Martin's Axiom and Choice Principles

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Abstract

Martin's Axiom (MA) is the statement that for every well-ordered cardinal number $\kappa < 2^{\aleph_0}$, $\mathsf{MA}(\kappa)$ holds, where $\mathsf{MA}(\kappa)$ is "if $(P, \leq)$ is a c.c.c. quasi order and $\mathcal{D}$ is a family of $\leq \kappa$ dense sets in P, then there is a $\mathcal{D}$-generic filter of P". The fragment $\mathsf{MA}(\aleph_0)$ of MA is provable in ZFC, but is *not* provable in ZF.

In set theory without the Axiom of Choice (AC), we discuss the interrelation between $\mathsf{MA}(\aleph_0)$, $\mathsf{MA}(\aleph_0)$ *restricted to complete Boolean algebras* and various choice principles, and also clarify the status of the above instances of MA in certain ZFA (i.e. ZF with the Axiom of Extensionality modified in order to allow atoms) and ZF models that fail AC.

In the choiceless context, it makes sense to drop the requirement that the cardinal κ be well-ordered and we can define for any (not necessarily well-ordered) cardinal $\mathfrak{p}$ the statement $\mathsf{MA}_{\mathfrak{p}}$ to be "if $(P, \leq)$ is a c.c.c. quasi order with $|P| \leq \mathfrak{p}$, and $\mathcal{D}$ is a family of $\leq \mathfrak{p}$ dense sets in P, then there is a $\mathcal{D}$-generic filter of P". We then define MA^* to be the statement that for every (not necessarily well-ordered) cardinal $\mathfrak{p} < 2^{\aleph_0}$, $\mathsf{MA}_{\mathfrak{p}}$ holds. In ZFC, MA^* is equivalent to MA. We show that $\mathsf{MA}^* + \neg\mathsf{MA}$ is relatively consistent with ZFA.