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Note

A Short Proof of Hindman's Theorem

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Recently Hindman [1] proved the following theorem, which was conjectured by Graham and Rothschild:

THEOREM 1. *Let N be the set of nonnegative integers and suppose N is partitioned into sets $A_1, ..., A_k$. Then there exist i and $X = \{x_n : n \geq 1\} \subseteq A_i$ such that every sum of the form $x_{i_1} + \cdots + x_{i_n}$, where $i_1 < \cdots < i_n$, lies in A_i.*

It is not difficult to see that Theorem 1 is equivalent to:

THEOREM 2. *Let F be the set of all finite nonempty subsets of N. Suppose F is partitioned into sets $A_1, ..., A_k$. Then there exist i and $D \subseteq A_i$ such that D is infinite and all its elements are pairwise disjoint, and every finite union of members of D lies in A_i.*

To see that Theorem 1 follows from Theorem 2, define a function $f: F \to N$ by $f(\{i_1, ..., i_n\}) = 2^{i_1} + \cdots + 2^{i_n}$, and observe that if x and y are disjoint members of F, then $f(x \cup y) = f(x) + f(y)$.

Now we give a short proof of Theorem 2. It should be stressed that most of the ideas in this proof are implicitly contained in Hindman's original proof.

DEFINITIONS. A set $D \subseteq F$ is a *disjoint collection* iff D is infinite and its elements are pairwise disjoint.

If $D \subseteq F$ then $FU(D)$ is the set of all finite unions of elements of D (the empty union is excluded).

If $X \subseteq F$ and D is a disjoint collection, then X is *large for* D iff for every disjoint collection $D' \subseteq FU(D)$, $FU(D') \cap X \neq 0$.

LEMMA 1. (a) *If X is large for D and $X = Y \cup Z$, then there is a disjoint collection $D' \subseteq FU(D)$ such that either Y or Z is large for D'.*

(b) *If X is large for D then for every $n \geqslant 0$, $\{x \in X: \min(x) > n\}$ is large for D.*

Proof. Part (b) is trivial. Suppose part (a) is false. Since Y is not large for D, there is a disjoint collection $D' \subseteq FU(D)$ such that $FU(D') \cap Y = 0$. Since Z is not large for D', there is a disjoint collection $D'' \subseteq FU(D')$ such that $FU(D'') \cap Z = 0$. But now $FU(D'') \cap X = 0$, contradicting the assumption that X is large for D.

LEMMA 2. *Suppose X is large for D. Then there is a finite set $E \subseteq FU(D)$ such that for all $x \in FU(D)$, if $x \cap \cup E = 0$ then there exists $d \in FU(E)$ such that $x \cup d \in X$.*

Proof. Suppose the lemma is false. Then it is easy to see that we may obtain inductively a sequence $x_0, x_1, \ldots, x_n, \ldots,$ of pairwise disjoint elements of $FU(D)$ such that for all $n \geqslant 1$ and all $d \in FU(\{x_i : i < n\})$, $x_n \cup d \notin X$. (The first elements x_0 may be chosen arbitrarily.) Let $y_n = x_{2n} \cup x_{2n+1}$ for each n, and let $D' = \{y_n : n \geqslant 0\}$. Then D' is a disjoint collection, $D' \subseteq FU(D)$, and it is easy to see that $FU(D') \cap X = 0$, contradicting the fact that X is large for D.

LEMMA 3. *Suppose X is large for D. Then there is $d \in FU(D)$ so that $\{x \in X: x \cup d \in X\}$ is large for some $D' \subseteq FU(D)$.*

Proof. Let E be as in Lemma 2 and let $D_1 \subseteq FU(D)$ be a disjoint collection such that $x \cap \bigcup E = 0$ for all $x \in FU(D_1)$. Note that $X \cap FU(D_1)$ must be large for D_1. For each $d \in FU(E)$, let $X_d = \{x \in X: x \cup d \in X\}$. By Lemma 2,

$$X \cap FU(D_1) \subseteq \bigcup \{X_d : d \in FU(E)\}.$$

By repeated application of Lemma 1 (a), there is a disjoint collection $D' \subseteq FU(D_1)$ and a fixed d so that X_d is large for D'.

LEMMA 4. *If X is large for D then there is a disjoint collection $D' \subseteq FU(D)$ such that $FU(D') \subseteq X$.*

Proof. First note that by Lemma 3 we can easily construct sequences d_n, D_n, X_n so that:

1. $D_0 = D$ and $X_0 = X$,
2. $d_n \in FU(D_n)$,
3. $X_{n+1} \subseteq X_n$ and $D_{n+1} \subseteq FU(D_n)$,

4. X_n is large for D_n,
5. if $x \in X_{n+1}$ then $x \cup d_n \in X_n$,
6. $d_m \cap d_n = 0$ if $m \neq n$.

Let $\bar{D} = \{d_n : n \geqslant 0\}$. Now we define pairwise disjoint $x_n \in FU(\bar{D})$, $n \geqslant 0$, by induction so that:

(7) if $k_n = \max\{k\colon d_k \subseteq \bigcup_{1 \leqslant i < m} x_i\}$ then $x_n \in X_{k_n+1}$.

(Let x_0 be an arbitrary element of $FU(\bar{D}) \cap X$.) This is possible since $\bar{D} - \{d_i : i \leqslant k_n\} \subseteq FU(D_{k_n+1})$ and X_{k_n+1} is large for D_{k_n+1} by (4).

We claim $D' = \{x_n : n \geqslant 0\}$ works. Clearly $D' \subseteq X$. We must show $FU(D') \subseteq X$. Let $x \in FU(D')$. Say

$$x = x_{i_1} \cup \cdots \cup x_{i_n} \cup x_r,$$

where $i_1 < \cdots < i_n < r$. Suppose

$$x_{i_1} \cup \cdots \cup x_{i_n} = d_{j_1} \cup \cdots \cup d_{j_m},$$

where $j_1 < \cdots < j_m$. Then $j_m \leqslant k_r$. Since $x_r \in X_{k_r+1}$ we know $x_r \in X_{j_m+1}$ so $x_r \cup d_{j_m} \in X_{j_m}$ by (5). But then $x_r \cup d_{j_m} \in X_{j_{m-1}+1}$ by (3) so

$$x_r \cup d_{j_m} \cup d_{j_{m-1}} \in X_{j_{m-1}}$$

by (5) again. Repeating the argument, we see

$$x_r \cup d_{j_m} \cup \cdots \cup d_{j_1} \in X_{j_1} \subseteq X$$

as desired. This proves Lemma 4.

Theorem 2 now follows immediately. Since F is large for any D, by Lemma 1 (a) some A_i is large for some disjoint collection D. By Lemma 4, there is a disjoint collection $D' \subseteq FU(D)$ so that $FU(D') \subseteq A_i$.

Reference

1. N. Hindman, Finite sums from sequences within cells of a partition of N, *J. Comb. Theory* (*A*) **17** (1974), 1–11.