Spectra of $\mathcal{L}_{\omega_1,\omega}$ sentences

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Outline

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Introduction

Löwenheim-Skolem theorem is one of the basic results in mathematical logic.

Theorem

Let φ be a first order sentences. If φ has an infinite model then it has a model of cardinality κ, for any infinite cardinal κ.

This result in general fails for non first order logics. We shall consider the simplest infinitary logic.

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Definition

Let τ be a countable vocabulary. The collection of $\mathcal{L}_{\omega_1,\omega}$ formulas over τ is obtained by closing the collection of atomic formulas under:

- *countable conjunctions, disjunctions and negation,*
- *quantification over finitely many variables.*

$\mathcal{L}_{\omega_1,\omega}$ satisfies the downward Löwenheim-Skolem theorem. We would like to understand to what extent the upward LS theorem holds.

For an $\mathcal{L}_{\omega_1,\omega}$ sentence φ and infinite cardinal κ we let $\mathrm{Mod}_\kappa(\varphi)$ will denote the collection of models of φ of size κ.

Definition

*Let P be a property. For an $\mathcal{L}_{\omega_1,\omega}$ sentence φ we let the P-**spectrum** $\mathrm{spec}_P(\varphi)$ denote the class: $\{\kappa :$ $\mathrm{Mod}_\kappa(\varphi)$ satisfies $P\}$.*

Examples:

- $\mathrm{spec}(\varphi) = \{\kappa : \mathrm{Mod}_\kappa(\varphi) \neq \emptyset\}$,
- $\mathrm{spec}_{\mathrm{cat}}(\varphi) = \{\kappa : |\mathrm{Mod}_\kappa(\varphi)/\simeq| = 1\}$
- $\mathrm{spec}_{\max}(\varphi) = \{\kappa :$ there is a maximal model in $\mathrm{Mod}_\kappa(\varphi)\}$
- $\mathrm{spec}_{\mathrm{AP}}(\varphi) = \{\kappa : \mathrm{Mod}_\kappa(\varphi)$ satisfies AP$\}$, where AP stands for the Amalgamation Property
- ...

Definition

The ***Hanf number*** *for a class* $\mathcal{K}$ *of sentences in a language* $\mathcal{L}$ *is the least cardinal* κ *such that, for every* $\varphi \in \mathcal{K}$*, if* $\sup \operatorname{spec}(\varphi) \geq \kappa$ *then* $\sup \operatorname{spec}(\varphi) = \infty$.

Theorem (Lopez-Escobar)

The Hanf number for the class of all sentences of $\mathcal{L}_{\omega_1,\omega}$ *is* $\beth_{\omega_1}$.

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Definition

We say that a sentence φ ***characterizes*** *a cardinal κ if* $\kappa = \max \operatorname{spec}(\varphi)$.

Question

What are the characterizable cardinals?

Theorem (folklore)

The $\aleph_\alpha$, for $\alpha < \omega_1$, are characterizable.

This settles the problem under GCH. What happens if we do not assume GCH?

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Conjecture

For every $\mathcal{L}_{\omega_1,\omega}$ sentence φ, $\sup \mathrm{spec}(\varphi) \notin [\aleph_{\omega_1}, 2^{\aleph_0})$.

Remark

If Conjecture is true and $2^{\aleph_0} > \aleph_{\omega_1}$ then no cardinal in the interval $[\aleph_{\omega_1}, 2^{\aleph_0})$ is characterizable.

Theorem (Shelah; Hrushovski - V.)

If we starts with a model of GCH and add $> \aleph_{\omega_1}$ Cohen reals, the Conjecture holds in the generic extension.

What can be said in ZFC alone?

Theorem (folklore)

The set of cardinals characterizable by $\mathcal{L}_{\omega_1,\omega}$ sentences is closed under:

1. $\kappa \mapsto \kappa^+$
2. $\kappa \mapsto 2^\kappa$
3. *countable sums*
4. *countable products.*

Question

Let C be the least set of cardinals containing $\aleph_0$ and closed under $(1) - (4)$. Is C the set of characterizable cardinals?

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Theorem (Souldatos, Sinapova)

The set of cardinals characterizable by $\mathcal{L}_{\omega_1,\omega}$ sentences is also closed under the following operations:

1. $(\kappa, \lambda) \mapsto \kappa^\lambda$
2. $\kappa \mapsto \mathrm{Ded}(\kappa)$
3. $\kappa \mapsto \max\{\lambda : \textit{ there is a } \kappa\textit{-Kurepa tree with } \lambda \textit{ branches}\}$.

Remark

There is a model of ZFC in which the set C above is not closed under the last operation.

Absoluteness

Question

Given an $\mathcal{L}_{\omega_1,\omega}$ sentence φ and a cardinal κ is the question $\kappa \in \text{spec}(\varphi)$ decidable in ZFC?

Theorem (Friedman,Hyttinen, Koerwien)

1. $\aleph_1 \in \text{spec}(\varphi)$ *is absolute for models of* ZFC.
2. $\aleph_\alpha \in \text{spec}(\varphi)$ *is not absolute for models of* ZFC, *for* $1 < \alpha < \omega_1$.
3. *Assuming the existence of uncountably many inaccessibles the question* $\aleph_{\alpha+2} \in \text{spec}(\varphi)$ *is not absolute for models of* ZFC + GCH.
4. *Let* $\alpha < \omega_1$ *be a limit ordinal. Assume there is a supercompact cardinal. Then there is a sentence* φ *such that* $\aleph_{\alpha+1} \in \text{spec}(\varphi)$ *is not absolute in* ZFC + GCH.

Theorem (Friedman,Hyttinen, Koerwien)

Let α be a limit ordinal with $\omega < \alpha < \omega_1$. Assume there is a supercompact cardinal. Then there is a $\mathcal{L}_{\omega_1,\omega}$ sentence φ such that $\aleph_\alpha \in \mathrm{spec}(\varphi)$ is not absolute for models of ZFC + GCH.

Remark

The question remains open for $\alpha = \omega$.

Theorem (Grossberg, Shelah)

Let φ be an $\mathcal{L}_{\omega_1,\omega}$ sentence. The question whether $\sup \mathrm{spec}(\varphi) = \infty$ is decidable in ZFC.

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Scott sentences

Theorem (Scott)

Let M be a countable structure in a countable vocabulary τ. Then there is a sentence φ_M such that for every other countable model N in vocabulary τ, if $N \models \varphi_M$ then $N \simeq M$.

Remark

φ_M is called the **Scott sentence** of M.

Proposition

*Suppose M is a countable structure in a countable vocabulary τ and N is **any** structure in the same vocabulary such that $N \models \varphi_M$. Then $N \equiv_{\mathcal{L}_{\infty,\omega}} M$.*

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Definition

An $\mathcal{L}_{\omega_1,\omega}$ sentence is called ***complete*** *if it has a unique countable model.*

We now consider the above questions for complete sentences. If M is a countable structure we say that a cardinal κ is **characterized** by M if $\kappa = \max \operatorname{spec}(\varphi_M)$.

Question

What are the cardinals that can be characterized by countable structures?

Theorem (Malitz assuming GCH, Baumgartner in ZFC)

The $\beth_\alpha$, for $\alpha < \omega_1$, can be characterized by countable structures, i.e. by complete $\mathcal{L}_{\omega_1,\omega}$ sentences.

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Theorem (Knight (1977))

There is a complete $\mathcal{L}_{\omega_1,\omega}$ sentence characterizing $\aleph_1$.

How about the $\aleph_\alpha$, for $\alpha < \omega_1$? This took about 25 years to settle.

Theorem (Hjorth (2002))

For every $\alpha < \omega_1$ there is a complete $\mathcal{L}_{\omega_1,\omega}$ sentence φ_α characterizing $\aleph_\alpha$.

Why was this difficult? We do it by induction. The limit case is no problem, for the successor case we need to start with a bit more at $\aleph_\alpha$ to get a complete sentence characterizing $\aleph_{\alpha+1}$.

Definition

Let $\mathcal{A}$ be a structure in a vocabulary τ which includes a unary predicate P. We say that $P^{\mathcal{A}}$ is ***completely homogenous*** *for $\mathcal{A}$ if it is infinite and totally indiscernible for $\mathcal{A}$, i.e. every permutation π of $P^{\mathcal{A}}$ extends to an automorphism of $\mathcal{A}$.*

In a sense $P^{\mathcal{A}}$ is a **pure set**, the only thing that matters is its cardinality.

Definition

We say that a cardinal κ is ***homogeneously characterized*** *by a complete $\mathcal{L}_{\omega_1,\omega}$ sentence φ if there is a predicate P in the vocabulary of φ such that:*

1. *φ has no model of size $> \kappa$.*
2. *if $\mathcal{A}$ is the unique countable model of φ then $P^{\mathcal{A}}$ is completely homogeneous for $\mathcal{A}$.*
3. *there is a model $\mathcal{A}_\kappa$ of φ such that $P^{\mathcal{A}_\kappa}$ has cardinality κ.*

Proposition

Let κ be a cardinal. Suppose φ is a complete $\mathcal{L}_{\omega_1,\omega}$ sentence that homogeneously characterizes κ. Then there is a complete sentence φ^+ characterizing κ^+.

The vocabulary of φ^+ contains the vocabulary of φ. In addition it will two unary predicates U, V and a binary relation $<$. The sentence φ^+ says that U and V are disjoint, U is a model of φ, call it $\mathcal{U}$, $<$ is a dense linear ordering without endpoints on V and for any proper initial segment of V we have an injection into $P^{\mathcal{U}}$. We need $P^{\mathcal{U}}$ to be complete indiscernible so that the choice of the injections does not affect the resulting $\mathcal{L}_{\omega_1,\omega}$ theory. However, we lose homogeneity by going from φ to φ^+, so we cannot proceed with the induction.

Hjorth's proof is quite involved. Given $\alpha < \omega_1$, he first defines a complete $\mathcal{L}_{\omega_1,\omega}$ sentence $\varphi_{\alpha+1}$ and shows that:

1. $\varphi_{\alpha+1}$ has a homogeneous model of size $\aleph_\alpha$.
2. $\varphi_{\alpha+1}$ dos not have a model of size $> \aleph_{\alpha+1}$.

Now, if $\varphi_{\alpha+1}$ has a model of size $\aleph_{\alpha+1}$ then we are done. Otherwise, we can do the above stepping-up procedure and obtain another complete sentence $\psi_{\alpha+1}$ characterizing $\aleph_{\alpha+1}$.

We now describe an alternative simpler approach. It involves a Fraïssé type construction. We do it for the $\aleph_n$. For the general case $\aleph_\alpha$, for $\alpha < \omega_1$ we need some additional ideas.

Definition

*Let I be an index set. A function $f : [I]^{n+1} \to [I]^n$ is an n-**selector** if $f(A) \subseteq A$, for all $A \in [I]^{n+1}$.*

Definition

Given an index set I we let $\mathbb{P}_I$ be the set of pairs $p = (I_p, f_p)$, where I_p is a finite subset of I and f_p is an n-selector on I_p. We let $p \leq q$ if

1. $I_p \subseteq I_q$,
2. $f_p = f_q \restriction [I_p]^{n+1}$,
3. $f_q^{-1}(T) \subseteq [I_p]^{n+1}$, *for every $T \in [I]^n$.*

Definition

Given a set I and an n-selector f on I we say that f is ***generic*** *if:*

1. f *is locally finite, i.e. for every finite* $J \subseteq I$ *there is finite* $\bar{J}$ *such that* $J \subseteq \bar{J} \subseteq I$ *such that* $(\bar{J}, f \restriction [\bar{J}]^{n+1}) \leq (I, f)$*. We say that* $\bar{J}$ *is* ***closed*** *in* I*.*
2. *for every finite closed* $J \subseteq I$ *and any* n*-selector* (K, g) *such that* K *is finite and* $(J, f \restriction [J]^{n+1}) \leq (K, g)$ *there is* $h : K \to I$ *such that*
   1. $h \restriction J = \mathrm{id}_J$,
   2. $h[K]$ *is closed in* I*,*
   3. $g = f \circ h$*.*

Remark

Being a generic n-selector can be expressed by an $\mathcal{L}_{\omega_1,\omega}$ sentence, say φ_{n-1}.

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Proposition

Any two generic n-selectors are $\mathcal{L}_{\infty,\omega}$-equivalent. In particular, φ_{n-1} is a complete $\mathcal{L}_{\omega_1,\omega}$-sentence.

Proposition

Suppose $n > 0$ and (I, f) is a locally finite n-selector with $\mathrm{card}(I) < \aleph_{n-1}$. Let $J \subseteq I$ be a closed finite set. Let (K, g) be a finite n-selector such that $(J, f \restriction [J]^{n+1}) \leq (K, g)$. Then there is $h : K \to I$ such that

1. $h \restriction J = \mathrm{id}_J$,
2. $h[K]$ *is closed in* I,
3. $g = f \circ h$.

Corollary

There is a generic n-selector on a set I of size $\aleph_{n-1}$.

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The main point is that there is **no** locally finite n-selector on a set of size $\aleph_n$. This follows from the following.

Theorem (Kuratowski)

Suppose $F : [I]^n \to [I]^{<\omega}$ *is a finite set mapping on a set* I *of cardinality* $\aleph_n$*. Then there is a* ***free*** *set* J *of size* $n+1$*. This means:* $x \notin F(J \setminus \{x\})$*, for every* $x \in J$*.*

Indeed, if (I, f) is a locally finite n-selector on a set I of size $\aleph_n$ we can let $F(J) = \{x \in I \setminus J : f(J \cup \{x\}) = J)$, for all $J \in [I]^n$. Then F would violate Kuratowski's free set mapping theorem.

Conclusion:

The sentence φ_k characterizes $\aleph_k$, for all k. $\square$

Closure properties of characterizable cardinals

Definition

- *Let $\mathcal{CH}$ be the set of cardinals that can be characterized by a complete $\mathcal{L}_{\omega_1,\omega}$ sentence.*
- *Let $\mathcal{HCH}$ be the set of cardinals that can be characterized by a complete homogeneous $\mathcal{L}_{\omega_1,\omega}$ sentence.*

What are the closure properties of $\mathcal{CH}$ and $\mathcal{HCH}$?

Theorem (Hjorth)

1. *$\mathcal{CH}$ is closed under successors and limits of countable sequences.*
2. *If $\kappa \in \mathcal{CH}$ then either κ or κ^+ are in $\mathcal{HCH}$.*

Theorem (Baumgartner)

If κ belongs to $\mathcal{HCH}$ then so does 2^κ.

Theorem (Souldatos)

If $\lambda \in \mathcal{CH}$ then $\lambda^\omega \in \mathcal{HCH}$. In particular, $\mathcal{CH}$ and $\mathcal{HCH}$ are closed under countable products.

Outline

Open problems

Question

1. If a cardinal κ is characterizable, is it characterizable by a complete sentence?
2. Is $\mathrm{spec}(\varphi)$ always closed in the order topology?
3. If φ is a complete $\mathcal{L}_{\omega_1,\omega}$ sentence and $\alpha < \omega_1$ is $\aleph_\alpha \in \mathrm{spec}(\varphi)$ absolute?

Thank You!

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