Combinatorics, Probability and Computing (2005) 00, 000–000. © 2005 Cambridge University Press
DOI: 10.1017/S0000000000000000 Printed in the United Kingdom

A Multidimensional Central Sets Theorem

M. BEIGLBOECK†

Department of Discrete Mathematics and Geometry, TU Vienna,
1040 Vienna, Austria, mathias.beiglboeck@tuwien.ac.at

The Theorems of Hindman and van der Waerden belong to the classical theorems of partition Ramsey Theory. The Central Sets Theorem is a strong simultaneous extension of both theorems that applies to general commutative semigroups. We give a common extension of the Central Sets Theorem and Ramsey's Theorem.

1. Introduction

Van der Waerden's Theorem ([9]) states that for any partition of the positive integers $\mathbb{N}$ one of the cells of the partition contains arbitrarily long arithmetic progressions.

To formulate Hindman's Theorem ([5]) and the Central Sets Theorem we set up some notation. By $\mathcal{P}_f(\omega)$ we denote the set of all finite nonempty subsets of $\omega = \mathbb{N}\cup\{0\}$. For a sequence $\langle x_n\rangle_{n=0}^{\infty}$ in $\mathbb{N}$ we put $FS(\langle x_n\rangle_{n=0}^{\infty}) := \{\sum_{t\in\alpha} x_t : \alpha \in \mathcal{P}_f(\omega)\}$. A set $A \subseteq \mathbb{N}$ is called an *IP-set* iff there exists a sequence $\langle x_n\rangle_{n=0}^{\infty}$ in $\mathbb{N}$ such that $FS(\langle x_n\rangle_{n=0}^{\infty}) \subseteq A$. (This definitions make perfect sense in any semigroup $(S,\cdot)$ and we indeed plan to use them in this context. FS is an abbriviation of *finite sums* and will be replaced by FP if we use multiplicative notation for the semigroup operation.) Now Hindman's Theorem states that in any finite partition of $\mathbb{N}$ one of the cells is an IP-set.

K. Milliken and A. Taylor ([7, 8]) found a quite natural common extension of the Theorems of Hindman and Ramsey: For a sequence $\langle x_n\rangle_{n=0}^{\infty}$ in $\mathbb{N}$ and $k \geq 1$ put $[FS(\langle x_n\rangle_{n=0}^{\infty})]^k_< := \left\{\left\{\sum_{t\in\alpha_1} x_t, \ldots, \sum_{t\in\alpha_k} x_t\right\} : \alpha_1 < \ldots < \alpha_k \in \mathcal{P}_f(\omega)\right\}$, where we write $\alpha < \beta$ for $\alpha, \beta \in \mathcal{P}_f(\omega)$ iff $\max\alpha < \min\beta$. For an arbitrary set S let $[S]^k$ be the set of all finite subsets of S consisting of exactly k elements. If $[\mathbb{N}]^k = \bigcup_{i=1}^r A_i$ then there exist $i \in \{1, 2, \ldots, r\}$ and a sequence $\langle x_n\rangle_{n=0}^{\infty}$ in $\mathbb{N}$ such that $FS[\langle x_n\rangle_{n=0}^{\infty}]^k_< \subseteq A_i$.

Let Φ be the set of all functions $f : \omega \to \omega$ such that $f(n) \leq n$ for all $n \in \omega$. Then our main theorem may be stated as follows:

Theorem 1.1. *Let $(S,\cdot)$ be a commutative semigroup and assume that there exists a non principal minimal idempotent in βS. For each $l \in \mathbb{N}$, let $\langle y_{l,n}\rangle_{n=0}^{\infty}$ be a sequence in S. Let $k, r \geq 1$ and let $[S]^k = \bigcup_{i=1}^r A_i$. There exist $i \in \{1, 2, \ldots, r\}$, a*

† The author thanks the Austrian Science Foundation FWF for its support through Project no. S8312. and Project no. P17627-N12

sequence $\langle a_n \rangle_{n=0}^{\infty}$ *in* S *and a sequence* $\alpha_0 < \alpha_1 < \ldots$ *in* $\mathcal{P}_f(\omega)$ *such that for each* $g \in \Phi$, $\left[FP\left(\left\langle a_n \prod_{t\in\alpha_n} y_{g(n),t}\right\rangle_{n=0}^{\infty}\right)\right]_<^k \subseteq A_i$.

We will review some properties of the Stone-Čech compactification as well as the definition of a minimal idempotent in the next chapter. In the case $k = 1$ the somewhat odd assumption that βS should contain a non principal minimal idempotent is not needed. In general this condition will be satisfied if S is *weakly (left) cancellative*, i.e. for all $u, v \in S$ the set $\{s \in S : us = v\}$ is finite and S itself is infinite (see [6], Theorem 4.3.7). In particular the conclusion of Theorem 1.1 holds in the semigroups $(\mathbb{N}, +), (\mathbb{N}, \cdot), (\mathcal{P}_f(\omega), \cup)$.

The case $k = 1$ of Theorem 1.1 is exactly the Central Sets Theorem. (More precisely this is the version stated in [6], Corollary 14.12. A discussion on the origin of the Central Sets Theorem can also be found there.) By further specifying $(S, \cdot) = (\mathbb{N}, +)$ and $\langle y_{l,n} \rangle_{n=0}^{\infty} = \langle l, l, \ldots \rangle$ we get that all finite sums of elements of the arithmetic progressions $a_n, a_n + |\alpha_n|, \ldots, a_n + n|\alpha_n|, n \geq 0$ are guaranteed to be monochrome.

Theorem 1.1 may be seen as a generalization of the Central Sets Theorem in the same sense as the Milliken-Taylor Theorem is a multidimensional version of Hindman's Theorem.

2. Preliminaries on ultrafilters

For a set S let βS be the set of all ultrafilters on S. For $s \in S$ we will identify s with the principal ultrafilter of all subsets of S that contain s. If $(S, \cdot)$ is a semigroup, the operation $\cdot\,\cdot\,\cdot$ may be extended to a semigroup operation on βS by defining

$$A \in p \cdot q \;:\Leftrightarrow\; \{s \in S : s^{-1}A \in q\} \in p.^1 \tag{2.1}$$

If βS is properly topologized it turns out to be the Stone-Čech compactification of S (where we regard S to be a discrete space). It can be shown that the operation $\cdot\,\cdot\,\cdot : \beta S \times \beta S \to \beta S$ defined in (2.1) is the unique extension of $\cdot\,\cdot\,\cdot : S \times S \to S$, such that for each $s \in S$ and each $q \in \beta S$ the functions $\lambda_s, \rho_q : \beta S \to \beta S$ defined by $\lambda_s(r) := sr, \rho_q(r) := rq$ are continuous.

Applications of the algebraic structure of βS in partition Ramsey Theory are abundant. Examples are simple proofs of the theorems of Hindman and van der Waerden:

Idempotent ultrafilters (i.e. ultrafilters $e \in \beta S$ satisfying $ee = e$) turn out to be tightly connected with IP-sets in S: A subset A of S is an IP-set iff there is an idempotent $e \in \beta S$ such that $A \in e$. By a theorem of Ellis βS always contains an idempotent ultrafilter e and by the ultrafilter properties of e for any partition $A_1, A_2, \ldots, A_r$ of S there exists an $i \in \{1, 2, \ldots, r\}$ such that $A_i \in e$. Thus A_i is an IP-set.

βS always has a smallest (two-sided) ideal which will be denoted by $K(\beta S)$. It turns out that for $(S, \cdot) = (\mathbb{N}, +)$ the elements of $K(\beta\mathbb{N})$ are well suited for van der Waerden's Theorem.

Idempotents in $K(\beta S)$ (which are always present) are called *minimal idempotents*. Not at all surprisingly minimal idempotents are particularly interesting for combinatorial applications. Subsets of S which are contained in some minimal idempotent are called *central sets* and that these sets satisfy the conclusion of the Central Sets Theorem reveals the source of the theorem's name.

See [6] for an elementary introduction to the semigroup βS as well as for the combinatorial applications mentioned in this section.

If S is an infinite set an arbitrary non principal ultrafilter $p \in \beta S$ may be used to

[1] S is a semigroup, so s might not have an inverse. We may avoid this obstacle by defining $s^{-1}A := \{t \in S : st \in A\}$.

give a proof of Ramsey's Theorem. (This proof is by now classical. See [2] p.39 for a discussion of its origins.) It's an idea of V. Bergelson and N. Hindman that in the case $S = \mathbb{N}$, something might be gained by using an ultrafilter with special algebraic properties. Via this approach in [1] a short proof of the Milliken-Taylor Theorem is given and a very strong simultaneous generalization of Ramsey's Theorem and numerous single-dimensional Ramsey-type Theorems (including van der Waerden's Theorem) is obtained. Our proof is a variation on this idea.

3. The proof of the main theorem

The following Lemma is the basic tool in the ultrafilter proof of Ramsey's theorem:

Lemma 3.1. *Let S be a set, let $e \in \beta S \setminus S$, let $k, r \geq 1$, and let $[S]^k = \bigcup_{i=1}^r A_i$. For each $i \in \{1, 2, \ldots, r\}$, each $t \in \{1, 2, \ldots, k\}$ and each $E \in [S]^{t-1}$, define $B_t(E, i)$ by downward induction on t:*

(1) For $E \in [S]^{k-1}, B_k(E, i) := \{y \in S \setminus E : E \cup \{y\} \in A_i\}$.

(2) For $1 \leq t < k$ and $E \in [S]^{t-1}$, $B_t(E, i) := \{y \in S \setminus E : B_{t+1}(E \cup \{y\}, i) \in e\}$.

Then there exists some $i \in \{1, 2, \ldots, r\}$ such that $B_1(\emptyset, i) \in e$.

Proof. For each $E \in [S]^{k-1}$ one has $S = E \cup \bigcup_{i=1}^r B_k(E, i)$, so there exists $i \in \{1, 2, \ldots, r\}$ such that $B_k(E, i) \in e$. Next let $E \in [S]^{k-2}$ and $y \in S \setminus E$. Then there exists $i \in \{1, 2, \ldots, r\}$ such that $B_k(E \cup \{y\}, i) \in e$. Thus $S = E \cup \bigcup_{i=1}^r B_{k-1}(E, i)$. After iterating this argument $k - 1$ times we achieve $S = \emptyset \cup \bigcup_{i=1}^r B_1(\emptyset, i)$ which clearly proves the statement. □

To formulate our key lemma we need to introduce some notation: Let S be a set and put $S^{<\omega} = \bigcup_{n=0}^\infty S^{\{0,\ldots,n-1\}}$. A non empty set $T \subseteq S^{<\omega}$ is a *tree* in S iff for all $f \in S^{<\omega}$, $g \in T$ such that *dom* $f \subseteq$ *dom* g, $g_{\restriction dom\ f} = f$ one has $f \in T$. We will identify a function $f \in S^{\{0,1,\ldots,n-1\}}$ with the sequence $\langle f(0), f(1), \ldots, f(n-1)\rangle$. If $s \in S$ then $f^\frown s := \langle f(0), f(1), \ldots, f(n-1), s\rangle$. For $f \in S^{<\omega}$ we put $T(f) := \{s \in S : f^\frown s \in T\}$.

Lemma 3.2. *Let $(S, \cdot)$ be a semigroup such that there exists an idempotent $e \in \beta S \setminus S$, let $k, r \geq 1$ and let $[S]^k = \bigcup_{i=1}^r A_i$. Then there exist $i \in \{1, 2, \ldots, r\}$ and a tree $T \subseteq S^{<\omega}$ such that for all $f \in T$ and $\alpha_1 < \alpha_2 < \ldots < \alpha_k \subseteq$ dom $f, \alpha_i \in \mathcal{P}_f(\omega)$ one has:*

(1) $T(f) \in e$.

(2) $\left\{\prod_{t\in\alpha_1} f(t), \prod_{t\in\alpha_2} f(t), \ldots, \prod_{t\in\alpha_k} f(t)\right\} \in A_i$.

In the proof we will employ some basic properties of idempotent ultrafilters. For a set $A \subseteq S$ we have $A \in e = ee$ iff $\{s \in S : s^{-1}A \in e\} \in e$ by the definition of the multiplication in βS. For $A \in e$ let

$$A^\star := \{s \in A : s^{-1}A \in e\} = A \cap \{s \in S : s^{-1}A \in e\} \in e$$

Then $t^{-1}A^\star \in e$ for all $t \in A^\star$:
It is clear that for $t \in A^\star, t^{-1}A \in e$. Furthermore

$$t^{-1}\{s \in S : s^{-1}A \in e\} = \{s \in S : s^{-1}(t^{-1}A) \in e\} \in e.$$

Thus in fact $t^{-1}A^\star = t^{-1}A \cap t^{-1}\{s \in S : s^{-1}A \in e\} \in e$. (This is [6], Lemma 4.14.)

Proof. Let $i \in \{1, 2, \ldots, r\}$ be such that $B_1(\emptyset, i) \in e$. (We use the notation of Lemma 3.1. Since i will be fixed in the rest of the proof, we will suppress it and write

$B_r(E)$ instead of $B_r(E,i)$.) We will inductively construct an increasing sequence of trees $\langle T_n\rangle_{n=0}^\infty$, satisfying for each $n\geq 0$, $T_n=\{f_{\restriction\{1,2,\dots,n-1\}}: f\in T_{n+1}\}$ such that the for each $f\in T_n$ the following holds:

(i) If *dom* $f\subseteq\{0,1,\dots,n-2\}$ then $T_n(f)\in e$.

(ii) If $\alpha_1,\alpha_2,\dots,\alpha_r\in\mathcal{P}_f(\omega), r\in\{1,2,\dots,k\}$ satisfy $\alpha_1<\alpha_2<\dots<\alpha_r\subseteq dom\ f$ and if $x_i=\prod_{t\in\alpha_i}f(t)$ then $x_r\in B_r(\{x_1,x_2,\dots,x_{r-1}\})^{\star 2}$.

Trivially we may put $T_0=\{\emptyset\}$. Assume now that $T_0,T_1,\dots,T_n$ have already been defined. Fix $f\in T_n$ with *dom* $f=\{0,1,\dots,n-1\}$. For $\alpha_1<\alpha_2<\dots<\alpha_r\subseteq$ *dom* f let $x_i=\prod_{t\in\alpha_i}f(t)$. By assumption $x_r\in B_r(\{x_1,x_2,\dots,x_{r-1}\})$ and thus $B_{r+1}(\{x_1,x_2,\dots,x_r\})\in e$ for $r\in\{1,2,\dots,k-1\}$. Since $x_r\in B_r(\{x_1,x_2,\dots,x_{r-1}\})^\star$ we have $x_r^{-1}B_r(\{x_1,x_2,\dots,x_{r-1}\})^\star\in e$ for $r\in\{0,1,\dots,k\}$. Define $T_n(f)$ to be the intersection of all sets $B_{r+1}(\{x_1,x_2,\dots,x_r\})^\star$, $r\in\{0,1,\dots,k-1\}$ and $x_r^{-1}B_r(\{x_1,x_2,\dots,x_{r-1}\})^\star, r\in\{0,1,\dots,k\}$ such that indeed $T_n(f)\in e$. Using this put $T_{n+1}=T_n\cup\{f^\frown t: f\in T_n, dom\ f=\{0,1,\dots,n-1\}, t\in T_n(f)\}$. It is not hard to verify that this implies that the inductive construction can be continued: This is only interesting for *dom* $f=\{0,1,\dots,n\}$ and $n\in\alpha_r$ (where $r\in\{1,2,\dots,k\}$). Fix $f':\{0,1,\dots,n-1\}\to S$ such that $f'^\frown f(n)=f$. If $\alpha_r=\{n\}$, $x_r=f(n)\in T_n(f')\subseteq B(\{x_1,x_2\dots,x_{r-1}\})^\star$ so we are done. If $\alpha_r=\alpha_r'\cup\{n\}$ for some non empty $\alpha_r'\subseteq\{0,1,\dots,n-1\}$ we have $f(n)\in T_n(f')\subseteq\left(\prod_{t\in\alpha_r'}f'(t)\right)^{-1}B_r(\{x_1,x_2,\dots,x_{r-1}\})^\star$ and this implies $x_r=\prod_{t\in\alpha_r}f(t)\in B_r(\{x_1,x_2,\dots,x_{r-1}\})^\star$.

Finally put $T=\bigcup_{n=0}^\infty T_n$. Obviously $T(f)\in e$ for all $f\in T$. Since $\prod_{t\in\alpha_k}f(t)\in B_k\left(\left\{\prod_{t\in\alpha_1}f(t),\dots,\prod_{t\in\alpha_{k-1}}f(t)\right\}\right)$ for all $f\in T$ and $\alpha_1<\alpha_2<\dots<\alpha_k\subseteq$ *dom* f we see that (2) holds. □

From this Lemma one may directly derive the following strong version of the Milliken-Taylor Theorem:

Corollary 3.3. *Let $k,r\geq 1$, let $(S,\cdot)$ be a semigroup, let $\langle x_n\rangle_{n=0}^\infty$ be a sequence in S and let $[S]^k=\bigcup_{i=1}^r A_i$. Assume that for every idempotent $s\in S$ there exists some $m\in\mathbb{N}$ such that $s\notin FP(\langle x_n\rangle_{n=m}^\infty)$. Then there exist $i\in\{1,2,\dots,r\}$ and a sequence $\alpha_0<\alpha_1<\dots$ in $\mathcal{P}_f(\omega)$ such that $\left[FP\left(\left\langle\prod_{t\in\alpha_n}x_t\right\rangle_{n=0}^\infty\right)\right]_<^k\subseteq A_i$.*

Proof. By [6], Lemma 5.11 there exists an idempotent $e\in\beta S$, such that for all $m\geq 0$, $FP(\langle x_n\rangle_{n=m}^\infty)\in e$ and by our assumption we have $e\in\beta S\setminus S$. Let $i\in\{1,2,\dots,r\}$ and $T\subseteq S^{<\omega}$ be as provided by Lemma 3.2. We have $T(\emptyset)\cap FP(\langle x_n\rangle_{n=0}^\infty)\in e$. In particular this set is not empty, so we may choose $\alpha_0\in\mathcal{P}_f(\omega)$ such that $\prod_{t\in\alpha_0}x_t\in T(\emptyset)$. Let $m_0:=\max\alpha_0$. As before $T\left(\left(\prod_{t\in\alpha_0}x_t\right)\right)\cap FP\left(\langle x_n\rangle_{n=m_0}^\infty\right)\in e$, so we find $\alpha_1>\alpha_0,\alpha_1\in\mathcal{P}_f(\omega)$ such that $\prod_{t\in\alpha_1}x_t\in T\left(\left\langle\prod_{t\in\alpha_0}x_t\right\rangle\right)$. By continuing in this fashion we achieve a sequence with the required properties. □

We remark that our restriciton on the idempotents contained in $FP(\langle x_n\rangle_{n=0}^\infty)$ cannot be dropped: Consider for example $(S,\cdot)=(\mathbb{Z},+)$ and $\langle x_n\rangle_{n=0}^\infty=\langle 0,0,\dots\rangle$: In this case $[FP(\langle x_n\rangle_{n=0}^\infty)]_<^k=\{\{0\}\}$ for any $k\in\mathbb{N}$.

Another possibility to avoid this difficulty is presented in [6], Corollary 18.9: Instead of partitions of $[S]^k$, partitions of $\bigcup_{i=1}^k[S]^i$ are considered there.

In the proof of Theorem 1.1 we will require the following:

[2] For $r=1$ this is meant to be $B_1(\emptyset)^\star$.

Theorem 3.4. *Let $(S,\cdot)$ be a commutative semigroup, let $A \in e \in K(\beta S)$, let $l \in \mathbb{N}$ and for each $j \in \{0,1,\dots,l-1\}$ let $\langle y_{j,n}\rangle_{n=0}^{\infty}$ be a sequence in S. Then there exist $a \in S$ and $\alpha \in \mathcal{P}_f(\omega)$ such that $a\prod_{t\in\alpha} y_{j,t} \in A$ for each $j \in \{0,1,\dots,l-1\}$.*

Theorem 3.4 is a special case of the Central Sets Theorem and may easily be derived from the Hales-Jewett Theorem ([4]).

We are now able to prove our main Theorem:

Proof of Theorem 1.1. Fix a minimal idempotent $e \in \beta S\setminus S$. Let $i \in \{1,2,\dots,r\}$ and $T \subseteq S^{<\omega}$ be as provided by lemma 3.2. We will inductively construct sequences $\langle a_n\rangle_{n=0}^{\infty}$ in S and $\alpha_0 < \alpha_1 < \dots$ in $\mathcal{P}_f(\omega)$ such that for all $n \in \mathbb{N}$ and all $g \in \Phi$:

$$\left\langle a_0\prod_{t\in\alpha_0} y_{g(0),t}, a_1\prod_{t\in\alpha_1} y_{g(1),t},\dots,a_{n-1}\prod_{t\in\alpha_{n-1}} y_{g(n-1),t}\right\rangle \in T. \tag{3.1}$$

By the properties of T this is sufficient to proof the Theorem.

Assume that $a_0,a_1,\dots,a_{n-1} \in S$ und $\alpha_0 < \dots < \alpha_{n-1} \in \mathcal{P}_f(\omega)$ have already been constructed such that (3.1) is true for all $g \in \Phi$. We have

$$G_n := \bigcap_{g\in\Phi} T\left(\left\langle a_0\prod_{t\in\alpha_0} y_{g(0),t},\ a_1\prod_{t\in\alpha_1} y_{g(1),t},\dots,a_{n-1}\prod_{t\in\alpha_{n-1}} y_{g(n-1),t}\right\rangle\right) \in e.$$

Let $m := \max\alpha_{n-1}$. By applying Theorem 3.4 to the set G_n and the sequences $\langle y_{0,k}\rangle_{k=m}^{\infty}, \langle y_{1,k}\rangle_{k=m}^{\infty},\dots,\langle y_{n,k}\rangle_{k=m}^{\infty}$ we find $a_n \in S$ und $\alpha_n \in \mathcal{P}_f(\mathbb{N})$, $\alpha_n > \alpha_{n-1}$ such that $a_n\prod_{t\in\alpha_n} y_{0,t}, a_n\prod_{t\in\alpha_n} y_{1,t},\dots,a_n\prod_{t\in\alpha_n} y_{n,t} \in G_n$.

Thus for all $g \in \Phi$, $\left\langle a_0\prod_{t\in\alpha_0} y_{g(0),k}, a_1\prod_{t\in\alpha_1} y_{g(1),k},\dots,a_n\prod_{t\in\alpha_n} y_{g(n),k}\right\rangle \in T$, as we wanted to show. □

We conclude this section by giving a strengthening of Theorem 1.1 that applies to partitions of the spaces $[S]^1,[S]^2,\dots,[S]^k$ simultaenously. It is not hard to verify that a similar extension of Corollary 3.3 is also valid. To avoid confusion about indices we use colourings instead of partitions.

Corollary 3.5. *Let $(S,\cdot)$ be a commutative semigroup and assume that there exists a non principal minimal idempotent in βS. For each $l \in \mathbb{N}$, let $\langle y_{l,n}\rangle_{n=0}^{\infty}$ be a sequence in S. Let $k \geq 1$ and assume that for each $m \in \{1,2,\dots,k\}$, $[S]^m$ is finitely coloured. There exist a sequence $\langle a_n\rangle_{n=0}^{\infty}$ in S, a sequence $\alpha_0 < \alpha_1 < \dots$ in $\mathcal{P}_f(\omega)$ and for each $m \in \{1,2,\dots,k\}$ a monochrome set $A^{(m)}$ such that for each $g \in \Phi$ and each $m \in \{1,2,\dots,k\}$, $\left[FP\left(\left\langle a_n\prod_{t\in\alpha_n} y_{g(n),t}\right\rangle_{n=0}^{\infty}\right)\right]_<^m \subseteq A^{(m)}$.*

Proof. We describe two ways to prove Corollary 3.5:

Fix a linear ordering $\prec$ on S. For $m \in \{1,2,\dots,k\}$ let $f^{(m)} : [S]^m \to \{1,2,\dots,r_m\}$ be the colouring at hand. Define a colouring $g^{(m)} : [S]^k \to \{1,2,\dots,r_m\}$ by letting $g^{(m)}(\{x_1,x_2,\dots,x_k\}) = f(m)(\{x_1,x_2,\dots,x_m\})$, where $\{x_1,x_2,\dots,x_m\}$ are the m smallest elements of $\{x_1,x_2,\dots,x_k\}$ with respect to $\prec$. Then apply Theorem 1.1 to the colouring

$$\begin{aligned} f : [S]^k &\to \{1,2,\dots,r_1\}\times\{1,2,\dots,r_2\}\times\dots\times\{1,2,\dots,r\}_k \\ E &\mapsto (g^{(1)}(E),g^{(2)}(E),\dots,g^{(k)}(E)). \end{aligned}$$

It is clear that the resulting sequences $\langle a_n\rangle_{n=0}^{\infty}$ and $\langle \alpha_n\rangle_{n=0}^{\infty}$ satisfy the conclusion of Corollary 3.5.

The more complicated way to prove Corollary 3.5 is to start by extending Lemma 3.2. Pick a minimal idempotent $e \in \beta S \setminus S$. Choose by Lemma 3.2 for each $m \in \{1,2,\dots,k\}$ a monochrome set $A^{(m)} \subseteq [S]^m$ and a tree $T^{(m)} \subseteq S^{<\omega}$ such that for all $f \in T^{(m)}$ and all $\alpha_1 < \alpha_2 < \dots < \alpha_m \subseteq dom\ f, \alpha_i \in \mathcal{P}_f(\omega)$ one has $T^{(m)}(f) \in e$ and

$\{\prod_{t\in\alpha_1} f(t), \prod_{t\in\alpha_2} f(t), \ldots, \prod_{t\in\alpha_m} f(t)\} \in A^{(m)}$. But then $T := \bigcap_{m=1}^{k} T^{(m)}$ is a tree such that for all $f \in T, T(f) \in e$ and for all $m \in \{1, 2, \ldots, k\}$ and all $\alpha_1 < \alpha_2 < \ldots < \alpha_m \subseteq dom\ f, \alpha_i \in \mathcal{P}_f(\omega)$, $\{\prod_{t\in\alpha_1} f(t), \prod_{t\in\alpha_2} f(t), \ldots, \prod_{t\in\alpha_m} f(t)\} \in A^{(m)}$. By performing the proof of Theorem 1.1 with this tree T we again see that Corollary 3.5 is valid. □

4. Conclusion

When applying ultrafilters to Ramsey theory one typically establishes that a set is non empty by showing that it is actually large, i.e. contained in a certain ultrafilter e. The Milliken-Taylor Theorem mentioned in the introduction states that for any partition $A_1, A_2, \ldots, A_r$ of $\mathbb{N}$ there exist i and a sequence $\langle x_n\rangle_{n=0}^{\infty}$ such that $[FS(\langle x_n\rangle_{n=0}^{\infty})]_<^k \subseteq A_i$. In the spirit of the principle stated above, one could expect that after constructing the first n elements $x_0, x_1, \ldots, x_{n-1}$ of the sequence, the set of possible choices of the element x_n is contained in an ultrafilter e. Lemma 3.2 gives this idea a rigorous meaning. The combinatorial gain is that the sequence $\langle x_n\rangle_{n=0}^{\infty}$ can be forced to satisfy additional properties: In our generalizations 3.3 of the Milliken-Taylor Theorem the sequence may be chosen from a predefined IP-set in a quite general semigroup. In appropriate commutative semigroups the variety of possible sequences is large enough to achieve the multidimensional extension 1.1 of the Central Sets Theorem.

Acknowledgements

The author thanks Vitaly Bergelson, Neil Hindman and Dona Strauss for useful hints and remarks. He also thanks the referee whose valueable suggestions led to a significantly more concise and elegant presentation.

References

[1] V. Bergelson and N. Hindman, *Ultrafilters and multidimensional Ramsey theorems*, Combinatorica **9** (1989), 1-7.

[2] W. Comfort, *Some recent applications of ultrafilters to topology*, in: Proceedings of the Fourth Prague Topological Symposium, 1976, J. Novak (ed.), Lecture Notes in Math. 609 (1977), 34-42.

[3] H. Furstenberg, *Recurrence in ergodic theory and combinatorial number theory*, Princton University Press, Princeton, 1981.

[4] A. Hales and R. Jewett, *Regularity and positional games*, Trans. Amer. Math. Soc. **106** (1963), 222-229.

[5] N. Hindman, *Finite sums from a sequences within cells of a partition of* ℕ, J. Combin. Theory Ser. A **17** (1974), 1-11.

[6] N. Hindman and D. Strauss, *Algebra in the Stone-Čech compactification: theory and applications*, de Gruyter, Berlin, 1998.

[7] K. Milliken, *Ramsey's Theorem with sums or unions*, J. Combin. Theory Ser. A **18** (1975), 276-290.

[8] A. Taylor, *A canonical partition relation for finite subsets of* ω, J. Combin. Theory Ser. A **21** (1976), 137-146.

[9] B. van der Waerden, *Beweis einer Bedautschen Vermutung*, Nieuw Arch. Wiskunde **19** (1927), 212-216.