Wadge hierachies versus generalised Wadge hierarchies

Riccardo Camerlo

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The single most important and best studied space from the point of view of Wadge reducibility is Baire space $\mathbb{N}^{\mathbb{N}}$.

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Fact. $A \leq_W^{\mathbb{N}^{\mathbb{N}}} B$ iff player *II* has a winning strategy in $G_W(A, B)$.

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The Wadge duality

Using Wadge games and Martin's Borel determinacy, the structure of $\leq_W^{\mathbb{N}^\mathbb{N}}$ restricted to Borel sets becomes transparent. Most notably, $\leq_W^{\mathbb{N}^\mathbb{N}}$ satisfies the *Wadge duality principle* on Borel subsets: given $A, B \subseteq \mathbb{N}^\mathbb{N}$,

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Remark (ZFC, probably folklore)**.**

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2. this breaks at the level of F_σ and G_δ: if $A \in B(\mathbb{N}^\mathbb{N}) \setminus \mathbf{\Delta}^0_2(\mathbb{N}^\mathbb{N})$ and B is a Bernstein set, then $A \not\leq^{\mathbb{N}^\mathbb{N}}_W B$

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moreover, sets in $\Delta(D_2)$ precede every other set.
Conjecture: The structure is the same as in Baire space up to $\boldsymbol{\Delta}^0_2$ sets; the similarity breaks at the level of F_σ and G_δ sets.

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- P. Schlicht showed that if X is a positive dimensional metric space, then there is $\leq_W^X$ has an antichain of size the continuum, consisting of sets in D_2.

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- It satisfies the Wadge duality principle on Borel subsets of *Borel representable spaces*
- It coincides with $\leq^X_W$ for zero-dimensional spaces

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  - every $\rho^{-1}(\{x\})$ is a G_δ subset of $\mathbb{N}^{\mathbb{N}}$

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Question. (Pequignot 2015) Is there an intrinsic characterisation of relative continuous total relations (i.e. without reference to admissible representations)? Partial results by Brattka, Hertling (1994) and Pauly, Ziegler (2013).

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For $A, B \in \mathcal{P}(X)$, define $A \preceq_{TP}^{X} B$ if there exists an everywhere defined, relatively continuous relation $R \subseteq X^2$ s.t.

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Fact. Let X be second countable, T_0, and let $\rho : Z \to X$ be any admissible representation for X. Then

$$\forall A, B \in \mathcal{P}(X) \; (A \preceq^X_{TP} B \Leftrightarrow \rho^{-1}(A) \leq^Z_W \rho^{-1}(B))$$

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$$\begin{array}{lccccccccc} I & x_0 & & (skip) & & x_1 & & x_2 & & \ldots = x \\ II & & y_0 & & y_1 & & (skip) & & y_2 & \ldots = y \end{array}$$

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so producing sequences $x, y \in \mathbb{N}^{\leq\omega}$. Player *II* wins the run of the game iff

$$x \in A \Leftrightarrow y \in B$$

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Endow $\mathbb{N}^{\leq\omega}$ with the prefix topology.

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Theorem (Duparc, Fournier)

Endow $\mathbb{N}^{\leq\omega}$ with the prefix topology. Then

$$\begin{aligned} \leq_c &\neq \leq_W^{\mathbb{N}^{\leq\omega}} \\ \leq_c &= \preceq_{TP}^{\mathbb{N}^{\leq\omega}} \end{aligned}$$

The questions

Question. (Duparc, Fournier) Is there a topology τ on $\mathbb{N}^{\leq\omega}$ such that $\leq_c = \leq_W^\tau$?

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More general question. Given a second countable, T_0 space $X = (X, \mathcal{T})$, when there is a topology τ on X such that $\preceq_{TP}^{\mathcal{T}} = \leq_W^\tau$?

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Rather unexpectedly — at least to me — the answer seems to depend on an analysis of the separation axioms satisfied by X:

- Hausdorff spaces
- T_1, non-Hausdorff spaces
- non-T_1 spaces

Hausdorff spaces

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Let X be second countable, Hausdorff.
Then $\leq_W^X = \preceq_{TP}^X$ iff X is zero-dimensional.

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- for Borel representable spaces this was already known, by Schlicht's antichain
- if $\leq_W^X = \preceq_{TP}^X$ and X is not Hausdorff — and there are such spaces! — then $dim(X) > 0$

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Example. Let X be a countable space with the cofinite topology. Then $\leq_W^X = \preceq_{TP}^X$.

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As a consequence, $\mathrm{card}(\mathrm{X}) \leq \aleph_0$.

The specialisation order

Given a topological space X define the *specialisation partial order* $\leq$ on X by letting

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Given any partial order $\leq$ on a non-empty set X there is exactly one Alexandrov topology $\mathcal{T}$ on X such that $\leq$ is the specialisation order of $\mathcal{T}$: the open sets of $\mathcal{T}$ are the upward closed sets with respect to $\leq$.

Alexandrov topologies and wqo's

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Let X be endowed with an Alexandrov topology, with $\mathrm{card}(\mathrm{X}) \leq \aleph_0$. *Let* $\leq$ *be the specialisation order on X.*

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- *If there is $n \in \mathbb{N}$ such that all chains in $\leq$ have cardinality less than n, then $\leq_W^X = \preceq_{TP}^X$*

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- *If there is $n \in \mathbb{N}$ such that all chains in $\leq$ have cardinality less than n, then $\leq_W^X = \preceq_{TP}^X$*
- *If both ω and ω^* embed into $(X, \leq)$, then $\leq_W^X \neq \preceq_{TP}^X$*