \text{ and} \\ f \in \bigtimes_{t \in F}(B_t \cup C_t \cup B_t \cdot C_t) \Big\} . \end{array}$$

Then X is a finite intersection of members of q, so $X \in q$. Pick $B_{n+1} \in \mathcal{F}_{n+1}$ such that $B_{n+1} \subseteq X^\star$. Then $X \cap \bigcap_{x \in B_{n+1}} x^{-1}X \in q$, so pick $C_{n+1} \in \mathcal{G}_{n+1}$ such that $C_{n+1} \subseteq X \cap \bigcap_{x \in B_{n+1}} x^{-1}X$. □

Corollary 3.9. *Let $m, k \in \mathbb{N}$ and let $\mathbb{N} = \bigcup_{i=1}^m A_i$. Let $\mathcal{H} = \langle H_n \rangle_{n=1}^\infty$ be a Følner sequence in $(\mathbb{N},\cdot)$. Then there exist $i \in \{1,2,\ldots,m\}$, $a, d, b \in A_i$, and $r \in A_i \setminus \{1\}$ such that $\overline{d}(A_i) > 0$, $d^*_{\mathcal{H}}(A_i) > 0$, and*

$$\begin{gathered} \big\{br^s : s \in \{0,1,\ldots,k\}\big\} \cup \big\{a + td : t \in \{0,1,\ldots,k\}\big\} \cup \{rd\} \cup \\ \big\{r(a+td) : t \in \{0,1,\ldots,k\}\big\} \cup \big\{bdr^s : s \in \{0,1,\ldots,k\}\big\} \cup \\ \big\{br^s(a+td) : s,t \in \{0,1,\ldots,k\}\big\} \subseteq A_i \,. \end{gathered}$$

Proof. Let $\mathcal{F} = \Big\{ \big\{br^s : s \in \{0,1,\ldots,k\}\big\} \cup \{r\} : b, r \in \mathbb{N} \Big\}$ and let

$$\mathcal{G} = \Big\{ \big\{a + td : t \in \{0,1,\ldots,k\}\big\} \cup \{d\} : a, d \in \mathbb{N} \Big\} .$$

By applying Theorem 2.6 to $(\mathbb{N},\cdot)$ and to $(\mathbb{N},+)$ one concludes that every multiplicatively central set contains a member of $\mathcal{F}$ and that every additively central set contains a member of $\mathcal{G}$. Thus we may apply Theorem 3.8. By assigning 1 to its own cell one may ensure that $r \neq 1$. □

Corollary 3.10. *Let $m, k \in \mathbb{N}$ and let $\mathbb{N} = \bigcup_{i=1}^m A_i$. Let $\mathcal{H} = \langle H_n \rangle_{n=1}^\infty$ be a Følner sequence in $(\mathbb{N}, \cdot)$. Then there exist $i \in \{1, 2, \ldots, m\}$, $a, d \in A_i$, and $r \in A_i \setminus \{1\}$ such that $\overline{d}(A_i) > 0$, $d^*_{\mathcal{H}}(A_i) > 0$, and*

$$\big\{r^s(a + td) : s, t \in \{0, 1, \ldots, k\}\big\} \cup \{dr^s : s \in \{0, 1, \ldots, k\}\} \subseteq A_i\,.$$

Proof. Let i, a, b, d, and r be as in Corollary 3.9. Put $a_1 = ab$ and $d_1 = db$. Then $\big\{r^s(a_1 + td_1) : s, t \in \{0, 1, \ldots, k\}\big\} \cup \big\{d_1 r^s : s \in \{0, 1, \ldots, k\}\big\} \subseteq A_i$. $\square$

The following proposition states that the geoarithmetic structure in the conclusion of Corollary 3.10 can be found in any multiplicatively piecewise syndetic IP set.

Theorem 3.11. *Let A be a piecewise syndetic IP-set in $(\mathbb{N}, \cdot)$ with $1 \notin A$ and let $k \in \mathbb{N}$. Then there exist $a, d \in A$ and $r \in A \setminus \{1\}$ such that*

$$\big\{r^s(a + td) : s, t \in \{0, 1, \ldots, k\}\big\} \cup \{dr^s : s \in \{0, 1, \ldots, k\}\} \subseteq A_i\,.$$

Proof. Let $\mathcal{F} = \big\{\big\{br^s : s \in \{0, 1, \ldots, k\}\big\} : b \in \mathbb{N} \text{ and } r \in A \setminus \{1\}\big\}$ and let

$$\mathcal{G} = \big\{\{d\} \cup \big\{a + td : t \in \{0, 1, \ldots, k\}\big\} : a, d \in \mathbb{N}\big\}\,.$$

By Corollary 2.7, $\mathcal{F}$ and $\mathcal{G}$ are partition regular. By Theorem 1.1 this holds for $\mathcal{H} = \{B \cdot C : B \in \mathcal{F} \text{ and } C \in \mathcal{G}\}$ as well. Since for any $t \in \mathbb{N}$ and $H \in \mathcal{H}$, $tH \in \mathcal{H}$ we may apply Lemma 2.3 and pick some $B \in \mathcal{F}$ and $C \in \mathcal{G}$ such that $B \cdot C \subseteq A$. Pick $b \in \mathbb{N}$ and $r \in A$ such that $B = \big\{br^s : s \in \{0, 1, \ldots, k\}\big\}$ and pick $a_1, d_1 \in \mathbb{N}$ such that $C = \{d_1\} \cup \big\{a_1 + td_1 : t \in \{0, 1, \ldots, k\}\big\}$. Let $a = a_1 b$ and $d = d_1 b$. $\square$

4. Extensions of geoarithmetic progressions

A *geoarithmetic progression* is a set of the form $\big\{r^j(a + id) : i, j \in \{0, 1, \ldots, k\}\big\}$ where $a, d, k \in \mathbb{N}$ and $r \in \mathbb{N} \setminus \{1\}$. We shall be concerned in this section with finding certain generalizations of geoarithmetic progressions in one cell of a finite partition of $\mathbb{N}$.

Our first result in this direction (Corollary 4.3) replaces r in a geometric progression by multiples of members of any partition regular family of finite sets. For that result, one needs to add a multiplier b because, for example, one can certainly not expect to find a set of the form $\{r, r^2\}$ for $r > 1$ in one cell of an arbitrary finite partition of $\mathbb{N}$. Indeed one may assign the members of $\mathbb{N} \setminus \big\{x^2 : x \in \mathbb{N} \setminus \{1\}\big\}$ to A_1 or A_2 at will, and then assign x^2 to the cell that x is not in, x^4 to the cell x^2 is not in, and so on.

To establish Theorem 4.2 we need the following algebraic result which is of interest in its own right. We let $\omega = \mathbb{N} \cup \{0\}$. The case $(S, +) = (\omega, +)$ of Theorem 4.1 follows from [15, Theorem 2.10]. Given a semigroup S, a set $C \subseteq S$ is said to be *central** if and only if for every central subset B of S, $C \cap B \neq \emptyset$. (Equivalently, $S \setminus C$ is not central.) Notice in particular that S is always central* so that if all first entries of a first entries matrix A are equal to 1, the requirement in the following theorem that $1S$ be central* is automatically satisfied.

Theorem 4.1. *Let $u, v \in \mathbb{N}$ and let A be a $u \times v$ first entries matrix with entries from ω. Let $(S, +)$ be a commutative semigroup with identity 0 and let C be a central subset of S. If for every first entry c of A, cS is central*, then $\{\vec{x} \in S^v : A\vec{x} \in C^u\}$ is central in S^v.*

Proof. Pick a minimal idempotent e of βS such that $C \in e$. Define $\varphi : S^v \to S^u$ by $\varphi(\vec{x}\,) = A\vec{x}$ and let $\widetilde{\varphi} : \beta(S^v) \to (\beta S)^u$ be its continuous extension. Let $M = \{p \in \beta(S^v) : \widetilde{\varphi}(p) = (e, e, \ldots, e)^T\}$. By [16, Corollary 4.22], $\widetilde{\varphi}$ is a homomorphism, so to see that M is a subsemigroup, it suffices to show that $M \neq \emptyset$.

For each $B \in e$, pick by [16, Theorem 15.5], $\vec{x}_B \in S^v$ such that $\varphi(\vec{x}_B) \in B^u$. Direct e by reverse inclusion and let q be a limit point in $\beta(S^v)$ of the net $\langle \vec{x}_B \rangle_{B \in e}$. Then $q \in M$.

Since M is a compact right topological semigroup, pick a minimal idempotent r of M. We claim that r is minimal in $\beta(S^v)$. To see this, let p be an idempotent of $\beta(S^v)$ such that $p \leq r$. Then $\widetilde{\varphi}(p) \leq \widetilde{\varphi}(r) = (e, e, \ldots, e)^T$ and $(e, e, \ldots, e)^T$ is minimal in $(\beta S)^u$ by [16, Theorem 2.23], so $\widetilde{\varphi}(p) = (e, e, \ldots, e)^T$. Thus $p \in M$ and so $p = r$.

Pick $X \in r$ such that $\widetilde{\varphi}[\,\overline{X}\,] \subseteq (\,\overline{B}\,)^u$. Then $X \subseteq \{\vec{x} \in S^v : A\vec{x} \in B^u\}$. □

Notice that all we need in the proof of the following theorem is that $\{(b, r) \in S^2 : \{r, b, br, \ldots, br^k\} \subseteq C\}$ is piecewise syndetic, which we establish by (algebraically) showing that it is central. We do not have, nor do we think it is likely to be easy to find, an elementary proof of this fact.

Theorem 4.2. *Let $(S, \cdot)$ be a commutative semigroup with identity and let C be a central subset of S. If $\mathcal{F}$ is a partition regular family of finite subsets of S and $k \in \mathbb{N}$, then there exist $b, r \in S$ and $F \in \mathcal{F}$ such that $rF \cup \big\{b(rx)^j : x \in F$ and $j \in \{0, 1, \ldots, k\}\big\} \subseteq C$.*

Proof. Let $k \in \mathbb{N}$ and let

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \\ \vdots & \vdots \\ 1 & k \end{pmatrix}.$$

Then A is a first entries matrix with all first entries equal to 1, so by Theorem 4.1, $\{(b, r) \in S^2 : \{b, r, br, \ldots, br^k\} \subseteq C\}$ is central in S^2 and is in particular piecewise syndetic. Let $\mathcal{G} = \big\{\{b\} \times F : b \in S$ and $F \in \mathcal{F}\big\}$. Then $\mathcal{G}$ is a partition regular family of finite subsets of S^2, so pick by Lemma 2.3, $F \in \mathcal{F}$, $c \in S$, and $(s, r) \in S^2$ such that $(s, r) \cdot (\{c\} \times F) \subseteq \{(b, r) \in S^2 : \{b, r, br, \ldots, br^k\} \subseteq C\}$. Let $b = sc$. □

Notice that, if in the above proof, the matrix A is replaced by a matrix whose set of rows is $\{(0, 0, 1)\} \cup \big\{(0, 1, j) : j \in \{0, 1, \ldots, k\}\big\} \cup \big\{(1, i, j) : i, j \in \{0, 1, \ldots, k\}\big\}$, then the conclusion of Theorem 4.2 becomes "there exist $b, c, r \in S$ and $F \in \mathcal{F}$ such that $rF \cup \big\{b(rx)^j : x \in F$ and $j \in \{0, 1, \ldots, k\}\big\} \cup \big\{cb^i(rx)^j : x \in F$ and $i, j \in \{0, 1, \ldots, k\}\big\} \subseteq C$." Of course additional strengthenings can be obtained using first entries matrices with all first entries equal to 1 and additional columns.

We see now that, given any central subset C of $(\mathbb{N}, \cdot)$ we can get sets of the form $\big\{b(a + id)^j : i, j \in \{0, 1, \ldots, k\}\big\}$ together with the multiplier, the increment, and the arithmetic progression in C.

Corollary 4.3. *Let C be a central subset of $(\mathbb{N}, \cdot)$ and let $k \in \mathbb{N}$. There exist $a, b, d \in \mathbb{N}$ such that*

$$\begin{gathered}\big\{b(a + id)^j : i, j \in \{0, 1, \ldots, k\}\big\} \cup \big\{bd^j : j \in \{0, 1, \ldots, k\}\big\} \\ \cup \big\{a + id : i \in \{0, 1, \ldots, k\}\big\} \cup \{d\} \subseteq C\,.\end{gathered}$$

Proof. Let $\mathcal{F} = \big\{\{d, a, a+d, \ldots, a+kd\} : a, d \in \mathbb{N}\big\}$. Pick by Theorem 4.2, $b, r \in S$ and $F \in \mathcal{F}$ such that $rF \cup \big\{b(rx)^j : x \in F \text{ and } j \in \{0, 1, \ldots, k\}\big\} \subseteq C$. Pick $c, s \in \mathbb{N}$ such that $F = \{c, s, s+c, \ldots, s+kc\}$. Let $d = rc$ and $a = rs$. □

Again note that if the stronger version of Theorem 4.2 that we mentioned after its proof is used, the conclusion of Corollary 4.3 becomes "There exist $a, b, c, d \in \mathbb{N}$ such that

$$\begin{gathered}\big\{cb^i(a+td)^j : t, i, j \in \{0, 1, \ldots, k\}\big\} \cup \big\{cb^i d^j : i, j \in \{0, 1, \ldots, k\}\big\} \\ \cup \big\{b(a+td)^j : t, j \in \{0, 1, \ldots, k\}\big\} \cup \big\{bd^j : j \in \{0, 1, \ldots, k\}\big\} \\ \cup \big\{a+td : t \in \{0, 1, \ldots, k\}\big\} \cup \{d\} \subseteq C\,.\text{"}\end{gathered}$$

We remark also that Corollary 4.3 could also be stated in terms of an arbitrary commutative ring with no change in the proof.

The following result is stronger than Corollary 4.3. We state it separately because its formulation is more involved and the proof requires more theoretical background.

Corollary 4.4. *Let S be an infinite set with operations $+$ and $\cdot$ such that $(S, +)$ is a commutative semigroup with identity 0, $(S \setminus \{0\}, \cdot)$ is a commutative semigroup with identity 1, and $\cdot$ distributes over $+$. Let C be a central subset of $(S \setminus \{0\}, \cdot)$, let $k \in \mathbb{N}$, and let G be a finite subset of $S \setminus \{0\}$. Then there exist $a, b, d \in C$ such that*

$$\begin{gathered}\big\{b(a+di)^j : i \in G \text{ and } j \in \{0, 1, \ldots, k\}\big\} \cup \big\{bd^j : j \in \{0, 1, \ldots, k\}\big\} \\ \cup \{a+di : i \in G\} \subseteq C\,.\end{gathered}$$

Proof. We observe first that $S \setminus \{0\}$ is central in $(S, +)$. To see this, suppose instead that 0 is a minimal idempotent of $(\beta S, +)$. Then by [16, Theorem 2.9], $\beta S = 0 + \beta S = \beta S + 0$ is a group and in particular $(S, +)$ is cancellative. But then by [16, Theorem 4.36], $\beta S \setminus S$ is an ideal of $(\beta S, +)$ and so $0 \in \beta S \setminus S$, a contradiction.

Let $\mathcal{F} = \big\{\{a, d\} \cup \{a + dj : j \in G\} : a, d \in S\big\}$. We claim that $\mathcal{F} \cap \mathcal{P}(S \setminus \{0\})$ is partition regular in $S \setminus \{0\}$. So let $r \in \mathbb{N}$ and let $S \setminus \{0\} = \bigcup_{i=1}^r D_i$. Pick $i \in \{1, 2, \ldots, r\}$ such that D_i is central in $(S, +)$. Let $\langle d_n \rangle_{n=1}^\infty$ be a sequence such that $FS(\langle d_n \rangle_{n=1}^\infty) \subseteq D_i$. Theorem 2.6 applied to the sequences $\langle jd_n \rangle_{n=1}^\infty$ for $j \in G$ yields that there exist $a \in D_i$ and $F \in \mathcal{P}_f(\mathbb{N})$ such that $a + \sum_{t \in F} jd_t \in D_i$ for all $j \in G$. If we let $d = \sum_{t \in F} d_t$ we see that $\{a, d\} \cup \{a + dj : j \in G\} \subseteq D_i$.

Pick by Theorem 4.2 $b, r \in S \setminus \{0\}$ and $F \in \mathcal{F} \cap \mathcal{P}(S \setminus \{0\})$ such that $rF \cup \big\{b(rx)^j : x \in F \text{ and } j \in \{0, 1, \ldots, k\}\big\} \subseteq C$. Pick $c, s \in S$ such that $F = \{c, s\} \cup \{s + ic : i \in G\}$. Let $d = rc$ and $a = rs$. Since $a, d \in rF$, we have $a, d \in C$. Also $b = ba^0$, so $b \in C$. □

Suppose that the semigroup S satisfies the hypotheses of Corollary 4.4 and that $0 \cdot x = 0$ for every $x \in S$. Then, by [4, Theorem 4.4] first entry matrices over S whose first entries are all 1 can be used to prove Corollary 4.4 as well as a sequence of successively stronger theorems. For example, the theorem stated in the remark following Theorem 4.2 is valid in S if C is any central subset of $(S \setminus \{0\}, \cdot)$, G is any given finite subset of S and $F = \{f\} \cup \{d + tf : t \in G\} \cup \{a + sd + tf : s, t \in G\}$ for some a, d, and f in $S \setminus \{0\}$.

The following corollary is also a consequence of [1, Theorem 3.15].

Corollary 4.5. *Let $k \in \mathbb{N}$, and let A be piecewise syndetic in $(\mathbb{N}, \cdot)$. Then there exist $a, b, d \in \mathbb{N}$ such that $\big\{b(a+id)^j : i, j \in \{0, 1, \ldots, k\}\big\} \subseteq A$.*

Proof. Pick $t \in \mathbb{N}$ such that $t^{-1}A$ is central and apply Corollary 4.3. $\square$

Corollary 4.3 extends the conclusion of [1, Theorem 3.15] (i.e., the one given in Corollary 4.5) in the sense that the arithmetic progression of the ratios is contained in the same cell as the geoarithmetic structure. Moreover it replaces arithmetic progressions by arithmetic progressions together with the common difference. The strongest natural generalization of this kind of structure is perhaps given by Deuber's (m,p,c)-sets.

Definition 4.6. Let $m,p,c \in \mathbb{N}$. A set $F \subseteq \mathbb{N}$ is an (m,p,c)-set if and only if there exists $\vec{x} \in \mathbb{N}^m$ such that

$$F = \{cx_m\} \cup \bigcup_{k=1}^{m-1} \left\{cx_k + \sum_{i=k+1}^{m} \lambda_i x_i : \{\lambda_{k+1}, \lambda_{k+2}, \ldots, \lambda_m\} \subseteq \{0,1,\ldots,p\}\right\}.$$

Thus a set of the form $\{a, a+d, \ldots, a+pd, d\}$, $a,d \in \mathbb{N}$, is precisely a $(2,p,1)$-set. Note that (m,p,c)-sets are very closely related to first entries matrices: Let A be a matrix whose set of rows is $\{(\lambda_1, \lambda_2, \ldots, \lambda_m) :$ there is some $j < m$ such that $\lambda_j = c$, $\lambda_i = 0$ for $i < j$, and $\lambda_i \in \{0,1,\ldots,p\}$ for $i > j\}$. Then A is a first entries matrix and a set $F \subseteq \mathbb{N}$ is an (m,p,c)-set if and only if there exist $\vec{x} \in \mathbb{N}^m$ such that F is the set of entries of $A\vec{x}$. By the results on first entries matrices cited above, the family of all (m,p,c)-sets is partition regular for all $m,p,c \in \mathbb{N}$. In fact, Deuber's Theorem [5, Satz 3.1] states that for all $m,p,c,r \in \mathbb{N}$ there exist $n,q,d \in \mathbb{N}$ such that whenever A is an (n,q,d)-set and $A = \bigcup_{i=1}^{r} B_i$, there exist $i \in \{1,2,\ldots,r\}$ and an (m,p,c)-set F such that $F \subseteq B_i$.

Since for each (m,p,c)-set F and each $r \in \mathbb{N}$ the set rF is again an (m,p,c)-set we have the following immediate corollary of Theorem 4.2 which extends Corollary 4.3.

Corollary 4.7. *Let C be a central subset of $(\mathbb{N},\cdot)$ and let $k,m,p,c \in \mathbb{N}$. There exist $b \in \mathbb{N}$ and an (m,p,c)-set F such that*

$$F \cup \left\{bx^j : x \in F, j \in \{1,2,\ldots,k\}\right\} \subseteq C.$$

One might hope that, in analogy with Deuber's Theorem, configurations of the form $\{ba^j : j \in \{0,1,\ldots,N\}, a \in A\}$ where A is an (m,p,c)-set and $m,p,c,N,b \in \mathbb{N}$ are strongly partition regular as well. We shall see in Theorem 4.9 that this is not the case.

Lemma 4.8. *Let $N \in \mathbb{N}$ and let $\gamma, \rho \in \mathbb{R}$ with $0 < \gamma$ and $1 < \rho^N < 2$. Let $\alpha > \max\{2\gamma, 2\rho^N, \frac{\rho^{N-1}}{2-\rho^N}, \frac{\gamma}{2-\rho}\}$. If $a,d \in \mathbb{R}$, $a > 0$, $d > 0$, and $\{a, a+d, a+2d\} \subseteq [0,\gamma] \cup \bigcup_{j=1}^{N} [\alpha^j, (\alpha\rho)^j]$, then either $\{a, a+d, a+2d\} \subseteq [0,\gamma]$ or there is some $j \in \{1,2,\ldots,N\}$ such that $\{a, a+d, a+2d\} \subseteq [\alpha^j, (\alpha\rho)^j]$.*

Proof. If $a+d \leq \gamma$, then since $\alpha > 2\gamma$, we have that $\{a, a+d, a+2d\} \subseteq [0,\gamma]$. So assume that $a+d > \gamma$. Pick $j \in \{1,2,\ldots,N\}$ such that $a+d \in [\alpha^j, (\alpha\rho)^j]$. Since $\alpha > 2\rho^N$, we have that $a+2d \leq 2(\alpha\rho)^j \leq 2\alpha^j\rho^N < \alpha^{j+1}$. We conclude that $\{a+d, a+2d\} \subseteq [\alpha^j, (\alpha\rho)^j]$; hence $a > 2\alpha^j - \alpha^j\rho^j = \alpha^j(2-\rho^j)$. If $j > 1$, then since $\alpha > \frac{\rho^{N-1}}{2-\rho^N} \geq \frac{\rho^{j-1}}{2-\rho^j}$, we have that $\alpha^j(2-\rho^j) > (\alpha\rho)^{j-1}$. If $j = 1$, then since $\alpha > \frac{\gamma}{2-\rho}$ we have that $\alpha(2-\rho) > \gamma$. $\square$

Theorem 4.9. *Let $m,p,c,N \in \mathbb{N}$. There exist an (m,p,c)-set A and $B_1, B_2 \subseteq \mathbb{N}$ such that $B_1 \cup B_2 = \left\{a^j : a \in A \text{ and } j \in \{0,1,\ldots,N\}\right\}$ and there do not exist $a,b,d \in \mathbb{N}$ and $t \in \{1,2\}$ such that $\left\{b(a+id)^j : i \in \{0,1,2\} \text{ and } j \in \{1,2\}\right\} \subseteq B_t$.*

Proof. Fix $\rho \in \mathbb{R}$ with $1 < \rho^N < 2$. Let $x_m = 1$. We define $x_{m-1}, x_{m-2}, \ldots, x_1$ by downward induction. So let $k \in \{1, 2, \ldots, m-1\}$ and assume x_i has been chosen for $i \in \{k+1, k+2, \ldots, m\}$. For $i \in \{k+1, k+2, \ldots, m\}$ and $j \in \{1, 2, \ldots, N\}$, let $A_{i,j} = \big\{(x_i c + \sum_{k=i+1}^m x_k \lambda_k)^j : \{\lambda_{i+1}, \lambda_{i+2}, \ldots, \lambda_m\} \subseteq \{0, 1, \ldots, p\}\big\}$. Assume that for each $r \in \{k+1, k+2, \ldots, m\}$ the following induction hypotheses are satisfied.

(1) If $r < m$, then $\max \bigcup_{i=r+1}^m \bigcup_{j=1}^N A_{i,j} < \min \bigcup_{j=1}^N A_{r,j}$.

(2) If $a, b, d \in \mathbb{N}$ and $\big\{b(a+sd)^t : s \in \{0,1,2\}$ and $t \in \{1,2\}\big\} \subseteq \bigcup_{i=r}^m \bigcup_{j=1}^N A_{i,j}$, then either $\big\{b(a+sd)^t : s \in \{0,1,2\}$ and $t \in \{1,2\}\big\} \subseteq \bigcup_{i=r+1}^m \bigcup_{j=1}^N A_{i,j}$ or $\{ba, b(a+d), b(a+2d)\} \subseteq A_{r,j}$ for some $j \in \{1, 2, \ldots, N\}$.

(3) If $a, b, d \in \mathbb{N}$ and $\big\{b(a+sd)^t : s \in \{0,1,2\}$ and $t \in \{1,2\}\big\} \subseteq \bigcup_{j=1}^N A_{r,j}$, then there exists some $j \in \{1, 2, \ldots, \lfloor \frac{N}{2} \rfloor\}$ such that $\{ba, b(a+d), b(a+2d)\} \subseteq A_{r,j}$ and $\{ba^2, b(a+d)^2, b(a+2d)^2\} \subseteq A_{r,2j}$.

Now $A_{m,1} = \{c\}$, so all hypotheses hold vacuously for $r = m$.

Let $\gamma = \max \bigcup_{i=k+1}^m \bigcup_{j=1}^N A_{i,j}$, let $D = p \sum_{i=k+1}^m x_i$, and choose $x_k \in \mathbb{N}$ such that $cx_k > \max\{2\gamma, 2\rho^N, \frac{\rho^{N-1}}{2-\rho^N}, \frac{\gamma}{2-\rho}, \gamma^2, D^N \rho^{2N}, \frac{D}{\rho-1}\}$. Put $\alpha = cx_k$. Observe that $A_{k,1} \subseteq cx_k + \{0, 1, \ldots, D\} \subseteq [\alpha, \alpha\rho]$ because $\alpha = cx_k \geq \frac{D}{\rho-1}$. Consequently, for each $j \in \{1, 2, \ldots, N\}$, $A_{k,j} \subseteq [\alpha^j, (\alpha\rho)^j]$.

To verify (2), let $a, b, d \in \mathbb{N}$ such that $\big\{b(a+sd)^t : s \in \{0,1,2\}$ and $t \in \{1,2\}\big\} \subseteq \bigcup_{i=k}^m \bigcup_{j=1}^N A_{i,j}$. Now

$$\bigcup_{i=k}^m \bigcup_{j=1}^N A_{i,j} = \bigcup_{i=k+1}^m \bigcup_{j=1}^N A_{i,j} \cup \bigcup_{j=1}^N A_{k,j} \subseteq [0, \gamma] \cup \bigcup_{j=1}^N [\alpha^j, (\alpha\rho)^j],$$

so Lemma 4.8 applied to $\{ba, ba+bd, ba+2bd\}$ yields that either $\{ba, ba+bd, ba+2bd\} \subseteq [0, \gamma]$ or $\{ba, ba+bd, ba+2bd\} \subseteq [\alpha^j, (\alpha\rho)^j]$ for some $j \in \{1, 2, \ldots, N\}$. In the second case (2) holds directly, so assume that $ba + 2bd \leq \gamma$. Then $b(a+2d)^2 \leq \gamma^2 < cx_k = \alpha$ and thus $\big\{b(a+sd)^t : s \in \{0,1,2\}$ and $t \in \{1,2\}\big\} \subseteq \bigcup_{i=k+1}^m \bigcup_{j=1}^N A_{i,j}$.

To verify (3), let $a, b, d \in \mathbb{N}$ and assume that $\big\{b(a+sd)^t : s \in \{0,1,2\}$ and $t \in \{1,2\}\big\} \subseteq \bigcup_{j=1}^N A_{k,j}$. Again applying Lemma 4.8, pick $j \in \{1, 2, \ldots, N\}$ such that $\{ba, ba+bd, ba+2bd\} \subseteq A_{k,j}$. Pick $w, z \in A_{k,1}$ such that $ba = w^j$ and $ba + bd = z^j$ and let $b_1 = \gcd(w, z)$. Then ${b_1}^j = \gcd(ba, ba+bd)$ so $b | {b_1}^j$ so $b \leq {b_1}^j$. Choose a_1, a_2 such that $w = b_1 a_1$ and $z = b_1 a_2$. Since $w, z \in cx_k + \{0, 1, \ldots, D\}$ and $a_2 \geq a_1 + 1$, we have $b_1 \leq D$, so $b \leq D^j$. From $cx_k > D^N \rho^{2N}$ we deduce that $\frac{(cx_k)^{2j}}{D^j} > (cx_k)^{2j-1} \rho^{2j-1}$. Thus

$$ba^2 \geq \frac{(ba)^2}{D^j} \geq \frac{(cx_k)^{2j}}{D^j} > (cx_k \rho)^{2j-1}.$$

All elements of $A_{k,1}$ are smaller than $cx_k \rho = \alpha\rho$. Thus for $l \in \{1, 2, \ldots, N\}$ the set $A_{k,l}$ is bounded by $(cx_k\rho)^l$. Hence ba^2 cannot be an element of $A_{k,l}$ if $l \leq 2j - 1$. Pick $l, r \in \{1, 2, \ldots, N\}$ such that $ba^2 \in A_{k,l}$ and $b(a+2d)^2 \in A_{k,r}$. We have seen that $l \geq 2j$. Also $\alpha^r \leq b(a+2d)^2 \leq b^2(a+2d)^2 \leq (\alpha\rho)^{2j} < \alpha^{2j+1}$, so $r \leq 2j$ and thus $l = r = 2j$ and (3) is established.

We take $A = \bigcup_{k=1}^m A_{k,1}$. To define the sets B_1 and B_2, choose a partition $\{I_1, I_2\}$ of $\mathbb{N}$ such that for $d \in \mathbb{N}$, $d \in I_1$ if and only if $2d \in I_2$. Let

$$\begin{aligned} B_1 &= \{1\} \cup \bigcup_{i=1}^m \bigcup \{A_{i,j} : j \in \{1, 2, \ldots, N\} \cap I_1\} \text{ and} \\ B_2 &= \bigcup_{i=1}^m \bigcup \{A_{i,j} : j \in \{1, 2, \ldots, N\} \cap I_1\}\,. \end{aligned}$$

Suppose we have $r \in \{1,2\}$ and $a,b,d \in \mathbb{N}$ such that $\big\{b(a+sd)^t : s \in \{0,1,2\}$ and $t \in \{1,2\}\big\} \subseteq B_r$. Consider first the possibility that $b = a = 1$, in which case $r = 1$. Then $b(a+d) = 1+d \in A_{i,j}$ for some $i \in \{1,2,\ldots,m\}$ and $j \in I_1$ and $b(a+d)^2 = (1+d)^2 \in A_{i,2j}$, while $2j \in I_2$. Now assume that $ba > 1$. Pick the largest k such that $\big\{b(a+sd)^t : s \in \{0,1,2\}$ and $t \in \{1,2\}\big\} \subseteq \bigcup_{i=k}^m \bigcup_{j=1}^N A_{i,j}$. Then by (2), $\{ba, b(a+d), b(a+2d)\} \subseteq A_{k,j}$ for some $j \in \{1,2,\ldots,N\}$. Since $\max \bigcup_{i=k+1}^m \bigcup_{j=1}^N A_{i,j} < \min \bigcup_{j=1}^N A_{k,j}$, one has in fact that $\big\{b(a+sd)^t : s \in \{0,1,2\}$ and $t \in \{1,2\}\big\} \subseteq \bigcup_{j=1}^N A_{k,j}$, so (3) applies and we are done. $\square$

Now, as we promised in the introduction, we turn our attention to extensions of the following result from [1].

Theorem 4.10. *Let $k \in \mathbb{N}$. For each $i \in \{0,1,\ldots,k\}$ let $\langle x_{i,t}\rangle_{t=1}^\infty$ and $\langle y_{i,t}\rangle_{t=1}^\infty$ be sequences in $\mathbb{N}$, and let A be a multiplicatively large subset of $\mathbb{N}$. Then there exist $F, G \in \mathcal{P}_f(\mathbb{N})$ and $a, b \in \mathbb{N}$ such that $\big\{b(a+\sum_{t\in F} x_{i,t}) \cdot (\prod_{t\in G} y_{j,t}) : i,j \in \{0,1,\ldots,k\}\big\} \subseteq A$.*

Proof. See [1, Theorem 3.13] $\square$

We now show that this result can be strengthened to guarantee $F = G$. The proof uses the very deep and powerful *Density Hales-Jewett Theorem* which we now state.

Theorem 4.11 (Furstenberg and Katznelson)**.** *Let L be a finite alphabet and let $\epsilon > 0$. There exists $n \in \mathbb{N}$ such that, if S_n is the set of length n words over L and $B \subseteq S_n$ such that $|B| \geq \epsilon \cdot |S_n|$, then there is a variable word w of length n over L such that $\{\theta_a(w) : a \in L\} \subseteq B$.*

Proof. See [8, Theorem E]. $\square$

Theorem 4.12. *Let $k \in \mathbb{N}$ and for each $i \in \{0,1,\ldots,k\}$ let $\langle x_{i,t}\rangle_{t=1}^\infty$ and $\langle y_{i,t}\rangle_{t=1}^\infty$ be sequences in $\mathbb{N}$. Let A be a multiplicatively large subset of $\mathbb{N}$. Then there exist $F \in \mathcal{P}_f(\mathbb{N})$ and $a, b \in \mathbb{N}$ such that*

$$\begin{aligned}&\{ba\} \cup \big\{b(a+\textstyle\sum_{t\in F} x_{i,t}) : i \in \{0,1,\ldots,k\}\big\} \cup \\ &\big\{ba \cdot \textstyle\prod_{t\in F} y_{j,t} : j \in \{0,1,\ldots,k\}\big\} \cup \\ &\big\{b(a+\textstyle\sum_{t\in F} x_{i,t}) \cdot (\prod_{t\in F} y_{j,t}) : i,j \in \{0,1,\ldots,k\}\big\} \subseteq A\,.\end{aligned}$$

Proof. Pick a Følner sequence $\langle H_n\rangle_{n=1}^\infty$ in $(\mathbb{N},\cdot)$ such that $\limsup\limits_{n\to\infty} \dfrac{|A\cap H_n|}{|H_n|} > 0$. By thinning the sequence we may presume that we have $\epsilon > 0$ such that for each $n \in \mathbb{N}$, $|A\cap H_n| > \epsilon\cdot|H_n|$. Let $x_{k+1,t} = 0$ and $y_{k+1,t} = 1$ for all t. Let $L = \{0,1,\ldots,k+1\}^2$. By Theorem 4.11, choose $n \in \mathbb{N}$ such that whenever $B \subseteq S_n$ and $|B| \geq \frac{\epsilon}{2}\cdot|S_n|$ there must exist some variable word w such that $\{\theta_{(i,j)}(w) : (i,j) \in L\} \subseteq B$.

Define $f : S_n \to \mathbb{N}$ as follows. For $w \in S_n$ and $t \in \{1,2,\ldots,n\}$, let $w_1(t) = \pi_1\big(w(t)\big)$ and $w_2(t) = \pi_2\big(w(t)\big)$, so that $w(t) = \big(w_1(t), w_2(t)\big)$. (We are treating the members of S_n as functions from $\{1,2,\ldots,n\}$ to L.) Let

$$f(w) = (1 + \textstyle\sum_{t=1}^n x_{w_1(t),t}) \cdot \prod_{t=1}^n y_{w_2(t),t}\,.$$

We claim that there is some $b \in \mathbb{N}$ such that $|\{w \in S_n : b\cdot f(w) \in A\}| \geq \frac{\epsilon}{2}\cdot|S_n|$. To this end, since $\langle H_m\rangle_{m=1}^\infty$ is a Følner sequence, pick $m \in \mathbb{N}$ such that for all $w \in S_n$,

$|H_m \setminus f(w) \cdot H_m| < \frac{\epsilon}{2} \cdot |H_m|$. Then $|A \cap H_m| \subseteq (A \cap f(w) \cdot H_m) \cup (H_m \setminus f(w) \cdot H_m)$, so $|A \cap f(w) \cdot H_m| \geq |A \cap H_m| - |H_m \setminus f(w) \cdot H_m| > \epsilon \cdot |H_m| - \frac{\epsilon}{2} \cdot |H_m|$; thus

$$\begin{array}{rcl} |S_m| \cdot \frac{\epsilon}{2} \cdot |H_m| & \leq & \sum_{w \in S_n} |A \cap f(w) \cdot H_m| \\ & = & \sum_{w \in S_n} \sum_{b \in H_m} \chi_A(f(w) \cdot b) \\ & = & \sum_{b \in H_m} \sum_{w \in S_n} \chi_A(f(w) \cdot b). \end{array}$$

Therefore, for some $b \in H_m$, $\sum_{w \in S_n} \chi_A(f(w) \cdot b) \geq \frac{\epsilon}{2}$ and thus we may pick $b \in H_m$ such that $|\{w \in S_n : b \cdot f(w) \in A\}| \geq \frac{\epsilon}{2} \cdot |S_n|$ as required. Let $B = \{w \in S_n : b \cdot f(w) \in A\}$. Pick a variable word w such that $\{\theta_{(i,j)}(w) : (i,j) \in L\} \subseteq B$. Letting v be the variable, let $F = \{t \in \{1, 2, \ldots, n\} : w(t) = v\}$ and let $G = \{1, 2, \ldots, n\} \setminus F$. For $t \in G$ and $(i,j) \in L$, $\pi_1\big(\theta_{(i,j)}(w)(t)\big) = w_1(t)$ and $\pi_2\big(\theta_{(i,j)}(w)(t)\big) = w_2(t)$. For $t \in F$ and $(i,j) \in L$, $\pi_1\big(\theta_{(i,j)}(w)(t)\big) = i$ and $\pi_2\big(\theta_{(i,j)}(w)(t)\big) = j$. Let $b' = b \cdot \prod_{t \in G} y_{w_2(t),t}$ and let $a = 1 + \sum_{t \in G} x_{w_1(t),t}$. Then for any $i, j \in \{0, 1, \ldots, k\}$, $b' \cdot (a + \sum_{t \in F} x_{i,t}) \cdot (\prod_{t \in G} x_{j,t}) \in A$. □

Let $\mathcal{F} = \langle F_n \rangle_{n=1}^{\infty}$ be a Følner sequence in $(\mathbb{N}, \cdot)$. By Lemma 3.2, $\Delta^*_{\mathcal{F}}$ is a two sided ideal of $(\beta\mathbb{N}, \cdot)$ and consequently, any piecewise syndetic set A has $d^*_{\mathcal{F}}(A) > 0$, and is in particular multiplicatively large. Notice that, if one wants the conclusion of Theorem 4.12 only for piecewise syndetic sets, one can get by with an appeal to the (standard) Hales-Jewett Theorem (Theorem 2.5), using Lemma 2.3.

We have just seen that we can take $F = G$ in the partition theoretic version of Theorem 4.10 (Corollary 1.4), and we will show in Corollary 4.15(a) that the multiplier b may be eliminated. We show in Corollary 4.19, however, that one cannot simultaneously take $F = G$ and eliminate b.

Lemma 4.13. *Let $(S, \cdot)$ be a commutative semigroup, let L be a minimal left ideal of $(\beta S, \cdot)$, and let $k \in \mathbb{N}$. Let $\mathcal{F}$ be a family of finite subsets of S such that the family $\{bF : F \in \mathcal{F}$ and $b \in S\}$ is partition regular. Let $A \subseteq S$ such that $\overline{A} \cap L \neq \emptyset$. Then there exists $F \in \mathcal{F}$ such that $L \cap \bigcap_{y \in F} \overline{y^{-1}A} \neq \emptyset$.*

Proof. Pick $v \in \overline{A} \cap L$. Pick a minimal right ideal R of $(\beta S, \cdot)$ such that $v \in R$ and pick an idempotent $u \in R$. Then $v = uv$, so $B = \{x \in S : x^{-1}A \in v\} \in u$. In particular B is central, so pick by Lemma 2.3, some $b \in S$ and $F \in \mathcal{F}$ such that $bF \subseteq B$. So for each $y \in F$, $(by)^{-1}A \in v$. Equivalently for each $y \in F$, $y^{-1}A \in bv$. Since $bv \in L$, we are done. □

We have by Lemma 3.6 that if $D = \{q \in \Delta : q$ is a minimal idempotent of $(\beta\mathbb{N}, +)\}$, then $c\ell D$ is a left ideal of $(\beta\mathbb{N}, \cdot)$ and consequently $c\ell D \cap K(\beta\mathbb{N}, \cdot) \neq \emptyset$. Given any $p \in \beta S$ and any finite partition $\{A_1, \ldots, A_m\}$ there is at least one cell A_i such that $A_i \in p$. Consequently, the partition versions of Theorem 4.14 and Corollary 4.15 are also valid.

Theorem 4.14. *Let $D = \{q \in \Delta : q$ is a minimal idempotent of $(\beta\mathbb{N}, +)\}$ and let A be a subset of $\mathbb{N}$ such that $\overline{A} \cap c\ell D \cap K(\beta\mathbb{N}, \cdot) \neq \emptyset$. Let $\mathcal{F}$ be a family of finite subsets of $\mathbb{N}$ such that the family $\{bF : F \in \mathcal{F}$ and $b \in \mathbb{N}\}$ is partition regular and let $\mathcal{G}$ be a family of subsets of $\mathbb{N}$ such that any set which is central in $(\mathbb{N}, +)$ contains a member of $\mathcal{G}$. Let $\mathcal{H} = \langle H_n \rangle_{n=1}^{\infty}$ be a Følner sequence in $(\mathbb{N}, \cdot)$. Then there exist $F \in \mathcal{F}$ and $G \in \mathcal{G}$ such that $\overline{d}(\bigcap_{y \in F} y^{-1}A) > 0$, $d^*_{\mathcal{H}}(\bigcap_{y \in F} y^{-1}A) > 0$ and $FG \subseteq A$.*

Proof. Pick a minimal left ideal L of $(\beta\mathbb{N},\cdot)$ such that $\overline{A}\cap c\ell D\cap L\neq\emptyset$. Since $c\ell D$ is a left ideal of $(\beta\mathbb{N},\cdot)$, $L\subseteq c\ell D$. Pick, by Lemma 4.13, $F\in\mathcal{F}$ such that $L\cap\bigcap_{y\in F}\overline{y^{-1}A}\neq\emptyset$. Since $L\subseteq K(\beta\mathbb{N},\cdot)\subseteq\Delta^*_{\mathcal{H}}$ by Lemma 3.2, $d^*_{\mathcal{H}}(\bigcap_{y\in F}y^{-1}A)>0$. Since $L\subseteq c\ell D$, pick $q\in\Delta$ such that q is a minimal idempotent of $(\beta\mathbb{N},+)$ and $\bigcap_{y\in F}y^{-1}A\in q$. Then this set is central in $(\mathbb{N},+)$, so pick $G\in\mathcal{G}$ such that $G\subseteq\bigcap_{y\in F}y^{-1}A$. Since $q\in\Delta$, $\overline{d}(\bigcap_{y\in F}y^{-1}A)>0$. □

Corollary 4.15. *Let $D=\{q\in\Delta : q$ is a minimal idempotent of $(\beta\mathbb{N},+)\}$, let A be a subset of $\mathbb{N}$ such that there is a multiplicative idempotent $p\in\overline{A}\cap c\ell D\cap K(\beta\mathbb{N},\cdot)$, and let $k\in\mathbb{N}$. Let $\mathcal{H}=\langle H_n\rangle_{n=1}^\infty$ be a Følner sequence in $(\mathbb{N},\cdot)$.*
(a) For each $i\in\{1,2,\ldots,k\}$ let $\langle x_{i,t}\rangle_{t=1}^\infty$ and $\langle y_{i,t}\rangle_{t=1}^\infty$ be sequences in $\mathbb{N}$. Then there exist $H,K\in\mathcal{P}_f(\mathbb{N})$ and $a\in A$ such that $\overline{d}(A\cap\bigcap_{j=1}^k(\prod_{t\in H}y_{j,t})^{-1}A)>0$, $d^*_{\mathcal{H}}(A\cap\bigcap_{j=1}^k(\prod_{t\in H}y_{j,t})^{-1}A)>0$, and

$$\begin{aligned}\big\{a+\sum_{t\in K}x_{i,t}:i\in\{1,2,\ldots,k\}\big\}&\cup\big\{a\cdot\prod_{t\in H}y_{j,t}:j\in\{1,2,\ldots,k\}\big\}\\ &\cup\big\{(a+\sum_{t\in K}x_{i,t})\cdot\prod_{t\in H}y_{j,t}:i,j\in\{1,2,\ldots,k\}\big\}\subseteq A\,.\end{aligned}$$

(b) There exist $a,r,d\in A$ such that $r>1$, $\overline{d}(\bigcap_{j=0}^k(r^j)^{-1}A)>0$, $d^*_{\mathcal{H}}(\bigcap_{j=0}^k(r^j)^{-1}A)>0$, and $\big\{(a+id)r^j:i,j\in\{0,1,\ldots,k\}\big\}\cup\big\{dr^j:j\in\{0,1,\ldots,k\}\big\}\subseteq A$.
(c) There exist $a,r,d\in A$ such that $r>1$, $\overline{d}(\bigcap_{j=1}^k(j^r)^{-1}A)>0$, $d^*_{\mathcal{H}}(\bigcap_{j=0}^k(j^r)^{-1}A)>0$, and $\big\{dj^r:j\in\{1,2,\ldots,k\}\big\}\cup\big\{(a+id)j^r:i\in\{0,1,\ldots,k\}$ and $j\in\{1,2,\ldots,k\}\big\}\cup\big\{a+id:i\in\{0,1,\ldots,k\}\big\}\subseteq A$.

Proof. Since 1 is not an element of any minimal left ideal of $(\beta\mathbb{N},\cdot)$, by considering $A\setminus\{1\}$ instead of A we may assume that $1\notin A$. Let

$$\begin{aligned}\mathcal{F}_1&=\big\{\{1\}\cup\big\{\textstyle\prod_{t\in H}y_{i,t}:i\in\{1,2,\ldots,k\}\big\}:H\in\mathcal{P}_f(\mathbb{N})\big\}\,,\\ \mathcal{G}_1&=\big\{\{a\}\cup\{a+\textstyle\sum_{t\in K}x_{i,t}:i\in\{1,2,\ldots,k\}:K\in\mathcal{P}_f(\mathbb{N})\text{ and }a\in\mathbb{N}\big\}\,,\\ \mathcal{F}_2&=\big\{\big\{r^i:i\in\{0,1,\ldots,k\}\big\}:r\in A\big\}\,,\\ \mathcal{G}_2&=\big\{\{d\}\cup\{a+id:i\in\{0,1,\ldots,k\}\}:a,d\in\mathbb{N}\big\}\,,\end{aligned}$$

and put $\mathcal{F}_i'=\{bF:b\in\mathbb{N}$ and $F\in\mathcal{F}_i\}$ for $i\in\{1,2\}$. By applying Theorem 2.6 and Corollary 2.7 to the semigroup $(\mathbb{N},\cdot)$ we see that the families $\mathcal{F}_1'$ and $\mathcal{F}_2'$ are partition regular. Similarly by Theorem 2.6 and Corollary 2.7 applied to the semigroup $(\mathbb{N},+)$, every subset of $\mathbb{N}$ that is central in $(\mathbb{N},+)$ contains a member of $\mathcal{G}_1$ and a member of $\mathcal{G}_2$. Thus we get (a) by applying Theorem 4.14 to $\mathcal{F}_1$ and $\mathcal{G}_1$ and (b) by applying Theorem 4.14 to $\mathcal{F}_2$ and $\mathcal{G}_2$.

We will prove (c) by using Theorem 4.14 with $\mathcal{F}_1$ and $\mathcal{G}_2$, where we define the sequences $\langle y_{i,n}\rangle_{n=1}^\infty$, $i\in\{1,2,\ldots,k\}$ appropriately. Since A is central in $(\mathbb{N},+)$, choose a sequence $\langle r_n\rangle_{n=1}^\infty$ such that $FS(\langle r_n\rangle_{n=1}^\infty)\subseteq A$. Using this put $y_{i,n}=i^{r_n}$ for $i\in\{1,2,\ldots,k\}$ and $n\in\mathbb{N}$. By Theorem 4.14 we find $a,d\in A$ and $H\in\mathcal{P}_f(\mathbb{N})$ such that $G=\{d\}\cup\{a+id:i\in\{0,1,\ldots,k\}\}$ and $F=\{1\}\cup\big\{\prod_{t\in H}y_{j,t}:j\in\{1,2,\ldots,k\}\big\}$ satisfy the conclusion of Theorem 4.14. Let $r=\sum_{t\in H}r_t\in A$. Then for $j\in\{1,2,\ldots,k\}$, $\prod_{t\in H}y_{j,t}=\prod_{t\in H}j^{r_t}=j^r$. Thus we see that (c) is valid. □

We now turn our attention to showing that one cannot simultaneously let $F=G$ and eliminate the multiplier b in Theorem 4.10.

The following theorem is of interest in its own right. Recall from Corollary 2.7 that when $\mathbb{N}$ is finitely colored, one can find arbitrarily long monochromatic arithmetic progressions with increments chosen from any IP-set. This theorem tells us that at least relatively thin sequences cannot replace IP-sets.

Theorem 4.16. *Let $\langle d_n\rangle_{n=1}^{\infty}$ be a sequence in $\mathbb{N}$ such that for all $n \in \mathbb{N}$, $3d_n \leq d_{n+1}$. There exists a partition $\{A_0, A_1, A_2, A_3\}$ of $\mathbb{N}$ such that there do not exist $s \in \{0, 1, 2, 3\}$ and $a, k \in \mathbb{N}$ with $\{a, a + d_k\} \subseteq A_s$.*

Proof. For $\alpha \in \mathbb{T} = \mathbb{R}/\mathbb{Z}$ we denote by $\|\alpha\|$ the distance to the nearest integer. We will not distinguish strictly between equivalance classes and their representatives in $[0, 1)$. To complete the proof we need the following lemma.

Lemma 4.17. *There exists $\alpha \in \mathbb{T}$ such that $\|\alpha d_n\| \geq 1/4$ for each $n \in \mathbb{N}$.*

Proof. For each $n \in \mathbb{N}$ put $R_n = \{\alpha \in \mathbb{T} : \|\alpha d_n\| \geq 1/4\}$. Each R_n consists of intervals of length $\dfrac{1}{2d_n}$ which are separated by gaps of the same length. Since $d_{n+1} \geq 3d_n$, every interval of R_n is 3 times longer than an interval or a gap of R_{n+1}. Thus any interval of R_n contains an interval of R_{n+1}. This shows that for each $N \in \mathbb{N}$, $\bigcap_{n=1}^{N} R_n \neq \emptyset$. By compactness of $\mathbb{T}$ there exists $\alpha \in \bigcap_{n=1}^{\infty} R_n$. $\square$

Let $\alpha \in \mathbb{T}$ such that $d_n\alpha \in [1/4, 3/4]$ for each $n \in \mathbb{N}$. For $i \in \{0, 1, 2, 3\}$ put $A_i = \{m \in \mathbb{N} : m\alpha \in [i/4, (i + 1)/4)\}$. Then for any $a, k \in \mathbb{N}$, $\alpha(a + d_k) = \alpha a + \beta$ for some $\beta \in [1/4, 3/4]$ and thus αa and $\alpha(a + d_k)$ must not lie in the same quarter of $[0, 1)$. Equivalently there exists no $s \in \{0, 1, 2, 3\}$ such that $\{a, a+d_k\} \subseteq A_s$. $\square$

We remark that Lemma 4.17 is well known. Under the much weaker assumption that the growth rate of the sequence $\langle d_n\rangle_{n=1}^{\infty}$ is bounded from below by some $q > 1$, B. de Mathan [17] and A. Pollington [18] independently proved that there exists some $\alpha \in \mathbb{T}$ such that $\{\alpha d_n : n \in \mathbb{N}\}$ is not dense in $\mathbb{T}$. In order to give a self contained proof we have chosen to go with the weaker statement. The loss is that we have to make an additional step to show that any growth rate $q > 1$ is sufficient to avoid monochromatic arithmetic progressions with some d_k as increment.

Corollary 4.18. *Let $q \in \mathbb{R}$ with $q > 1$ and assume that $\langle d_n\rangle_{n=1}^{\infty}$ is a sequence in $\mathbb{N}$ such that for all $n \in \mathbb{N}$, $qd_n \leq d_{n+1}$. There exists a finite partition $\{A_1, A_2, \ldots, A_r\}$ of $\mathbb{N}$ such that there do not exist $s \in \{1, 2, \ldots, r\}$ and $a, k \in \mathbb{N}$ with $\{a, a+d_k\} \subseteq A_s$.*

Proof. Pick $m \in \mathbb{N}$ such that $q^m \geq 3$. For $t \in \{0, 1, \ldots, m - 1\}$ and $n \in \mathbb{N}$, let $c_{t,n} = d_{nm-t}$. Given $t \in \{0, 1, \ldots, m\}$ one has that $3c_{t,n} \leq c_{t,n+1}$ for each n, so pick by Theorem 4.16 some $\{B_{t,0}, B_{t,1}, B_{t,2}, B_{t,3}\}$ of $\mathbb{N}$ such that there do not exist $s \in \{0, 1, 2, 3\}$ and $a, k \in \mathbb{N}$ with $\{a, a + c_{t,k}\} \subseteq B_{t,i}$. Let $r = 4^m$ and define a partition $\{A_1, A_2, \ldots, A_r\}$ of $\mathbb{N}$ with the property that x and y lie in the same cell of the partition if and only if $x \in B_{t,i} \Leftrightarrow y \in B_{t,i}$ for each $t \in \{0, 1, \ldots, m - 1\}$ and each $i \in \{0, 1, 2, 3\}$. $\square$

Corollary 4.19. *There exist sequences $\langle x_{0,n}\rangle_{n=1}^{\infty}$, $\langle x_{1,n}\rangle_{n=1}^{\infty}$, and $\langle y_n\rangle_{n=1}^{\infty}$ in $\mathbb{N}$ and a partition $\{A_0, A_1, A_2, A_3\}$ of $\mathbb{N}$ such that there do not exist $s \in \{0, 1, 2, 3\}$, $F \in \mathcal{P}_f(\mathbb{N})$, and $a \in \mathbb{N}$ with $\big\{(a + \sum_{n\in F} x_{i,n}) \cdot \prod_{n\in F} y_n : i \in \{0, 1\}\big\} \subseteq A_s$.*

Proof. For each $t \in \mathbb{N}$, let $x_{0,t} = 1$, $x_{1,t} = 2$ and $y_{i,t} = 3$. For each $n \in \mathbb{N}$, let $d_n = n3^n$. Pick A_0, A_1, A_2, A_3 as guaranteed by Theorem 4.16. Suppose one has $F \in \mathcal{P}_f(\mathbb{N})$ and $a \in \mathbb{N}$ with $\big\{(a + \sum_{t\in F} x_{i,t}) \cdot \prod_{t\in F} y_t : i \in \{0, 1\}\big\} \subseteq A_s$. Let

$n = |F|$. Then $(a + \sum_{t\in F} x_{1,t}) \cdot \prod_{t\in F} y_t = d_n + (a + \sum_{t\in F} x_{0,t}) \cdot \prod_{t\in F} y_t$, a contradiction. $\square$

We have just shown that one cannot simultaneously take $F = G$ and eliminate the multiplier b from Corollary 1.4. We show now that this multiplier cannot be eliminated from Theorem 4.10 (and consequently cannot be eliminated from Theorem 4.12), even if different F and G are allowed. Recall that thick sets in any semigroup are also piecewise syndetic, in fact central. Consequently, they are also multiplicatively large.

Theorem 4.20. *There exists a set A which is thick in $(\mathbb{N},\cdot)$ and a sequence $\langle x_n\rangle_{n=1}^\infty$ in $\mathbb{N}$ with the property that there do not exist $a \in \mathbb{N}$ and $d \in FS(\langle x_n\rangle_{n=1}^\infty)$ with $\{a, a + d\} \subseteq A$.*

Proof. Let $A = \bigcup_{n=1}^\infty \{(3n)!, 2(3n)!, \ldots, n(3n)!\}$ and for each n, let $x_n = (3n + 1)!$. Observe that A is thick in $(\mathbb{N},\cdot)$. Let $a \in A$ and let $d \in FS(\langle x_n\rangle_{n=1}^\infty)$. We shall show that $a + d \notin A$. Pick $n \in \mathbb{N}$ and $k \in \{1, 2, \ldots, n\}$ such that $a = k(3n)!$. Pick $F \in \mathcal{P}_f(\mathbb{N})$ such that $d = \sum_{t\in F} x_t$ and let $m = \max F$. Then $(3m + 1)! \le d < (3m + 2)!$.

If $m < n$ we have $k(3n)! < a + d < (k + 1)(3n)!$, so $a + d \notin A$. If $m \ge n$, then $a < (3m + 1)!$, so $(3m + 1)! < a + d < (3m + 3)!$ and thus $a + d \notin A$. $\square$

It was shown in [1, Theorem 1.3] that the fact that a subset A of $\mathbb{N}$ is multiplicatively large is enough to guarantee that A contains arbitrarily large geoarithmetic progressions. However, consider the set $A = \{x \in \mathbb{N} : \text{the number of terms in the prime factorization of } x \text{ is odd}\}$. It is not hard to show that $\overline{d}_{\mathcal{F}}(A) = \frac{1}{2}$ for any Følner sequence $\mathcal{F}$ in $(\mathbb{N},\cdot)$. Consequently, the fact that A is multiplicatively large is not enough to guarantee geoarithmetic progressions together with the common ratio r, nor together with both b and a.

As is well known among afficionados, geoarithmetic progressions are *strongly partition regular*. That is, for each $m, k \in \mathbb{N}$ there exists $K \in \mathbb{N}$ such that whenever $A, B, D \in \mathbb{N}$, $R \in \mathbb{N}\backslash\{1\}$, and $\{BR^s(A+tD) : s, t \in \{0, 1, \ldots, K\}\} = \bigcup_{i=1}^m C_i$, there exist $i \in \{1, 2, \ldots, m\}$, $a, b, d \in \mathbb{N}$, and $r \in \mathbb{N}\backslash\{1\}$ such that $\{br^s(a+td) : s, t \in \{0, 1, \ldots, k\}\} \subseteq A_i$. (The easiest way to see this is to use the Grünwald/Gallai Theorem[1] [11, Theorem 2.8]. Color the pair $(s, t) \in \{0, 1, \ldots, K\} \times \{0, 1, \ldots, K\}$ according to the color of $BR^s(A + tD)$.)

We now present an easy proof that even very limited configurations of the sort produced by Corollary 3.9 are not strongly partition regular.

Theorem 4.21. *There is a set $C \subseteq \mathbb{N}$ such that for each $k \in \mathbb{N}$ there exist $b, a, d \in \mathbb{N}$ and $r \in \mathbb{N} \setminus \{1\}$ such that $\{br^n(a + td) : n, t \in \{0, 1, \ldots, k\}\} \cup \{br^n : n \in \{0, 1, \ldots, k\}\} \cup \{a + td : t \in \{0, 1, \ldots, k\}\} \subseteq C$ and there exist sets A_1 and A_2 such that $C = A_1 \cup A_2$ and there do not exist $i \in \{1, 2\}$, $c, a, d \in \mathbb{N}$, and $s \in \mathbb{N} \setminus \{1\}$ such that $\{cs, cs^2, cs(a + d), cs^2(a + d), cs(a + 2d)\} \subseteq A_i$.*

Proof. Let $r_1 = 5$. Inductively choose a prime $r_{k+1} > \big(r_k^{k+1}(2k + 1)\big)^2$. For each $k \in \mathbb{N}$, let $B_k = \{{r_k}^n x : n \in \{1, 2, \ldots, k + 1\} \text{ and } x \in \{k + 1, k + 2, \ldots, 2k + 1\}\}$ and let $B = \bigcup_{k=1}^\infty B_k$.

[1] This theorem was never published by its author. Its first publication was in [19] where it was referred to as Grünwald's Theorem, Grünwald being the original name of the author. During the period surrounding World War II Grünwald changed his name to Gallai.

Lemma 4.22. *If $a, d \in \mathbb{N}$ and $\{a+d, a+2d\} \subseteq B$, then there exist $k \in \mathbb{N}$ and $n \in \{1, 2, \ldots, k+1\}$ such that $\{a+d, a+2d\} \subseteq \{{r_k}^n x : x \in \{k+1, k+2, \ldots, 2k+1\}\}$.*

Proof. Pick $k \in \mathbb{N}$, $n \in \{1, 2, \ldots, k+1\}$, and $x \in \{k+1, k+2, \ldots, 2k+1\}$ such that $a + d = {r_k}^n x$. Then $a + 2d < 2(a + d) = 2{r_k}^n x$. Also $2{r_k}^n x < {r_k}^{n+1}(k + 1)$ and $2{r_k}^n x < r_{k+1}(k+2)$. The first member of B larger than ${r_k}^n(2k+1)$ is ${r_k}^{n+1}(k+1)$ (if $n \leq k$) or $r_{k+1}(k+2)$ (if $n = k+1$). Thus there is some $y \in \{x+1, x+2, \ldots, k+1\}$ such that $a + 2d = {r_k}^n y$. □

Lemma 4.23. *If $c \in \mathbb{N}$, $s \in \mathbb{N} \setminus \{1\}$, and $\{cs, cs^2\} \subseteq B$, then there exist $k \in \mathbb{N}$, $n \in \{0, 1, \ldots, k\}$, $t \in \{1, 2, \ldots, k + 1 - n\}$, and $y \in \{k + 1, k + 2, \ldots, 2k + 1\}$ such that $c = {r_k}^n y$ and $s = {r_k}^t$.*

Proof. Pick $k \leq m$, $\delta \in \{1, 2, \ldots, k + 1\}$, $\nu \in \{1, 2, \ldots, m + 1\}$, $y \in \{k + 1, k + 2, \ldots, 2k + 1\}$, and $z \in \{m + 1, m + 2, \ldots, 2m + 1\}$ such that $cs = {r_k}^\delta y$ and $cs^2 = {r_m}^\nu z$.

Now $s \leq {r_k}^\delta y \leq {r_k}^{k+1}(2k + 1)$ and $s = \dfrac{{r_m}^\nu z}{{r_k}^\delta y} > \dfrac{r_m}{{r_k}^{k+1}(2k + 1)}$, so

$$r_m < \left({r_k}^{k+1}(2k + 1)\right)^2 < r_{k+1}$$

and so $m \leq k$ and thus $m = k$. Therefore $s = {r_k}^{\nu-\delta}\dfrac{z}{y}$. Since r_k is a prime which does not divide y, we must have that y divides z and therefore that $y = z$. Let $t = \nu - \delta$. Since $c{r_k}^{\nu-\delta} = cs = {r_k}^\delta y$ we have $c = {r_k}^{2\delta-\nu} y$. Let $n = 2\delta - \nu$. Since $c = {r_k}^n y$ and $s = {r_k}^t$ we have that $n \geq 0$ and $t \geq 1$. Since $n + t = \delta$ we have that $n + t \leq k + 1$. □

To complete the proof of the theorem, let $A_1 = B$, let $A_2 = \{{r_k}^n : k \in \mathbb{N}$ and $n \in \{1, 2, \ldots, k + 1\}\}$, and let $C = A_1 \cup A_2$. Given $k \in \mathbb{N}$, let $a = r_k(k + 1)$ and let $d = b = r = r_k$. Then for $t, n \in \{0, 1, \ldots, k - 1\}$ one has $br^n = {r_k}^{n+1} \in A_2$, $a + td = r_k(k + t + 1) \in A_1$, and $br^n(a + td) = {r_k}^{n+2}(k + t + 1) \in A_1$.

It is trivial that A_2 does not contain $\{cs(a+d), cs(a+2d)\}$ as the latter element is less than twice the former. Suppose we have some $c, a, d \in \mathbb{N}$ and some $s \in \mathbb{N}\setminus\{1\}$ such that

$$\{cs, cs^2, cs(a + d), cs^2(a + d), cs(a + 2d)\} \subseteq A_1\,.$$

Pick by Lemma 4.23 some $k \in \mathbb{N}$, $n \in \{0, 1, \ldots, k\}$, $t \in \{1, 2, \ldots, k + 1 - n\}$, and $y \in \{k + 1, k + 2, \ldots, 2k + 1\}$ such that $c = {r_k}^n y$ and $s = {r_k}^t$. Again invoking Lemma 4.23, pick some $k' \in \mathbb{N}$, $m \in \{0, 1, \ldots, k'\}$, $t' \in \{1, 2, \ldots, k' + 1 - m\}$, and $z \in \{k' + 1, k' + 2, \ldots, 2k' + 1\}$ such that $c(a + d) = {r_{k'}}^m z$ and $s = {r_{k'}}^{t'}$.

Since ${r_{k'}}^{t'} = s = {r_k}^t$ we have $k = k'$ and $t = t'$. Pick by Lemma 4.22, $k'' \in \mathbb{N}$ and $\nu \in \{1, 2, \ldots, k'' + 1\}$ such that

$$\{cs(a + d), cs(a + 2d)\} \subseteq \left\{{r_{k''}}^\nu w : w \in \{k'' + 1, k'' + 2, \ldots, 2k'' + 1\}\right\}.$$

Since $cs(a + d) = {r_k}^{t+m} z$ we have $k'' = k$ and $\nu = t + m$. Since $cs = {r_k}^{t+n} y$ we have $a + d = {r_k}^{m-n}\dfrac{z}{y}$. Since r_k is a prime which does not divide y, we have that y divides z, so $y = z$ and thus $a + d = {r_k}^{m-n}$.

Pick $w \in \{k + 1, k + 2, \ldots, 2k + 1\}$ such that $cs(a + 2d) = {r_k}^{t+m} w$. Then $a + 2d = {r_k}^{m-n}\dfrac{w}{y}$, so w divides y and thus $a + 2d = {r_k}^{m-n}$. Therefore $d = 0$, a contradiction. □

5. Algebra in $(\beta\mathbb{N},+)$ and $(\beta\mathbb{N},\cdot)$: Extending the central sets theorem

In attempting to derive results about geoarithmetic progressions, the approach that one might try first after a little experience in deriving Ramsey theoretic consequences of the algebra of $\beta\mathbb{N}$ would be to choose an appropriate idempotent q in $(\beta\mathbb{N},\cdot)$ and show that if $A \in q$, then there is some r, preferably in A, such that $\bigcap_{s=0}^{k}(r^s)^{-1}A \in q$. We show first that such an approach is doomed to failure.

Theorem 5.1.
(a) For all $q \in \beta\mathbb{N}$, there exists a partition $\{A_0, A_1\}$ of $\mathbb{N}$ such that for all $i \in \{0,1\}$ and all $x \in \mathbb{N}$, $(-x + A_i) \cap (-2x + A_i) \notin q$. In particular there exists $A \in q$ such that for all $x \in \mathbb{N}$, either $-x + A \notin q$ or $-2x + A \notin q$.
(b) There does not exist $q \in \beta\mathbb{N}$ such that for each $A \in q$ there is some $r \in \mathbb{N} \setminus \{1\}$ with $r^{-1}A \in q$ and $(r^2)^{-1}A \in q$.

Proof. (a) Let $q \in \beta\mathbb{N}$. Then $q + \beta\mathbb{N}$ is a right ideal of $(\beta\mathbb{N},+)$, so there is an additive idempotent in $q + \beta\mathbb{N}$. Pick $r \in \beta\mathbb{N}$ such that $q + r$ is an idempotent in $(\beta\mathbb{N},+)$. Then $q + r \in \bigcap_{n=1}^{\infty} c\ell(\mathbb{N}2^n)$ by [16, Lemma 6.6].

Define $f : \mathbb{N} \to \omega$ by $f(n) = \min F$ where $F \in \mathcal{P}_f(\omega)$ and $n = \sum_{t\in F} 2^t$. Then f has a continuous extension $\widetilde{f} : \beta\mathbb{N} \to \beta\omega$. For $i \in \{0,1\}$ let $A_i = \{x \in \mathbb{N} : (2\mathbb{N} - i) \in \widetilde{f}(x + r)\}$.

Let $i \in \{0,1\}$ and let $x \in \mathbb{N}$ and suppose that $(-x + A_i) \cap (-2x + A_i) \in q$. Pick $j, k \in \omega$ such that $x = 2^j(2k+1)$. Denote addition of z on the left in $\beta\mathbb{N}$ by λ_z and addition of z on the right by ρ_z. Then $\widetilde{f} \circ \lambda_x$ is constantly equal to $f(x)$ and $\widetilde{f} \circ \lambda_{2x}$ is constantly equal to $f(x)+1$ on $\mathbb{N}2^{j+2}$, which is a member of $q + r$. So $\widetilde{f}(x + q + r) = f(x)$ and $\widetilde{f}(2x + q + r) = f(x) + 1$. Therefore $\widetilde{f} \circ \lambda_x \circ \rho_r(q) = f(x)$ and $\widetilde{f} \circ \lambda_{2x} \circ \rho_r(q) = f(x) + 1$, so

$$\{y \in \mathbb{N} : \widetilde{f}(x + y + r) = f(x) \text{ and } \widetilde{f}(2x + y + r) = f(x) + 1\} \in q.$$

Therefore, pick $y \in (-x + A_i) \cap (-2x + A_i)$ such that $\widetilde{f}(x + y + r) = f(x)$ and $\widetilde{f}(2x + y + r) = f(x) + 1$.

Since $x + y \in A_i$, we have that $2\mathbb{N} - i \in \widetilde{f}(x + y + r) = f(x)$, so $f(x) + i \in 2\mathbb{N}$. (Recall that we are identifying points of $\mathbb{N}$ with the principle ultrafilters they generate.) Since $2x + y \in A_i$, we have that $2\mathbb{N} - i \in \widetilde{f}(2x + y + r) = f(x) + 1$, so $f(x) + i + 1 \in 2\mathbb{N}$, a contradiction.

(b) For $x \in \mathbb{N} \setminus \{1\}$, let $\ell(x)$ be the number of terms in the prime factorization of x. Then ℓ is a homomorphism from $(\mathbb{N} \setminus \{1\}, \cdot)$ onto $(\mathbb{N},+)$ and so its continuous extension $\widetilde{\ell} : (\beta\mathbb{N} \setminus \{1\}, \cdot) \to (\beta\mathbb{N},+)$ is also a homomorphism by [16, Corollary 4.22]. $\square$

We know that there exist multiplicative idempotents in the closure of the set of additive idempotents in $\beta\mathbb{N}$. In fact, there exist minimal multiplicative idempotents in the closure of the set of minimal additive idempotents in $\beta\mathbb{N}$, and we used one such in the proof of Theorem 3.8. In particular we know that $c\ell K(\beta\mathbb{N},+) \cap K(\beta\mathbb{N},\cdot) \neq \emptyset$. In the following we shall assume that geometric progressions have integer common ratios, though the lemma would remain valid with the more liberal definition.

Lemma 5.2. *Let* $D = \{q \in \beta\mathbb{N} :$ for all $A \in q$, A contains arbitrarily long geometric progressions$\}$. *Then D is a closed two sided ideal of* $(\beta\mathbb{N}, \cdot)$. *In particular* $c\ell K(\beta\mathbb{N}, \cdot) \subseteq D$.

Proof. Trivially D is closed. Let $q \in D$, let $s \in \beta\mathbb{N}$, let $A \in qs$, and let $B \in sq$. Let $n \in \mathbb{N}$. We need to show that A and B contain length n geometric progressions. Now $\{x \in \mathbb{N} : x^{-1}A \in s\} \in q$, so pick $a \in \mathbb{N}$ and $r \in \mathbb{N} \setminus \{1\}$ such that $\{a, ar, ar^2, \ldots, ar^{n-1}\} \subseteq \{x \in \mathbb{N} : x^{-1}A \in s\}$. Then $\bigcap_{t=0}^{n-1}(ar^t)^{-1}A \in s$, so pick $b \in \bigcap_{t=0}^{n-1}(ar^t)^{-1}A$. Then $\{ba, bar, bar^2, \ldots, bar^{n-1}\} \subseteq A$. Also $\{x \in \mathbb{N} : x^{-1}B \in q\} \in s$, so pick $x \in \mathbb{N}$ such that $x^{-1}B \in q$. Pick $c \in \mathbb{N}$ and $d \in \mathbb{N} \setminus \{1\}$ such that $\{c, cd, cd^2, \ldots, cd^{n-1}\} \subseteq x^{-1}B$. Then $\{xc, xcd, xcd^2, \ldots, xcd^{n-1}\} \subseteq B$. □

We will see now that there would be interesting Ramsey theoretic consequences of the existence of an additive idempotent in the set D defined above. (Compare the conclusion with those of Theorem 3.8.)

Theorem 5.3. *Let* $D = \{q \in \beta\mathbb{N} :$ for all $A \in q$, A contains arbitrarily long geometric progressions$\}$ *and assume that there exists $q \in D$ such that $q + q = q$. Then whenever $r \in \mathbb{N}$ and $\mathbb{N} = \bigcup_{i=1}^{r} A_i$, there exist $i \in \{1, 2, \ldots, r\}$ and a sequence $\langle H_n\rangle_{n=1}^{\infty}$ such that for each $n \in \mathbb{N}$, H_n is a length n geometric progression and for every $F \in \mathcal{P}_f(\mathbb{N})$, one has $\sum_{n\in F} H_n \subseteq A_i$.*

Proof. Pick $q \in D$ such that $q+q = q$. Given $B \in q$, let $B^\star = \{x \in B : -x+B \in q\}$. Then by [16, Lemma 4.14], whenever $x \in B^\star$ one has $-x + B^\star \in q$.

Pick $i \in \{1, 2, \ldots, r\}$ such that $A_i \in q$. Pick $x \in A_i{}^\star$ and let $H_1 = \{x\}$. Let $n \in \mathbb{N}$ and assume that $\langle H_t\rangle_{t=1}^{n}$ have been chosen so that for any F with $\emptyset \neq F \subseteq \{1, 2, \ldots, n\}$ and any $f \in \times_{t\in F} H_t$, $\sum_{t\in F} f(t) \in A_i{}^\star$. Let

$$B = A_i{}^\star \cap \bigcap\{-\textstyle\sum_{t\in F} f(t) + A_i{}^\star : F \in \mathcal{P}_f(\{1, 2, \ldots, n\}) \text{ and } f \in \times_{t\in F} H_t\}\,.$$

Then $B \in q$, so pick a length $n + 1$ geometric progression $H_{n+1} \subseteq B$. □

We now turn our attention to deriving an extension, Theorem 5.8, of the Central Sets Theorem for countable commutative semigroups [16, Theorem 14.11]. The Central Sets Theorem for $(\mathbb{N}, +)$ is due to Furstenberg [7, Proposition 8.21]. See [16, Part III] for numerous combinatorial applications of the Central Sets Theorem. Theorem 5.8 has several earlier theorems as immediate corollaries. To establish this theorem we shall use the notion of *partial semigroup* introduced in [2].

Definition 5.4. (a) A *partial semigroup* is a set S together with an operation $\cdot$ that maps a subset of $S \times S$ into S and satisfies the associative law $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ in the sense that if either side is defined, then so is the other and they are equal.

(b) Given a partial semigroup $(S, \cdot)$ and $x \in S$, $\varphi(x) = \{y \in S : x \cdot y \text{ is defined}\}$.

(c) Given a partial semigroup $(S, \cdot)$, $x \in S$, and $A \subseteq S$, $x^{-1}A = \{y \in \varphi(x) : x \cdot y \in A\}$.

(d) A partial semigroup $(S, \cdot)$ is *adequate* if and only if for each $F \in \mathcal{P}_f(S)$, $\bigcap_{x\in F} \varphi(x) \neq \emptyset$.

(e) Given an adequate partial semigroup $(S, \cdot)$, $\delta S = \bigcap_{x\in S} c\ell_{\beta S}\varphi(x)$.

Lemma 5.5. *Let $(S, \cdot)$ be an adequate partial semigroup and for $p, q \in \delta S$ define $p\cdot q = \big\{A \subseteq S : \{x \in S : x^{-1}A \in q\} \in p\big\}$. Then, with the relative topology inherited from βS, $(\delta S, \cdot)$ is a compact right topological semigroup.*

Proof. See [2, Proposition 2.6]. $\square$

Lemma 5.6. *Let $(S,\cdot)$ and $(T,*)$ be adequate partial semigroups and let $f: S \xrightarrow{\text{onto}} T$ have the property that for all $x \in S$ and all $y \in \varphi_S(x)$, $f(y) \in \varphi_T\big(f(x)\big)$ and $f(x\cdot y) = f(x) * f(y)$. Let $\widetilde{f} : \beta S \to \beta T$ be the continuous extension of f. Then the restriction of $\widetilde{f}$ to δS is a homomorphism from $(\delta S, \cdot)$ to $(\delta T, *)$.*

Proof. See [2, Proposition 2.8]. $\square$

Definition 5.7. $\Phi = \{f : \mathbb{N} \to \mathbb{N} : f(n) \le n$ for all $n \in \mathbb{N}\}$.

Theorem 5.8. *Let $k \in \mathbb{N}$. For each $i \in \{1,2,\ldots,k\}$, let E_i be a countable commutative semigroup with identity e_i. For each $i \in \{1,2,\ldots,k\}$ and $j \in \mathbb{N}$, let $\langle z_{i,j,t}\rangle_{t=1}^{\infty}$ be a sequence in E_i. We assume that, for every $i \in \{1,2,\ldots,k\}$, $z_{i,1,t} = e_i$ for every $t \in \mathbb{N}$, and that $\langle z_{i,2,t}\rangle_{t=1}^{\infty}$ is a sequence which contains every element of E_i infinitely often. Let ψ be an arbitrary function mapping $E_1 \times E_2 \times \cdots \times E_k$ to a set X and let C_i be a central set in E_i for each $i \in \{1,2,\ldots,k\}$. Then, for any finite coloring of X, there exist a sequence $\langle H_n\rangle_{n=1}^{\infty}$ in $\mathcal{P}_f(\mathbb{N})$, a sequence $\langle c_{i,n}\rangle_{n=1}^{\infty}$ in E_i for each $i \in \{1,2,\ldots,k\}$ and a monochromatic subset A of X such that the following statements hold for every $G \in \mathcal{P}_f(\mathbb{N})$, every $i \in \{1,2,\ldots,k\}$ and all $f_1, f_2, \ldots, f_k \in \Phi$:*
(i) $\psi\big(\prod_{n\in G} c_{1,n}\cdot\prod_{t\in H_n} z_{1,f_1(n),t}, \ldots, \prod_{n\in G} c_{k,n}\cdot\prod_{t\in H_n} z_{k,f_k(n),t}\big) \in A$ and
(ii) $\prod_{n\in G} c_{i,n}\cdot\prod_{t\in H_n} z_{i,f_i(n),t} \in C_i$.

Proof. Let $L = \mathbb{N}^k$ and let v be a "variable" not in L. A *located word* over L is a function w from a nonempty finite subset D_w of $\mathbb{N}$ to L. Let S_0 be the set of located words over L and let S_1 be the set of located variable words over L, that is, the set of words over $L \cup \{v\}$ in which v occurs. Let $S = S_0 \cup S_1$. Given $u, w \in S$, if $\max D_u < \min D_w$, then define $u\cdot w$ by $D_{u\cdot w} = D_u \cup D_w$ and for $t \in D_{u\cdot w}$,

$$(u\cdot w)(t) = \begin{cases} \theta_t(u) & \text{if } t \in D_u, \\ \theta_t(w) & \text{if } t \in D_w\,. \end{cases}$$

With this operation S, S_0, and S_1 are adequate partial semigroups, so by Lemma 5.5, δS, δS_0, and δS_1 are compact right topological semigroups. Also δS_1 is an ideal of δS. (The verification of this latter statement is an easy exercise and a good chance for the reader to see whether she has grasped the definition of the operation.) Notice that for $j \in \{1,2\}$ and $p \in \beta S_j$, one has that $p \in \delta S_j$ if and only if for each $n \in \mathbb{N}$, $\{w \in S_j : \min D_w > n\} \in p$.

We take for each $w \in S$, $D_{\theta_a(w)} = D_w$. We have that for each $a \in L$, $\theta_a : S \to S_0$ (where θ_a is the identity on S_0). Denote also by θ_a its continuous extension taking βS to βS_0 and notice that θ_a is the identity on βS_0.

For each $i \in \{1,2,\ldots,k\}$, define $g_i : S_0 \to E_i$ by $g_i(w) = \prod_{t\in D_w} z_{i,\pi_i(w(t)),t}$ for each $w \in S_0$. We shall also use g_i to denote the continuous function from βS_0 to βE_i which extends g_i.

We claim that, if $b_i \in E_i$ for each $i \in \{1,2,\ldots,k\}$ and if $n \in \mathbb{N}$, there exists $w \in S_0$ such that $g_i(w) = b_i$ for every $i \in \{1,2,\ldots,k\}$ and $\min(D_w) > n$. To see this, observe that we can choose $n_1, n_2, \ldots, n_k$ in $\mathbb{N}$ such that $n < n_1 < n_2 < \ldots < n_k$ and $z_{i,2,n_i} = b_i$ for every $i \in \{1,2,\ldots,k\}$. We can then define w by putting $D_w = \{n_1, n_2, \ldots, n_k\}$ and $w(n_i) = (1,1,\ldots,1,2,1,\ldots,1)$, with 2 occurring as the ith term in this k-tuple, for each $i \in \{1,2,\ldots,k\}$.

In particular each $g_i : S_0 \to E_i$ is surjective and so, by Lemma 5.6, the restriction of g_i to δS_0 is a homomorphism to $\delta E_i = \beta E_i$.

For each $i \in \{1, 2, \ldots, k\}$, let p_i be a minimal idempotent in βE_i for which $C_i \in p_i$. We shall first show that we can choose a minimal idempotent $q \in \delta S_0$ and an idempotent $r \in \delta S_1$ such that $q \leq r$, $g_i(q) = p_i$ for every $i \in \{1, 2, \ldots, k\}$, and $\theta_a(r) = q$ for every $a \in L$.

Given $(X_1, X_2, \ldots, X_k, n) \in p_1 \times p_2 \times \cdots \times p_k \times \mathbb{N}$ we choose $w(X_1, X_2, \ldots, X_k, n) \in S_0$ such that $\min(D_{w(X_1,X_2,\ldots,X_k,n)}) > n$ and $g_i\big(w(X_1, X_2, \ldots, X_k, n)\big) \in X_i$ for each $i \in \{1, 2, \ldots, k\}$. We give $p_1 \times p_2 \times \cdots \times p_k \times \mathbb{N}$ a directed set ordering by stating that $(X_1, X_2, \ldots, X_k, n) \prec (X'_1, X'_2, \ldots, X'_k, n')$ if and only if $X'_i \subseteq X_i$ for each $i \in \{1, 2, \ldots, k\}$ and $n < n'$. If x is any limit point of the net $\langle w(X_1, X_2, \ldots, X_k, n)\rangle$ in βS_0, we have $x \in \delta S_0$ and $g_i(x) = p_i$ for every $i \in \{1, 2, \ldots, k\}$. (That $x \in \delta S_0$ follows from the fact that $\min(D_{w(X_1,X_2,\ldots,X_k,n)}) > n$. To see that $g_i(x) = p_i$, let $A \in p_i$ and suppose $g_i(x) \notin \overline{A}$. Pick $B \in x$ such that $g_i[\,\overline{B}\,] \cap \overline{A} = \emptyset$. Let $X_i = A$ and for $j \neq i$ let $X_j = E_j$. Pick $(X'_1, X'_2, \ldots, X'_k, n') \succ (X_1, X_2, \ldots, X_k, 1)$ such that $w(X'_1, X'_2, \ldots, X'_k, n') \in \overline{B}$. But $g_i\big(w(X'_1, X'_2, \ldots, X'_k, n')\big) \in X'_i \subseteq X_i = A$, a contradiction.)

Let $C = \big\{x \in \delta S_0 : g_i(x) = p_i \text{ for all } i \in \{1, 2, \ldots, k\}\big\}$. We have just seen that C is nonempty, and so it is a compact subsemigroup of δS_0. Let q be a minimal idempotent in C. Then q is minimal in δS_0, because if q' is any idempotent of δS_0 satisfying $q' \leq q$, we have $g_i(q') \leq g_i(q) = p_i$ for every $i \in \{1, 2, \ldots, k\}$. This implies that $g_i(q') = p_i$ for every $i \in \{1, 2, \ldots, k\}$. So $q' \in C$ and thus $q' = q$.

Let r be any idempotent in the intersection of the right ideal $q\delta S_1$ and the left ideal $\delta S_1 q$ of δS_1. Then $r \leq q$. For any $a \in L$, we have $\theta_a(r) \leq \theta_a(q) = q$ and hence $\theta_a(r) = q$.

We define $\gamma : S_0 \to X$ by $\gamma(w) = \psi\big(g_1(w), g_2(w), \ldots, g_k(w)\big)$. We can choose a monochromatic subset A of X such that $\gamma^{-1}[A] \in q$. Let $Q = \gamma^{-1}[A] \cap \bigcap_{i=1}^{k} g_i{}^{-1}[C_i]$. Then $Q \in q$. Let $Q^\star = \{w \in Q : w^{-1}Q \in q\}$. Then $Q^\star \in q$ and $w^{-1}Q^\star \in q$ for every $w \in Q^\star$ by [16, Lemma 4.14].

We shall inductively choose a sequence $\langle w_n\rangle_{n=1}^{\infty}$ in S_1 such that for each $n \in \mathbb{N}$,
(a) if $n > 1$, then $\min D_{w_n} > \max D_{w_{n-1}}$ and
(b) for every nonempty $F \subseteq \{1, 2, \ldots, n\}$ and every choice of $a_t \in \{1, 2, \ldots, t\}^k$ for $t \in F$, $\prod_{t\in F} \theta_{a_t}(w_t) \in Q^\star$.

We first choose $w_1 \in S_1$ such that $\theta_a(w_1) \in Q^\star$, where a denotes the k-tuple $(1, 1, \ldots, 1)$. This is possible because $\theta_a{}^{-1}[Q^\star] \in r$ and so $\theta_a{}^{-1}[Q^\star] \neq \emptyset$. Now let $n \in \mathbb{N}$ and assume that $w_1, w_2, \ldots, w_n$ have been chosen. Let

$$U = \big\{ \textstyle\prod_{t\in F} \theta_{a_t}(w_t) : \emptyset \neq F \subseteq \{1, 2, \ldots, n\} \text{ and for all } t \in F\,,\ a_t \in \{1, 2, \ldots, t\}^k \big\}\,.$$

By our assumption (b), $U \subseteq Q^\star$, so $\bigcap_{u\in U} u^{-1}Q^\star \in q$. We observe that, for any $V \in q$ and any $a \in L$, $\theta_a{}^{-1}[V] \in r$ and that $\{w \in S_1 : \min(D_w) > \max(D_{w_n})\} \in r$. Thus we can choose w_{n+1} such that $\min(D_{w_{n+1}}) > \max(D_{w_n})$, and

$$w_{n+1} \in \bigcap \big\{\theta_a{}^{-1}\left[Q^\star \cap \textstyle\bigcap_{u\in U} u^{-1}Q^\star\right] : a \in \{1, 2, \ldots, n+1\}^k\big\}\,.$$

We can now conclude the proof. For each $n \in \mathbb{N}$ and $i \in \{1, 2, \ldots, k\}$, let $H_n = \{t \in D_{w_n} : w_n(t) = v\}$ and let $c_{i,n} = \prod_{t\in D_{w_n}\setminus H_n} z_{i,\pi_i(w_n(t)),t}$. Then, if $a \in L$, we have $g_i\big(\theta_a(w_n)\big) = c_{i,n} \cdot \prod_{t\in H_n} z_{i,\pi_i(a),t}\,$.

Suppose now that $f_1, f_2, \ldots, f_k \in \Phi$ and $G \in \mathcal{P}_f(\mathbb{N})$. For each $n \in \mathbb{N}$, define $a_n \in \{1, 2, \ldots, n\}^k$ by $\pi_i(a_n) = f_i(n)$ for each $i \in \{1, 2, \ldots, k\}$. Then for each

$i \in \{1, 2, \ldots, k\}$, we have

$$\begin{aligned} \prod_{n\in G} c_{i,n} \cdot \prod_{t\in H_n} z_{i,f_i(n),t} &= \prod_{n\in G} c_{i,n} \cdot \prod_{t\in H_n} z_{i,f_i(n),t} \\ &= \prod_{n\in G} g_i\big(\theta_{a_n}(w_n)\big) \\ &= g_i\big(\prod_{n\in G} \theta_{a_n}(w_n)\big)\,. \end{aligned}$$

Since $\prod_{n\in G} \theta_{a_n}(w_n) \in Q$, $\gamma[Q] \subseteq A$, and $g_i[Q] \subseteq C_i$ for each $i \in \{1, 2, \ldots, k\}$ the conclusions of the theorem hold. □

Corollary 5.9. *Let $m, k \in \mathbb{N}$. Let C_1 be central in $(\mathbb{N}, +)$ and let C_2 be central in $(\mathbb{N}, \cdot)$. For each $i \in \{0, 1, \ldots, k\}$ let $\langle x_{i,t}\rangle_{t=1}^\infty$ and $\langle y_{i,t}\rangle_{t=1}^\infty$ be sequences in $\mathbb{N}$. Let $\mathbb{N} = \bigcup_{s=1}^m A_s$. Then there exist $s \in \{1, 2, \ldots, m\}$, $F \in \mathcal{P}_f(\mathbb{N})$, and $a, b \in \mathbb{N}$ such that*

$$\begin{aligned} &\{ba\} \cup \big\{b(a + \textstyle\sum_{t\in F} x_{i,t}) : i \in \{0, 1, \ldots, k\}\big\} \cup \\ &\big\{ba \cdot \textstyle\prod_{t\in F} y_{j,t} : j \in \{0, 1, \ldots, k\}\big\} \cup \\ &\big\{b(a + \textstyle\sum_{t\in F} x_{i,t}) \cdot (\prod_{t\in F} y_{j,t}) : i, j \in \{0, 1, \ldots, k\}\big\} \subseteq A_s\,, \\ &\{a\} \cup \big\{a + \textstyle\sum_{t\in F} x_{i,t} : i \in \{0, 1, \ldots, k\}\big\} \subseteq C_1\,, \ \textit{and} \\ &\{b\} \cup \big\{b \cdot \textstyle\prod_{t\in F} y_{j,t} : j \in \{0, 1, \ldots, k\}\big\} \subseteq C_2\,. \end{aligned}$$

Proof. Let $E_1 = (\omega, +)$ and let $E_2 = (\mathbb{N}, \cdot)$. Define $\psi : E_1 \times E_2 \to \omega$ by $\psi(a, b) = ab$. For $t \in \mathbb{N}$ let $z_{1,1,t} = 0$ and $z_{2,1,t} = 1$. For $i \in \{1, 2\}$ let $\langle z_{i,2,t}\rangle_{t=1}^\infty$ be a sequence which contains every element of E_i infinitely often. For $j \in \{0, 1, \ldots, k\}$ and $t \in \mathbb{N}$ let $z_{1,j+3,t} = x_{j,t}$ and $z_{2,j+3,t} = y_{j,t}$. (For $j > k + 3$ we do not care what $z_{1,j,t}$ and $z_{2,j,t}$ are.)

Since $\mathbb{N}$ is an ideal of $(\omega, +)$, C_1 is central in E_1. Pick $\langle H_n\rangle_{n=1}^\infty$, $\langle c_{1,n}\rangle_{n=1}^\infty$, $\langle c_{2,n}\rangle_{n=1}^\infty$, and A as guaranteed by Theorem 5.8. Pick $s \in \{1, 2, \ldots, m\}$ such that $A \subseteq A_s$. Let $n = k+3$. (We choose $n = k+3$ rather than $n = 1$ so that there will be functions f_1 and f_2 in Φ with the properties required of them below.) Let $a = c_{1,n}$, let $b = c_{2,n}$, and let $F = H_n$. If $f_1(n) = 1$, then $c_{1,n} + \sum_{t\in H_n} z_{1,f_1(n),t} = a$. If $f_1(n) = j+3$ for some $j \in \{0, 1, \ldots, k\}$, then $c_{1,n} + \sum_{t\in H_n} z_{1,f_1(n),t} = a + \sum_{t\in F} x_{j,t}$. If $f_2(n) = 1$, then $c_{2,n} \cdot \prod_{t\in H_n} z_{2,f_2(n),t} = b$. If $f_2(n) = j + 3$ for some $j \in \{0, 1, \ldots, k\}$, then $c_{2,n} \cdot \prod_{t\in H_n} z_{2,f_2(n),t} = b \cdot \prod_{t\in F} y_{j,t}$. □

We conclude with a simple variation on the proof of Theorem 5.8, which applies in case the semigroups are all the same.

Theorem 5.10. *Let $k \in \mathbb{N}$, let E be a countable commutative semigroup with identity e, let $R_1, R_2, \ldots, R_k$ be IP-sets in E, and let C be a central subset of E. There exist $r_i \in R_i$ and $b_i \in E$ for each $i \in \{1, 2, \ldots, k\}$ such that whenever $f : \{1, 2, \ldots, k\} \to \{1, 2, \ldots, k\}$, $h : \{1, 2, \ldots, k\} \to \{0, 1, \ldots, k\}$, and $\emptyset \neq F \subseteq \{1, 2, \ldots, k\}$, one has $\prod_{i\in F} b_i \cdot (r_{f(i)})^{h(i)} \in C$.*

Proof. Let $L = \{1, 2, \ldots, k^2 + k + 2\}^k$ and let v, S_0, S_1, S, $\langle D_w\rangle_{w\in S}$, and $\langle \theta_a\rangle_{a\in L}$ be as in the proof of Theorem 5.8. For $j \in \{1, 2, \ldots, k\}$ pick a sequence $\langle x_{j,t}\rangle_{t=1}^\infty$ such that $FP(\langle x_{j,t}\rangle_{t=1}^\infty) \subseteq R_j$. For $m \in \{0, 1, \ldots, k\}$, $j \in \{1, 2, \ldots, k\}$, and $t \in \mathbb{N}$, let $z_{2+mk+j,t} = (x_{j,t})^m$. Let $z_{1,t} = e$ for each t and let $\langle z_{2,t}\rangle_{t=1}^\infty$ be a sequence in E which takes on each member of E infinitely often.

For $i \in \{1, 2, \ldots, k\}$, define $g_i : S_0 \to E$ by $g_i(w) = \prod_{t\in D_w} z_{\pi_i(w(t)),t}$. For $F \in \mathcal{P}_f(\{1, 2, \ldots, k\})$, define $\gamma_F : S_0 \to E$ by $\gamma_F(w) = \prod_{i\in F} g_i(w)$ (so $\gamma_{\{i\}} = g_i$). Denote also by γ_F the continuous extension taking βS_0 to βE.

As in the proof of Theorem 5.8 we see that given any $b_1, b_2, \ldots, b_k \in E$ there is some $w \in S_0$ such that $g_i(w) = b_i$ for each $i \in \{1, 2, \ldots, k\}$. In particular each γ_F

is a surjective homomorphism, so by Lemma 5.6 the restriction of γ_F to δS_0 is a homomorphism to βE.

Pick a minimal idempotent $p \in \beta E$ such that $C \in p$. We claim that for any $B \in p$ and any $n \in \mathbb{N}$ there exists $w_{B,n} \in S_0$ such that for all $F \in \mathcal{P}_f(\{1,2,\ldots,k\})$, $\gamma_F(w_{B,n}) \in B$. To see this pick $b_1, b_2, \ldots, b_k$ such that $FP(\langle b_t\rangle_{t=1}^k) \subseteq B$, which one may do because p is an idempotent. Pick $w_{B,n}$ such that $g_i(w_{B,n}) = b_i$ for each $i \in \{1,2,\ldots,k\}$.

Direct $\mathcal{D} = \{(B,n) : B \in p \text{ and } n \in \mathbb{N}\}$ by $(B,n) \prec (B',n')$ if and only if $B' \subseteq B$ and $n < n'$. Let u be a limit point of the net $\langle w_{B,n}\rangle_{(B,n)\in\mathcal{D}}$ in βS_0. We see as in the proof of Theorem 5.8 that $u \in \delta S_0$ and $\gamma_F(u) = p$ for all $F \in \mathcal{P}_f(\{1,2,\ldots,k\})$. Let $J = \{w \in \delta S_0 : \gamma_F(w) = p \text{ for all } F \in \mathcal{P}_f(\{1,2,\ldots,k\})\}$. Then J is a compact subsemigroup of δS_0 since each γ_F is a continuous homomorphism. Pick a minimal idempotent q of J. Given any idempotent $q' \in \delta S_0$ such that $q' \leq q$, for each $F \in \mathcal{P}_f(\{1,2,\ldots,k\})$, $\gamma_F(q') \leq \gamma_F(q) = p$, so $\gamma_F(q') = p$. Thus $q' \in J$ and so $q' = q$. That is, q is minimal in δS_0.

Now we claim that we may choose $w \in S_1$ such that $\gamma_F\big(\theta_a(w)\big) \in C$ for every $a \in L$ and every $F \in \mathcal{P}_f(\{1,2,\ldots,k\})$. To see this, pick an idempotent r in $q\delta S_1 \cap \delta S_1 q$. Then $r \leq q$, so for each $a \in L$, $\theta_a(r) \leq \theta_a(q) = q$ and so $\theta_a(r) = q$ and thus for each $F \in \mathcal{P}_f(\{1,2,\ldots,k\})$, $\gamma_F\big(\theta_a(r)\big) = \gamma_F(q) = p$. Pick $w \in S_1 \cap \bigcap\{(\gamma_F \circ \theta_a)^{-1}[C] : a \in L \text{ and } F \in \mathcal{P}_f(\{1,2,\ldots,k\})\}$.

Let $H = \{t \in D_w : w(t) = v\}$. For $i \in \{1,2,\ldots,k\}$, let $b_i = \prod_{t\in D_w\setminus H} z_{\pi_i(w(t)),t}$ and let $r_i = \prod_{t\in H} x_{i,t}$. Now let $f : \{1,2,\ldots,k\} \to \{1,2,\ldots,k\}$, $h : \{1,2,\ldots,k\} \to \{0,1,\ldots,k\}$, and $\emptyset \neq F \subseteq \{1,2,\ldots,k\}$. Let

$$a = \big(2 + h(1)k + f(1), 2 + h(2)k + f(2), \ldots, 2 + h(k)k + f(k)\big)\,.$$

Then for $i \in F$,

$$\begin{aligned} b_i(r_{f(i)})^{h(i)} &= b_i \cdot \big(\textstyle\prod_{t\in H} (x_{f(i),t})^{h(i)}\big) \\ &= b_i \cdot \textstyle\prod_{t\in H} z_{\pi_i(a),t} \\ &= g_i\big(\theta_a(w)\big), \end{aligned}$$

so $\prod_{i\in F} b_i(r_{f(i)})^{h(i)} = \gamma_F\big(\theta_a(w)\big) \in C$. □

References

1. V. Bergelson, *Multiplicatively large sets and ergodic Ramsey theory*, Israel J. Math. **148** (2005), 23-40. MR2191223
2. V. Bergelson, A. Blass, and N. Hindman, *Partition theorems for spaces of variable words*, Proc. London Math. Soc. **68** (1994), 449-476. MR1262304 (95i:05107)
3. V. Bergelson and A. Leibman, *Polynomial extensions of van der Waerden's and Szemerédi's Theorems*, J. Amer. Math. Soc. **9** (1996), 725-753. MR1325795 (96j:11013)
4. T. Carlson, N. Hindman, and D. Strauss, *Ramsey theoretic consequences of some new results about algebra in the Stone-Čech compactification*, Integers **5(2)** (2005) #A04, pp. 1-26. MR2192082 (2006k:54031)
5. W. Deuber, *Partitionen und lineare Gleichungssysteme,* Math. Zeit. **133** (1973), 109-123. MR0325406 (48:3753)
6. H. Furstenberg, *Ergodic behavior of diagonal measures and a theorem of Szemerédi on arithmetic progressions*, J. d'Analyse Math. **31** (1977), 204-256. MR0498471 (58:16583)
7. H. Furstenberg, *Recurrence in ergodic theory and combinatorical number theory*, Princeton University Press, Princeton, 1981. MR603625 (82j:28010)
8. H. Furstenberg and Y. Katznelson, *A density version of the Hales-Jewett theorem*, J. Anal. Math. **57** (1991), 64-119. MR1191743 (94f:28020)
9. H. Furstenberg, Y. Katznelson, and D. Ornstein, *The ergodic theoretical proof of Szemerédi's Theorem*, Bull. Amer. Math. Soc. **7** (1982), 527-552. MR670131 (84b:28016)

10. W. Gowers, *A new proof of Szemerédi's Theorem*, Geom. Funct. Anal. **11** (2001), 465–588. MR1844079 (2002k:11014)
11. R. Graham, B. Rothschild, and J. Spencer, *Ramsey theory*, Wiley, New York, 1990. MR1044995 (90m:05003)
12. A. Hales and R. Jewett, *Regularity and positional games*, Trans. Amer. Math. Soc. **106** (1963), 222-229. MR0143712 (26:1265)
13. N. Hindman, *Partitions and sums and products of integers*, Trans. Amer. Math. Soc. **247** (1979), 227-245. MR517693 (80b:10022)
14. N. Hindman, *Problems and new results in the algebra of βS and Ramsey theory*, in Unsolved problems on mathematics for the 21st Century, J. Abe and S. Tanaka, eds., IOS Press, Amsterdam (2001), 295-305. MR1896684 (2004a:22003)
15. N. Hindman, I. Leader, and D. Strauss, *Image partition regular matrices – bounded solutions and preservations of largeness*, Discrete Math. **242** (2002), 115-144. MR1874760 (2002j:05146)
16. N. Hindman and D. Strauss, *Algebra in the Stone-Čech compactification: Theory and applications*, de Gruyter, Berlin, 1998. MR1642231 (99j:54001)
17. B. de Mathan, *Numbers contravening a condition in density modulo* 1, Acta Math. Acad. Sci. Hungar. **36** (1980), 237-241. MR612195 (82e:10088)
18. A. Pollington, *On the density of the sequence* $\{n_k\xi\}$, Illinois J. Math. **23** (1979), 511-515. MR540398 (80i:10066)
19. R. Rado, *Verallgemeinerung Eines Satzes von van der Waerden mit Anwendungen auf ein Problem der Zahlentheorie*, Sonderausg. Sitzungsber. Preuss. Akad. Wiss. Phys.-Math. Klasse **17** (1933), 1-10.
20. R. Rado, *Studien zur Kombinatorik*, Math. Zeit. **36** (1933), 242-280.
21. E. Szemerédi, *On sets of integers containing no k elements in arithmetic progression*, Acta Arith. **27** (1975), 199-245. MR0369312 (51:5547)
22. B. van der Waerden, *Beweis einer Baudetschen Vermutung*, Nieuw Arch. Wiskunde **19** (1927), 212-216.

Institute of Discrete Mathematics and Geometry, Vienna University of Technology, Wiedner Hauptstr. 8-10, 1040 Wien, Austria
E-mail address: mathias.beiglboeck@tuwien.ac.at

Department of Mathematics, Ohio State University, Columbus, Ohio 43210
E-mail address: vitaly@math.ohio-state.edu

Department of Mathematics, Howard University, Washington, DC 20059
E-mail address: nhindman@aol.com

Mathematics Centre, University of Hull, Hull HU6 7RX, United Kingdom
E-mail address: d.strauss@hull.ac.uk