Recursive and/or iterative refinement for a superfast solver for real symmetric Toeplitz systems based on real trigonometric transformations

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Contents

- The building blocks of the new $O(n \log^2 n)$ algorithm.
- A factor 2 reduction of the computation in the symmetric Toeplitz case.
- A new inversion formula for Chebyshev-Hankel matrices and *Chebyshev-Hankel Bezoutians*.
- Rational interpolation interpretation of the *canonical fundamental system* of a CH-matrix.
- Numerical experiments.

The building blocks of the new $O(n \log^2 n)$ algorithm

1. Transformation ($O(n)$) of the problem $(T + H)x = b$ of dimension n to 2 new problems $Rx = b'$ of half dimension.

2. Computation of the inversion formula of R's in $O(n \log^2 n)$ by means of a superfast rational interpolation solver in the Chebyshev basis.

3. Computation of the final solution by means of fast trigonometric transformations in $O(n \log n)$.

Toeplitz-plus-Hankel centrosymmetric matrices

A Toeplitz-plus-Hankel matrix (TpH) A is a matrix defined as the sum of a Toeplitz and a Hankel matrix:

$$A = [t_{i-j} + h_{i+j}]_{i,j=0}^{n-1}.$$

A is a centrosymmetric TpH matrix if $J_n A J_n = A$, where J_n denotes the $n \times n$ counteridentity.

Remark 1 *A special case of TpH centrosymmetric matrices are the* symmetric Toeplitz *matrices.*

Symmetric-skewsymmetric splitting

Proposition 1 *The centrosymmetric TpH matrix* $A = T + J_n S$ *admits a representation*

$$A = Q_n^T \begin{bmatrix} R_- & 0 \\ 0 & R_+ \end{bmatrix} Q_n.$$

where

$$I_n = Q_n^T Q_n, \qquad Q_{2m} = \begin{bmatrix} -J_m & I_m \\ J_m & I_m \end{bmatrix}.$$

Relation between Symmetric and Skewsymmetric Parts of a Symmetric Toeplitz Matrix

Let us associate with $[x_0, \ldots, x_{n-1}]$ the polynomial $x(t)$, we can define the sets of symmetric and skewsymmetric vectors:

$$\mathbb{F}^n_+ : t^{n-1}x(\frac{1}{t}) = x(t)$$
$$\mathbb{F}^n_- : t^{n-1}x(\frac{1}{t}) = -x(t)$$

Let $T = [a_{|i-j|}]$ a symmetric Toeplitz matrix, n even, and let $T_\pm$ denote the restriction of T to $\mathbb{F}^n_\pm$.

Given a vector x let $x_+ = (x + Jx)/2$ and $x_- = (x - Jx)/2$.

Let W the transformation

$$(Wx_-)(t) = \frac{t+1}{t-1} x_-(t) \quad x_- \in \mathbb{F}^n_-.$$

W is an isomorphism between $\mathbb{F}^n_-$ and $\mathbb{F}^n_+$ and is given by:

$$W = \begin{bmatrix} -1 & & & \\ -2 & -1 & & \\ \vdots & \ddots & -1 & \\ -2 & \dots & -2 & -1 \end{bmatrix}.$$

Theorem 1 *The operators T_+ and T_- are related via*

$$T_+ W - W T_- = 2ea^T (Z - I_n)^{-1},$$

where $a = [a_{n-1}, \dots, a_0]^T, e = [1, 1, \dots, 1]^T$ and Z is the downshift matrix.

So now we can prove the following theorem.

Theorem 2 *Let T be a symmetric Toeplitz matrix. The solution of the system $Tx = b$ can be computed as follows:*

$$Tx_+ = b_+, \qquad Ty_+ = Wb_-, \qquad Tv_+ = e.$$

and

$$x = x_+ + W^{-1}(y_+ + 2cv_+)$$

for some constant c which can be found by one row equation of $T_-x_- = b_-$ where $x_- = W^{-1}(y_+ + 2cv_+)$.

The *Chebyshev-Hankel* matrices

Let the polynomial $f(t)$ be given in its first kind Chebyshev expansion:

$$f(t) = a_0 + 2\sum_{k=1}^{n-1} a_k T_k(t).$$

We consider the operator of multiplication:

$$\mathscr{M}_m(f)x(t) = f(t)x(t)$$

mapping $\mathbb{R}^m[t]$ to $\mathbb{R}^{m+n-1}[t]$.

We evaluate the matrices of $\mathscr{M}_m(f)$ with respect to different Chebyshev bases. Let
$e_k^1(t) = T_k(t), \quad e_k^2(t) = U_k(t), \quad e_k^3(t) = V_k(t), \quad e_k^4(t) = W_k(t)$ and $e_0^1(t) = \frac{1}{\sqrt{2}}$.

Proposition 2 *for* $\nu = 1, 2, 3, 4$, *the matrix* $M_m^\nu(f)$ *of* $\mathscr{M}_m(f)$ *with respect to the basis* $e_k^\nu(t)$ *is given by*

$$M_m^1 = D_{m+n-1}[a_{|i-j|} + a_{i+j}]D_m,$$

$$M_m^2 = [a_{|i-j|} - a_{i+j+2}],$$

$$M_m^3 = [a_{|i-j|} + a_{i+j+1}],$$

$$M_m^4 = [a_{|i-j|} - a_{i+j+1}],$$

for $i = 0, \ldots, m+n-2$, $j = 0, \ldots, m-1$, *where we set* $a_i = 0$ *for* $i \geq n$ *and* D_n *defined as in Proposition 1.*

We introduce $m \times m$ matrices $R_m^\nu(f)(\nu = 1, 2, 3, 4)$ by

$$R_m^\nu(f) = \begin{bmatrix} I_m & 0 \end{bmatrix} M_m^\nu(f).$$

The matrix $R_m^\nu(f)$ will be called *$m \times m$ CH-matrix of the νth kind with the symbol $f(t)$*.

Theorem 3 *Let* $A = T + J_n S$, $T = [t_{i-j}]_{i,j=0}^{n-1}$, $S = [s_{i-j}]_{i,j=0}^{n-1}$ *be a centrosymmetric TpH matrix,* $a_k^{\pm} = t_k \pm s_k$*, and*

$$f_{\pm} = a_0^{\pm} + 2\sum_{k=1}^{n-1} a_k^{\pm} T_k(t).$$

Then the matrices $R_{\pm}$ *of the Proposition 1 are CH-matrices given by*

$$R_{+} = R_m^3(f_{+}), \qquad R_{-} = R_m^4(f_{-}),$$

where $n = 2m$*.*

Inversion formulas for Chebyshev-Hankel matrices and *Chebyshev-Hankel Bezoutians*

Definition 1 *Let $u, v \in \mathbb{R}^{m+1}$. Then the $m \times m$ matrix $B = Bez^{\nu}(u, v)$ with*

$$B^{\nu}(t, s) = \frac{u(t)v(s) - v(t)u(s)}{t - s}$$

will be called CH-Bezoutian of the νth kind *(or* CH-Bezoutian*) of u and v.*

CH-Bezoutians and classical Bezoutians are connected as follows:

$$E^{\nu} Bez^{\nu}(u, v)(E^{\nu})^T = Bez(u, v).$$

We consider a nonsingular $R = R_m^\nu(f)(\nu \in 1, 2, 3, 4)$.

Suppose u', v' are the solution of

$$Ru' = -h, \qquad Rv' = e_m,$$

where e_m is the last unit vector and h the last column of $R_{m+1}^\nu(f)$ after cancelling the last component.

Then form the vectors

$$u = \begin{bmatrix} u' \\ 1 \end{bmatrix}, \qquad v = \begin{bmatrix} v' \\ 0 \end{bmatrix}.$$

We call u, v or $u^\nu(t), v^\nu(t)$ the *canonical fundamental system of* R.

Theorem 4 *Let R be a nonsingular CH-matrix of the νth kind and let u, v be the canonical fundamental system of R. Then*

$$R^{-1} = \frac{1}{2} Bez^{\nu}(u, v).$$

The importance of this theorem for practical calculus consists in the fact that CH-Bezoutians can be represented with the help of trigonometric transformations.

So a vector can be multiplied by a CH-Bezoutian with 6 real transforms of length m plus $O(m)$ operations.

Suppose that u, v are given. Then, 4 transforms of length m are required for preprocessing.

The last problem to solve to obtain a superfast algorithm, is computing the canonical fundamental system in a superfast way.

Rational interpolation interpretation of the *canonical fundamental system* of a CH-matrix

Let $u = (u_k)_{k=0}^{m} (u_m = 1)$ be then a part of the canonical fundamental system of a CH-matrix. In polynomial language

$$P_m^\nu f(t)u(t) = 0. \tag{1}$$

Proposition 3 *All solutions $u(t)$ of equation (1) also satisfy the homogenous interpolation conditions*

$$f(\tau_k)u(\tau_k) + w(\tau_k) = 0 \qquad (k = 0, \ldots, N-1).$$

with $w(t) \in \mathscr{P}_{m,N}^{\nu}$. And viceversa.

The same can be applied for the solution v.

We can rewrite the problem as follows:

$$\begin{bmatrix} f(\tau_k) & 1 \end{bmatrix} B^\nu(\tau_k) = \begin{bmatrix} 0 & 0 \end{bmatrix}, \qquad k = 0, 1, \ldots, N-1. \tag{2}$$

where

$$B^\nu(z) = \begin{bmatrix} \tilde{u}(z) & \tilde{v}(z) \\ w_1(z) & w_2(z) \end{bmatrix} \in R^\nu[z]^{2\times 2}$$

is a 2×2 matrix polynomial (in the Chebyshev base of νth kind). The degree of the first row should be equal to m but $w(t)$ should be in $\mathscr{P}^\nu_{m,N}$.

The problem is keeping the degree structure of $w(t)$.

We show the solution in the even case, the odd case is similar.

Let

$\tau_j = \cos(\frac{(2j+1)\pi}{2N}), (j = 0, \dots, N-1)$ the nodes,

$$V_h(t) = \frac{\sin((h+\frac{1}{2})\theta)}{\sin(\frac{\theta}{2})}, \quad W_k(t) = \frac{\cos((k+\frac{1}{2})\theta)}{\cos(\frac{\theta}{2})}, \quad t = \cos\theta$$

the Chebyshev polynomials of third and fourth kind.

It's easy to show that:

$$V_h(\tau_j) = (-1)^j \frac{\sin(\frac{(2j+1)\pi}{4N})}{\cos(\frac{(2j+1)\pi}{4N})} W_{N-h-1}(\tau_j).$$

So the problem (2) can be rewritten as

$$\begin{bmatrix} f(\tau_k) & (-1)^k \frac{\sin(\frac{(2k+1)\pi}{4N})}{\cos(\frac{(2k+1)\pi}{4N})} \end{bmatrix} B^\nu(\tau_k) = \begin{bmatrix} 0 & 0 \end{bmatrix}, \qquad k = 0, 1, \dots, 2N-1.$$

where the second components of $B^\nu(\tau_k)$ have to be in $\mathscr{P}^\nu_{0,N-m}$.

The iterative refinement

Let $Q = [q_i(t)]$ a basis in the polynomial space.

An interpolation problem can be written as:

$$Vx = y$$

where $V = [q_j(c_i)]$ is a polynomial Vandermonde matrix and $x(t)$ is the coefficient vector in the basis Q. The Lagrange formula can be rewritten with the help of Bezoutian matrices in this way:

$$V^{-1} = BV^T D$$

where $D = diag(u'(c_i))$ and B in the case of chebychev is a "J-upper triangular" Hankel matrix.

In the two dimensional case we have a similar formulation:

$$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} B_u & B_v \\ B_p & B_q \end{bmatrix} \begin{bmatrix} W^T D_{W_1} \\ W^T D_{W_2} \end{bmatrix} \begin{bmatrix} Y \end{bmatrix}$$

where all the computations can be done in $O(n \log n)$.

This reformulation gives a very fast algorithm ($O(n \log n)$) to refine the solution at each intermediate step of the divide and conquer.

But this is not enough to grant stability.

It is necessary, for big dimension to improve the solution using a sort of "bootstrap" technique:

The recursive refinement

Suppose we have solved a problem at a step i. We compute the residuals R_i then we can rewrite a linear homogeous interpolation problem:

$$F_i \tilde{B}(z_i) - R_i I = 0$$

that is

$$\begin{bmatrix} F_i & -R_i \end{bmatrix} \begin{bmatrix} I & \tilde{B}(z_i) \\ 0 & I \end{bmatrix} = 0.$$

This problem strongly refines the residuals.

Unfortunately it's too much expensive. This refinement has to be applied only for the largest sub-problems as shown in the numerical experiments.

Timings without refinements

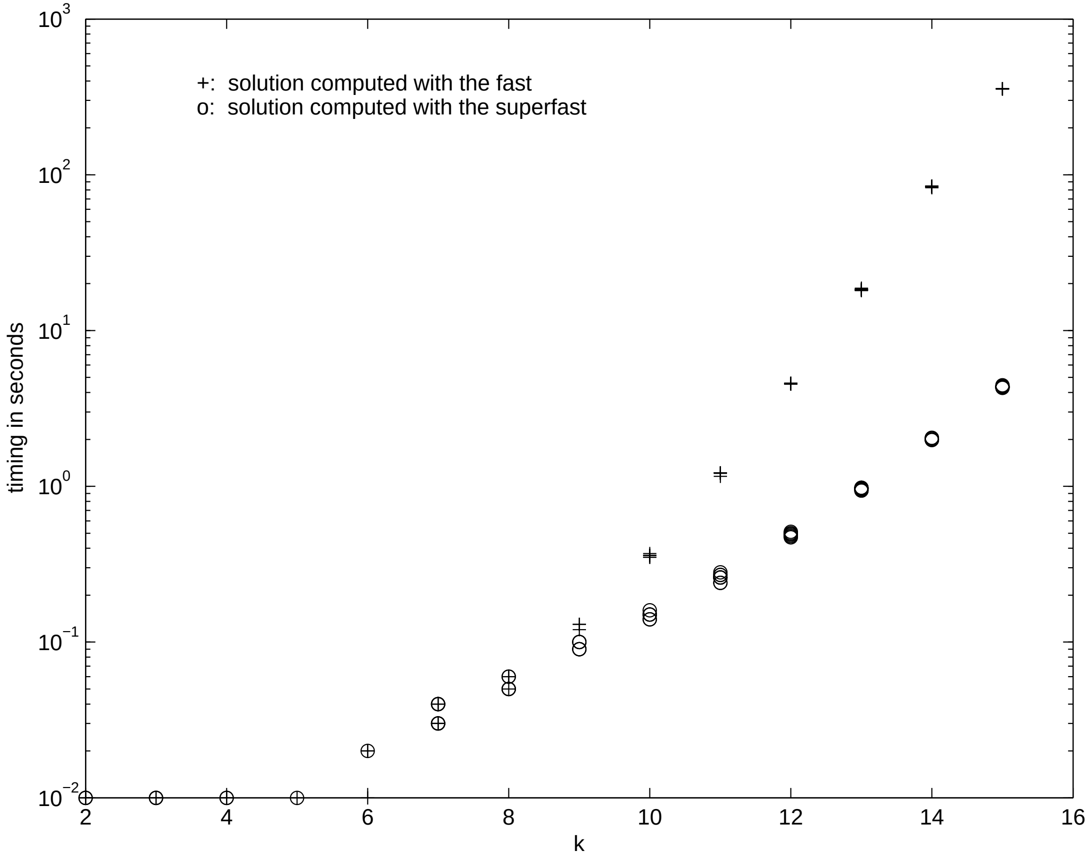

Figure 1: $n = 2^k$

Residuals without refinements

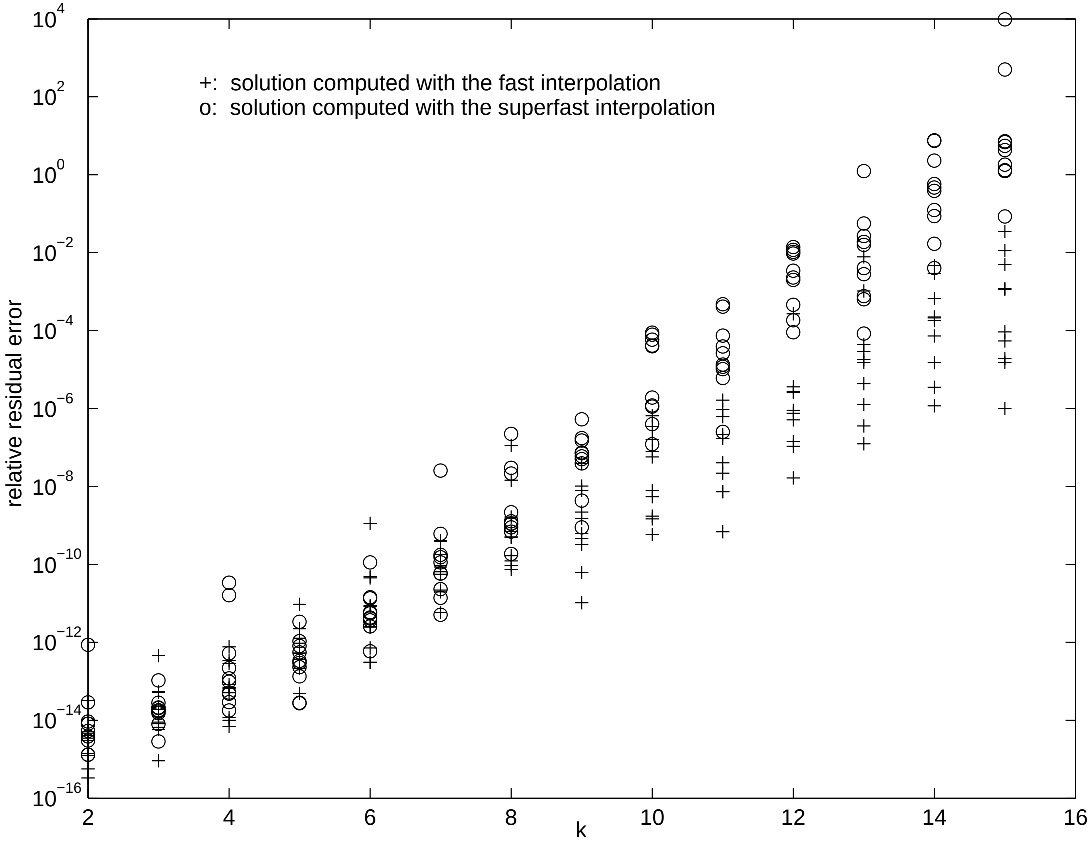

Figure 2: $n = 2^k$

Timings with refinements at EACH LEVEL

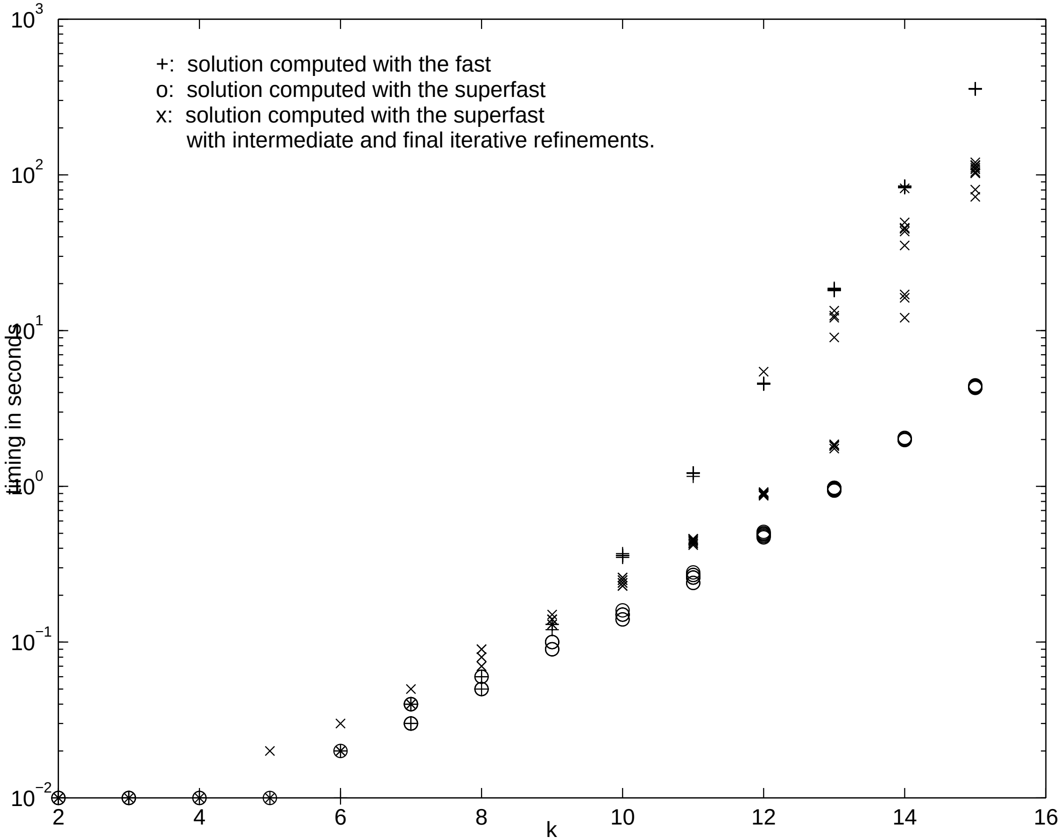

Figure 3: $n = 2^k$

Residuals with refinements at EACH LEVEL

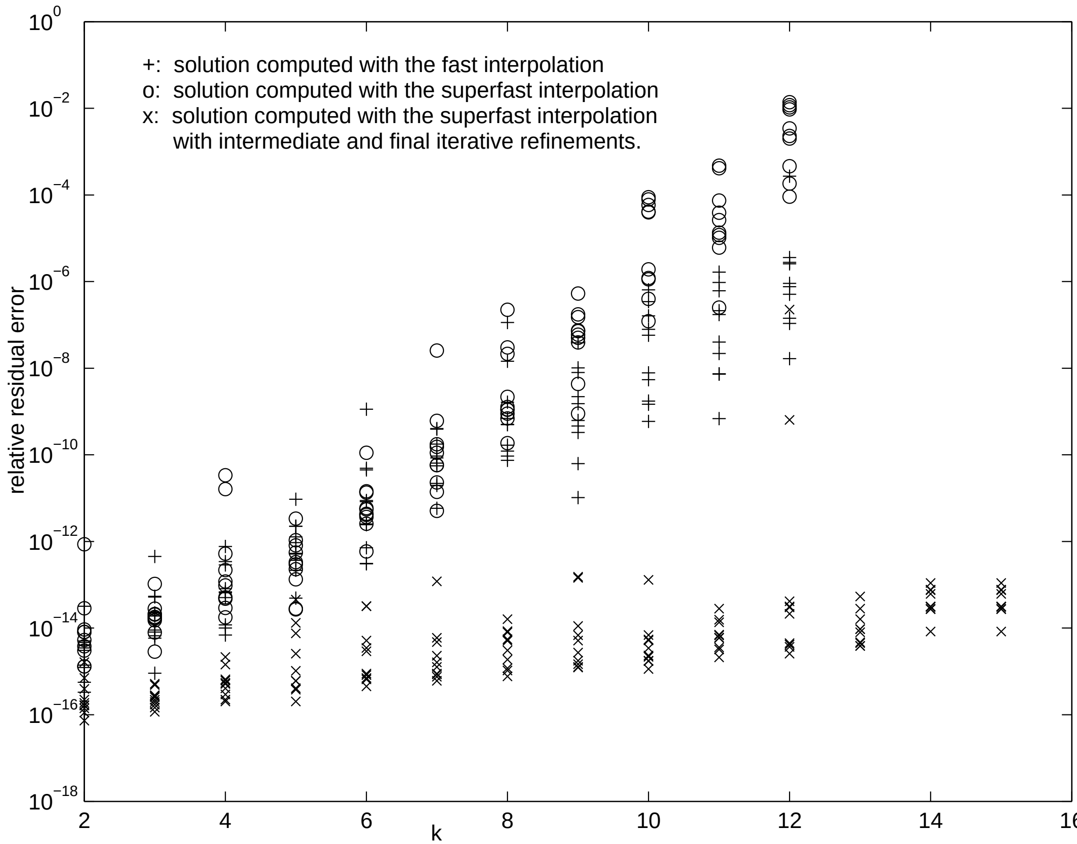

Figure 4: $n = 2^k$

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